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► **To cite this version:**

Vesselin Petkov, Luchezar Stoyanov. Correlations for pairs of periodic trajectories for open billiards. 2009. hal-00392274v1

HAL Id: hal-00392274

<https://hal.science/hal-00392274v1>

Preprint submitted on 6 Jun 2009 (v1), last revised 30 Sep 2009 (v3)

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CORRELATIONS FOR PAIRS OF PERIODIC TRAJECTORIES FOR OPEN BILLIARDS

VESELIN PETKOV AND LUCHEZAR STOYANOV

ABSTRACT. In this paper we prove two asymptotic estimates for pairs of closed trajectories for open billiards similar to those established by Pollicott and Sharp [PoS2] for closed geodesics on negatively curved compact surfaces. The first of these estimates holds for general open billiards in any dimension. The more intricate second estimate is established for open billiards satisfying the so called Dolgopyat type estimates. This class of billiards includes all open billiards in the plane and open billiards in \mathbb{R}^N ($N \geq 3$) satisfying some additional conditions.

2000 AMS Subject Classification: Primary: 37D50, Secondary: 58J50

Key words: open billiard, periodic reflecting rays, symbolic coding.

1. INTRODUCTION

In [PoS2] Pollicott and Sharp prove some interesting asymptotic estimates concerning the distribution of lengths $\lambda(\gamma)$ of closed geodesics γ on a compact surface V of negative curvature. Given a finite symmetric set S of generators of the fundamental group $\pi_1(V)$, for each closed geodesic γ on V let $|\gamma|$ denote the minimal number of elements of S needed to write down an elements of $\pi_1(V)$ conjugate to γ . Given real numbers $a < b$, the first asymptotic estimate in [PoS2] concerns the number $\pi(n, [a, b])$ of all pairs (γ, γ') of closed geodesics with $|\gamma|, |\gamma'| \leq n$ and $a \leq \lambda(\gamma) - \lambda(\gamma') \leq b$, and has the same form as that in Theorem 1 below except that the constant h_0 that appears in [PoS2] depends on the set of generators S . The second asymptotic in [PoS2] is much more delicate and involves a family of intervals $I_n = [z + \epsilon_n a, z + \epsilon_n b]$, $z \in \mathbb{R}$, where $\epsilon_n \rightarrow 0$ subexponentially (see the formula in Theorem 2 below which has the same form as the one in [PoS2]). In the proof of this a crucial role is played by Dolgopyat's estimates [D] which apply to any Anosov flow on a compact surface (and also to some Anosov flows on higher dimensional compact manifolds), and in particular to geodesic flows on compact surfaces of negative curvature.

In this paper we prove similar asymptotic estimates for the billiard flow in the exterior of several strictly convex bodies in \mathbb{R}^N ($N \geq 2$) having smooth boundaries and satisfying the so called no eclipse condition (H) defined below. In this case there is a natural coding for the closed trajectories using configurations (admissible sequences) of convex bodies and the constant $h_0 > 0$ that appears in the asymptotic formulae is just the topological entropy of the billiard ball map from boundary to boundary, and h_0 depends only on the number of obstacles (see Sect. 4). The first asymptotic estimate (Theorem 1) holds for any open billiard in any dimension. As in [PoS2], the second asymptotic (Theorem 2) relies heavily on Dolgopyat type estimates. For open billiards these estimates are available without any extra assumptions for $N = 2$ ([St2]) and under some

The first author was partially supported by the ANR project NONAA.

additional conditions for $N \geq 3$ ([St3]). On the other hand, Dolgopyat type estimates concern codings via Markov families, and therefore they are not readily applicable to the natural coding of billiard trajectories mentioned above. However a link between these two types of codings can be established which turns out to be sufficiently convenient, so that Dolgopyat type estimates can be applied in the situation described above and this is one of the purposes of this work.

We now proceed to state precisely the results in this paper.

Let K be a subset of \mathbb{R}^N ($N \geq 2$) of the form $K = K_1 \cup K_2 \cup \dots \cup K_{\kappa_0}$, where K_i are compact strictly convex disjoint domains in \mathbb{R}^N with C^r ($r \geq 3$) boundaries $\Gamma_i = \partial K_i$ and $\kappa_0 \geq 3$. Set $\Omega = \overline{\mathbb{R}^N} \setminus K$. Throughout this paper we assume that K satisfies the following (no-eclipse) condition:

$$(H) \quad \begin{cases} \text{for every pair } K_i, K_j \text{ of different connected components of } K \text{ the convex hull of} \\ K_i \cup K_j \text{ has no common points with any other connected component of } K. \end{cases}$$

With this condition, the *billiard flow* ϕ_t defined on the *cosphere bundle* $S^*(\Omega)$ in the standard way is called an open billiard flow. It has singularities, however its restriction to the *non-wandering set* Λ has only simple discontinuities at reflection points. Moreover, Λ is compact, ϕ_t is hyperbolic and transitive on Λ , and it follows from [St1] that ϕ_t is non-lattice and therefore by a result of Bowen [B], it is topologically weak-mixing on Λ .

Given a periodic billiard trajectory (ray) γ in Ω , denote by d_γ the period (return time) of γ , and by T_γ the primitive period (length) of γ . For any configuration $\mathbf{j} = (j_1, j_2, \dots, j_m)$ with $j_\mu \in \{1, \dots, \kappa_0\}$, $j_i \neq j_{i+1}$ and $|\mathbf{j}| = m$, there exists a unique periodic reflecting ray γ with reflecting points on $\Gamma_{j_1}, \dots, \Gamma_{j_m}$ following the configuration \mathbf{j} (see [I1], [PS1]) and we set $|\gamma| = |\mathbf{j}| = m$. We denote by m_γ the number of the reflections of γ and by P_γ the linear Poincaré map related to γ (see [PS1]). For $a < b$ consider

$$\pi(n, [a, b]) = \#\{(\gamma, \gamma') : |\gamma|, |\gamma'| \leq n, a \leq T_\gamma - T_{\gamma'} \leq b\}$$

and denote by $h_0 > 0$ the *topological entropy* of the billiard ball map related to ϕ_t which coincides with the topological entropy of the shift map σ on a naturally defined symbol space Σ_A (see Subsect. 2.2 for the notations). This implies $h_0 = \log \lambda$, where $\lambda > 1$ is the maximal positive simple eigenvalue of the matrix A , so h_0 depends only on the number of obstacles κ_0 . Then we have

$$\#\{\gamma : |\gamma| \leq n\} \sim \frac{e^{h_0}}{e^{h_0} - 1} \frac{e^{h_0 n}}{n}, \quad n \rightarrow +\infty.$$

Our first result is the following

Theorem 1. *There exists $\beta > 0$ such that for any $a < b$ we have*

$$\pi(n, [a, b]) \sim \frac{(b-a)e^{2h_0}}{(2\pi)^{1/2}\beta(e^{h_0}-1)^2} \frac{e^{2h_0 n}}{n^{5/2}}, \quad n \rightarrow +\infty.$$

This estimate is derived from Lemma 1 in Sect. 4 and the argument from the proof of the first result in [PoS2].

To obtain a more precise result we use the Dolgopyat type estimates (3.14) from Subsection 3.3 below. Consider a sequence of intervals $I_n(z) = [z + \epsilon_n a, z + \epsilon_n b]$, where $\epsilon_n \searrow 0$. We say that ϵ_n goes to 0 *subexponentially* if $\limsup_{t \rightarrow +\infty} |\log \epsilon_n|/n = 0$. Using the notation of Sect. 2 and 3, the second result in this paper is the following analogue of Theorem 2 in [PoS2].

Theorem 2. *Assume that for the Ruelle operator \mathcal{L}_{itr-h_0} , $|t| \geq t_0 > 1$ the estimates (3.14) hold and that the strong stable and the strong unstable laminations $\{W_\epsilon^s(x)\}_{x \in \Lambda}$ and $\{W_\epsilon^u(x)\}_{x \in \Lambda}$ are*

Lipschitz in $x \in \Lambda$. Then there exists $\beta > 0$ such that for any $a < b$ and for every sequence ϵ_n going to 0 subexponentially, we have

$$\lim_{n \rightarrow +\infty} \sup_{z \in \mathbb{R}} \left| \frac{\beta n^{5/2}}{\epsilon_n e^{2h_0 n}} \pi(n, I_n(z)) - \frac{(b-a)e^{2h_0}}{(2\pi)^{1/2}(e^{h_0}-1)^2} e^{-z^2/2\beta^2 n} \right| = 0.$$

In particular, for any fixed $z \in \mathbb{R}$ we get

$$\pi(n, I_n(z)) \sim \frac{(b-a)e^{2h_0}\epsilon_n}{(2\pi)^{1/2}\beta(e^{h_0}-1)^2} \frac{e^{2h_0 n}}{n^{5/2}}, \quad n \rightarrow +\infty.$$

Notice that the assumptions of Theorem 2 are satisfied for $N = 2$ without any additional geometric conditions. The central point in the proof of Theorem 2 is Lemma 4 in Sect. 4, where the Dolgopyat type estimates are used. We should remark that Lemma 4 deals with Ruelle transfer operators defined by means of the symbolic coding Σ_A^+ using the connected components K_j of K (see Sect. 2 for the notation). This coding is very natural and most convenient for a variety of problems where the open billiard flow is involved. However Σ_A^+ does not have the properties of a coding by means of a Markov family and therefore Dolgopyat type estimates do not automatically apply to it. One needs to make a transition from one coding to the other, and this is done in Sects. 2 and 3 below. As a result of this one identifies a class of functions on the symbolic model Σ_A^+ to which the estimates (3.14) can be applied. Unfortunately, this class of functions is not of the form $\mathcal{F}_\theta(\Sigma_A^+)$ for some $\theta > 0$, and it is very doubtful that Dolgopyat type estimates would apply to all functions in $\mathcal{F}_\theta(\Sigma_A^+)$.

This rather serious difficulty appears also in the analysis of the approximation of the cut-off resolvent of the Dirichlet Laplacian in [PS2], where we deal with phases and amplitudes related to the connected components K_j of K (and not to Markov sections R_i) to build an approximation. To overcome it, we use a link between the Ruelle operators $L_{-s\tilde{f}+\tilde{g}}$ and \mathcal{L}_{-sr+hg} corresponding to each other via the transition from one coding to the other one. The dynamical zeta function related to the billiard flow is independent of the choice of coding and by using thermodynamic formalism (see e.g. [PP]) one can show that the eigenvalues of the operators $L_{-s\tilde{f}+\tilde{g}}$ and $\mathcal{L}_{-sr+\hat{g}}$ coincide with their multiplicities. However this does not imply similar estimates for the norms of the iterations of these operators. To get such estimates we use the explicit link between $L_{-s\tilde{f}+\tilde{g}}$ and $\mathcal{L}_{-sr+\hat{g}}$ established in Proposition 3 below for a special class of functions. From the analysis in Sect. 4 we need estimates for the operators $L_{-s\tilde{f}-h_0}^n$ and this corresponds to the case when $\tilde{g} = -h_0$. It is important to note that we may count the periodic rays using the coding with the Markov family, but in this case the number of reflexions of periodic trajectories will depend of the choice of the Markov family and the leading term of the asymptotic will involve the topological entropy $h'_0 > 0$ of the shift in the Markov coding. The choice of Markov family is not unique and working with such coding it is difficult to have a result depending on the topological entropy of billiard ball map.

In the proofs of Theorems 1 and 2 (see Sect. 4) we use the analytic arguments in [PoS2] with minor changes, so we omit the details. Most of the arguments below concern the natural symbolic model for the open billiard flow. On the other hand, our result in Sect. 3 shows that there exists a class of functions for which the Dolgopyat estimates hold for the Ruelle operator related to the coding with obstacles and this has been applied in [PS2] in the analysis of the analytic continuation of the cut-off resolvent of the Laplacian.

In Sect. 5 we discuss some open problems related to correlations of pairs of periodic trajectories when ϵ_n goes to 0 faster than $e^{-h_0 n}$ and their relationship with some separation conditions.

2. SYMBOLIC CODINGS

In this section we compare the Ruelle transfer operators related to two different codings of the billiard flow ϕ_t on Λ – the first of these is related to a Markov family for Λ , while the second is using the boundary components ∂K_i .

Fix a *large ball* B_0 containing K in its interior. For any $x \in \Gamma = \partial K$ we will denote by $\nu(x)$ the *outward unit normal* to Γ at x .

It follows from Lemma 3.1 in [I1] that there exist $\delta_1 > 0$ and $0 < d_0 < \frac{1}{2} \min_{i \neq j} \text{dist}(K_i, K_j)$ such that for any $i = 1, \dots, Q$, $x \in \Gamma_i$ and $\xi \in \mathbb{S}^{N-1}$ with $0 \leq \langle \xi, \nu(x) \rangle \leq \delta_1$, the ray $\{x + t\xi : t \geq 0\}$ has no common points with $\cup_{j \neq i} B(K_j, d_0)$, where

$$B(A, d_0) = \{y + w : y \in A, w \in \mathbb{R}^N, \|w\| \leq d_0\}.$$

Let $z_0 = (x_0, u_0) \in S^*(\Omega)$. Denote by

$$X_1(z_0), X_2(z_0), \dots, X_m(z_0), \dots$$

the successive *reflection points* (if any) of the *forward trajectory*

$$\gamma_+(z_0) = \{\text{pr}_1(\phi_t(z_0)) : 0 \leq t\}.$$

Similarly, we will denote by $\gamma_-(z_0)$ the *backward trajectory* determined by z_0 and by

$$\dots, X_{-m}(z_0), \dots, X_{-1}(z_0), X_0(z_0)$$

its backward reflection points (if any). If $\gamma(z_0) = \gamma_+(z_0) \cup \gamma_-(z_0)$ is bounded (i.e. it has infinitely many reflection points both forwards and backwards), we will say that it *has an itinerary* $\eta = (\eta_j)_{j=-\infty}^{\infty}$ (or that it follows the configuration η) if $X_j(z_0) \in \partial K_{\eta_j}$ for all $j \in \mathbb{Z}$. We will say that the itinerary η is *admissible* if $\eta_j \neq \eta_{j+1}$ for all j .

