



HAL
open science

N=4 Yang–Mills theory as a complexification of the N=2 theory

Laurent Baulieu

► **To cite this version:**

Laurent Baulieu. N=4 Yang–Mills theory as a complexification of the N=2 theory. 2009. hal-00392666

HAL Id: hal-00392666

<https://hal.science/hal-00392666>

Preprint submitted on 8 Jun 2009

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

$\mathcal{N} = 4$ Yang–Mills theory as a complexification of the $\mathcal{N} = 2$ theory

Laurent Baulieu*

Theory Division, CERN, Switzerland[†]

LPTHE, Universités Pierre et Marie Curie, Paris, France[‡] §

Abstract

A complexification of the twisted $\mathcal{N} = 2$ theory allows one to determine the $\mathcal{N} = 4$ Yang–Mills theory in its third twist formulation. The imaginary part of the gauge symmetry is used to eliminate two scalars fields and create gauge covariant longitudinal components for the imaginary part of the gauge field. The latter becomes the vector field of the thirdly twisted $\mathcal{N} = 4$ theory. Eventually, one gets a one to one correspondence between the fields of both theories. Analogous complexifications can be done for topological 2d-gravity and topological sigma models.

*email address: baulieu@lpthe.jussieu.fr

[†] 1211-Geneve 23, Switzerland

[‡] 4 place Jussieu, F-75252 Paris Cedex 05, France.

[§]Based on lectures given in Cargèse, June 16-28, 2008.

1 Introduction

The $\mathcal{N} = 4, d = 4$ super-Yang–Mills theory can be expressed under three different twisted versions [1][2]. The third twisted version was originally found by Marcus [1]. Kapustin and Witten have beautifully interpreted part of the Langland program in terms of its quantum field theory [3].

The information about the $\mathcal{N} = 4$ theory in flat space can be encoded in any one of its twisted or untwisted versions, since these theories are related by linear field transformations. These formulations give different insights for the $\mathcal{N} = 4$ supersymmetry. Their links are most likely preserved by radiative corrections. Here, it will be shown that the formalism for the thirdly twisted $N = 4$ supersymmetric Yang–Mills theory is very analogous to the apparently much simpler one of the twisted $\mathcal{N} = 2$ theory [4][5], modulo a complexification of the latter theory, followed by a gauge-fixing of the imaginary part of its gauge symmetry. From the point of view of the Poincaré supersymmetry, it comes as a surprise that such a simple link exists between the $\mathcal{N} = 2$ and $\mathcal{N} = 4$ cases. This might suggest the existence of a possibly interesting new twisted superfield description of the $N = 4$ theory, with a superspace with $5=4+1$ anticommuting directions, as in [7].

The general idea is as follows. The twisted $\mathcal{N} = 2$ theory expresses the vector supersymmetry multiplet with eight fermionic degrees of freedom as

$$A_\mu, \Psi_\mu, \chi_{\mu\nu^-}, \eta, \Phi, \bar{\Phi} \quad (1)$$

The corresponding quantum field theory and its algebraic structure are now quite well understood, both in twisted superspace and component formalism [6][7]. An advantage of using the twisted formulation is that it isolates an interesting sub-sector of the $\mathcal{N} = 2$ supersymmetry, made of $5 = 1+4$ generators, a scalar and a vector. This fact generalizes itself to the case of extended supersymmetry, and gives an interesting way of exploring its off-shell structure.

As, we will show, the twisted $\mathcal{N} = 2$ theory accommodates quite well a complexification of the fields in (1), as follows

$$A_\mu + iV_\mu, \quad \Psi_\mu + i\tilde{\Psi}_\mu, \quad \chi_{\mu\nu^-} + i\chi_{\mu\nu^+}, \quad \eta + i\tilde{c}, \quad \Phi + i\tilde{\Phi}, \quad \bar{\Phi} + i\tilde{\bar{\Phi}} \quad (2)$$

and it yields the information about the $\mathcal{N} = 4$ theory, expressed in its third twist.

The complexified multiplet (2) is covariant under complex infinitesimal gauge transformations, where the parameters have been also complexified. It permits one to define a "complexified $\mathcal{N} = 2$ supersymmetric gauge theory", which, however, seems quite different than the $\mathcal{N} = 4$ theory. Its field content is made of 16 fermions as in the $\mathcal{N} = 4$

theory, but there are troublesome features: the complex gauge invariance introduces non-unitarity questions; there are 4 scalar fields instead of 6, and one has an imaginary part to the gauge field. However, on-shell, the good point is that the counting sounds right, since, by combining the complex part of the gauge invariance and the on-shell condition, V_μ has two degrees of freedom, which, added to the 4 degrees of freedom carried by the fields $\Phi, \tilde{\Phi}, \bar{\Phi}, \tilde{\bar{\Phi}}$, give a number of 6 on-shell bosonic degrees of freedom that may represent the 6 scalar fields of the standard $\mathcal{N} = 4$ theory. In fact, this gauge theory has not yet been studied in details, as one could attempt by a standard covariant gauge-fixing of the complex gauge field $A + iV$ ¹. Here, we will propose in a rather unorthodox way that one can gauge-fix the imaginary part of the gauge symmetry in a supersymmetric way, which also preserves the real part of the gauge symmetry. In this way $\bar{\Phi}, \tilde{\bar{\Phi}}$ become equal to zero and V^μ becomes “covariantly gauge-fixed”, by addition of a term $(D_A^\mu V_\mu)^2$ in the action, which gives a propagating longitudinal part that is covariant under the real part of the gauge symmetry. Then, V^μ becomes a vector field that only transforms tensorially under the gauge symmetry, and effectively describes 4 degrees of freedom, by mean of an action $V^\mu D_A^2 V_\mu + \text{interacting terms}$. Quite surprisingly, the $\mathcal{N} = 4$ theory is then recovered, in the third twist formulation of Marcus [1]. Moreover, some of the $\mathcal{N} = 4$ transformation laws can be set under a form that might permit an analogous geometrical interpretation as in earlier works about the Donaldson–Witten theory [5].

This presentation is organized as follows. In a first section, there is a heuristic derivation of the thirdly twisted $\mathcal{N} = 4$ supersymmetry and of its relationship with the twisted $\mathcal{N} = 2$ supersymmetry, modulo a complexification. This indirect derivation actually provides many of the final formula, but it may appear as quite formal. In fact, it can be skipped, and one can proceed directly to the next sections, where the construction of the $\mathcal{N} = 4$ theory is explained in a self-contained way, starting from a complexification of the $\mathcal{N} = 2$ theory and of its horizontality equation. In the last section, there are analogous considerations for the topological sigma-model and the topological 2d-gravity.

¹ Notice that [12] has already pointed out the interest of the Chern-Simons theory with a complex gauge invariance.

