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Construction of Hierarchic Families of Finite Elements by Localization of Their Degrees of Freedom

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Hierarchic families of finite elements are widely used in higher order finite-element methods. Several hierarchic sets of basis functions for nodal, edge, and face elements are proposed by different authors in scientific literature. However, no general methodology exists for their construction. This paper presents an analysis of hierarchic families of finite elements which leads to the localization of their degrees of freedom. From this localization, we derive an algebraic approach for the construction of hierarchic families of nodal, edge, and face elements. The case of tetrahedral finite elements is discussed in detail.

***Index Terms*—Basis functions, degrees of freedom, hierarchic families, higher order finite-element methods.**

I. Introduction

THE use of hierarchic families in higher order finite-element methods presents several advantages. First, they reduce the computational cost of the construction of sparse linear systems of equations. Second, they also often lead to the setting of better conditioned problems [13]. Third, they allow the use of block linear solvers, such as hybrid and multilevel methods [9], [14]. Unfortunately, the construction of such families is not an easy task even if several sets of hierarchic basis functions are proposed in the scientific literature. Most of them are built from geometric properties of basis functions [15], [16] or from three-term recurrences satisfied by orthogonal polynomials [1], [13]. However, these approaches are heuristic ones. Beside, some authors proposed to localize degrees of freedom for the construction of hierarchic sets of basis functions [4]–[6], [9]. Unfortunately, they do not provide any methodology for the computation of these degrees of freedom. The work proposed in this paper is directly inspired from this strategy but we derive a simple methodology for the computation of hierarchic families of finite elements from it. The layout of this paper is the following. In Section II, we recall some basic aspects of finite elements and we explain in detail the motivations of our paper. In Section III, an abstract theory of hierarchic families of finite elements is proposed. We apply the results of our analysis to the description of hierarchic families of nodal, edge, and face finite elements on tetrahedra. Finally, we propose a symbolic computing-based approach for the automatic generation of hierarchic families of finite elements.

II. Preliminaries

A. Basic Aspects of Finite Elements

Let $\mathbb{N}_+$ be the set of positive integers, Ω be a bounded domain in $\mathbb{R}^n$, $n \in \mathbb{N}_+$, and $\mathcal{T}$ be a conforming mesh of Ω. A unisolvent finite element [2] is a triple (K, V, Σ) where K is a cell in $\mathcal{T}$, V is a vector space of real-valued functions on K of dimension $m < \infty$, and Σ is a basis of V^* (the dual space of V). The duality product between V and V^* is denoted by $\langle \cdot, \cdot \rangle$. The elements of Σ are called the degrees of freedom of the finite element (d.o.f. for short). We set

$$\Sigma = \{\sigma_1, \ldots, \sigma_m\}. \tag{1}$$

This definition implies the existence of a unique basis

$$\mathcal{B} = \{\phi_1, \ldots, \phi_m\} \tag{2}$$

in V such that

$$\langle \sigma_i, \phi_j \rangle = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

for $1 \leq i, j \leq m$. The elements of $\mathcal{B}$ are called the basis functions of (K, V, Σ). To each function u of a functional space F, one can associate a unique interpolate $\Pi_K(u) \in V$ on K so that

$$\sigma(u) = \sigma\left(\Pi_K(u)\right), \quad \forall \sigma \in \Sigma. \tag{4}$$

This interpolate is defined by

$$\Pi_K(u) = \sum_{i=1}^{m} \sigma_i(u)\phi_i \tag{5}$$

under the following assumptions:

1) $V \subset F$;
2) the definition of the d.o.f. in Σ can be extended to F.