The following is a consequence of the hyperbolicity of the billiard flow in the exterior of K and can be derived from the works of Sinai on general dispersing billiards ([Si1], [Si2]) and from Ikawa's papers on open billiards ([I1]; see also [Bu]). In this particular form it can be found in [Sj] (see also Ch. 10 in [PS1]).

Proposition 1. *There exist global constants $C > 0$ and $\alpha \in (0, 1)$ such that for any admissible configuration $\iota = (i_0, i_1, i_2, \dots, i_m)$ and any two billiard trajectories in Ω with successive reflection points x_0, x_1, \dots, x_m and y_0, y_1, \dots, y_m , both following the configuration ι , we have*

$$\|x_j - y_j\| \leq C(\alpha^j + \alpha^{m-j}) \quad , \quad 0 \leq j \leq m.$$

Moreover, C and α can be chosen so that if $(x_0, (x_1 - x_0)/\|x_1 - x_0\|)$ and $(y_0, (y_1 - y_0)/\|y_1 - y_0\|)$ belong to the same unstable manifold of the billiard flow, then

$$\|x_j - y_j\| \leq C \alpha^{m-j} \quad , \quad 0 \leq j \leq m.$$

As a consequence of this one obtains that there is an one-to-one correspondence between the bounded (in both directions) billiard trajectories in Ω and the set of admissible itineraries η . In particular this implies that the intersections of the non-wandering set Λ with cross-sections to the billiard flow ϕ_t are Cantor sets.

For $x \in \Lambda$ and a sufficiently small $\epsilon > 0$ let

$$W_\epsilon^s(x) = \{y \in M : d(\phi_t(x), \phi_t(y)) \leq \epsilon \text{ for all } t \geq 0, d(\phi_t(x), \phi_t(y)) \rightarrow_{t \rightarrow \infty} 0\},$$

$$W_\epsilon^u(x) = \{y \in M : d(\phi_t(x), \phi_t(y)) \leq \epsilon \text{ for all } t \leq 0, d(\phi_t(x), \phi_t(y)) \rightarrow_{t \rightarrow -\infty} 0\}$$

be the (strong) *stable* and *unstable manifolds* of size ϵ , and let $E^u(x) = T_x W_\epsilon^u(x)$ and $E^s(x) = T_x W_\epsilon^s(x)$. For any $A \subset M$ and $I \subset \mathbb{R}$ denote $\phi_I(A) = \{ \phi_t(y) : y \in A, t \in I \}$.

It follows from the hyperbolicity of Λ (cf. e.g. [KH]) that if $\epsilon > 0$ is sufficiently small, there exists $\delta > 0$ such that if $x, y \in \Lambda$ and $d(x, y) < \delta$, then $W_\epsilon^s(x)$ and $\phi_{[-\epsilon, \epsilon]}(W_\epsilon^u(y))$ intersect at exactly one point $[x, y] \in \Lambda$. That is, $[x, y] \in W_\epsilon^s(x)$ and there exists a unique $t \in [-\epsilon, \epsilon]$ such that $\phi_t([x, y]) \in W_\epsilon^u(y)$. Setting $\Delta(x, y) = t$, one defines the so called *temporal distance function* $\Delta(x, y)$. For $x, y \in \Lambda$ with $d(x, y) < \delta$, define

$$\pi_y(x) = [x, y] = W_\epsilon^s(x) \cap \phi_{[-\epsilon, \epsilon]}(W_\epsilon^u(y)) .$$

Thus, for a fixed $y \in \Lambda$, $\pi_y : W \rightarrow \phi_{[-\epsilon, \epsilon]}(W_\epsilon^u(y))$ is the *projection* along local stable manifolds defined on a small open neighborhood W of y in Λ .

2.1. Coding via a Markov family. Given $E \subset \Lambda$ we will denote by $\text{Int}_\Lambda(E)$ and $\partial_\Lambda E$ the *interior* and the *boundary* of the subset E of Λ in the topology of Λ , and by $\text{diam}(E)$ the *diameter* of E . We will say that E is an *admissible subset* of $W_\epsilon^u(z) \cap \Lambda$ ($z \in \Lambda$) if E coincides with the closure of its interior in $W_\epsilon^u(z) \cap \Lambda$. Admissible subsets of $W_\epsilon^s(z) \cap \Lambda$ are defined similarly. Following [D], a subset R of Λ will be called a *rectangle* if it has the form

$$R = [U, S] = \{ [x, y] : x \in U, y \in S \} ,$$

where U and S are admissible subsets of $W_\epsilon^u(z) \cap \Lambda$ and $W_\epsilon^s(z) \cap \Lambda$, respectively, for some $z \in \Lambda$. For such R , given $\xi = [x, y] \in R$, we will denote $W_R^u(\xi) = \{ [x', y] : x' \in U \}$ and $W_R^s(\xi) = \{ [x, y'] : y' \in S \} \subset W_\epsilon^s(x)$.

Let $\mathcal{R} = \{ R_i \}_{i=1}^k$ be a family of rectangles with $R_i = [U_i, S_i]$, $U_i \subset W_\epsilon^u(z_i) \cap \Lambda$ and $S_i \subset W_\epsilon^s(z_i) \cap \Lambda$, respectively, for some $z_i \in \Lambda$. Set

$$R = \cup_{i=1}^k R_i .$$

The family \mathcal{R} is called *complete* if there exists $T > 0$ such that for every $x \in \Lambda$, $\phi_t(x) \in R$ for some $t \in (0, T]$. The *Poincaré map* $\mathcal{P} : R \rightarrow R$ related to a complete family \mathcal{R} is defined by $\mathcal{P}(x) = \phi_{\tau(x)}(x) \in R$, where $\tau(x) > 0$ is the smallest positive time with $\phi_{\tau(x)}(x) \in R$. The function τ is called the *first return time* associated with \mathcal{R} . Notice that τ is constant on each of the set $W_{R_i}^s(x)$, $x \in R_i$. A complete family $\mathcal{R} = \{ R_i \}_{i=1}^k$ of rectangles in Λ is called a *Markov family* of size $\chi > 0$ for the flow ϕ_t if $\text{diam}(R_i) < \chi$ for all i and:

(a) for any $i \neq j$ and any $x \in \text{Int}_\Lambda(R_i) \cap \mathcal{P}^{-1}(\text{Int}_\Lambda(R_j))$ we have

$$\mathcal{P}(\text{Int}_\Lambda(W_{R_i}^s(x))) \subset \text{Int}_\Lambda(W_{R_j}^s(\mathcal{P}(x))) \quad , \quad \mathcal{P}(\text{Int}_\Lambda(W_{R_i}^u(x))) \supset \text{Int}_\Lambda(W_{R_j}^u(\mathcal{P}(x))) ;$$

(b) for any $i \neq j$ at least one of the sets $R_i \cap \phi_{[0, \chi]}(R_j)$ and $R_j \cap \phi_{[0, \chi]}(R_i)$ is empty.

The existence of a Markov family \mathcal{R} of an arbitrarily small size $\chi > 0$ for ϕ_t follows from the construction of Bowen [B] (cf. also Ratner [Ra]). Taking χ sufficiently small, we may assume that each rectangle R_i is ‘between two boundary components’ Γ_{p_i} and Γ_{q_i} of K , that is for any $x \in R_i$, the first backward reflection point of the billiard trajectory γ determined by x belongs to Γ_{p_i} , while the first forward reflection point of γ belongs to Γ_{q_i} .

Moreover, using the fact that the intersection of Λ with each cross-section to the flow ϕ_t is a Cantor set, we may assume that the Markov family \mathcal{R} is chosen in such a way that

(c) for any $i = 1, \dots, k$ we have $\partial_\Lambda U_i = \emptyset$.

Finally, partitioning every R_i into finitely many smaller rectangles, cutting R_i along some unstable leaves, and removing some rectangles from the family formed in this way, we may assume that

(d) for every $x \in R$ the billiard trajectory of x from x to $\mathcal{P}(x)$ makes exactly one reflection.

From now on we will assume that $\mathcal{R} = \{R_i\}_{i=1}^k$ is a fixed Markov family for ϕ_t of size $\chi < \epsilon_0/2$ satisfying the above conditions (a)–(d). Set

$$U = \cup_{i=1}^k U_i .$$

The *shift map* $\tilde{\sigma} : U \rightarrow U$ is given by $\tilde{\sigma} = \pi^{(U)} \circ \mathcal{P}$, where $\pi^{(U)} : R \rightarrow U$ is the *projection* along stable leaves.

Let $\mathcal{A} = (\mathcal{A}_{ij})_{i,j=1}^k$ be the matrix given by $\mathcal{A}_{ij} = 1$ if $\mathcal{P}(\text{Int}_\Lambda(R_i)) \cap \text{Int}_\Lambda(R_j) \neq \emptyset$ and $\mathcal{A}_{ij} = 0$ otherwise. Consider the symbol space

$$\Sigma_{\mathcal{A}} = \{(i_j)_{j=-\infty}^{\infty} : 1 \leq i_j \leq k, \mathcal{A}_{i_j i_{j+1}} = 1 \text{ for all } j\},$$

with the product topology and the *shift map* $\sigma : \Sigma_{\mathcal{A}} \rightarrow \Sigma_{\mathcal{A}}$ given by $\sigma((i_j)) = ((i'_j))$, where $i'_j = i_{j+1}$ for all j . As in [B] one defines a natural map

$$\Psi : \Sigma_{\mathcal{A}} \rightarrow R .$$

Namely, given any $\iota = (i_j)_{j=-\infty}^{\infty} \in \Sigma_{\mathcal{A}}$ there is exactly one point $x \in R_{i_0}$ such that $\mathcal{P}^j(x) \in R_{i_j}$ for all integers j . We then set $\Psi((i_j)) = x$. One checks that $\Psi \circ \sigma = \mathcal{P} \circ \Psi$ on R . It follows from the condition (c) above that **the map π is a bijection**. Moreover Ψ is Lipschitz if $\Sigma_{\mathcal{A}}$ is considered with the *metric* d_θ for some appropriately chosen $\theta \in (0, 1)$, where $d_\theta(\xi, \eta) = 0$ if $\xi = \eta$ and $d_\theta(\xi, \eta) = \theta^m$ if $\xi_i = \eta_i$ for $|i| < m$ and m is maximal with this property. Replacing θ by an appropriate $\rho \in (\theta, 1]$, makes Ψ^{-1} a Lipschitz map.

In a similar way one deals with the one-sided subshift of finite type

$$\Sigma_{\mathcal{A}}^+ = \{(i_j)_{j=0}^{\infty} : 1 \leq i_j \leq k, \mathcal{A}_{i_j i_{j+1}} = 1 \text{ for all } j \geq 0\},$$

where the *shift map* $\sigma : \Sigma_{\mathcal{A}}^+ \rightarrow \Sigma_{\mathcal{A}}^+$ is defined in a similar way: $\sigma((i_j)) = ((i'_j))$, where $i'_j = i_{j+1}$ for all $j \geq 0$. The metric d_θ on $\Sigma_{\mathcal{A}}^+$ is defined as above. One checks that there exists a unique map $\psi : \Sigma_{\mathcal{A}}^+ \rightarrow U$ such that $\psi \circ \sigma = \pi^{(U)} \circ \Psi$, where $\pi : \Sigma_{\mathcal{A}} \rightarrow \Sigma_{\mathcal{A}}^+$ is the *natural projection*.

Notice that the *root function* $r : \Sigma_{\mathcal{A}} \rightarrow [0, \infty)$ defined by $r(\xi) = \tau(\Psi(\xi))$ depends only on the forward coordinates of $\xi \in \Sigma_{\mathcal{A}}$. Indeed, if $\xi_+ = \eta_+$, where $\xi_+ = (\xi_j)_{j=0}^{\infty}$, then for $x = \Psi(\xi)$ and $y = \Psi(\eta)$ we have $x, y \in R_i$ for $i = \xi_0 = \eta_0$ and $\mathcal{P}^j(x)$ and $\mathcal{P}^j(y)$ belong to the same R_{i_j} for all $j \geq 0$. This implies that x and y belong to the same local stable fibre in R_i and therefore $\tau(x) = \tau(y)$. Thus, $r(\xi) = r(\eta)$. So, we can define a *root function* $r : \Sigma_{\mathcal{A}}^+ \rightarrow [0, \infty)$ such that $r \circ \pi = \tau \circ \Psi$.