2 Some remarks about the horizontality conditions for the $\mathcal{N} = 4$ theory

2.1 First twist horizontality conditions

Maximal supersymmetry in 8 dimensions is a useful theory for understanding the various twists of the $\mathcal{N} = 4, d = 4$ theory. It has a twisted formulation [5], which is analogous to that of the $\mathcal{N} = 2, d = 4$ theory, but originates from triality, with twisted scalar and vector generators [7] [10]. Its dimensional reduction indicates that, for the $N = 4$ supersymmetric Yang–Mills theory in euclidean flat space, one has four differential operators $s, \bar{s}, \delta, \bar{\delta}$, which are defined by graded horizontality conditions with Bianchi identities, as follows [6]

$$\begin{aligned} & (d + s + \bar{s} + \delta + \bar{\delta})(A + c + \bar{c} + i_\kappa \gamma_1 + i_{\bar{\kappa}} \bar{\gamma}_1) + (A + c + \bar{c} + i_\kappa \gamma_1 + i_{\bar{\kappa}} \bar{\gamma}_1)^2 \\ & = F + \Psi + \bar{\Psi} + g(\kappa)\eta + g(\bar{\kappa})\bar{\eta} + g(J_I \kappa)\chi^I + g(J_I \bar{\kappa})\bar{\chi}^I + (1 + \kappa \cdot \bar{\kappa})(\Phi + L + \bar{\Phi}) \end{aligned} \quad (3)$$

$$(d_A + s_c + \bar{s}_{\bar{c}} + \delta + \bar{\delta})h^I = d_A h^I + \bar{\chi}^I - \chi^I + i_{J^I \kappa}(\bar{\Psi} - \Psi) \quad (4)$$

$$\begin{aligned} & (d + s + \delta - i_\kappa)(A + c + i_\kappa \gamma_1) + (A + c + i_\kappa \gamma_1)^2 \\ & = F + \Psi + g(\kappa)\eta + g(J_I \kappa)\chi^I + \Phi + |\kappa|^2 \bar{\Phi} \end{aligned} \quad (5)$$

$$(d_A + s + \delta_\gamma - i_\kappa)h^I + [A + c + i_\kappa \gamma_1, h^I] = d_A h^I + \bar{\chi}^I + i_{J^I \kappa} \bar{\Psi} \quad (6)$$

The space is Euclidean. $A = A_\mu dx^\mu$ is the 1-form gauge connection. The anticommuting fields $\Psi_\mu, \bar{\Psi}_\mu, \chi^I, \bar{\chi}^I, \eta$ and $\bar{\eta}$ represent the four Majorana spinors of the $N = 4$ theory in the first twist formulation, for a total of sixteen (off-shell) anticommuting degrees of freedom. The 1-forms $\Psi \equiv \Psi_\mu dx^\mu$, and $\bar{\Psi} \equiv \bar{\Psi}_\mu dx^\mu$ are commuting objects because for instance the anticommuting field Ψ_μ is multiplied by dx^μ . The bosonic scalar fields $\Phi, L, \bar{\Phi}, h^I$ are the first twist transforms of the 6-dimensional R -symmetry multiplet scalar field of the $N = 4$ theory [8][9]. In this twist, the $R \times SO(4)$ symmetry has been reduced to a smaller but large enough subgroup, by taking the diagonal of one $SU(2)$ factor of the $R = SO(5, 1)$ euclidean R -symmetry with one $SU(2)$ factor of the original $SO(4)$ Lorentz symmetry. This redefines a new Lorentz symmetry $SO(4)'$ for the twisted fields, so that the remaining global invariance is a $SO(4)' \times SL(2, R)$ symmetry. The $SL(2, R)$ triplet index $I = 1, 2, 3$ can be identified as a self-dual index of $SO(4)'$, by mean of Kahler invariant forms.

In the above equations, κ and $\bar{\kappa}$ are arbitrarily given constant vectors, each one with 4 commuting components κ^μ and $\bar{\kappa}^\mu$. They can be considered as a set of 8 independent parameters. Here and elsewhere $g(\kappa)$ is the one form $g(\kappa) = g_{\mu\nu}dx^\nu$ and i_κ is the contraction operator along κ , eg, $i_\kappa dx^\mu = \kappa^\mu$.

The graded operator δ and $\bar{\delta}$ are in correspondence with vector anticommuting differential operators δ_μ and $\bar{\delta}_\mu$, with $\delta \equiv \kappa^\mu \delta_\mu$ and $\bar{\delta} \equiv \bar{\kappa}^\mu \bar{\delta}_\mu$. The transformation laws of all fields under the action of $s, \bar{s}, \delta, \bar{\delta}$ is obtained by decomposition of the above equations on all possible polynomials in κ and $\bar{\kappa}$.

The anticommuting scalar fields c, \bar{c} are called scalar shadows and γ_1 and $\bar{\gamma}_1$ are commuting 1-form shadows. The scalars c, \bar{c} and $i_\kappa \gamma, i_\kappa \bar{\gamma}$ must not be confused with Faddeev–Popov ghosts. They have very different transformation laws, and they play very different roles : they are used for keeping track of supersymmetry at the quantum level [7]. In fact, the introduction of shadows allows one to have no gauge transformations occurring in the squared of the differentials $s, \bar{s}, \delta, \bar{\delta}$.

As shown in [6], it is only for $\kappa = \bar{\kappa}$ that no equation of motion occurs in the commutators of these differential operators. This can be verified by solving the equations and computing the action of $s, \bar{s}, \delta, \bar{\delta}$ on all fields. With this restriction $\kappa = \bar{\kappa}$, the above horizontality condition determine six generators of the $\mathcal{N} = 4$ theory, made of two scalars and one vector, under the form of differential operators.. It was shown that the invariance under this twisted supersymmetry with 6 generators unambiguously determines the $\mathcal{N} = 4$ action.

The equation (4) for the field h^I seems however to escape any kind of a geometrical interpretation, contrarily to that satisfied by the field A in (3), which is a curvature equation. This motivates a further change of variable for getting much better looking equations, the result being that one must switch from the first twist of [9] to the third one, which was found in [1] ².

2.2 Heuristic switch to the third twist horizontality conditions

To proceed, we have to do a slight digression, coming back to the untwisted formalism. To make a bridge between twisted and untwisted fields, one can first define the N=4 su-

²Let us recall that, from a purely 4-dimensional point of view, a systematic analysis of all possible way of extracting $SU(2)$ factors from the $R = SO(5, 1)$ symmetry of the $\mathcal{N} = 4$ theory allows one to find all its possible independent twists.

persymmetry “almost-differential” operator \mathcal{Q} for the untwisted fields, as follows [6]

$$(d + \mathcal{Q} - i_{\bar{\epsilon}\gamma^\mu\epsilon})(A + c) + (A + c)^2 = F + \delta^{Susy} A + \bar{\epsilon}[\phi]\epsilon \quad (7)$$

Here, there is a single scalar shadow field c , and the generic parameter ϵ of supersymmetry appears explicitly. This equation defines a \mathcal{Q} -transformation of fields in function of the scalar shadow c and the supersymmetry commuting parameter ϵ . The later is made of 4 Majorana spinors, that count for a total of 16 real parameters, and $\mathcal{Q}\epsilon = 0$. In fact, Eq. (7) shows that the \mathcal{Q} transformation of a classical field can be identified as a supersymmetry transformation $\delta^{Susy}(\epsilon)$ minus a compensating gauge transformation $\delta^{\text{gauge}}(c)$, with a local anticommuting parameter equal to the shadow field. The four gluinos determine a $SL(2, \mathbb{H})$ -Majorana spinor, as well as ϵ . The six scalar fields build a real antisymmetric representation of $SL(2, \mathbb{H})$ whose Lie algebra is isomorphic to $SO(5, 1)$.