This enables defining a global interpolate Π so that

$$(\Pi(u))_{|K} = \Pi_K(u_{|K}), \quad \forall K \in \mathcal{T} \tag{6}$$

for any function u defined on Ω. This global interpolate is said to be conforming to a functional space H (or H-conforming for short) if, for any two adjacent cells $K_1, K_2 \in \mathcal{T}$, the function defined by $\Pi_{K_1}(u_{|K_1})$ on K_1 and $\Pi_{K_2}(u_{|K_2})$ on K_2 belongs to H. We may give some precisions about the space H. The space of measurable functions $f : \Omega \to \mathbb{R}$ satisfying

$$\int_\Omega |f(x)|^2 \, \mathrm{dx} < \infty \tag{7}$$

is denoted by $L^2(\Omega)$. For $m \in \mathbb{N}_+$, we use the notation $L^2(\Omega)^m$ to denote the space $(L^2(\Omega))^m$. The use of nodal, edge, and face

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finite elements corresponds to different choices for the space H and different continuity properties between adjacent mesh cells. Let Γ be the interface between two adjacent cells K_1 and K_2 in $\mathcal{T}$. The use of nodal finite elements corresponds to the choice

$$H = H^1(\Omega) = \left\{u \in L^2(\Omega) | \mathrm{grad}(u) \in L^2(\Omega)^3\right\}. \quad (8)$$

A function $u \in L^2(K_1) \cup L^2(K_2)$ is in the space $H^1(K_1 \cup K_2 \cup \Gamma)$ if and only if u is the same on each side of Γ. The use of edge finite elements corresponds to the choice

$$H = H(\mathrm{curl}; \Omega) = \left\{u \in L^2(\Omega)^3 | \mathrm{curl}(u) \in L^2(\Omega)^3\right\}. \quad (9)$$

A function $u \in H^1(K_1)^3 \cup H^1(K_2)^3$ is in the space $H(\mathrm{curl}; K_1 \cup K_2 \cup \Gamma)$ if and only if the trace $u \times n$ is the same on each side of Γ where n is normal to Γ and $\times$ denotes the cross product between vectors. Finally, the use of face finite elements corresponds to the choice

$$H = H(\mathrm{div}; \Omega) = \left\{u \in L^2(\Omega)^3 | \mathrm{div}(u) \in L^2(\Omega)\right\}. \quad (10)$$

A function $u \in H^1(K_1)^3 \cup H^1(K_2)^3$ is in the space $H(\mathrm{div}; K_1 \cup K_2 \cup \Gamma)$ if and only if the trace $u \cdot n$ is the same on each side of Γ where "$\cdot$" denotes the Euclidean scalar product.

B. Motivations of This Work

Several families of finite elements in H^1, $H(\mathrm{curl})$ and $H(\mathrm{div})$ exist in the literature [10]–[12] but they are not hierarchic [7]. Several authors propose hierarchic sets of basis functions for different types of cells in various functional spaces. Some works are based on the combination of polynomial and geometrical approaches [15], [16]. Other authors build their basis functions from three-term recurrences satisfied by some orthogonal polynomials [1], [13]. Unfortunately, the genericity of nonhierarchic families is lost and these approaches are only heuristic ones. The main reason for this is that these approaches are often dedicated to projection-based interpolations [3]. In projection-based interpolation, finite elements are not defined as triples (K, V, Σ) but as triples (K, V, Π_K). In other words, the d.o.f. of hierarchic families of finite elements is not described contrary to the case of classical families [10]–[12]. In addition, corresponding basis functions do not always lead to optimally conditioned problems, and a supplementary numerical orthogonalization of some higher order basis functions is often necessary to minimize the condition number of linear systems [17]. Nevertheless, in [4]–[6], some properties of the d.o.f. of hierarchic families of finite elements are derived but no technique for their computation is proposed. This method is also used in [9]. Inspired from [4], our contribution provides a general methodology for the computation of d.o.f. of hierarchic families of finite elements. By identifying higher order d.o.f. to some scalar products, we succeed in describing hierarchic families of finite elements for the standard functional spaces H^1, $H(\mathrm{curl})$ and $H(\mathrm{div})$.