Setting $r_n(\xi) = r(\xi) + r(\sigma\xi) + \dots + r(\sigma^{n-1}\xi)$ for any integer $n \geq 0$ and any $\xi \in \Sigma_{\mathcal{A}}^+$, we have the following

Proposition 2. *There exists a bijection $\psi : \Sigma_{\mathcal{A}}^+ \rightarrow U$ such that $\psi \circ \sigma = \tilde{\sigma} \circ \psi$ and a function $r : \Sigma_{\mathcal{A}}^+ \rightarrow [0, \infty)$ such that for any integer $n \geq 0$ and any $\xi \in \Sigma_{\mathcal{A}}^+$ we have $r_n(\xi) = \tau_n(\psi(\xi))$, i.e. this is the length of the billiard trajectory in Ω determined by $\psi(\xi)$ from $\psi(\xi)$ to the n th intersection with a rectangle from \mathcal{R} .*

Let $B(\Sigma_{\mathcal{A}}^+)$ be the *space of bounded functions* $g : \Sigma_{\mathcal{A}}^+ \rightarrow \mathbb{C}$ with its standard norm $\|g\|_0 = \sup_{\xi \in \Sigma_{\mathcal{A}}^+} |g(\xi)|$. Given a function $g \in B(\Sigma_{\mathcal{A}}^+)$, the *Ruelle transfer operator* $\mathcal{L}_g : B(\Sigma_{\mathcal{A}}^+) \rightarrow B(\Sigma_{\mathcal{A}}^+)$ is defined by

$$(\mathcal{L}_g h)(\eta) = \sum_{\sigma(\eta)=\xi} e^{g(\eta)} h(\eta) .$$

Let $\mathcal{F}_\theta(\Sigma_{\mathcal{A}}^+)$ denote the space of d_θ -Lipschitz functions $g : \Sigma_{\mathcal{A}}^+ \rightarrow \mathbb{C}$ with the norm

$$\|f\|_\theta = \|f\|_0 + \|f\|_\theta ,$$

where

$$\|f\|_\theta = \sup \left\{ \frac{|f(\xi) - f(\eta)|}{d_\theta(\xi, \eta)} : \xi, \eta \in \Sigma_{\mathcal{A}}^+, \xi \neq \eta \right\} .$$

If $g \in \mathcal{F}_\theta(\Sigma_{\mathcal{A}}^+)$, then \mathcal{L}_g preserves the space $\mathcal{F}_\theta(\Sigma_{\mathcal{A}}^+)$.

2.2. Coding using boundary components. Here we assume that K is as in Sect. 1. Denote by A the $\kappa_0 \times \kappa_0$ matrix with entries $A(i, j) = 1$ if $i \neq j$ and $A(i, i) = 0$ for all i , and set

$$\Sigma_A = \{(\dots, \eta_{-m}, \dots, \eta_{-1}, \eta_0, \eta_1, \dots, \eta_m, \dots) : 1 \leq \eta_j \leq \kappa_0, \eta_j \in \mathbb{N}, \eta_j \neq \eta_{j+1} \text{ for all } j \in \mathbb{Z}\} ,$$

$$\Sigma_A^+ = \{(\eta_0, \eta_1, \dots, \eta_m, \dots) : 1 \leq \eta_j \leq \kappa_0, \eta_j \in \mathbb{N}, \eta_j \neq \eta_{j+1} \text{ for all } j \geq 0\} ,$$

$$\Sigma_A^- = \{(\dots, \eta_{-m}, \dots, \eta_{-1}, \eta_0) : 1 \leq \eta_j \leq \kappa_0, \eta_j \in \mathbb{N}, \eta_{j-1} \neq \eta_j \text{ for all } j \leq 0\} .$$

The shift operators¹ $\sigma : \Sigma_A \rightarrow \Sigma_A$ and $\sigma : \Sigma_A^+ \rightarrow \Sigma_A^+$ are defined as before.

We will now define two important functions f and g on Σ_A . For the second one in particular we need some preliminary information.

A *phase function* on an open set \mathcal{U} in \mathbb{R}^N is a smooth (C^r) function $\varphi : \mathcal{U} \rightarrow \mathbb{R}$ such that $\|\nabla\varphi\| = 1$ everywhere in \mathcal{U} . For $x \in \mathcal{U}$ the level surface

$$\mathcal{C}_\varphi(x) = \{y \in \mathcal{U} : \varphi(y) = \varphi(x)\}$$

has a unit normal field $\pm\nabla\varphi(y)$.

The phase function φ defined on \mathcal{U} is said to satisfy the *condition (P)* on Γ_j if:

(i) the normal curvatures of \mathcal{C}_φ with respect to the normal field $-\nabla\varphi$ are non-negative at every point of \mathcal{C}_φ ;

(ii) $\mathcal{U}^+(\varphi) = \{y + t\nabla\varphi(y) : t \geq 0, y \in \mathcal{U} \cap \Gamma_j\} \supset \cup_{i \neq j} K_i$.

A natural extension of φ on $\mathcal{U}^+(\varphi)$ is obtained by setting $\varphi(y + t\nabla\varphi(y)) = \varphi(y) + t$ for $t \geq 0$ and $y \in \mathcal{U} \cap \Gamma_j$.

For any $\delta > 0$ and $V \subset \Omega$ denote by $S_\delta^*(V)$ the set of those $(x, u) \in S^*(\Omega)$ such that $x \in V$ and there exist $y \in \Gamma$ and $t \geq 0$ with $y + tu = x$, $y + su \in \mathbb{R}^N \setminus K$ for all $s \in (0, t)$ and $\langle u, \nu_\Gamma(y) \rangle \geq \delta$.

Notice that the condition (H) implies the existence of a constant $\delta_0 > 0$ depending only on the obstacle K such that any $(x, u) \in S^*(\Omega)$ whose backward and forward billiard trajectories both have common points with Γ belongs to $S_{\delta_0}^*(\Omega)$.

The following proposition is derived by using some tools from [I1] (see Proposition 4 in [PS2] for details).

¹We keep the same notation as for Σ_A and Σ_A^+ ; it will be clear from the context which of these we mean in each particular case.

Proposition 3. *There exists a constant $\epsilon_0 > 0$ such that for any $z_0 = (x_0, u_0) \in S_{\delta_0}^*(\Omega \cap B_0)$ whose backward trajectory $\gamma_-(z_0)$ has an infinite number of reflection points $X_j = X_j(z_0)$ ($j \leq 0$) and $\eta \in \Sigma_A^-$ is its itinerary, the following hold:*

(a) *There exists a smooth (C^r) phase function $\psi = \psi_\eta$ satisfying the condition (P) on $\mathcal{U} = B(x_0, \epsilon_0) \cap \Omega$ such that $\psi(x_0) = 0$, $u_0 = \nabla\psi(x_0)$, and such that for any $x \in C_\psi(x_0) \cap \mathcal{U}^+(\psi)$ the billiard trajectory $\gamma_-(x, \nabla\psi(x))$ has an itinerary η and therefore $d(\phi_t(x, \nabla\psi(x)), \phi_t(z_0)) \rightarrow 0$ as $t \rightarrow -\infty$. That is,*

$$W_{loc}^u(z_0) = \{(x, \nabla\psi(x)) : x \in C_\psi(x_0) \cap \mathcal{U}^+(\psi)\}$$

is the local unstable manifold of z_0 .

(b) *If $(y, v) \in S^*(\Omega \cap B_0)$ is such that $y \in C_\psi(x_0)$ and $\gamma_-(y, v)$ has the same itinerary η , then $v = \nabla\psi(y)$, i.e. $(y, v) \in W_{loc}^u(z_0)$.*

Notice that $W_{loc}^u(z_0)$ coincides locally near z_0 with $W_\epsilon^u(z_0)$ defined above.

In what follows we will use the notation $C_\eta(z_0) = C_\psi(x_0)$. Denote by $G_\eta(z_0)$ the Gauss curvature of $C_\eta(z_0)$ at x_0 .

Given $\xi \in \Sigma_A$, let

$$\dots, P_{-2}(\xi), P_{-1}(\xi), P_0(\xi), P_1(\xi), P_2(\xi), \dots$$

be the successive reflection points of the unique billiard trajectory in the exterior of K such that $P_j(\xi) \in K_{\xi_j}$ for all $j \in \mathbb{Z}$. Set

$$f(\xi) = \|P_0(\xi) - P_1(\xi)\| ,$$

and define the map

$$\Phi : \Sigma_A \longrightarrow \Lambda_{\partial K} = \Lambda \cap S_\Lambda^*(\Omega)$$

by $\Phi(\xi) = (P_0(\xi), (P_1(\xi) - P_0(\xi))/\|P_1(\xi) - P_0(\xi)\|)$. Then Φ is a bijection such that $\Phi \circ \sigma = B \circ \Phi$, where $B : \Lambda_{\partial K} \longrightarrow \Lambda_{\partial K}$ is the billiard ball map.

Next, set $\xi_- = (\dots, \xi_{-m}, \xi_{-m+1}, \dots, \xi_{-1}, \xi_0) \in \Sigma_A^-$ and choose an arbitrary point x_0 on the segment $[P_0(\xi), P_1(\xi)]$ such that $z_0 = (x_0, u_0) \in S_{\delta_0}^*(\Omega)$, where $u_0 = (P_1(\xi) - P_0(\xi))/\|P_1(\xi) - P_0(\xi)\|$. Let $C_{\xi_-}(z_0) = C_\psi(x_0)$ for some phase function ψ . Setting $t_- = \|x_0 - P_0(\xi)\|$ and $t_+ = \|x_0 - P_1(\xi)\|$, consider the surfaces

$$C_\xi^-(P_0(\xi)) = \{x - t_- \nabla\psi(x) : x \in C_{\xi_-}(z_0)\} \quad , \quad C_\xi^+(P_1(\xi)) = \{x + t_+ \nabla\psi(x) : x \in C_{\xi_-}(z_0)\} .$$

Clearly $C_\xi^-(P_0(\xi))$ is the surface passing through $P_0(\xi)$ obtained by shifting $C_{\xi_-}(z_0)$ in free space \mathbb{R}^N , t_- units backwards along its normal field, while $C_\xi^+(P_1(\xi))$ is the surface passing through $P_1(\xi)$ obtained by shifting $C_{\xi_-}(z_0)$ in free space \mathbb{R}^N , t_+ units forwards along its normal field. Let $G_\xi^-(P_0(\xi))$ and $G_\xi^+(P_1(\xi))$ be the Gauss curvatures of $C_\xi^-(P_0(\xi))$ at $P_0(\xi)$ and that of $C_\xi^+(P_1(\xi))$ at $P_1(\xi)$, respectively. Set

$$g(\xi) = \frac{1}{N-1} \ln \frac{G_\xi^+(P_1(\xi))}{G_\xi^-(P_0(\xi))} .$$

This defines a function $g : \Sigma_A \longrightarrow \mathbb{R}$.

Choosing appropriately $\theta \in (0, 1)$, we can define as in Subsection 2.1 above the space $\mathcal{F}_\theta(\Sigma_A)$ and we get $f, g \in \mathcal{F}_\theta(\Sigma_A)$ (see e.g. [I1]). By Sinai's Lemma (see e.g. [PP]), there exist functions $\tilde{f}, \tilde{g} \in \mathcal{F}_\theta(\Sigma_A)$ depending on future coordinates only and $\chi_f, \chi_g \in \mathcal{F}_\theta(\Sigma_A)$ such that

$$f(\xi) = \tilde{f}(\xi) + \chi_f(\xi) - \chi_f(\sigma\xi) \quad , \quad g(\xi) = \tilde{g}(\xi) + \chi_g(\xi) - \chi_g(\sigma\xi)$$

for all $\xi \in \Sigma_A$. As in the proof of Sinai's Lemma, for any $k = 1, \dots, \kappa_0$ choose and fix an arbitrary sequence $\eta^{(k)} = (\dots, \eta_{-m}^{(k)}, \dots, \eta_{-1}^{(k)}, \eta_0^{(k)}) \in \Sigma_a^-$ with $\eta_0^{(k)} = k$. Then for any $\xi \in \Sigma_A$ (or $\xi \in \Sigma_A^+$) set

$$e(\xi) = (\dots, \eta_{-m}^{(\xi_0)}, \dots, \eta_{-1}^{(\xi_0)}, \eta_0^{(\xi_0)} = \xi_0, \xi_1, \dots, \xi_m, \dots) \in \Sigma_A .$$

Then we have

$$\chi_f(\xi) = \sum_{n=0}^{\infty} [f(\sigma^n(\xi)) - f(\sigma^n e(\xi))] .$$

A similar formula holds for χ_g .

As in Subsection 2.1, given any function $V \in B(\Sigma_A^+)$, the *Ruelle transfer operator* $L_V : B(\Sigma_A^+) \longrightarrow B(\Sigma_A^+)$ is defined by

$$(L_V W)(\eta) = \sum_{\sigma(\eta)=\xi} e^{V(\eta)} W(\eta) .$$

2.3. Another coding related to the Markov family. Here we define another coding which uses the symbolic model Σ_A^+ . We will then define representatives \hat{f} and \hat{g} of the functions f and g on Σ_A^+ and consider the corresponding Ruelle operators $\mathcal{L}_{-sf+\hat{g}}$.

Let V_0 be the set of those $(p, u) \in S^*(\Omega)$ such that $p = q + t u$ and $(p, u) = \phi_t(q, u)$ for some $(q, u) \in S_{\partial K}^*(\Omega)$ with $\langle u, \nu(q) \rangle > 0$ and some $t \geq 0$. Clearly V_0 is an open subset of $S^*(\Omega)$ containing Λ . Setting $\omega(p, u) = (q, u)$, we get a smooth map $\omega : V_0 \longrightarrow S_{\partial K}^*(\Omega)$.