Thank’s to the shadow dependence, no gauge transformation occurs in the expression of \mathcal{Q}^2 . However for a general ϵ , \mathcal{Q}^2 closes on translations only modulo some equations of motion of a \mathcal{Q} -invariant lagrangian. To get rid of equations of motion in the closure relations, one must restrict the supersymmetry parameters. This can be done consistently when one twists the fields, since this operation allows one to lower the size of the symmetry. In fact, the twisted representations for the supersymmetry parameters (and more generally all $SO(5, 1)$ -Majorana spinor-tensors) become reducible, and can be separated in shorter multiplets. It becomes possible to consistently reduce the size of the $\mathcal{N} = 4$ supersymmetry, by projecting ϵ in these reduced representations, while retaining some of its non-trivial features.

As shown in [1] there are in fact three possible twists in the $\mathcal{N} = 4$ superPoincaré theory, due the three possibilities of selecting a $SU(2)$ subgroup in the Euclidian $SO(5, 1)$ R -symmetry, and then taking a diagonal of this $SU(2)$ with one $SU(2)$ factor of the Lorenz $SO(4)$ symmetry [1]. When one compares the fields of the $\mathcal{N} = 4$ theory in its four possible formulations (untwisted, first twisted, second twisted or third twisted formulation), they are thus related by linear mappings, which, basically, map spinor indices onto Lorentz ones, using the algebra of Pauli matrices. We now show directly how the third twist can be obtained form the first one.

The first twist is that we used for writing the horizontality conditions in the precedent section, and that we justified by straightforward dimensional reduction from the 8-dimensional theory. It is

$$\lambda \rightarrow (\Psi_\mu, \bar{\Psi}_\mu, \chi_{\mu\nu^-}, \bar{\chi}_{\mu\nu^-}, \eta, \bar{\eta}) \quad (8)$$

$$\phi \rightarrow (\Phi, \bar{\Phi}, L, h_{\mu\nu^-}) \quad (9)$$

The 16 generators of the susy algebra which compose \mathcal{Q} and their parameters ϵ are respectively twisted into $Q_0, \bar{Q}_0, Q_\mu, \bar{Q}_\mu, Q_{\mu\nu^-}, \bar{Q}_{\mu\nu^-}$ and

$$\epsilon \rightarrow (\epsilon_0, \bar{\epsilon}_0, \epsilon_\mu, \bar{\epsilon}_\mu, \epsilon_{\mu\nu^-}, \bar{\epsilon}_{\mu\nu^-}) \quad (10)$$

where we have identified the index I with a self dual index $\mu\nu^-$. To suppress all equations of motion in \mathcal{Q}^2 , one must have a symmetry with not more than 9 generators [6]. To define the third twist from the first one, we can heuristically choose a restricted 7-dimensional family of parameters, made of the 4 scalars u, \bar{u}, v, \bar{v} and a vector κ , where κ^μ is of norm 1. It reads

$$\epsilon_0 = u \quad \bar{\epsilon}_0 = \bar{u} \quad \epsilon_\mu = v\kappa_\mu, \quad \bar{\epsilon}_\mu = \bar{v}\kappa_\mu \quad (11)$$

and $\epsilon_{\mu\nu^-} = \bar{\epsilon}_{\mu\nu^-} = 0$. One has a nilpotent differential $d + \mathcal{Q} - i_{(u\bar{v}+v\bar{u})\kappa}$. The expression for the action of \mathcal{Q} on A and c is obtained by decomposing the following graded equation in form degree

$$\begin{aligned} \mathcal{F} \equiv & (d + \mathcal{Q} - i_{(u\bar{v}+v\bar{u})\kappa})(A + c) + (A + c)^2 = F \\ & + u\Psi + \bar{u}\bar{\Psi} + \bar{v}(g(\kappa)\eta + i_\kappa\chi_-) + v(g(\kappa)\bar{\eta} + i_\kappa\bar{\chi}_-) \\ & + (u^2 + v^2)\Phi + (\bar{u}^2 + \bar{v}^2)\bar{\Phi} + (u\bar{u} + v\bar{v})L \end{aligned} \quad (12)$$

The action of \mathcal{Q} on all fields in the right hand side of Eq. (12) is obtained by expanding in form degree the Bianchi identity $(d + \mathcal{Q} - i_{(u\bar{v}+v\bar{u})\kappa})\mathcal{F} = -[A, \mathcal{F}]$. One has on all fields

$$\mathcal{Q}^2 = \mathcal{L}_{(u\bar{v}+v\bar{u})\kappa} \quad (13)$$

If we put $\bar{u} = \bar{v} = 0$, we have a symmetry with a family of 5 =2+3 parameters, namely u, v and κ . Then, the restricted differential operator \mathcal{Q} is nilpotent

$$\mathcal{Q}^2 = 0 \quad (14)$$

The \mathcal{Q} -transformations are however still κ -dependent, according to

$$(d + \mathcal{Q})(A + c) + (A + c)^2 = F + u\Psi + v(g(\kappa)\bar{\eta} + i_\kappa\bar{\chi}_-) + (u^2 + v^2)\Phi \quad (15)$$

Notice that that, by this restriction on the parameters, the fields χ_- and η have disappeared from equations, so that their transformation laws are now unconstrained by the curvature condition for $A + c$.

We now redefine

$$\tilde{\Psi} \equiv g(\kappa)\bar{\eta} + i_\kappa\bar{\chi}_- \quad (16)$$

The troublesome dependence on κ can now be forgotten, by considering $\tilde{\Psi}$ as an independent new 1-form field, $(\bar{\eta}, \bar{\chi}_-) \rightarrow \tilde{\Psi}_\mu$, so that the curvature condition becomes

$$(d + \mathcal{Q})(A + c) + (A + c)^2 = F + u\Psi + v\tilde{\Psi} + (u^2 + v^2)\Phi \quad (17)$$

Since it now depends only on scalar parameters, the last equation can be extended in curved space. \mathcal{Q} has become a true differential operator, which transforms fields into expressions depending on both scalar real parameters u and v . The intriguing fact, which we shall shortly exploit, is the analogy between Eq. (17) and the geometrical horizontality condition of the $\mathcal{N} = 2$ theory (which is exactly reproduced for $u = 1, v = 0$, as in [5]).

There is another remarkable feature. If one looks at the transformation laws of h_- and L , as obtained from Eq. (4) and its Bianchi identity Eq. (3), and if one defines the following one-form

$$V \equiv g(\kappa)L + i_\kappa h_- \quad (18)$$

we get another κ -independent equation satisfied by V , as follows

$$(d_A + \mathcal{Q})V + [A + c, V] = d_A - v\Psi + u\tilde{\Psi} \quad (19)$$

By combining Eqs. (17) and (19), we get the following complex equation, where, for the moment, u and v , as well as all fields, must be considered as real

$$(d + \mathcal{Q})(A + iV + c) + (A + iV + c)^2 = F_{A+iV} + (u - iv)(\Psi + i\tilde{\Psi}) + (u^2 + v^2)\Phi \quad (20)$$

By analogy with Eq. (18), we redefine the 1-form $\bar{\Psi}$ as a an antiself-dual 2-form $\chi_{\mu\nu+}$ and a scalar form χ as follows

$$\begin{aligned} \chi &\equiv i_\kappa \bar{\Psi} \\ \chi_{\mu\nu+} &\equiv \kappa_{[\mu} \bar{\Psi}_{\nu]+} \end{aligned} \quad (21)$$

These changes of variables have transformed the fields $\Psi_\mu, \bar{\Psi}_\mu, \chi_{\mu\nu-}, \eta, \bar{\chi}_{\mu\nu-}, \bar{\eta}, \Phi, \bar{\Phi}, L, h_{\mu\nu-}$ of the first twist into the set of fields

$$A_\mu, V_\mu, \Psi_\mu, \tilde{\Psi}_\mu, \chi_{\mu\nu-}, \chi_{\mu\nu+}, \chi, \eta, \bar{\Phi}, \Phi \quad (22)$$

which is nothing but the set of fields that Marcus found in its direct third twist of the $\mathcal{N} = 4$ formulation. Here, this set has been found by the requirement of getting a geometrically meaningful curvature equation involving all fields of the $\mathcal{N} = 4$ theory.