III. Abstract Theory of Hierarchic Families

In this section, we propose localizing the d.o.f. of hierarchic families of finite elements for any kind of conformity. For $p \in \mathbb{N}_+$, we consider a family of unisolvent finite elements

$$\mathcal{F}^{(p)} = ((K, V_k, \Sigma_k))_{1 \le k \le p} \quad (11)$$

satisfying

$$V_k \subset V_{k+1}, 1 \le k < p. \quad (12)$$

We set $m_k = \dim(V_k) = \dim(\Sigma_k)$, $1 \le k \le p$. The corresponding d.o.f. is denoted by

$$\Sigma_k = \left\{\sigma_1^{(k)}, \ldots, \sigma_{m_k}^{(k)}\right\}, \quad 1 \le k \le p. \quad (13)$$

The corresponding sets of basis of functions are

$$\mathcal{B}_k = \left\{\phi_1^{(k)}, \ldots, \phi_{m_k}^{(k)}\right\}, \quad 1 \le k \le p. \quad (14)$$

We assume that the family $\mathcal{F}^{(p)}$ is hierarchic, that is, to say it satisfies

$$\mathcal{B}_k \subset \mathcal{B}_{k+1}, \quad 1 \le k < p. \quad (15)$$

In other words, first elements in $\mathcal{B}_{k+1}$ are the elements of $\mathcal{B}_k$ since

$$\phi_j^{(k+1)} = \phi_j^{(k)}, \quad 1 \le k < p \quad (16)$$

for $j = 1, \ldots, m_k$. We assume that a canonical basis $\{e_1, \ldots, e_{m_p}\}$ of V_p is known. From (12), it is clear that

$$V_k = \mathrm{span}\left\{e_1, \ldots, e_{m_k}\right\}, \quad 1 \le k \le p \quad (17)$$

and any basis function $\phi_j^{(k)} \in \mathcal{B}_k$ can be uniquely written in the form

$$\phi_j^{(k)} = \sum_{l=1}^{m_k} a_{lj} e_l. \quad (18)$$

Let $k < p$. From (3) and (18), we obtain

$$\left\langle \sigma_i^{(k+1)}, e_l \right\rangle = \left\langle \sigma_i^{(k)}, e_l \right\rangle, \quad l = 1, \ldots, m_k \quad (19)$$

for $1 \le i \le m_k$. This is equivalent to

$$\forall p \in V_k, \left\langle \sigma_i^{(k+1)}, p \right\rangle = \left\langle \sigma_i^{(k)}, p \right\rangle \quad (20)$$

for $1 \le i \le m_k$. This implies that the first m_k d.o.f. of Σ_{k+1} are the respective extensions to V_{k+1} of the m_k d.o.f. in Σ_k. Therefore, we only need to describe the last $m_{k+1} - m_k$ d.o.f. in Σ_{k+1}. From (3) and (15), we obtain

$$\left\langle \sigma_i^{(k+1)}, \phi_j^{(k)} \right\rangle = 0 \quad (21)$$

for $i = m_k + 1, \ldots, m_{k+1}$ and $j = 1, \ldots, m_k$. Since $\mathcal{B}_k$ is a basis of V_k, (21) is equivalent to

$$\forall p \in V_k, \quad \left\langle \sigma_i^{(k+1)}, p \right\rangle = 0 \quad (22)$$

$i = m_k + 1, \ldots, m_{k+1}$. Thus, if a family $\mathcal{F}^{(p)}$ of unisolvent finite elements is hierarchic, conditions (20) and (22) are necessarily satisfied. Conversely, it can be proved that conditions (20) and (22) are sufficient for the hierarchy of $\mathcal{F}^{(p)}$. Condition (22) was already mentioned and used for the construction of hierarchic families in [4]–[6], [9]. However, these authors do not mention how to choose d.o.f. and they do not provide any general definition of hierarchic families of finite elements in the spaces H^1, $H(\mathrm{curl})$, and $H(\mathrm{div})$. In the next section, we give a general description of each family by identifying the d.o.f. of classical families of finite element to scalar products.