Consider the bijection $\mathcal{S} = \Phi^{-1} \circ \omega \circ \Psi : \Sigma_A \longrightarrow \Sigma_A$. It induces a bijection $\mathcal{S} : \Sigma_A^+ \longrightarrow \Sigma_A^+$. Moreover $\mathcal{S} \circ \sigma = \sigma \circ \mathcal{S}$.

Define the functions $f', g' : \Sigma_A \longrightarrow \mathbb{R}$ by $f'(\underline{i}) = f(\mathcal{S}(\underline{i}))$ and $g'(\underline{i}) = g(\mathcal{S}(\underline{i}))$.

Next, repeating a part of the previous subsection, for any $i = 1, \dots, k$ choose

$$\hat{\underline{j}}^{(i)} = (\dots, j_{-m}^{(i)}, \dots, j_{-1}^{(i)})$$

such that $(\hat{\underline{j}}^{(i)}, i) \in \Sigma_A^-$. It is convenient to make this choice in such a way that $\hat{\underline{j}}^{(i)}$ **corresponds to the local unstable manifold** $U_i \subset \Lambda \cap W_\epsilon^u(z_i)$ (see the beginning of Subsection 2.1), i.e. the backward itinerary of every $z \in U_i$ coincides with $\hat{\underline{j}}^{(i)}$.

Now for any $\underline{i} = (i_0, i_1, \dots) \in \Sigma_A^+$ (or $\underline{i} \in \Sigma_A$) set

$$\hat{e}(\underline{i}) = (\hat{\underline{j}}^{(i_0)}; i_0, i_1, \dots) \in \Sigma_A .$$

According to the choice of $\hat{\underline{j}}^{(i_0)}$, we have $\Psi(\hat{e}(\underline{i})) = \psi(\underline{i}) \in U_{i_0}$. (Notice that without the above special choice we would only have that $\Psi(\hat{e}(\underline{i}))$ and $\psi(\underline{i}) \in U_{i_0}$ lie on the same stable leaf in R_{i_0} .)

Next, define

$$\hat{\chi}_f(\underline{i}) = \sum_{n=0}^{\infty} [f'(\sigma^n(\underline{i})) - f'(\sigma^n \hat{e}(\underline{i}))] \quad , \quad \underline{i} \in \Sigma_A ,$$

and

$$\hat{\chi}_g(\underline{i}) = \sum_{n=0}^{\infty} [g'(\sigma^n(\underline{i})) - g'(\sigma^n \hat{e}(\underline{i}))] \quad , \quad \underline{i} \in \Sigma_A .$$

As before, the functions $\hat{f}, \hat{g} : \Sigma_A \longrightarrow \mathbb{R}$ given by

$$\hat{f}(\underline{i}) = f'(\underline{i}) - \hat{\chi}_f(\underline{i}) + \hat{\chi}_f(\sigma \underline{i}) \quad , \quad \hat{g}(\underline{i}) = g'(\underline{i}) - \hat{\chi}_g(\underline{i}) + \hat{\chi}_g(\sigma \underline{i})$$

depend on future coordinates only, so they can be regarded as functions on $\Sigma_{\mathcal{A}}^+$.

3. RELATIONSHIP BETWEEN RUELLE OPERATORS

3.1. Relationship between $\mathcal{L}_{-sr+\hat{g}}$ and $\mathcal{L}_{-sf+\hat{g}}$. We will now describe a natural relationship between the operators $\mathcal{L}_V : B(\Sigma_{\mathcal{A}}^+) \rightarrow B(\Sigma_{\mathcal{A}}^+)$ and $L_v : B(\Sigma_{\mathcal{A}}^+) \rightarrow B(\Sigma_{\mathcal{A}}^+)$, where $V = v \circ \mathcal{S}$. Let $\mathcal{R} = \{R_i\}_{i=1}^k$ be a Markov family as in Subsection 2.1. We define a map

$$\Gamma : B(\Sigma_{\mathcal{A}}) \rightarrow B(\Sigma_{\mathcal{A}})$$

by

$$\Gamma(v) = v \circ \Phi^{-1} \circ \omega \circ \Psi = v \circ \mathcal{S} \quad , \quad V \in B(\Sigma_{\mathcal{A}}) . \quad (3.1)$$

Since by property (d) of the Markov family, $\omega : R \rightarrow \Lambda_{\partial K}$ is a bijection, it follows that Γ is a bijection and $\Gamma^{-1}(V) = V \circ \Psi^{-1} \circ \omega^{-1} \circ \Phi$.

Moreover Γ induces a bijection $\Gamma : B(\Sigma_{\mathcal{A}}^+) \rightarrow B(\Sigma_{\mathcal{A}}^+)$. Indeed, assume that $v \in B(\Sigma_{\mathcal{A}})$ depends on future coordinates only. Then $v \circ \Phi^{-1}$ is constant on local stable manifolds in $S_{\Lambda}^*(\Omega)$. Hence $v \circ \Phi^{-1} \circ \omega$ is constant on local stable manifolds on R , and therefore $\Gamma(v) = v \circ \Phi^{-1} \circ \omega \circ \Psi$ depends on future coordinates only.

Next, let $v, w \in B(\Sigma_{\mathcal{A}}^+)$ and let $V = \Gamma(v)$, $W = \Gamma(w)$. Given $\underline{i}, \underline{j} \in \Sigma_{\mathcal{A}}^+$ with $\sigma(\underline{j}) = \underline{i}$, setting $\xi = \mathcal{S}(\underline{i})$ and $\eta = \mathcal{S}(\underline{j})$, we have $\sigma(\eta) = \xi$. Thus,

$$\begin{aligned} \mathcal{L}_W V(\underline{i}) &= \sum_{\sigma(\underline{j})=\underline{i}} e^{W(\underline{j})} V(\underline{j}) = \sum_{\sigma(\underline{j})=\underline{i}} e^{w(\mathcal{S}(\underline{j}))} v(\mathcal{S}(\underline{j})) \\ &= \sum_{\sigma(\eta)=\xi} e^{w(\eta)} v(\eta) = L_w v(\xi) \end{aligned}$$

for all $\underline{i} \in \Sigma_{\mathcal{A}}^+$. This shows that

$$(L_w v) \circ \mathcal{S} = \mathcal{L}_{\Gamma(w)} \Gamma(v) . \quad (3.2)$$

Notice that the functions $r : \Sigma_{\mathcal{A}}^+ \rightarrow [0, \infty)$ and $\tilde{f} : \Sigma_{\mathcal{A}}^+ \rightarrow [0, \infty)$ do not correspond to each other via \mathcal{S} , and neither do r and $\hat{f} : \Sigma_{\mathcal{A}}^+ \rightarrow [0, \infty)$. To compensate the difference between the latter two, define $\hat{\lambda} : \Sigma_{\mathcal{A}} \rightarrow [0, \infty)$ by $\hat{\lambda}(\underline{i}) = t > 0$, where for the point $x = \Psi(\underline{i}) \in R$ we have $\phi_{-t}(x) = \omega(x) \in S_{\partial K}^*(\Omega)$. With the same notation, define $\lambda(x) = t = \hat{\lambda}(\underline{i})$. This defines a function $\lambda : R \rightarrow [0, \infty)$ so that $\lambda \circ \Psi = \hat{\lambda}$.

Before continuing, notice that

$$\hat{\lambda}(\hat{e}(\underline{i})) = \hat{\lambda}(\underline{i}) - \hat{\chi}_f(\underline{i}) \quad , \quad \underline{i} \in \Sigma_{\mathcal{A}} . \quad (3.3)$$

Indeed, given $\underline{i} = (\dots; i_0, i_1, \dots) \in \Sigma_{\mathcal{A}}$, let $x = \Psi(\underline{i}) \in R_{i_0}$ and $y = \Psi(\hat{e}(\underline{i})) \in R_{i_0}$. Since, \underline{i} and $\hat{e}(\underline{i})$ have the same forward coordinates, it follows that x and y lie on the same local stable manifold. Thus, $\tau^n(x) = \tau^n(y)$ and so $r_n(\underline{i}) = r_n(\hat{e}(\underline{i}))$ for all integers $n \geq 1$. Moreover, the m th reflection points of the billiard trajectories determined by x and y are $\text{Const} \theta^m$ -close for some global constants $\text{Const} > 0$ and $\theta \in (0, 1)$. Thus, $|\hat{\lambda}(\sigma^m(\underline{i})) - \hat{\lambda}(\sigma^m(\hat{e}(\underline{i})))| \leq \text{Const} \theta^m$, too (possibly with a different global constant $\text{Const} > 0$).

Set $t = \hat{\lambda}(\underline{i})$ and $t' = \hat{\lambda}(\hat{e}(\underline{i}))$. Given an integer $m \geq 1$, consider

$$A_m = \sum_{n=0}^m [f'(\sigma^n(\underline{i})) - f'(\sigma^n(\hat{e}(\underline{i})))] .$$

Then $\hat{\chi}_f(\underline{i}) = \lim_{m \rightarrow \infty} A_m$. Moreover,

$$\begin{aligned} A_m &= [r_m(\underline{i}) + t - \hat{\lambda}(\sigma^m(\underline{i}))] - [r_m(\hat{e}(\underline{i})) + t' - \hat{\lambda}(\sigma^m(\hat{e}(\underline{i})))] \\ &= (t - t') - [\hat{\lambda}(\sigma^m(\underline{i})) - \hat{\lambda}(\sigma^m(\hat{e}(\underline{i})))] = \hat{\lambda}(\underline{i}) - \hat{\lambda}(\hat{e}(\underline{i})) + O(\theta^m). \end{aligned}$$

and letting $m \rightarrow \infty$, we obtain (3.3).

Next, for any $\underline{i} \in \Sigma_{\mathcal{A}}$ and any $m \geq 1$ we have

$$\begin{aligned} f'_m(\underline{i}) &= f'(\underline{i}) + f'(\sigma \underline{i}) + \dots + f'(\sigma^{m-1} \underline{i}) \\ &= [\hat{f}(\underline{i}) + \hat{\chi}_f(\underline{i}) - \hat{\chi}_f(\sigma \underline{i})] + [\hat{f}(\sigma \underline{i}) + \hat{\chi}_f(\sigma \underline{i}) - \hat{\chi}_f(\sigma^2 \underline{i})] + \dots \\ &\quad + [\hat{f}(\sigma^{m-1} \underline{i}) + \hat{\chi}_f(\sigma^{m-1} \underline{i}) - \hat{\chi}_f(\sigma^m \underline{i})] \\ &= \hat{f}_m(\underline{i}) + \hat{\chi}_f(\underline{i}) - \hat{\chi}_f(\sigma^m \underline{i}). \end{aligned}$$

Since $r_m(\underline{i}) = f'_m(\underline{i}) + \hat{\lambda}(\sigma^m \underline{i}) - \hat{\lambda}(\underline{i})$, combining the above and (3.3), it follows that

$$r_m(\underline{i}) = \hat{f}_m(\underline{i}) + \hat{\chi}_f(\underline{i}) - \hat{\chi}_f(\sigma^m \underline{i}) + \hat{\lambda}(\sigma^m \underline{i}) - \hat{\lambda}(\underline{i}) = \hat{f}_m(\underline{i}) + \hat{\lambda}(\hat{e}(\sigma^m \underline{i})) - \hat{\lambda}(\hat{e}(\underline{i})).$$

Thus,

$$r_m(\underline{i}) = \hat{f}_m(\underline{i}) + \hat{\lambda}(\hat{e}(\sigma^m \underline{i})) - \hat{\lambda}(\hat{e}(\underline{i})) \quad , \quad \underline{i} \in \Sigma_{\mathcal{A}} \quad , \quad m \geq 1. \quad (3.4)$$

Notice that $\hat{f}_m(\underline{i}) = \hat{f}_m(\hat{e}(\underline{i}))$, since \hat{f} depends on future coordinates only.

Now we will find a relationship between the powers of the operators $\mathcal{L}_{-sr+\hat{g}}$ and $\mathcal{L}_{-s\hat{f}+\hat{g}}$. Given $s \in \mathbb{C}$, consider the function $h_s : U \rightarrow \mathbb{R}$ defined by

$$h_s(x) = e^{-s\lambda(x)}.$$

It gives rise to a function $\hat{h}_s : \Sigma_{\mathcal{A}}^+ \rightarrow \mathbb{R}$ defined by

$$\hat{h}_s(\underline{i}) = h_s(\psi(\underline{i})) = h_s(\Psi(\hat{e}(\underline{i}))) = e^{-s\hat{\lambda}(\hat{e}(\underline{i}))} \quad , \quad \underline{i} \in \Sigma_{\mathcal{A}}. \quad (3.5)$$

(See the remark after the definition of $\hat{e}(\underline{i})$ in Sect. 2.3 above.)