This construction is quite suggestive. Moreover, the complex horizontality condition (20) indicates a closer relationship between the $N = 2$ and $\mathcal{N} = 4$ theories. We will find that the $\mathcal{N} = 4$ theory, expressed in its third twist formulation is a mere complexification of the $\mathcal{N} = 2$ theory, associated to a suitable supersymmetric gauge-fixing of its imaginary part.

3 Complexification of the $\mathcal{N} = 2$ Yang–Mills supersymmetry

3.1 Complex gauge invariance with twisted $\mathcal{N} = 2$ supersymmetry

At this point, it is best to reconsider the problem, and come to the point of this presentation, by starting with a complex gauge field $A + iV$, within the context of $\mathcal{N} = 2$ supersymmetry. We therefore enter the not so familiar domain of a gauge theory where the infinitesimal parameter is complexified $\epsilon \rightarrow \epsilon + i\tilde{\epsilon}$. The Fadeev-Popov ghost is then complexified, $\Omega \rightarrow \Omega + i\tilde{\Omega}$, as well as the antighost and lagrange multiplier field, and one has the “complex” BRST symmetry

$$\begin{aligned} s(A + iV) &= -d(\Omega + i\tilde{\Omega}) - [A + iV, \Omega + i\tilde{\Omega}] \\ s(\Omega + i\tilde{\Omega}) &= -\frac{1}{2}[\Omega + i\tilde{\Omega}, \Omega + i\tilde{\Omega}] \end{aligned} \quad (23)$$

The symmetry equations have real and imaginary parts that must be considered independently, so that $sA = -d_A\Omega + [\tilde{\Omega}, V]$ and $sV = -d_A\tilde{\Omega} - [\Omega, V]$. The complex curvature has real part $dA + AA - VV$ and imaginary part d_AV . For $\tilde{\Omega} = 0$, one has the ordinary “real” BRST symmetry, for which V only transforms tensorially.

A complex gauge theory is problematic at the classical level. Indeed the Yang–Mills term $F_{A+iV}^* F_{A+iV}$ is invariant under the complexified symmetry, but it is complex. In this action, both quadratic parts for A and V are however transverse, and need gauge-fixing. On the other hand the real action $F_{A+iV}^* F_{A-iV}$ is only s -invariant when $\tilde{\Omega} = 0$. This action has purely transverse propagators too, both for A and V . We will shortly see that, within the context of the $\mathcal{N} = 2$ supersymmetrization of the complex gauge theory, the condition $\tilde{\Omega} = 0$ can be obtained from a gauge-fixing that eliminates some of the fields, in a supersymmetric way.

As in the case of the standard $\mathcal{N} = 2$ supersymmetric theory, one may try to construct five generators made of a scalar and a vector differential operator Q and Q_μ , with $Q_\kappa \equiv \kappa^\mu Q_\mu$. Here κ is a commuting vector field with 4 real components κ^μ . Q is made of two independent scalar generators Q_1 and Q_2 , with $Q = uQ_1 + vQ_2$, where u and v are two independent parameters. Eventually, these parameters should be interpreted as those that one can conventionally obtain by twisting the Poincaré $\mathcal{N} = 4$ supersymmetry. But, here, their introduction is motivated from a another point of view. Complexified scalar

and vector shadows must be defined,

$$c \rightarrow c + i\tilde{c} \quad \gamma_{1\mu} dx^\mu \rightarrow (\gamma_{1\mu} + i\tilde{\gamma}_{1\mu}) dx^\mu \quad (24)$$

One has the definition

$$\mathcal{A} \equiv A + iV + (u + iv)(c + i\tilde{c}) + i_\kappa(\gamma_1 + i\tilde{\gamma}_1) + \Omega + i\tilde{\Omega} \quad (25)$$

where $\Omega + i\tilde{\Omega}$ is the complex Faddeev–Popov ghost.

The goal is is of building Q and Q_μ , with

$$Q^2 = Q_\mu Q_\nu + Q_\nu Q_\mu = 0 \quad QQ_\mu + Q_\mu Q \sim \partial_\mu \quad (26)$$

on all complexified fields. Here will reduce ourselves to the definition of the scalar operator Q , whose action involves the two parameters u and v .

Thus, from now on we set $\kappa = 0$, which amounts to eliminate the dependence in the vector ghosts, and we have the restricted definition

$$\mathcal{A} \equiv A + iV + (u + iv)(c + i\tilde{c}) + \Omega + i\tilde{\Omega} \quad (27)$$

The matter fields of the twisted $\mathcal{N} = 2$ theory are extended as follows

$$\Psi \rightarrow \Psi + i\tilde{\Psi} \quad \Phi \rightarrow \Phi + i\tilde{\Phi} \quad \bar{\Phi} \rightarrow \bar{\Phi} + i\tilde{\bar{\Phi}} \quad \eta \rightarrow \eta + i\tilde{\eta} \quad \chi_- \rightarrow \chi_- + i\chi_+ \quad (28)$$

The standard horizontality condition [5] of the $\mathcal{N} = 2$ theory is “naturally” complexified as follows

$$(d + s + Q)\mathcal{A} + \frac{1}{2}[\mathcal{A}, \mathcal{A}] = F_{A+iV} + (u - iv)(\Psi + i\tilde{\Psi}) + (u^2 + v^2)(\Phi + i\tilde{\Phi}) \quad (29)$$

with the Bianchi identity

$$(d + s + Q + [\mathcal{A}, \])(F_{A+iV} + (u - iv)(\Psi + i\tilde{\Psi}) + (u^2 + v^2)(\Phi + i\tilde{\Phi})) = 0 \quad (30)$$

These equations are completely analogous to those of the ordinary case of the $\mathcal{N} = 2$ theory. They determine the action of both s and Q , by expansion in form degree, so that nilpotency is warranted because of the Bianchi identities for all values of u and v . In this complexified case one has to further split the equations in their real and imaginary parts (assuming for instance that the Lie algebra matrices are real, as well as u and v). This yields the differential operator Q whose action on the classical fields A, V, Ψ can be

identified as the sum of two scalar supersymmetries with parameters u and v and gauge transformations with parameters equal to the shadows c and \tilde{c} .