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IV. Imposing Hierarchy to Existing Families

For any open subset $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}_+$, the scalar product in $L^2(\Omega)^m$, $m \in \mathbb{N}_+$, is defined by

$$(f,g)_{L^2(\Omega)^m} = \int_\Omega f(x)\cdot g(x)\mathrm{dx}, \quad \forall f,g \in L^2(\Omega)^m. \quad (23)$$

In the next subsection, we describe classical families of finite elements by identifying their d.o.f. to such scalar products. They exactly correspond to those described in [10]–[12]. We introduce some notations for this task. Let K be a mesh cell. The d.o.f. of finite elements may be associated with each vertex v, each edge e, each face f of K, or K itself. The length of an edge e is denoted by $\int_e dl$ and the unit vector along e is denoted by τ_e. The area of a face f is denoted by $\int_f ds$, and n_f is the unit outward normal vector on a face f. Finally, the volume of K is denoted by $\int_K \mathrm{dx}$.

A. Classical Families on Tetrahedra

Families of finite elements in the spaces H^1, $H(\mathrm{curl})$, and $H(\mathrm{div})$ can be described in a generic fashion. We only consider some finite elements on tetrahedra but similar considerations can be made for the families proposed in [10]–[12]. Let $K = \mathcal{K}_T$ be a tetrahedron in the mesh $\mathcal{T}$ and $\mathcal{E}_k = (\mathcal{K}_T, V_k, \Sigma_k)$, $1 < k \le p$, be a unisolvent finite element belonging to a family $\mathcal{F}_p$. The choice of the space V_k depends on the choice of the space H. Following [10] and [11], we define several spaces of polynomials. The space of polynomials of a degree less than or equal to k is denoted by P_k, and $\tilde{P}_k$ is the space of homogenenous polynomials of a degree equal to k. We set

$$D_k = (P_{k-1})^3 \oplus \tilde{P}_{k-1}\cdot x \quad (24)$$

and

$$R_k = (P_{k-1})^3 \oplus S_k \quad (25)$$

where

$$S_k = \left\{p \in (\tilde{P}_k)^3 | \forall x \in \mathbb{R}^3, p(x)\cdot x = 0\right\}. \quad (26)$$

Let $p \in V_k$. The d.o.f. of classical families in the spaces H^1, $H(\mathrm{curl})$, and $H(\mathrm{div})$ will be described. They involve some polynomial spaces which are given in Table I. If $\mathcal{E}_k$ is H^1-conforming, then $V_k = P_k$ and the d.o.f. in Σ_k can be expressed in the following form:

$$\left\langle \sigma_v^{(1)}, p\right\rangle = p(v) \quad (27)$$

$$\left\langle \sigma_e^{(k)}, p\right\rangle = (p,q)_{L^2(e)}, q \in \mathcal{N}_k^e \quad (28)$$

$$\left\langle \sigma_f^{(k)}, p\right\rangle = (p,q)_{L^2(f)}, q \in \mathcal{N}_k^f \quad (29)$$

$$\left\langle \sigma_K^{(k)}, p\right\rangle = (p,q)_{L^2(K)}, q \in \mathcal{N}_k^K. \quad (30)$$

If $\mathcal{E}_k$ is $H(\mathrm{curl})$-conforming, then $V_k = R_k$ and the corresponding d.o.f. are

$$\left\langle \sigma_e^{(k)}, p\right\rangle = (p\cdot\tau_e, q)_{L^2(e)}, q \in \mathcal{E}_k^e \quad (31)$$

$$\left\langle \sigma_f^{(k)}, p\right\rangle = (p\times n_f, q)_{L^2(f)^2}, q \in \mathcal{E}_k^f \quad (32)$$

$$\left\langle \sigma_K^{(k)}, p\right\rangle = (p,q)_{L^2(K)^3}, q \in \mathcal{E}_k^K. \quad (33)$$