It now follows from (3.4) and (3.5) that for any function $V : \Sigma_{\mathcal{A}}^+ \rightarrow \mathbb{C}$, any $s \in \mathbb{C}$ and any integer $n \geq 1$ we have

$$\begin{aligned} \mathcal{L}_{-sr+\hat{g}}^{n+1}(\hat{h}_s \cdot V)(\underline{i}) &= \sum_{\sigma^{n+1} \underline{k} = \underline{i}} e^{-sr_{n+1}(\underline{k}) + \hat{g}_{n+1}(\underline{k})} e^{-s\hat{\lambda}(\hat{e}(\underline{k}))} V(\underline{k}) \\ &= \sum_{\sigma^{n+1} \underline{k} = \underline{i}} e^{-s[\hat{f}_{n+1}(\underline{k}) + \hat{\lambda}(\hat{e}(\sigma^{n+1} \underline{k})) - \hat{\lambda}(\hat{e}(\underline{k}))] + \hat{g}_{n+1}(\underline{k}) - s\hat{\lambda}(\hat{e}(\underline{k}))} V(\underline{k}) \\ &= \sum_{\sigma^{n+1} \underline{k} = \underline{i}} e^{-s\hat{f}_{n+1}(\underline{k}) - s\hat{\lambda}(\hat{e}(\sigma^{n+1} \underline{k})) + \hat{g}_{n+1}(\underline{k})} V(\underline{k}) \\ &= e^{s\hat{\lambda}(\hat{e}(\underline{i}))} \sum_{\sigma^{n+1} \underline{k} = \underline{i}} e^{-s\hat{f}_{n+1}(\underline{k}) + \hat{g}_{n+1}(\underline{k})} V(\underline{k}) = \hat{h}_s(\underline{i}) \cdot \left(\mathcal{L}_{-s\hat{f}+\hat{g}}^{n+1} V \right)(\underline{i}). \end{aligned}$$

Thus,

$$\frac{1}{\hat{h}_s} \cdot \mathcal{L}_{-sr+\hat{g}}^n(\hat{h}_s \cdot V) = \mathcal{L}_{-s\hat{f}+\hat{g}}^n V. \quad (3.6)$$

3.2. Relationship between $\mathcal{L}_{-s\hat{f}+\hat{g}}$ and $L_{-s\tilde{f}+\tilde{g}}$. Here we will use arguments similar to these in Subsect. 3.1 to find a relationship between the operators $\mathcal{L}_{-s\hat{f}+\hat{g}}^n$ and $L_{-s\tilde{f}+\tilde{g}}^n$.

For any $p = 1, 2, \dots, k$ fix an arbitrary point $y_p \in \Lambda \cap S_{\delta_0}^*(\Omega)$ such that $\eta^{(p)} \in \Sigma_{\mathcal{A}}^-$ **corresponds to the unstable manifold** $W^u(y_p)$, i.e. the backward itinerary of every $z \in W^u(y_p) \cap V_0$ coincides with $\eta^{(p)}$. Define $\kappa : \Sigma_{\mathcal{A}}^+ \rightarrow \mathbb{R}$ by

$$\kappa(\underline{i}) = [\Psi(\hat{e}(\underline{i})), y_{\xi_0}] \quad , \quad \xi = \mathcal{S}(\underline{i}) \in \Sigma_{\mathcal{A}}^+ .$$

We will now prove that

$$\hat{\chi}_f(\underline{i}) - \chi_f(\mathcal{S}(\underline{i})) = \lambda(\kappa(\underline{i})) - \hat{\lambda}(\hat{e}(\underline{i})) \quad , \quad \underline{i} \in \Sigma_{\mathcal{A}} . \quad (3.7)$$

It is important that the right-hand-side of (3.7) depends only on the future coordinates of \underline{i} . Indeed, given $\underline{i} = (\dots; i_0, i_1, \dots) \in \Sigma_{\mathcal{A}}$, set $x = \Psi(\hat{e}(\underline{i})) \in U_{i_0}$, $\xi = \mathcal{S}(\underline{i})$, $\xi_0 = p$ and $y = \kappa(\underline{i}) = [x, y_p]$. Notice that $y \in W^s(x)$, and so the forward billiard trajectories of y and x converge. On the other hand, $\phi_\tau(y) \in W^u(y_p)$ for some $\tau \in \mathbb{R}$ (in fact, $\tau = \Delta(x, y_p)$; see e.g. [D] or Sect. 2 in [St1] for the definition of Δ), so the billiard trajectory determined by y has backward itinerary $\eta^{(p)} \in \Sigma_{\mathcal{A}}^-$ and forward itinerary $(\xi_0, \xi_1, \dots) \in \Sigma_{\mathcal{A}}^+$, i.e. this is the trajectory determined by $e(\xi)$.

Since $y \in W^s(x)$, the j th reflection points p_j and q_j ($j \geq 0$) of the billiard trajectories determined by x and y , respectively, are $\text{Const} \theta^j$ -close for some global constants $\text{Const} > 0$ and $\theta \in (0, 1)$.

Set $t = \hat{\lambda}(\hat{e}(\underline{i})) = \lambda(x)$ and $t' = \lambda(y) = \lambda(\kappa(\underline{i}))$. Given an integer $m \geq 1$, consider

$$A_m = \sum_{n=0}^{m-1} [f'(\sigma^n(\underline{i})) - f'(\sigma^n \hat{e}(\underline{i}))] - \sum_{n=0}^{m-1} [f(\sigma^n(\xi)) - f(\sigma^n e(\xi))] .$$

Then $\hat{\chi}_f(\underline{i}) - \chi_f(\mathcal{S}(\underline{i})) = \lim_{m \rightarrow \infty} A_m$. Since $f'(\sigma^n(\underline{i})) = f(\sigma^n(\xi))$, we have

$$A_m = - \sum_{n=0}^{m-1} f'(\sigma^n \hat{e}(\underline{i})) + \sum_{n=0}^{m-1} f(\sigma^n e(\xi)) .$$

The first sum in this expression is the length of the billiard trajectory determined by x from p_0 till p_m , while the second is the length of the billiard trajectory determined by y from q_0 till q_m . Since $\|q_m - p_m\| \leq \text{Const} \theta^m$ and $y \in W^s(x)$, it now follows that $A_m = -t + t' + O(\theta^m)$, and letting $m \rightarrow \infty$ proves (3.7).

In a similar way, using the definition of the function g , one derives that

$$\hat{\chi}_g(\underline{i}) - \chi_g(\xi) = \frac{1}{N-1} \ln G_{\hat{e}(\underline{i})}^-(P_0(\hat{e}(\underline{i}))) - \frac{1}{N-1} \ln G_{e(\xi)}^-(P_0(e(\xi))) \quad , \quad \underline{i} \in \Sigma_{\mathcal{A}} . \quad (3.8)$$

Notice that $G_{\hat{e}(\underline{i})}^-(P_0(\hat{e}(\underline{i})))$ is the Gauss curvature of a shift of U_{i_0} backwards along the corresponding billiard trajectory at $P_0(\hat{e}(\underline{i}))$ (so this is uniquely determined by the forward coordinates of \underline{i}), while $G_{e(\xi)}^-(P_0(e(\xi)))$ is the Gauss curvature of the shift of $W^u(y_{\xi_0})$ at $P_0(e(\xi))$ (so again this is uniquely determined by the forward coordinates of \underline{i}).

Given $\underline{i} \in \Sigma_{\mathcal{A}}$, set $\xi = \mathcal{S}(\underline{i})$. Then (3.7) implies

$$\begin{aligned} \hat{f}_m(\underline{i}) - \tilde{f}_m(\xi) &= [f'_m(\underline{i}) - \hat{\chi}_f(\underline{i}) + \hat{\chi}_f(\sigma^m \underline{i})] - [f_m(\xi) - \chi_f(\xi) + \chi_f(\sigma^m \xi)] \\ &= -[\hat{\chi}_f(\underline{i}) - \hat{\chi}_f(\sigma^m \underline{i})] + [\chi_f(\xi) - \chi_f(\sigma^m \xi)] . \end{aligned} \quad (3.9)$$

Similarly,

$$\hat{g}_m(\underline{i}) - \tilde{g}_m(\xi) = -[\hat{\chi}_g(\underline{i}) - \hat{\chi}_g(\sigma^m \underline{i})] + [\chi_g(\xi) - \chi_g(\sigma^m \xi)]. \quad (3.10)$$

Given $s \in \mathbb{C}$, consider the functions $d_s : U \rightarrow \mathbb{R}$ and $\hat{d}_s : \Sigma_{\mathcal{A}}^+ \rightarrow \mathbb{R}$ defined by

$$d_s(\Psi(\underline{i})) = \hat{d}_s(\underline{i}) = e^{s[\lambda(\kappa(\underline{i})) - \hat{\lambda}(\hat{e}(\underline{i}))] - \frac{1}{N-1} \ln \frac{G_{\hat{e}(\underline{i})}^-(P_0(\hat{e}(\underline{i})))}{G_{e(\xi)}^-(P_0(e(\xi)))}} \quad , \quad \underline{i} \in \Sigma_{\mathcal{A}}^+ , \quad \xi = \mathcal{S}(\underline{i}). \quad (3.11)$$

Notice that by (3.7) and (3.8),

$$\hat{d}_s(\underline{i}) = e^{s[\hat{\chi}_f(\underline{i}) - \chi_f(\xi)] - [\hat{\chi}_g(\underline{i}) - \chi_g(\xi)]} \quad , \quad \xi = \mathcal{S}(\underline{i}). \quad (3.12)$$

Now for any function $V : \Sigma_{\mathcal{A}}^+ \rightarrow \mathbb{C}$, any $s \in \mathbb{C}$ and any integer $n \geq 1$, setting $\xi = \mathcal{S}(\underline{i})$ and $\zeta = \mathcal{S}(\underline{k})$ and using (3.9) and (3.10), we get

$$\begin{aligned} L_{-s\hat{f}+\hat{g}}^n \left((\hat{d}_s \circ \mathcal{S}^{-1}) \cdot (V \circ \mathcal{S}^{-1}) \right) (\xi) &= \sum_{\sigma^n \zeta = \xi} e^{-s\hat{f}_n(\zeta) + \hat{g}_n(\zeta)} \hat{d}_s(\underline{k}) V(\mathcal{S}^{-1}(\zeta)) \\ &= \sum_{\sigma^n \underline{k} = \underline{i}} e^{-s\hat{f}_n(\underline{k}) - s[\hat{\chi}_f(\underline{k}) - \hat{\chi}_f(\sigma^n \underline{k})] + s[\chi_f(\zeta) - \chi_f(\sigma^n \zeta)]} \\ &\quad \times e^{\hat{g}_n(\underline{k}) + [\hat{\chi}_g(\underline{k}) - \hat{\chi}_g(\sigma^n \underline{k})] - [\chi_g(\zeta) - \chi_g(\sigma^n \zeta)]} \\ &\quad \times e^{s[\hat{\chi}_f(\underline{k}) - \chi_f(\zeta)] - [\hat{\chi}_g(\underline{k}) - \chi_g(\zeta)]} V(\underline{k}) \\ &= \sum_{\sigma^n \underline{k} = \underline{i}} e^{-s\hat{f}_n(\underline{k}) + \hat{g}_n(\underline{k})} \times e^{s[\hat{\chi}_f(\underline{i}) - \chi_f(\xi)] - [\hat{\chi}_g(\underline{i}) - \chi_g(\xi)]} V(\underline{k}) \\ &= \hat{d}_s(\underline{i}) \cdot \left(\mathcal{L}_{-s\hat{f}+\hat{g}}^n V \right) (\underline{i}). \end{aligned}$$

Combining the latter with (3.6), yields

$$L_{-s\hat{f}+\hat{g}}^n \left((\hat{d}_s \circ \mathcal{S}^{-1}) \cdot (V \circ \mathcal{S}^{-1}) \right) (\xi) = \frac{\hat{d}_s(\underline{i})}{\hat{h}_s(\underline{i})} \cdot \mathcal{L}_{-sr+\hat{g}}^n (\hat{h}_s \cdot V) (\underline{i}).$$

Setting $u = (\hat{d}_s \circ \mathcal{S}^{-1}) \cdot (V \circ \mathcal{S}^{-1}) \in C(\Sigma_{\mathcal{A}}^+)$ in the above yields the following

Proposition 4. *For any $u \in C(\Sigma_{\mathcal{A}}^+)$ we have*

$$\left(L_{-s\hat{f}+\hat{g}}^n u \right) (\mathcal{S}(\underline{i})) = \frac{\hat{d}_s(\underline{i})}{\hat{h}_s(\underline{i})} \cdot \mathcal{L}_{-sr+\hat{g}}^n \left(\frac{\hat{h}_s}{\hat{d}_s} \cdot (u \circ \mathcal{S}) \right) (\underline{i}) \quad , \quad \underline{i} \in \Sigma_{\mathcal{A}}^+. \quad (3.13)$$

In particular, the eigenvalues of $\mathcal{L}_{-sr+\hat{g}}$ and $L_{-s\hat{f}+\hat{g}}$ coincide with their multiplicities.

3.3. Dolgopyat type estimates. To obtain Dolgopyat type estimates for the left-hand side of (3.13) we can use the Dolgopyat type estimates for $\mathcal{L}_{-sr+\hat{g}}$, provided $u \circ \mathcal{S}$ is determined by a Lipschitz function on R (with respect to the distance on R determined by the standard metric in $S^*(\Omega)$). To do so we also need to show that the functions d_s and h_s given by (3.11) and (3.5) are Lipschitz on U .