Following the method of [10], one introduces the BRST-partners of the shadows, to make the later ones parts of a BRST-exact doublets. This permits one to determine the action of the BRST symmetry operator s on the shadows and that of Q on the ghosts. One has also a complex BRST-exact doublet for the anti-ghost sector, which is made of $\bar{\Omega} + i\tilde{\Omega}$ and $H + i\tilde{H}$, with

$$s(\bar{\Omega} + i\tilde{\Omega}) = H + i\tilde{H} - [\Omega + i\tilde{\Omega}, \bar{\Omega} + i\tilde{\Omega}] \quad s(H + i\tilde{H}) = -[\Omega + i\tilde{\Omega}, H + i\tilde{H}] \quad (31)$$

The BRST exact doublets in the shadow sector are $\bar{c} + i\tilde{c}$, $\mu + i\tilde{\mu}$ and $\bar{\mu} + i\tilde{\mu}$, with

$$\begin{aligned} s(c + i\tilde{c}) &= \mu + i\tilde{\mu}, & s(\mu + i\tilde{\mu}) &= 0 \\ s(\bar{\mu} + i\tilde{\mu}) &= \bar{c} + i\tilde{c} & s(\bar{c} + i\tilde{c}) &= 0 \end{aligned} \quad (32)$$

It is useful to list the following Q -transformations of the fields, as given by the horizontality equations.

$$\begin{aligned} Q(A + iV) &= (u - iv)(\Psi + i\tilde{\Psi}) - (u - iv)d_{A+iV}(c + i\tilde{c}) \\ Q(c + i\tilde{c}) &= (u + iv)(\Phi + i\tilde{\Phi}) - (u - iv)(c + i\tilde{c})^2 \\ Q(\Psi + i\tilde{\Psi}) &= (u + iv)d_{A+iV}(\Phi + i\tilde{\Phi}) - (u - iv)[c + i\tilde{c}, \Psi + i\tilde{\Psi}] \\ Q(\Phi + i\tilde{\Phi}) &= -(u - iv)[c + i\tilde{c}, \Phi + i\tilde{\Phi}] \end{aligned} \quad (33)$$

$$\begin{aligned} Q_{\tilde{c}}(\Omega + i\tilde{\Omega}) &= -(u - iv)(\mu + i\tilde{\mu}) & Q_{\tilde{c}}(\mu + i\tilde{\mu}) &= -(u + iv)[\Phi, \Omega + i\tilde{\Omega}] \\ Q_{\tilde{c}}(\bar{\mu} + i\tilde{\mu}) &= (u - iv)(\bar{\Omega} + i\tilde{\Omega}) & Q_{\tilde{c}}(\bar{\Omega} + i\tilde{\Omega}) &= (u + iv)[\Phi, \bar{\mu} + i\tilde{\mu}] \\ Q_{\tilde{c}}(\bar{c} + i\tilde{c}) &= -(u - iv)(H + i\tilde{H}) & Q_{\tilde{c}}(H + i\tilde{H}) &= -(u + iv)[\Phi, \bar{c} + i\tilde{c}] \end{aligned} \quad (34)$$

where $Q_{\tilde{c}} \equiv Q - (u - iv)[c + i\tilde{c}, \]$, and thus $Q_{\tilde{c}}^2 = (u^2 + v^2)[\Phi + i\tilde{\Phi}, \]$. These equations must be decomposed in real and imaginary parts, which yields the action of Q_1 and Q_2 on all fields from the u and v dependence of the Q transformations.

In view of a the further determination of a Q invariant action, we need two introduce pairs of complex self-dual and antiself-dual anticommuting 2-forms χ_- and χ_+ with Lagrange multipliers H_{\pm} , as well a complex commuting scalar fields $\bar{\Phi} + i\tilde{\Phi}$ with fermionic lagrange multipliers $\eta + i\tilde{\eta}$, such that

$$\begin{aligned}
Q_{\tilde{c}}(\bar{\Phi} + i\tilde{\Phi}) &= (u - iv)(\eta + i\tilde{\eta}) & Q_{\tilde{c}}(\eta + i\tilde{\eta}) &= (u + iv)[\Phi, \bar{\Phi} + i\tilde{\Phi}] \\
Q_{\tilde{c}}\chi_{\pm} &= (u - iv)H_{\pm} & Q_{\tilde{c}}H_{\pm} &= (u + iv)[\Phi + i\tilde{\Phi}, \chi_{\pm}]
\end{aligned} \tag{35}$$

The introduction of the fields χ_{\pm} and $\bar{\Phi} + i\tilde{\Phi}$ would be more natural within the context of the vector symmetry transformations that we do not discuss here ³.

4 Equivariant gauge-fixing with complexified self-duality equation for $A + iV$

The full gauge symmetry includes imaginary and real parts, and we might look for the obtention of a Q -invariant action in the cohomology of the complex BRST symmetry. However, by doing, and using a Q -exact action, one gets an action of the type $\int(F_{A+iV} \wedge^* F_{A+iV} + \text{supersymmetric terms})$, which is interesting per se, but doesn't reach our goal of getting a unitary theory such as the $\mathcal{N} = 4$ theory. Rather, we will look for the obtention of a Q -invariant action in the cohomology of the real part of the BRST symmetry. This means that must take the condition $\tilde{c} = 0$ for the definition of Q as well as $\tilde{\Omega} = 0$ for that of s . In the next section, we will show that the restriction of the BRST symmetry to its real part can be done by a gauge-fixing that uses a Q -invariant action.

One wishes a Q -exact action whose u, v dependence is only an overall factor of $u^2 + v^2$, modulo boundary terms that can depend on u and v , at least after the elimination of auxiliary fields, as in [3]. This ensures the invariance of the action under both Q_1 and Q_2 .

The following action, which turns out to use complex self-dual conditions as a Q -

³There is no reality condition on χ_+ and χ_- , as well as on H_+ and H_- , but each one of these fields counts for 3 degrees of freedom, as a self-dual or antiself-dual field.

antecedent, satisfies such requests :

$$\begin{aligned}
I &= Re Q \int \text{Tr} \left((\chi_- - i\chi_+) \wedge (\mathcal{F} + \star \bar{\mathcal{F}} - \frac{u-iv}{2}(H_- + iH_+)) \right) \\
&= Q \int \text{Tr} \left(\chi_- \star (u(F_A - VV) - vd_AV)_- - \frac{u-iv}{2}H_- \right. \\
&\quad \left. + \chi_+ \star (v(F_A - VV) + ud_AV)_+ - \frac{u-iv}{2}H_+ \right) \\
&= \int \text{Tr} \left(-\frac{1}{2}|H|^2 + H_- \star (u(F_A - VV) - vd_AV)_- + H_+ \star (v(F_A - VV) + ud_AV)_+ \right. \\
&\quad \left. (u^2 + v^2)(\chi_- \star (d_A\Psi - [V, \tilde{\Psi}]) + \chi_+ \star (d_A\tilde{\Psi} + [V, \Psi]) - \Phi|\chi|^2) \right) \quad (36)
\end{aligned}$$

where $|\chi|^2 = \frac{1}{2}[\chi_+, \star\chi_+] + \frac{1}{2}[\chi_-, \star\chi_-]$. We have redefined H_{\pm} in such way that $Q\chi_{\pm} = (u-iv)H_{\pm}, QH_{\pm} = 0$.