TABLE I
Classical and New Higher Order d.o.f. for Tetrahedra

	Classical	New	Order
$\mathcal{N}_k^e$	$P_{k-2}(e)$	$(P_{k-1}(e))^\perp \cap P_k(e)$	$k>1$
$\mathcal{N}_k^f$	$P_{k-3}(f)$	$(P_{k-1}(f))^\perp \cap P_k(f)$	$k>2$
$\mathcal{N}_k^K$	$P_{k-4}(K)$	$(P_{k-1}(K))^\perp \cap P_k(K)$	$k>3$
$\mathcal{E}_k^e$	$P_{k-1}(e)$	$(P_{k-1}(e))^\perp \cap P_k(e)$	$k>1$
$\mathcal{E}_k^f$	$(P_{k-2}(f))^2$	$((P_{k-1}(f))^2)^\perp \cap (P_k(f))^2$	$k>2$
$\mathcal{E}_k^K$	$(P_{k-3}(K))^3$	$((P_{k-1}(K))^3)^\perp \cap P_k(K)^3$	$k>3$
F_k^f	$P_{k-1}(f)$	$(P_{k-1}(f))^\perp \cap P_k(f)$	$k>1$
F_k^K	$(P_{k-2}(K))^3$	$((P_{k-1}(K))^3)^\perp \cap (P_k(K))^3$	$k>2$

Finally, if $\mathcal{E}_k$ is $H(\mathrm{div})$-conforming, $V_k = D_k$ and its d.o.f. are

$$\left\langle \sigma_f^{(k)}, p\right\rangle = (p\cdot n_f, q)_{L^2(f)}, q \in \mathcal{F}_k^f \quad (34)$$

$$\left\langle \sigma_K^{(k)}, p\right\rangle = (p,q)_{L^2(K)^3}, q \in \mathcal{F}_k^K. \quad (35)$$

It is well known that these classical families are not hierarchic. For example, the first higher order basis functions described by the d.o.f. (31)–(33) are given explicitly in [7]. In the next subsection, we explain how to impose hierarchy to these families.

B. Hierarchic Families on Tetrahedra

The idea behind our approach is to modify the definition of the classical d.o.f. in order to impose (21). Since we identified these d.o.f. with scalar products in the space L^2, we proceed by orthogonalization to form new d.o.f. First, we consider nodal finite elements. For $k = 1$, the d.o.f. in (27) are the same. For $k > 1$ and any subset U of $\mathbb{R}^m$, we define $((P_{k-1}(U))^m)^\perp$ as the orthogonal space of $(P_{k-1}(U))^m$ in $L_2(U)^m$

$$\left\{q \in L_2(U)^m | \forall p \in (P_{k-1}(U))^m, (p,q)_{L^2(U)^m} = 0\right\}. \quad (36)$$

Let $p \in P_k$. The restriction of p on any edge e (respectively, face f) belongs to $P_k(e)$ (respectively, $P_k(f)$). Thus, choosing $\mathcal{N}_k^e = (P_{k-1}(e))^\perp \cap P_k(e)$, $\mathcal{N}_k^f = (P_{k-1}(f))^\perp \cap P_k(f)$, and $\mathcal{N}_k^K = P_{k-1}(K)^\perp \cap P_k(K)$, we satisfy (3), (20), and (22) at any order $k > 1$: the resulting family is unisolvent, hierarchic and H^1-conforming on $\mathcal{K}_T$. The generalization of this orthogonalization process in the spaces $H(\mathrm{curl})$ and $H(\mathrm{div})$ is obtained by following a similar approach. For example, let us consider the case of finite elements in the space $H(\mathrm{curl})$ and let $p \in R_k$. Of course, the first-order d.o.f. remain unchanged. The restriction of $p\cdot\tau_e$ (respectively, $p\times n_f$) on any edge e (respectively, face f) belongs to $P_k(e)$ (respectively, $(P_k(f))^2$). Thus, we choose $\mathcal{E}_k^e = (P_{k-1}(e))^\perp \cap P_k(e)$ and $\mathcal{E}_k^f = ((P_{k-1}(f))^2)^\perp \cap (P_k(f))^2$, etc. The complete description of our new d.o.f. is given in Table I. The construction process for other mesh cells is exacly the same, except that the definition of the space V_k is different.

V. Symbolic Construction

One difficulty in programming the finite-element method is the complexity of generating higher order basis functions. In this section, we explain how to generate them automatically. The idea is to use the property of hierarchy to allow the symbolic

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computing of d.o.f.: it requires the implementation of an orthogonalization process. Then, the symbolic computing of basis functions is straightforward considering (3) and only requires the implementation of a dense direct linear solver.