Lemma 1. *Assume that the strong stable and the strong unstable laminations $\{W_\epsilon^s(x)\}_{x \in \Lambda}$ and $\{W_\epsilon^u(x)\}_{x \in \Lambda}$ are Lipschitz in $x \in \Lambda$. Then the functions d_s and h_s are Lipschitz on U .*

Proof. Consider the function d_s on $U_{\tilde{i}}$ for some fixed $\tilde{i} = 1, \dots, k$. Given \tilde{i} , there exists $\tilde{p} = 1, \dots, k_0$ such that $\text{pr}_1(\omega(U_{\tilde{i}})) \subset \Gamma_{\tilde{p}}$. Fix an arbitrary $\tilde{z} \in \Lambda \cap S_{\delta_0}^*(\Omega)$ with a backward itinerary $\eta^{(\tilde{p})} \in \Sigma_A^-$ and $\text{pr}_1(\omega(\tilde{z})) \in \Gamma_{\tilde{p}}$ and an arbitrary $\tilde{y} \in U_{\tilde{i}}$ with backward itinerary $\hat{j}^{(\tilde{i})} \in \Sigma_A^-$. Consider the surface $C = C_{\eta^{(\tilde{p})}}(\tilde{z})$ (see Subsection 2.2 for the definition).

Notice that for any $\underline{i} = (i_0, i_1, \dots) \in \Sigma_A^+$ with $i_0 = \tilde{i}$ and any $y \in U_{\tilde{i}}$, locally the surface $C_{\hat{e}(\underline{i})}(y)$ coincides with $\text{pr}_1(\phi_t(U_{\tilde{i}}))$ for some $t = t(y)$ which is a Lipschitz function of $y \in U_{\tilde{i}}$ (in fact $t(y)$ extends smoothly to a neighborhood of $U_{\tilde{i}}$ in the local unstable manifold containing it). Thus, $G_{\hat{e}(\underline{i})}(y)$ is a Lipschitz function of $y \in U_{\tilde{i}}$. Similarly, for $\xi = (\xi_0, \xi_1, \dots) \in \Sigma_A^+$ with $\xi_0 = \tilde{p}$ and $z \in \Lambda$ with $\text{pr}_1(\omega(z)) \in \Gamma_{\tilde{p}}$, $G_{e(\xi)}(z)$ is a Lipschitz function of z .

Next, let $x \in U_{\tilde{i}}$. Then $x = \psi(\underline{i})$ for some $\underline{i} = (i_0, i_1, \dots) \in \Sigma_A^+$ with $i_0 = \tilde{i}$, and by the choice of $\hat{e}(\underline{i})$ (see Subsection 2.3), we have $x = \Psi(\hat{e}(\underline{i})) \in U_{\tilde{i}}$. Thus, $\hat{\lambda}(\hat{e}(\underline{i})) = \lambda(x)$, and the definition of κ gives $\kappa(\underline{i}) = [x, y_{\tilde{p}}]$. Moreover, $x = \Psi(\hat{e}(\underline{i}))$ shows that $P_0(\hat{e}(\underline{i})) = \text{pr}_1(\omega(x))$. Finally, notice that $P_0(e(\xi)) = \text{pr}_1(\omega([x, y_{\tilde{p}}]))$. Indeed, the point $z = [x, y_{\tilde{p}}]$ lies on $W^s(x)$, so its forward itinerary in the model Σ_A^+ is the same as that of $x = \psi(\underline{i})$, i.e. it is \underline{i} . Thus, the forward itinerary of z in the model Σ_A^+ is $\mathcal{S}(\underline{i}) = \xi$. On the other hand, $\phi_s(z) \in W^u(y_{\tilde{p}})$ for some (small) $s \in \mathbb{R}$, so z has the same backward itinerary in the model Σ_A as $y_{\tilde{p}}$, i.e. it is $\eta^{(\tilde{p})}$. Thus, z lies on the trajectory determined by $e(\xi) = (\eta^{(\tilde{p})}; \xi) \in \Sigma_A$, and therefore $P_0(e(\xi)) = \text{pr}_1(\omega(z)) = \text{pr}_1(\omega([x, y_{\tilde{p}}]))$.

It now follows from the above and (3.11) that

$$\ln d_s(x) = s [\lambda([x, y_{\tilde{p}}]) - \lambda(x)] - \frac{1}{N-1} \ln \frac{G_{\hat{e}(\underline{i})}^-(\text{pr}_1(\omega(x)))}{G_{e(\xi)}^-(\text{pr}_1(\omega([x, y_{\tilde{p}}])))}$$

Since $\lambda : R \rightarrow [0, \infty)$ is Lipschitz (it is actually smooth on a neighborhood of Λ in $S^*(\Omega)$) and $[\cdot, \cdot]$ is uniformly Lipschitz, it follows that $d_s(x)$ is Lipschitz with $\text{Lip}(d_s) \leq \text{Const} |s|$. When $\Re(s)$ is bounded, we can also write

$$\text{Lip}(d_s) \leq \text{Const} |\text{Im}(s)| \quad , \quad s \in \mathbb{C} \quad , \quad |\Re(s)| \leq \text{Const} .$$

The same argument applies to the function h_s . □

Denote by $C^{\text{Lip}}(U)$ the space of Lipschitz functions $v : U \rightarrow \mathbb{C}$. For such v let $\text{Lip}(v)$ denote the Lipschitz constant of v , and for $t \in \mathbb{R}$, $|t| \geq 1$, define

$$\|v\|_{\text{Lip}, t} = \|v\|_0 + \frac{\text{Lip}(v)}{|t|} \quad , \quad \|v\|_0 = \sup_{x \in U} |v(x)| .$$

Let $P(F)$ denote the topological pressure of F defined by

$$P(F) = \sup_{\mu \in \mathcal{M}_{\tilde{\sigma}}} [h(\mu) + \int_{\Sigma_A^+} F d\mu] ,$$

where $\mathcal{M}_{\tilde{\sigma}}$ is the set of all probability measures on Σ_A^+ invariant with respect to $\tilde{\sigma}$ and $h(\mu)$ is the measure theoretic entropy of $\tilde{\sigma}$ with respect to μ .

In the next section we will need an estimate for the iterations $L_{-s\tilde{f}-h_0}^n$. Below we deal with more general situation when \tilde{g} and \hat{g} are not necessarily constant functions. In particular, we study the case when g is defined as in Subsection 2.2 by Gauss curvatures at reflection points. This analysis is motivated by applications related to the dynamical zeta function (see [PS2]). The case $g = h_0$ is covered by the same argument.

We will apply Dolgopyat type estimates ([D]) established in the case of open billiard flows in [St2] for $N = 2$ and in [St3] for $N \geq 3$ under certain assumptions. We are now going to state these assumptions in details.

The following *pinching condition*² is one of the assumptions needed below:

(P): *There exist constants $C > 0$ and $\alpha > 0$ such that for every $x \in \Lambda$ we have*

$$\frac{1}{C} e^{\alpha_x t} \|u\| \leq \|d\phi_t(x) \cdot u\| \leq C e^{\beta_x t} \|u\| \quad , \quad u \in E^u(x) \quad , t > 0 \quad ,$$

for some constants $\alpha_x, \beta_x > 0$ depending on x but independent of u with $\alpha \leq \alpha_x \leq \beta_x$ and $2\alpha_x - \beta_x \geq \alpha$ for all $x \in \Lambda$.

Notice that when $N = 2$ this condition is always satisfied. For $N \geq 3$, (P) follows from certain estimates on the eccentricity of the connected components K_j of K . According to general regularity results ([PSW]), (P) implies that $W_\epsilon^u(x)$ and $W_\epsilon^s(x)$ are Lipschitz in $x \in \Lambda$.

Next, consider the following *non-flatness condition*:

(NF): *For every $x \in \Lambda$ there exists $\epsilon_x > 0$ such that there is no C^1 submanifold X of $W_{\epsilon_x}^u(x)$ of positive codimension with $\Lambda \cap W_{\epsilon_x}^u(x) \subset X$.*

Clearly this condition is always satisfied if $N = 2$, while for $N \geq 3$ it is at least generic. In the proof of the main result in [St3] this condition plays a technical role, and one would expect that a future refinement of the proof would remove it.

Next, we need some definitions from [St3]. Given $z \in \Lambda$, let $\exp_z^u : E^u(z) \rightarrow W_{\epsilon_0}^u(z)$ and $\exp_z^s : E^s(z) \rightarrow W_{\epsilon_0}^s(z)$ be the corresponding *exponential maps*. A vector $b \in E^u(z) \setminus \{0\}$ is called *tangent to Λ* at z if there exist infinite sequences $\{v^{(m)}\} \subset E^u(z)$ and $\{t_m\} \subset \mathbb{R} \setminus \{0\}$ such that $\exp_z^u(t_m v^{(m)}) \in \Lambda \cap W_\epsilon^u(z)$ for all m , $v^{(m)} \rightarrow b$ and $t_m \rightarrow 0$ as $m \rightarrow \infty$. It is easy to see that a vector $b \in E^u(z) \setminus \{0\}$ is tangent to Λ at z iff there exists a C^1 curve $z(t)$ ($0 \leq t \leq a$) in $W_\epsilon^u(z)$ for some $a > 0$ with $z(0) = z$, $\dot{z}(0) = b$, and $z(t) \in \Lambda$ for arbitrarily small $t > 0$. In a similar way one defines tangent vectors to Λ in $E^s(z)$.

Denote by $d\alpha$ the standard symplectic form on $T^*(\mathbb{R}^N) = \mathbb{R}^N \times \mathbb{R}^N$. The following condition says that $d\alpha$ is in some sense non-degenerate on the ‘tangent space’ of Λ near some of its points:

(ND): *There exist $z_0 \in \Lambda$, $\epsilon > 0$ and $\mu_0 > 0$ such that for any $\hat{z} \in \Lambda \cap W_\epsilon^u(z_0)$ and any unit vector $b \in E^u(\hat{z})$ tangent to Λ at \hat{z} there exist $\tilde{z} \in \Lambda \cap W_\epsilon^u(z_0)$ arbitrarily close to \hat{z} and a unit vector $a \in E^s(\tilde{z})$ tangent to Λ at \tilde{z} with $|d\alpha(a, b)| \geq \mu_0$.*

Clearly when $N = 2$ this condition is always satisfied. In fact, it seems very likely (and there is some evidence supporting it) that this condition is always satisfied for open billiard flows.

Given a real-valued function $q \in \mathcal{F}_\theta(\Sigma_{\mathcal{A}}^+)$, there exists a unique number $s(q) \in \mathbb{R}$ such that $P(-s(q)r + g) = 0$. The following is an immediate consequence of the main result in [St3].

Theorem 3. *Assume that the billiard flow ϕ_t over Λ satisfies the conditions (P), (NF) and (ND). Let $q \in \mathcal{F}_\theta(\Sigma_{\mathcal{A}}^+)$ be a real-valued function such that $q \circ \psi^{-1} \in C^{\text{Lip}}(U)$. Then for any $a > 0$ there exist constants $\sigma(g) < s(q)$, $C = C(a) > 0$ and $0 < \rho < 1$ so that for $s = \tau + \mathbf{i}t$ with*

²It appears that in the proof of the Dolgopyat type estimates in the case of open billiard flows (and some geodesic flows), one should be able to replace the condition (P) by just assuming Lipschitzness of the stable and unstable laminations – this will be the subject of some future work.

$\sigma(g) \leq \tau$, $|\tau| \leq a$, $|t| \geq 1$ and $n = p[\log |t|] + l$, $p \in \mathbb{N}$, $0 \leq l \leq [\log |t|] - 1$, for every function $v : \Sigma_{\mathcal{A}}^+ \rightarrow \mathbb{C}$ with $v \circ \psi^{-1} \in C^{\text{Lip}}(U)$ we have

$$\|(\mathcal{L}_{-sr+q}^n v) \circ \psi^{-1}\|_{\text{Lip},t} \leq C \rho^{p[\log |t|]} e^{lP(-\tau r+q)} \|v \circ \psi^{-1}\|_{\text{Lip},t}. \quad (3.14)$$

As mentioned above, the conditions (P), (NF) and (ND) are always satisfied for $N = 2$, so (3.14) hold for $N = 2$ without any additional assumptions.

Now combining (3.13) and Theorem 3, we obtain estimates for the iterations $L_{-s\tilde{f}+\tilde{g}}^n u$, provided $v = u \circ \mathcal{S}$ is determined by a Lipschitz function on R .

To relate the quantities $P(-\tau r + \hat{g})$ and $P(-\tau \tilde{f} + \tilde{g})$, consider

$$\underline{T}_n = \sum_{\sigma^n \dot{i} = \dot{i}} e^{-\tau r n(\dot{i}) + \hat{g}(\dot{i})} = \sum_{\sigma^n \dot{i} = \dot{i}} e^{-\tau d_{\gamma(\dot{i})} + g'_n(\dot{i})},$$

where $d_{\gamma(\dot{i})}$ is the length of the periodic trajectory $\gamma(\dot{i})$ determined by $\dot{i} \in \Sigma_{\mathcal{A}}^+$. Since $\mathcal{S} \circ \sigma = \sigma \circ \mathcal{S}$, we get

$$g'_n(\dot{i}) = g_n(\mathcal{S}\dot{i}) = \tilde{g}_n(\mathcal{S}\dot{i}).$$

Setting $\mathcal{S}\dot{i} = v \in \Sigma_{\mathcal{A}}^+$, we have $\sigma^n v = v$ and the periodic trajectory $\gamma(v)$ determined by v has length $d_{\gamma(v)} = d_{\gamma(\dot{i})}$. Thus

$$\underline{T}_n = \sum_{\sigma^n v = v} e^{-\tau d_{\gamma(v)} + \tilde{g}_n(v)} = \sum_{\sigma^n v = v} e^{-\tau \tilde{f}_n(v) + \tilde{g}_n(v)} = T_n.$$

On the other hand, it follows from a general property of the topological pressure that

$$P(-\tau r + \hat{g}) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \underline{T}_n$$

(see e.g. [R] or Theorem 20.3.7 in [KH] from which this property can be derived). Similarly,

$$P(-\tau \tilde{f} + \tilde{g}) = \lim_{n \rightarrow \infty} \frac{1}{n} \log T_n,$$

so we get

$$P(-\tau r + \hat{g}) = P(-\tau \tilde{f} + \tilde{g}). \quad (3.15)$$

Introduce the number $s_0 \in \mathbb{R}$ such that $P(-s_0 r + \hat{g}) = P(-s_0 \tilde{f} + \tilde{g}) = 0$.