We have introduced

$$\mathcal{F} \equiv (u+iv)F_{A+iv} \quad (37)$$

and one has

$$Q\mathcal{F} = (u^2 + v^2)(d_{A+iv}(\Psi + i\tilde{\Psi}) - [c + i\tilde{c}, \mathcal{F}]) \quad (38)$$

with $s\mathcal{F} = -[\Omega + i\tilde{\Omega}, \mathcal{F}]$.

When one integrates over H_{\pm} , $(u^2 + v^2)$ factorizes and one gets,

$$\begin{aligned}
I &= (u^2 + v^2) \int \text{Tr} \left(\frac{1}{4}(F_A - VV + id_AV) \star \overline{(F_A - VV + id_AV)} \right. \\
&\quad \left. + \chi_- \star (d_A\Psi - [V, \tilde{\Psi}]) + \chi_+ \star (d_A\tilde{\Psi} + [V, \Psi]) - \Phi|\chi|^2 \right. \\
&\quad \left. - Re((u+iv)^2 \frac{1}{4}(F_A - VV + id_AV) \wedge (F_A - VV + id_AV)) \right) \quad (39)
\end{aligned}$$

The desired result is present : there is not u, v -dependence in $I/(u^2 + v^2)$, but for the last term that is a topological term for the complex connection $A + iV$

$$I_{\text{top}} = \int \frac{1}{4} Re((u+iv)^2 \text{Tr} (F_A - VV + id_AV) \wedge (F_A - VV + id_AV)) \quad (40)$$

The latter topological term is defined in function of

$$\text{Tr} \mathcal{F} \wedge \mathcal{F} = (u+iv)^2 \text{Tr} F_{A+iv} \wedge F_{A+iv} = (u+iv)^2 \text{Tr} (F_A \wedge F_A + 2i \text{Tr} d(V \wedge F_A)) \quad (41)$$

where $V \wedge F_A$ is globally well defined when one restricts the gauge invariance to its real part. It can be thought of as a classical lagrangian, (in the spirit of TQFT's as in [5]),

such that one has locally

$$\begin{aligned} & \text{Tr} (F_A - VV + id_A V) \wedge (F_A - VV + id_A V) \\ = & d\text{Tr} ((A + iV)F_{A+iV} - \frac{1}{3}(A + iV)^3) = d\text{Tr} (AF_A - \frac{1}{3}A^3 + 2iVF_A) \end{aligned} \quad (42)$$

The action (39) is its gauge-fixing, using self-duality gauge conditions, analogously as in [5].

So, the “topological gauge functions” for $A + iV$ are

$$\begin{aligned} & (u(F_A - VV) - vd_A V)_- \\ & (v(F_A - VV) + ud_A V)_+ \end{aligned} \quad (43)$$

These gauge conditions have the following suggestive expression in complex notation

$$(u + iv)(F_A - VV + id_A V) - \overline{\star(u + iv)(F_A - VV + id_A V)} = 0 \quad (44)$$

that is

$$\mathcal{F} = \star\bar{\mathcal{F}} \quad (45)$$

which expresses a class of self-duality condition of $A + iV$, parametrized by the parameter u/v , and covariant under the real part of the gauge symmetry. The action is thus obtainable by a generalization of the method of [5], which was well-suited for the Donaldson-Witten theory, but here we have used a complexification that amounts to a doubling of all fields.

At this point we have the following observations.

The six components of the self-duality equations are such that I gives transverse propagators, both for A and V . Their gauge-fixing can be possibly done in a rather standard way, by choosing a Landau gauge for both A and V , which gives a theory that has yet no interpretation. In particular, it is not the $\mathcal{N} = 4$ theory. Moreover, at this stage, there is a gauge degeneracy for the propagators of Ψ and $\tilde{\Psi}$, and the propagation of the field Φ has not been ensured. If we count the number of on-shell degrees of freedom of the fields $V_\mu, \Phi, \bar{\Phi}, \tilde{\Phi}, \bar{\tilde{\Phi}}$, we find however $2+1+1+1+1=6$ degrees of freedom, which is also the number of scalar fields of the $\mathcal{N} = 4$ theory.

We now come to the point of gauge-fixing the imaginary part of the gauge symmetry for recovering the $\mathcal{N} = 4$ theory in its third twist formulation, and ensuring a standard propagation for all fields.

5 Q -invariant gauge-fixing for the imaginary part of the gauge symmetry

5.1 Elimination of the imaginary part of ghosts and shadows

The following quantities with shadow number -1 are invariant under the BRST symmetry of the real part of the gauge symmetry (with $D \equiv d_A$)

$$\tilde{\Phi}\tilde{c} \quad \tilde{\mu}\tilde{\Omega} \quad (a\tilde{\eta} + b\tilde{c})(D^\mu V_\mu + c\tilde{H}) \quad \tilde{\Phi}(\alpha D^\mu \Psi_\mu + \beta D^\mu \tilde{\Psi}_\mu - \delta[\Phi, \eta]) \quad (46)$$

where $a, b, c, \alpha, \beta, \gamma, \delta$ are numbers. The Q -invariant term

$$\begin{aligned} \mathcal{L}_{\text{Im}} &= Q(\tilde{\Phi}\tilde{c} + \tilde{\mu}\tilde{\Omega}) \\ &= \tilde{\Phi}\tilde{\Phi} + \tilde{\eta}\tilde{c} + \tilde{\mu}\tilde{\mu} + \tilde{\Omega}\tilde{\Omega} \end{aligned} \quad (47)$$

breaks the imaginary part of the gauge symmetry, but respects its real part. It yields algebraic equations of motion of action that enforce the condition

$$\tilde{\Omega} = \tilde{\tilde{\Omega}} = \tilde{\tilde{\Phi}} = \tilde{\Phi} = \tilde{\eta} = 0 \quad (48)$$

for having only the real part of the gauge symmetry.

After this elimination, the following supersymmetric term

$$Q\left(\tilde{c}(D^\mu V_\mu + \tilde{H}) + \tilde{\Phi}(\alpha D^\mu \Psi_\mu + \beta D^\mu \tilde{\Psi}_\mu + \delta[\Phi, \eta])\right) \quad (49)$$

yields an action with the following form

$$\tilde{H}^2 + \tilde{H}D^\mu V_\mu + \eta D^\mu \Psi_\mu + \tilde{c}D^\mu \tilde{\Psi}_\mu + \tilde{\Phi}D^2\Phi + \dots \quad (50)$$

It defines a longitudinal propagation of V through a term $\sim |D^\mu V_\mu|^2$, as well as a longitudinal propagation for Ψ and $\tilde{\Psi}$ by the terms $D^\mu V_\mu$ and $D^\mu \tilde{\Psi}_\mu$. There is some flexibility for the relative coefficients if one only requires Q and BRST invariance. However, the demand of vector symmetry for the sum of both action (49) and (39) fixes all the coefficients.

In fact \tilde{c} and η play the role of propagating fermionic Lagrange multipliers, and \mathcal{L}_{Im} is an action that enforces the equivariant (with respect to the real part of the gauge symmetry) topological gauge functions

$$D^\mu V_\mu + \dots = 0 \quad D^\mu \Psi_\mu + \dots = 0 \quad D^\mu \tilde{\Psi}_\mu + \dots = 0 \quad (51)$$

We have the following Q -quartet diagram

$$\begin{array}{c} \bar{\Phi} \\ \eta \quad \tilde{c} \\ \tilde{H} \end{array} \quad (52)$$

These fields transform tensorially under the s transformations, for $\tilde{\Omega} = 0$. The way the anti-shadow field \tilde{c} becomes associated to the field η after the gauge-fixing of the imaginary part of the gauge symmetry is quite interesting.