A. Degrees of Freedom

The d.o.f. of hierarchic families of finite elements depend on a polynomial q, which belongs to the orthogonal of a polynomial space in a L^2-type scalar product. Thus, computing d.o.f. is equivalent to computing a polynomial q in an orthogonal space. For example, let $\{p_1, \ldots, p_l\}$ be a set of linearly independent polynomials in $P_k(U)^m$, $l > 1$. An orthogonal basis $\{q_1, \ldots, q_l\}$ of $P_k(U)^m$ may be computed according to the scalar product $(\cdot,\cdot)_{L^2(U)^m}$ by applying the methods of Gram–Schmidt or Householder to the set $\{p_1, \ldots, p_l\}$ [8]. Assuming $\{p_1, \ldots, p_j\}$ spans $P_{k-1}(U)^m \subset P_k(U)^m$ and $\{p_{j+1}, \ldots, p_l\}$ spans $P_k(U)^m$ for some $j < l$, it follows that $\{q_{j+1}, \ldots, q_l\}$ forms a basis of $(P_{k-1}(U)^m)^\perp \cap P_k(U)^m$. Thus, q may be computed as a nontrivial linear combination of the elements q_i, $i = j+1, \ldots, l$. Let us remark that families of orthogonal polynomials may also be used directly if some are known.

B. Basis Functions

We use the notations introduced in Section III. Any basis function $\phi_j^{(k+1)} \in V_{k+1}, m_k + 1 \leq j \leq m_{k+1}$, can be uniquely written in the form

$$\phi_j^{(k+1)} = \sum_{l=1}^{m_k} a_{jl} e_l + \sum_{l=m_k+1}^{m_{k+1}} a_{jl} e_l = \varphi_j^{(k+1)} + \psi_j^{(k+1)} \quad (37)$$

where $\varphi_j^{(k+1)} \in V_k$ and $\psi_j^{(k+1)} \in V_{k+1} \setminus V_k$. Considering (3), we obtain that the coefficients appearing in functions $\psi_j^{(k+1)}$ are the unique solution of the block linear system of equations

$$S_1 A_1 = I, \quad (38)$$

where

$$S_1(i,l) = \left\langle \sigma_i^{(k+1)}, e_l \right\rangle \quad (39)$$

for $m_k + 1 \leq i, l \leq m_{k+1}$, $A_1(j,l) = a_{jl}$ for $m_k + 1 \leq j, l \leq m_{k+1}$ and I is the identity matrix. Then, coefficients appearing in the functions $\varphi_j^{(k+1)}$ are computed by solving another block linear system of equations

$$S_2 A_2 = -B \quad (40)$$

where

$$S_2(i,l) = \left(\left\langle \sigma_i^{(k)}, e_l \right\rangle \right) \quad (41)$$

for $1 \leq i, l \leq m_k$, $A_2(l,j) = a_{lj}$ for $1 \leq l \leq m_k$ and $m_k + 1 \leq j \leq m_{k+1}$. The block right-hand side is defined by $B(i,j) = < \sigma_i^{(k+1)}, \psi_j^{(k+1)} >$ for $1 \leq i \leq m_k$ and $m_k + 1 \leq j \leq m_{k+1}$. In practice, basis functions are computed for a small order p. Then, they may be implemented in a finite-element code by copying their respective expressions in a subprogram.

VI. Conclusion

In this paper, we describe hierarchic families of finite elements in a very general framework. Our analysis shows that hierarchy is obtained from a judicious localization of the d.o.f. of classical families of finite elements. As a consequence, the construction of hierarchic families in the spaces H^1, $H(\mathrm{curl})$, and $H(\mathrm{div})$ is simply performed by modifying the d.o.f. of the corresponding nonhierarchic families previously proposed. The localization of d.o.f. proposed in this paper leads to the setting of a symbolic process for the explicit construction of higher order basis functions. This process requires the implementation of an orthogonalization method and a linear system solution only. The characterization of d.o.f. satisfying additional properties (optimal conditioning, exact sequence property, etc.) is under study.

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