As a consequence of (3.15), Theorem 3, Lemma 1 and Proposition 4 we obtain the following

Theorem 4. *Assume that the billiard flow ϕ_t over Λ satisfies the conditions (P), (NF) and (ND). Then for any $a > 0$ there exist constants $\sigma_0 < s_0$, $C' = C'(a) > 0$ and $0 < \rho < 1$ so that for any $s = \tau + \mathbf{i}t \in \mathbb{C}$ with $\tau \geq \sigma_0$, $|\tau| \leq a$, $|t| \geq 1$, any integer $n = p[\log |t|] + l$, $p \in \mathbb{N}$, $0 \leq l \leq [\log |t|] - 1$, and any function $u : \Sigma_{\mathcal{A}}^+ \rightarrow \mathbb{R}$ such that $u \circ \mathcal{S} \circ \psi^{-1} \in C^{\text{Lip}}(U)$ we have*

$$\left\| \left(L_{-s\tilde{f}+\tilde{g}}^n u \right) \circ \mathcal{S} \circ \psi^{-1} \right\|_{\text{Lip},t} \leq C' \rho^{p[\log |t|]} e^{lP(-\tau \tilde{f} + \tilde{g})} \|u \circ \mathcal{S} \circ \psi^{-1}\|_{\text{Lip},t}. \quad (3.16)$$

Proof. Using Theorem 3 with $q = \hat{g}$, we find constants σ_0 , C and ρ satisfying (3.14) with $q = \hat{g}$. Given $a > 0$, let $s = \tau + \mathbf{i}t \in \mathbb{C}$ be such that $\tau \geq \sigma_0$, $|\tau| \leq a$ and $|t| \geq 1$, and let $n = p[\log |t|] + l$, $p \in \mathbb{N}$, $0 \leq l \leq [\log |t|] - 1$.

Consider an arbitrary function $u \in C(\Sigma_{\mathcal{A}}^+)$ such that $v = u \circ \mathcal{S} \circ \psi^{-1} \in C^{\text{Lip}}(U)$. It follows from Lemma 1 (and its proof) that there exists a constant $\text{Const} > 0$, depending on a , such that for $s \in \mathbb{C}$ with $|\text{Re}(s)| \leq a$ we have $\text{Lip}(h_s) \leq \text{Const} |t|$ and $\text{Lip}(d_s) \leq \text{Const} |t|$. From the definitions of these

functions we also have $\|h_s\|_0 \leq \text{Const}$, $\|d_s\|_0 \leq \text{Const}$, $\|d_s/h_s\|_0 \leq \text{Const}$ and $\|h_s/d_s\|_0 \leq \text{Const}$. This implies $\text{Lip}(h_s/d_s) \leq \text{Const}|t|$ and $\text{Lip}(d_s/h_s) \leq \text{Const}|t|$. Hence

$$\text{Lip}((h_s/d_s)v) \leq \|h_s/d_s\|_0 \text{Lip}(v) + \|v\|_0 \text{Lip}(h_s/d_s) \leq \text{Const}(\text{Lip}(v) + |t| \|v\|_0).$$

Thus,

$$\|(h_s/d_s)v\|_{\text{Lip},t} = \|(h_s/d_s)v\|_0 + \frac{\text{Lip}((h_s/d_s)v)}{|t|} \leq \text{Const} \left(\|v\|_0 + \frac{\text{Lip}(v)}{|t|} \right) = \text{Const} \|v\|_{\text{Lip},t}.$$

Similarly,

$$\left\| \frac{d_s}{h_s} \mathcal{L}_{-sr+\hat{g}}^n \left(\frac{h_s}{d_s} v \right) \right\|_{\text{Lip},t} \leq \text{Const} \left\| \mathcal{L}_{-sr+\hat{g}}^n \left(\frac{h_s}{d_s} v \right) \right\|_{\text{Lip},t}.$$

Using the above, (3.14), (3.13) and (3.15) we get

$$\begin{aligned} \left\| \left(L_{-s\tilde{f}+\tilde{g}}^n u \right) \circ \mathcal{S} \circ \psi^{-1} \right\|_{\text{Lip},t} &= \left\| \frac{d_s}{h_s} \mathcal{L}_{-sr+\hat{g}}^n \left(\frac{h_s}{d_s} v \right) \right\|_{\text{Lip},t} \leq \text{Const} \left\| \mathcal{L}_{-sr+\hat{g}}^n \left(\frac{h_s}{d_s} v \right) \right\|_{\text{Lip},t} \\ &\leq \text{Const} \rho^{p[\log|t|]} e^{LP(-\tau r+\hat{g})} \|(h_s/d_s)v\|_{\text{Lip},t} \\ &\leq \text{Const} \rho^{p[\log|t|]} e^{LP(-\tau \tilde{f}+\tilde{g})} \|v\|_{\text{Lip},t}. \end{aligned}$$

This proves (3.16). \square

4. PROOFS OF THEOREMS 1 AND 2

Consider the space Σ_A and the function $f : \Sigma_A \rightarrow \mathbb{R}^+$ introduced in subsection 2.2. Let \mathcal{M}_σ be the *space of all probability measures on Σ_A invariant with respect to σ* . For a continuous function $G : \Sigma_A \rightarrow \mathbb{R}$, the *pressure* $P(G)$ is defined by

$$P(G) = \sup \left[h(\mu) + \int G d\mu : \mu \in \mathcal{M}_\sigma \right],$$

where $h(\mu)$ is the measure theoretic entropy of σ with respect to μ . The measure of maximal entropy μ_0 for σ is determined by

$$h(\mu_0) = \sup \{ h(\mu) : \mu \in \mathcal{M}_\sigma \} = P(0) = h_0,$$

$h_0 > 0$ being the topological entropy of σ . Since σ is conjugated to the billiard ball map $B : \Lambda_{\partial K} \rightarrow \Lambda_{\partial K}$, introduced in Sect. 2, h_0 coincides with the topological entropy of B . On the other hand, the matrix A related to the symbolic codings with obstacles has an unique maximal simple eigenvalues $\lambda > 1$ and $h_0 = \log \lambda$. Notice that if we consider, as in Section 2, the function $\tilde{f} \in \mathcal{F}_\theta(\Sigma_A)$ depending only on future coordinates, then $P(f) = P(\tilde{f})$.

Next, consider the space $\overline{\Sigma_A} = \Sigma_A \times \Sigma_A$ and the shift operator $\tilde{\sigma}(x, y) = (\sigma x, \sigma y)$. Notice that $\overline{\Sigma_A}$ is a subshift of finite type with matrix

$$\tilde{A}((i, j), (i', j')) = A(i, j)A(i', j').$$

Given a function $G : \overline{\Sigma_A} \rightarrow \mathbb{R}$, we define the *pressure* $P(G)$ by

$$P(G) = \sup \left[h(m) + \int G dm : m \in \mathcal{M}_{\tilde{\sigma}} \right],$$

$\mathcal{M}_{\tilde{\sigma}}$ being the space of probability measures on $\overline{\Sigma_A}$ invariant with respect to $\tilde{\sigma}$. As in [PoS2], the measure of maximal entropy m_0 for $\tilde{\sigma}$ is equal to $\mu_0 \times \mu_0$ and the topological entropy of $\tilde{\sigma}$ is $2h_0$.

Consider the function $F(x, y) = \tilde{f}(x) - \tilde{f}(y)$, $x, y \in \Sigma_A^+$, and notice that $F(sR)|_{s=0} = 2h_0$,

$$\frac{d}{ds}F(sR)|_{s=0} = \int F(x, y) dm_0(x, y) = \int \tilde{f}(x)d\mu_0(x) - \int \tilde{f}(y)d\mu_0(y) = 0.$$

Here we used the fact that

$$\frac{d}{ds}P(sf)|_{s=0} = \int \tilde{f}(x)d\mu_0.$$

Lemma 2. *There do not exist constants $a > 0$ and $c \in \mathbb{R}$ and continuous functions $\psi : \overline{\Sigma_A} \rightarrow \mathbb{R}$ and $M : \overline{\Sigma_A} \rightarrow a\mathbb{Z}$ such that*

$$F = \psi \circ \bar{\sigma} - \psi + M + c.$$

Proof. Let $d = \text{dist}(K_1, K_2) > 0$. Assume that the above equation holds for some $a > 0$, $c \in \mathbb{R}$ and continuous functions $\psi : \overline{\Sigma_A} \rightarrow \mathbb{R}$ and $M : \overline{\Sigma_A} \rightarrow a\mathbb{Z}$. Fix a point $y \in \Sigma_A$ with $\sigma^2 y = y$ corresponding to a periodic trajectory with 2 reflection points and length $2d$. Then for every $n = 2k \geq 2$ and $\sigma^n x = x$, we get

$$F^n(x, y) = \tilde{f}_{2k}(x) - \tilde{f}_{2k}(y) = M^{2k}(x, y) + 2kc$$

so

$$\tilde{f}_{2k}(x) = M^{2k}(x, y) + 2kq, \quad \forall k \in \mathbb{N}$$

with $q = c + d$. Now we will exploit the construction in Lemma 5.2 in [St1], where configurations

$$\alpha_k = \underbrace{\{1, 2, 1, 2, \dots, 1, 2, 3, 1\}}_{4k \text{ terms}}, \quad k \in \mathbb{N}$$

with $4k+2$ terms has been considered. Let T_k be the periods of the primitive periodic rays following the configurations α_k . It was shown in [St1] that T_k satisfy the estimate

$$T_{k-1} + 4d < T_k < T_{k-1} + 4d + C\delta^{2k-4}$$

with some global constants $0 < \delta < 1$, $C > 0$ independent on k . Then

$$T_k = az_k + (4k + 2)q, \quad T_{k-1} = az_{k-1} + (4k - 2)q, \quad 2d = az_0 + 2q$$

with some integers $z_0, z_{k-1}, z_k \in \mathbb{Z}$. This implies

$$z_{k-1} + 2z_0 < z_k < z_{k-1} + 2z_0 + \frac{C}{a}\delta^{2k-4}.$$

Letting $k \rightarrow \infty$, we deduce that $z_k = 0$ for $k \geq k_0$. This implies $T_k = (4k + 2)q$, $k \geq k_0$ and we get

$$2az_0 + (4k + 2)q < (4k + 2)q < 2az_0 + (4k + 2)q + C\delta^{2k-4}, \quad k \geq k_0 + 1$$

which leads to a contradiction. □

An application of Lemma 2 shows (see [PP]) that there exists $\beta > 0$ such that

$$\frac{d^2}{ds^2}P(sF)|_{s=0} = \beta^2 > 0.$$

Repeating the proof of Lemma 1.4 and Lemma 1.5 in [PoS2], we obtain the following

Lemma 3. *The function $t \rightarrow e^{itF}$ has a Taylor expansion*

$$e^{itF} = e^{2h_0} \left(1 - \frac{\beta^2 t^2}{2} + \mathcal{O}(|t|^3) \right)$$

with $\mathcal{O}(|t|^3)$ uniform on bounded intervals. Moreover, there exists a change of coordinates $v = v(t)$ such that for $|t| \leq \epsilon$ we have $e^{itR} = e^{2h_0}(1 - v^2)$.

We say that $x \in \Sigma_A^+$ is a prime point if $\sigma^n x = x$ for some $n \geq 2$ and there are no integers $m < n$ with the property $\sigma^m x = x$. Next, as in [PoS2], given a continuous non-negative function $\chi : \mathbb{R} \rightarrow \mathbb{R}$ with compact support, we introduce the function

$$\rho_N(\chi) = \sum_{\substack{\gamma, \gamma' \\ |\gamma|, |\gamma'| \leq N}} \chi(T_\gamma - T_{\gamma'}).$$

Then

$$\begin{aligned} \rho_N(\chi) &= \sum_{n,m=1}^N \sum_{|\gamma|=n} \sum_{|\gamma'|=m} \chi(T_\gamma - T_{\gamma'}) \\ &= \Xi_N(\chi) + \mathcal{O}(\|\chi\|_\infty (\log N)^2 e^{3h_0 N/2}) \end{aligned}$$

with

$$\begin{aligned} \Xi_N(\chi) &= \sum_{|\gamma|=n} \sum_{|\gamma'|=m} \frac{1}{nm} \sum_{\substack{\sigma^n x=x, \sigma^m y=y \\ x,y \text{ prime points}}} \chi(T_\gamma - T_{\gamma'}) \\ &= \sum_{|\gamma|=n} \sum_{|\gamma'|=m} \frac{1}{nm} \sum_{\substack{\sigma^n x=x, \sigma^m y=y \\ x,y \text{ prime points}}} \chi(\tilde{f}_n(x) - \tilde{f}_m(y)). \end{aligned}$$

Next the *proof of Theorem 1* follows without any change that of Theorem 1 in [PoS2] and we omit the details.