The property $Q_c^2 = (u^2 + v^2)[\Phi, \]$ can be enforced as

$$\begin{aligned} Q_c(\eta + i\tilde{c}) &= (u - iv)([\Phi, \bar{\Phi}] + i\tilde{H}) \\ Q_c([\Phi, \bar{\Phi}] + i\tilde{H}) &= (u + iv)[\Phi, \eta + i\tilde{c}] \end{aligned} \quad (53)$$

provided one does field rescalings. The first equation gives $Q\tilde{c} = uH + \dots$, and the second one gives $Q\bar{\Phi} = u\eta + \dots$. The conjugate operator \bar{Q} that anticommutes with Q is defined by

$$\begin{aligned} \bar{Q}_c(\eta + i\tilde{c}) &= i(u - iv)([\Phi, \bar{\Phi}] + i\tilde{H}) \\ \bar{Q}_c([\Phi, \bar{\Phi}] + i\tilde{H}) &= -i(u + iv)[\Phi, \eta + i\tilde{c}] \end{aligned} \quad (54)$$

These equations determine the Q and \bar{Q} transformations of the fields of the quartet $\bar{\Phi}, \eta, \tilde{c}, H$, by separation of their real and imaginary parts.

6 Recovering the $\mathcal{N} = 4$ theory

6.1 Restricted horizontality equation and twisted $\mathcal{N} = 4$

After the Q -invariant gauge-fixing of its imaginary part, the BRST symmetry is

$$\begin{aligned} s(A + iV) &= -d\Omega - [A + iV, \Omega] \\ s(\Omega) &= -\frac{1}{2}[\Omega, \Omega] \end{aligned} \quad (55)$$

It identifies V as a vector that only transform tensorially. By using the reduced unified field $\mathcal{A} = A + iV + (u + iv)c + i_\kappa \gamma_1 + \Omega$, we have obtained the following equation for the definition of Q

$$(d + s + Q+)\mathcal{A} + \frac{1}{2}[\mathcal{A}, \mathcal{A}] = F_{A+iV} + (u - iv)(\Psi + i\tilde{\Psi}) + (u^2 + v^2)\Phi \quad (56)$$

with its Bianchi identity

$$(d + s + Q + [\mathcal{A}, \])(F_{A+iV} + (u - iv)(\Psi + i\tilde{\Psi}) + (u^2 + v^2)\Phi) = 0 \quad (57)$$

These equations give the two scalar transformation laws the $\mathcal{N} = 4$ theory in the third twist, using the field $A + iV$.

The invariant action can be expressed in a most simple form, as the sum

$$\begin{aligned} (u^2 + v^2)I_T = & \int Q\text{Tr} \left(\tilde{\chi}_- \star (u(F_A - VV) - vd_AV)_- - \frac{1}{2}H_- \right) \\ & + \tilde{\chi}_+ \star (v(F_A - VV) + ud_AV)_+ - \frac{1}{2}H_+ \Big) \\ & + \int Q\bar{Q}\text{Tr} \left(\eta \star \tilde{c} + \bar{\Phi}d_A \star V \right) \quad (58) \end{aligned}$$

The $Q\bar{Q}$ exact term reproduces the Q invariant actions discussed in the last section for eliminating the imaginary part of ghosts and shadows and providing the longitudinal part of V . One can check that the action (58) reproduces the one originally found by Marcus for the $\mathcal{N} = 4$ super-Yang–Mills action in the third twist [1].

The last term of the action suggests the relevance in perturbative theory of the complex gauge function⁴

$$d_A(A + iV)_\mu = \partial_\mu A_\mu + id_AV_\mu \quad (59)$$

What we have done is the following. The complex self-duality equations count for 6 conditions. A seventh gauge condition was needed for producing a gauge-invariant longitudinal term of the form $|d_A \star V|^2$, since the $\mathcal{N} = 2$ action with complex gauge symmetry (39) only defines a transverse propagation of V . The last term in I_T provides such a term, as well as other needed terms for defining the propagation of the longitudinal parts of Ψ_μ and $\tilde{\Psi}_\mu$ and of all scalar fields of the complexified $\mathcal{N} = 2$ theory. We can reformulate the whole process by saying that all the needed fields come from at complexified version of the $\mathcal{N} = 2$ theory, followed by a supersymmetric gauge-fixing of the imaginary part of the gauge symmetry. One has a sort of transmutation between the imaginary parts of the scalar fields and the longitudinal degrees of freedom for the vector field V .

⁴The gauge -fixing of A can be done by a s exact term involving H, Ω and $\bar{\Omega}$. Moreover, it can also be made Q -exact using the shadow fields as in [10].

6.2 Supersymmetric observables

One can now simply observe that, with the condition $u - iv = 0$ on the analytically continued parameters u and v , all gauge-invariant observables $\mathcal{O}(A, F_A)$ determine Q -invariant quantities, that are nothing but $\mathcal{O}(A + iV, F_A - VV + id_AV)$. Indeed, for $u - iv = 0$, the extended curvature condition (56) becomes identical to that for the ordinary gauge invariance

$$(d + Q + s)(A + iV + c + \Omega) + (A + iV + c + \Omega)^2 = F_A - VV + id_AV \quad (60)$$

Having such a rich ensemble of supersymmetric observables has no equivalent in the twisted $N = 2$ theory. Kapustin and Witten discussed the Wilson and t'Hooft loops for the field $A + iV$ as supersymmetric observables.

Otherwise, for arbitrary values of u, v , the descent equations for the invariant polynomials of the complex curvature hold and give Q -cocycles depending on Ψ and Φ , which satisfy the usual relations of TQFT “topological observables”, as in [5][12].

We may notice here that the close relationship between the derivation both twisted $\mathcal{N} = 2$ and $\mathcal{N} = 4$ actions suggests a further relevance of the 3-dimensional theory with a “complex” Chern–Simons action as in [12]

$$\int_{M_3} \text{Tr} \left((A + iV)F_{A+iV} - \frac{1}{6}(A + iV)^3 \right) = \int_{M_3} \text{Tr} \left((AF_A - \frac{1}{6}A^3) + iVF_A - Vd_AV + \frac{i}{6}V^3 \right) \quad (61)$$

Natural observables of this 3-dimensional theory are Wilson loops of $A + iV$. One may question whether the twisted $N = 4$ theory can be derived from such a Chern–Simons theory, by constructing a supersymmetric Hamiltonian $H \sim [Q, \bar{Q}]$ in three dimensions, and extending it in four dimensions, as a generalization of the method used by Witten for originally constructing the topological twisted $\mathcal{N} = 2$ action that describes the Donaldson invariants [4].

7 Topological sigma-model and 2D-gravity

Analogous extensions that use the complexification of gauge symmetries can be done, for cases where horizontality conditions exist, such as the topological sigma-model and 2D-gravity [11]. We will display formula that illustrate the method, and will remain at a very formal level. We have chosen the most possible simple choices of using the imaginary part of the gauge symmetries, but more clever gauge choices might exist.