Proof of Theorem 2. The crucial point is an estimate for the iterations of the Ruelle operator

$$L_{it\tilde{f}-h_0} w(\xi) = \sum_{\sigma\eta=\xi} e^{it\tilde{f}(\eta)-h_0} w(\eta) = e^{-h_0} \sum_{\sigma\eta=\xi} e^{it\tilde{f}(\eta)} w(\eta).$$

It was shown in Section 3 that $P(-\tau r - h_0) = P(-\tau \tilde{f} - h_0)$. This shows that $P(-sr - h_0)|_{s=0} = h_0$, so for the Ruelle operator L_{-sr-h_0} and $s = it$, $|t| \geq 1$ we can apply the estimates (3.14) with $v = 1$. Next, the Ruelle operator $L_{it\tilde{f}-h_0}$ is conjugated to L_{itr-h_0} by (3.13), hence the estimates (3.14) can be applied to $L_{itr-h_0}^n$. More precisely, there exist $0 < \rho_1 < 1$ and $C > 0$ so that for $n = p[\log |t|] + l$, $0 \leq l \leq [\log |t|] - 1$ we have

$$\|L_{it\tilde{f}-h_0}^n 1\|_\infty \leq C^{1/2} \rho_1^p \rho_1^{[\log |t|]}, \quad n \geq 1.$$

This yields

$$\|L_{\pm it\tilde{f}}^n 1\|_\infty \leq C^{1/2} e^{h_0 n} \rho^p \rho^{[\log |t|]/2}, \quad n \geq 1,$$

with $0 < \rho = \rho_1^{1/2} < 1$.

Consider the transfer operator

$$(L_{itF} v)(x, y) = \sum_{\bar{\sigma}(x', y') = (x, y)} e^{itF(x', y')} v(x', y').$$

Since

$$\begin{aligned} (L_{itF}^n 1)(x_0, y_0) &= \sum_{\substack{\sigma^n x = x_0, \\ \sigma^n y = y_0}} e^{itF_n(x, y)} = \sum_{\sigma^n x = x_0} e^{it\tilde{f}_n(x)} \sum_{\sigma^n y = y_0} e^{-it\tilde{f}_n(y)} \\ &= L_{it\tilde{f}}^n 1(x_0) L_{-it\tilde{f}}^n 1(y_0), \end{aligned}$$

it follows that

$$\|L_{itF}^n 1\|_\infty \leq C e^{2h_0 n} \rho^{p|\log|t|} \leq C e^{2h_0 n} \rho^n \rho^{-\log|t|} \leq C e^{2h_0 n} \min\{\rho^n |t|^\alpha, 1\} \quad (4.1)$$

with $\alpha = |\log \rho|$. Thus we obtain the following

Lemma 4. *There exists $C > 0$, $0 \leq \rho \leq 1$ and $\alpha > 0$ such that for $|t| \geq 1 > 0$ we have*

$$\|L_{itF}^n 1\|_\infty \leq C e^{2h_0 n} \min\{\rho^n |t|^\alpha, 1\}.$$

Next, consider the function

$$\mathcal{S}_N(t) = \sum_{n, m=1}^N \sum_{\substack{\sigma^n x = x, \sigma^m x = x, \\ x, y \text{ prime points}}} e^{it(f^n(x) - f^m(y))}.$$

Let $\chi \in C_0^\infty(\mathbb{R})$ be a nonnegative function and let $\hat{\chi}$ be the Fourier transform of χ . Given a sequence $\epsilon_n = \mathcal{O}(e^{-\eta n})$, denote

$$A_2(N, z) = \left| \int_{|t| \geq \epsilon_n \beta \sqrt{N}} e^{\frac{itz}{\beta \sqrt{N}}} \left[e^{-2h_0 N} \left(\mathcal{S}_N \left(\frac{t}{\beta \sqrt{N}} \right) \hat{\chi} \left(\frac{\epsilon_n t}{\beta \sqrt{N}} \right) \right) \right] dt \right|.$$

As in [PoS2], applying Lemma 4 we obtain the following

Lemma 5. *For sufficiently small $\eta > 0$ and for $N \rightarrow +\infty$ we have $\sup_{z \in \mathbb{R}} A_2(N, z) = 0$.*

The rest of the proof of Theorem 2 follows closely arguments in [PoS2] and we omit the details.

By the same arguments we get

Proposition 5. *Let ϵ_n be a sequence such that $\epsilon_n = \mathcal{O}(e^{-\eta n})$, where $\eta > 0$ is a sufficiently small constant. Then for*

$$\omega(n, I_n(z)) = \#\{(\gamma, \gamma') : |\gamma| = |\gamma'| = n, z + \epsilon_n a \leq T_\gamma - T_{\gamma'} \leq z + \epsilon_n b\}$$

we have

$$\lim_{n \rightarrow +\infty} \sup_{z \in \mathbb{R}} \left| \frac{n^{5/2}}{\epsilon_n e^{2h_0 n}} \omega(n, I_n(z)) - \frac{b-a}{(2\pi)^{1/2} \beta} e^{-z^2/2\beta^2 n} \right| = 0.$$

5. SEPARATION CONDITION FOR THE LENGTHS OF THE PRIMITIVE PERIODIC RAYS

In this section we discuss some open problems related to the distribution of the lengths of primitive periodic rays. Let \mathcal{P} be the set of all primitive periodic rays in Ω . Let Π be the set of the lengths of rays $\gamma \in \mathcal{P}$ and let Ξ be the set of periods of all periodic rays in Ω . It is known that

$$\#\{\gamma \in \mathcal{P} : T_\gamma \leq x\} = \frac{e^{hx}}{hx} (1 + o(1)), \quad x \rightarrow +\infty, \quad (5.1)$$

where

$$h = \sup_{\mu \in \mathcal{M}_\sigma} \frac{h(m)}{\int f d\mu} > 0$$

is the *topological entropy* of the open billiard flow in Ω . In particular, if we have Dolgopyat type estimates for the Ruelle operator \mathcal{L}_{-sr} for $s = \tau + it$, $|t| \geq 1$ and $h - \epsilon \leq \tau \leq h$, we can obtain in (5.1) a sharper estimate with a remainder $\mathcal{O}(e^{-\epsilon x})$, $0 < \epsilon < h$ (see [St2], [St3]). It was proved in [PS1], Chapter 3, that for generic obstacles K the periods of the primitive rays are rationally independent, that is

$$\gamma \neq \delta \Rightarrow \frac{T_\gamma}{T_\delta} \notin \mathbb{Q}, \forall \gamma, \forall \delta \in \mathcal{P}.$$

Thus for generic obstacles we have $T_\gamma \neq T_\delta$ for any distinct elements γ and δ of \mathcal{P} .

In what follows we assume that the latter property is satisfied. Let T_γ , $\gamma \in \mathcal{P}$, be ordered as a sequence

$$T_1 < T_2 < \dots < T_n < \dots$$

Introduce the intervals

$$J(\gamma, \delta) = [T_\gamma - e^{-\delta T_\gamma}, T_\gamma + e^{-\delta T_\gamma}], \gamma \in \mathcal{P}, \delta > 0.$$

Obviously, the number of pairs (T_γ, T_δ) lying in $J(\gamma, \delta)$ decreases with δ .

Definition 1. *We say that the obstacle K satisfies the separation condition (S) if there exists $\delta > h > 0$ such that*

$$J(\gamma, \delta) \cap \Pi = T_\gamma, \forall \gamma \in \mathcal{P}. \quad (5.2)$$

We say that K satisfies the condition (S₂) if there exists $\delta > h > 0$ such that

$$\#\{\gamma \in \mathcal{P} : T_\gamma \leq x, J(\gamma, \delta) \cap \Pi = T_\gamma\} \sim e^{\frac{h}{2}x}, x \rightarrow +\infty. \quad (5.3)$$

The condition (S) was introduced in [P2], however we are not aware of any geometric conditions on K that would imply (S). The same can be said about (S₂). It is important to notice that Proposition 5 cannot be applied with the sequence $\epsilon_n = e^{-\delta T_n}$, $\delta > h$. On the other hand, it was shown in [P2] that under the condition (S) there exists a global constant $A_0 > 0$ such that

$$\#\{d_\gamma \in \Xi : d_\gamma \in J(\gamma, \delta)\} \leq A_0 T_\gamma, \forall \gamma \in \mathcal{P}.$$

Thus, the distribution of all periods in Ξ is such that we have no *clustering* of periods with big density in $J(\gamma, \delta)$. It is clear that the condition (S₂) is much weaker than (S).

Let $\chi(t) \in C_0^\infty(-1, 1)$ be a positive function such that $\chi(t) = 1$ for $|t| \leq \epsilon_0 < 1/2$ with Fourier transform $\hat{\chi}(\xi) \geq 0$, $\forall \xi \in \mathbb{R}$. Consider the function $\varphi_j(t) = \chi(e^{\delta T_j}(t - T_j))$. For the proof of the so called *Modified Lax-Phillips Conjecture (MLPC)* for the Dirichlet problem in $\mathbb{R} \times \Omega$ (see [I2]) it is necessary to find $\delta > h$ and a sequence $j \rightarrow +\infty$ so that

$$\left| \sum_\gamma (-1)^{|\gamma|} T_\gamma |\det(I - P_\gamma)|^{-1} \varphi_j(d_\gamma) \right| \geq \eta_0 e^{-\eta d_\gamma} \quad (5.4)$$

with $\eta_0 > 0$, $\eta > 0$ independent on j . Here P_γ is the *linear Poincaré map* related to γ introduced in Sect. 1. It is easy to see that under the condition (S₂) we can arrange (5.4) for a suitable sequence $j \rightarrow +\infty$, so (MLPC) follows from (S₂).

Indeed, assuming (S₂), consider the number of iterated periodic rays $\mu \notin \mathcal{P}$ with lengths $d_\mu \in J(\gamma, \delta)$, provided $\gamma \in \mathcal{P}$, $T_\gamma \leq x$. If $d_\mu = kT_\nu$, $k \geq 2$, then $2 \leq k \leq \frac{x+1}{d_0}$, where $d_0 =$

$2 \min_{i \neq j} \text{dist}(K_i, K_j)$. Thus, using (5.1), the number of such non primitive periodic rays μ for $x \geq M_0$ large enough is not greater than

$$\frac{C}{h(x+1)} \sum_{k=2}^{(x+1)/d_0} k e^{\frac{hx}{k}} \leq \frac{C_1 e^{\frac{h}{2}x}}{x+1} \left(1 + e^{-\frac{h}{6}x} \sum_{k=3}^{(x+1)/d_0} \frac{k}{2} \right) \leq \frac{1}{2} e^{\frac{h}{2}x}.$$

Consequently, taking into account (5.3), it is possible to find an infinite number of intervals $J(\gamma_j, \delta)$ with $\gamma_j \in \mathcal{P}$ such that $T_{\gamma_j} \rightarrow +\infty$ and

$$J(\gamma_j, \delta) \cap \Xi = T_{\gamma_j}.$$

For such rays the sum on the left hand side of (5.4) is reduced to the term $T_{\gamma_j} |\det(I - P_{\gamma_j})|^{-1/2}$ and the inequality (5.4) follows from the estimate

$$|\det(I - P_{\gamma_j})| \leq e^{cT_{\gamma_j}}$$

with some global constant $c > 0$ (see for instance, Appendix A.1 in [P1]). By the argument of Ikawa [I2] we conclude that (5.4) implies (MLPC).

The above analysis shows that (S_2) can be replaced by a weaker condition in order to satisfy (5.4). Let \mathcal{P}_e (resp. \mathcal{P}_o) be the set of primitive periodic rays with even (resp. odd) number of reflections and let Π_e (resp. Π_o) be the set of periods of $\gamma \in \mathcal{P}_e$ (resp. $\gamma \in \mathcal{P}_o$). We have ([G]) the following analog of (5.1)

$$\begin{aligned} \#\{\gamma \in \mathcal{P}_e : T_\gamma \leq x\} &\sim \frac{e^{hx}}{2hx}, \quad x \rightarrow +\infty, \\ \#\{\gamma \in \mathcal{P}_o : T_\gamma \leq x\} &\sim \frac{e^{hx}}{2hx}, \quad x \rightarrow +\infty. \end{aligned}$$

Definition 2. We say the obstacle K satisfies the condition (S_3) if there exists $\delta > h$ such that

$$\#\{\gamma \in \mathcal{P}_e : T_\gamma \leq x, J(\gamma, \delta) \cap \Pi_o = \emptyset\} \sim e^{\frac{h}{3}x}, \quad x \rightarrow +\infty. \quad (5.5)$$

Under the condition (S_3) in every interval $J(\gamma, \delta)$, $\gamma \in \mathcal{P}_e$, we may have an arbitrary clustering of periods in Π_e and this leads to a sum of terms with positive signs in (5.4). Assuming (S_3) fulfilled, it is not hard to deduce that there exists a sequence of $\gamma_j \in \mathcal{P}_e$, $T_{\gamma_j} \rightarrow +\infty$, such that

$$J(\gamma_j, \delta) \cap \Pi_o = \emptyset, \quad \forall j \in \mathbb{N},$$

and the latter implies (MLPC). We leave the details to the reader.

It is an open problem to investigate if the conditions (S_2) , (S_3) or some other similar condition are fulfilled.

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