7.1 Topological sigma-model

Let us first consider the topological sigma model. Given a world-sheet scalar field X^μ , in an appropriate target space with Kahler form J , $J^2 = -1$, its ordinary topological symmetry is [11]

$$(d + Q)X = dX + \Psi \quad (62)$$

The topological gauge function (holomorphic maps) is

$$\partial X^\mu = J_\nu^\mu \bar{\partial} X^\nu \quad (63)$$

and the action is

$$I \sim \int d^2x Q \left(\bar{\Psi}_\mu (H^\mu + \partial X^\mu - J_\nu^\mu \bar{\partial} X^\nu) \right) \quad (64)$$

One can introduce a new scalar \tilde{X}^μ , and extend $X \rightarrow X + i\tilde{X}$. Then one can generalize Q as a symmetry with two parameters u and v , with

$$(d + Q)(X + i\tilde{X}) = dX + id\tilde{X} + (u - iv)(\Psi + i\tilde{\Psi}) \quad (65)$$

One thus has a topological sigma-model, with topological gauge function

$$\partial(X + i\tilde{X})^\mu = J_\nu^\mu \bar{\partial}(X + i\tilde{X})^\nu \quad (66)$$

Q has 2 generators, and is governed by the 2 parameters u and v . One has descent equations, and the ordinary observables of the topological σ -model. However, for the values $u = iv$, the correlators of $dX + id\tilde{X}$ are Q -invariant.

7.2 2D-gravity

In topological gravity, the field is the Beltrami differential $\mu_{\bar{z}}^z$, its shadow is the anticommuting vector c^z , and we define

$$\mu^z = dz + \mu_{\bar{z}}^z d\bar{z} \quad (67)$$

Its topological symmetry involves the topological ghosts $\Psi_{\bar{z}}^z$ with

$$\Psi^z = \Psi_{\bar{z}}^z d\bar{z} \quad (68)$$

and ghost of ghost Φ^z . The ordinary topological BRST operator Q is given by

$$(d + Q)(\mu^z + c^z) + (\mu^z + c^z)\partial_z(\mu^z + c^z) = \Psi^z + \Phi^z \quad (69)$$

One extends

$$\mu^z = dz + \mu_{\bar{z}}^z d\bar{z} \rightarrow dz + (\mu_{\bar{z}}^z + iV_{\bar{z}}^z) d\bar{z} \quad (70)$$

and, as a generalization of the Yang–Mills case, $c^z \rightarrow c^z + i\tilde{c}^z$. Here we will chose $\tilde{c}^z = 0$. Then, one redefines Q into

$$\begin{aligned} (d + Q)(\mu^z + iV^z + c^z) + (\mu^z + iV^z + c^z)\partial_z(\mu^z + iV^z + c^z) = \\ = (u - iv)(\Psi^z + i\tilde{\Psi}^z) + (u^2 + v^2)\Phi^z \end{aligned} \quad (71)$$

so that

$$\begin{aligned} Q\mu_{\bar{z}}^z &= \partial_{\bar{z}}c^z + c^z\partial_z\mu_{\bar{z}}^z - \mu_{\bar{z}}^z\partial_zc^z + u\Psi^z - v\tilde{\Psi}^z \\ QV_{\bar{z}}^z &= c^z\partial_zV_{\bar{z}}^z - V_{\bar{z}}^z\partial_zc^z + v\Psi^z + u\tilde{\Psi}^z \\ Qc^z &= c^z\partial_zc^z + (u^2 + v^2)\Phi^z \\ Q_c(\Psi^z + i\tilde{\Psi}^z) &= (u - iv)\partial_{\bar{z}}\Phi^z \end{aligned} \quad (72)$$

A Q -exact action that gives a u - and v -independent action is

$$I = \frac{1}{u^2 + v^2} \int d^2z (Q(\bar{\Psi}_{zz}\mu_{\bar{z}}^z + \bar{Q}(\bar{\Phi}_{zz}V_{\bar{z}}^z))) \quad (73)$$

The elimination of the auxiliary fields sets the fields μ, V and all fermionic ghosts equal to zero. The only remaining propagating fields are $b_{zz} = Q\bar{\Phi}_{zz}$ and c^z , and the action is still the ordinary topological action

$$I \sim \int d^2z (b_{zz}\partial_{\bar{z}}c^z + \bar{\Phi}_{zz}\partial_{\bar{z}}\Phi^z) \quad (74)$$

One can define observables as Q -invariant correlators for $u - iv = 0$, which can be expressed in function of $\mu_{\bar{z}}^z + iV_{\bar{z}}^z$.

References

- [1] N. Marcus, “The Other topological twisting of $\mathcal{N} = 4$ Yang–Mills,” Nucl. Phys. B **452** (1995) 331 [arXiv:hep-th/9506002]; M. Blau and G. Thompson, “Aspects of $\mathcal{N}_T \geq 2$ topological gauge theories and D-branes,” Nucl. Phys. B **492** (1997) 545 [arXiv:hep-th/9612143].

- [2] J. M. F. Labastida and C. Lozano, “Mathai–Quillen formulation of twisted $\mathcal{N} = 4$ supersymmetric gauge theories in four dimensions,” Nucl. Phys. B **502** (1997) 741 [arXiv:hep-th/9702106].
- [3] A. Kapustin, E. Witten, ”Electric-Magnetic Duality And The Geometric Langlands Program,” hep-th/0604151.
- [4] E. Witten, “Topological Quantum Field Theory,” Commun. Math. Phys. 117:353, 1988.
- [5] L. Baulieu and I.M. Singer, “Topological symmetry,” Nucl. Phys. Proc. **5B** (1988) 12.
- [6] L. Baulieu, G. Bossard “New Results on N=4 SuperYang-Mills Theory,” Phys.Lett. B632 (2006) 131-137, [arXiv:hep-th/0507003]. ”Supersymmetric renormalization prescription in N = 4 super-Yang–Mills theory”, Phys.Lett. B643 (2006) 294-302, [arXiv:hep-th/0507003].
- [7] L. Baulieu, G. Bossard and A. Martin, “Twisted superspace”. [arXiv:hep-th/08021980], Phys.Lett.B663, (2008) 275-280.
- [8] J. P. Yamron, “Topological Actions From Twisted Supersymmetric Theories,” Phys. Lett. B **213** (1988) 325.
- [9] C. Vafa and E. Witten, “A Strong coupling test of S duality,” Nucl. Phys. B **431** (1994) 3 [arXiv:hep-th/9408074].
- [10] L. Baulieu, G. Bossard and S. P. Sorella, “Shadow fields and local supersymmetric gauges,” Nucl. Phys. B **753**, 273 (2006) [arXiv:hep-th/0603248]; ”Finiteness Properties of the N=4 Super-Yang–Mills Theory in Supersymmetric Gauge”, Nucl.Phys. B753 (2006) 252-272 hep-th/0605164
- [11] L. Baulieu, I.M. Singer, ”Conformally Invariant gauge-fixed Actions For $2d$ Topological Gravity”, Commun. Math. Phys. **135** (1991) 253; ”The Topological Sigma Model”, Commun. Math. Phys. **125** 227, 1989.
- [12] E. Witten, “ Quantization Of Chern-Simons Gauge Theory With Complex Gauge Group”, Commun.Math.Phys.137:29-66,1991