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Claude Bardos[1], Jasmine S. Linshiz[2a] and Edriss S. Titi[2,3]

[1] *Université Denis Diderot and Laboratory J. L. Lions*
Université Pierre et Marie Curie, Paris, France
bardos@ann.jussieu.fr
[2] *Department of Computer Science and Applied Mathematics*
Weizmann Institute of Science
Rehovot 76100, Israel
[a]jasmine.tal@weizmann.ac.il
[3] *Department of Mathematics and*
Department of Mechanical and Aerospace Engineering
University of California
Irvine, CA 92697-3875, USA
etiti@math.uci.edu *and* edriss.titi@weizmann.ac.il

Abstract

We present an α-regularization of the Birkhoff-Rott equation, induced by the two-dimensional Euler-α equations, for the vortex sheet dynamics. We show the convergence of the solutions of Euler-α equations to a weak solution of the Euler equations for initial vorticity being a finite Radon measure of fixed sign, which includes the vortex sheets case. We also show that, provided the initial density of vorticity is an integrable function over the curve with respect to the arc-length measure, (i) an initially Lipschitz chord arc vortex sheet (curve), evolving under the BR-α equation, remains Lipschitz for all times, (ii) an initially Hölder $C^{1,\beta}$, $0 \leq \beta < 1$, chord arc curve remains in $C^{1,\beta}$ for all times, and finally, (iii) an initially Hölder $C^{n,\beta}$, $n \geq 1$, $0 < \beta < 1$, closed chord arc curve remains so for all times. In all these cases the weak Euler-α and the BR-α descriptions of the vortex sheet motion are equivalent.

Keywords: inviscid regularization of Euler equations; Euler-α; Birkhoff-Rott; Birkhoff-Rott-α; vortex sheet.
Mathematics Subject Classification: 76B03, 35Q35, 76B47.

1 Introduction

The α-regularization of the Navier-Stokes equations (NSE) is one of the novel approaches for subgrid scale modeling of turbulence. The inviscid Euler-α model was originally introduced in the Euler-Poincaré variational framework in [38, 39]. In [13–15, 31, 32] the corresponding Navier-Stokes-α (NS-α) [also known as the viscous Camassa-Holm equations or the Lagrangian-averaged Navier-Stokes-α (LANS-α)] model, was obtained by introducing the appropriate viscous term into the Euler-α equations. The extensive research of the α-models (see, e.g., [2, 7, 10, 11, 13–18, 20, 31, 32, 34, 34–36, 40, 42, 43, 48, 49, 51, 52, 63, 77]) stems, on the one hand, from the successful comparison of their steady state solutions to empirical data, for a large range of huge Reynolds numbers, for turbulent flows in infinite channels and pipes [13–15]. On the other hand, the α-models can also be viewed as numerical regularizations of the original, Euler or Navier-Stokes, systems [7, 11, 44, 52]. The main practical question arising is that of the applicability of these regularizations to the correct predictions of the underlying flow phenomena.

In this paper we present some analytical results concerning the α-regularization of the two-dimensional (2D) Euler equations in the context of vortex sheet dynamics. The incompressible Euler equations are

$$\begin{aligned} &\frac{\partial v}{\partial t} + (v \cdot \nabla)v + \nabla p = 0, \\ &\nabla \cdot v = 0, \\ &v(x,0) = v^{in}(x), \end{aligned} \tag{1.1}$$

where v the fluid velocity field and p, the pressure are the unknowns, and v^{in} is the given initial velocity. A vortex sheet is a surface of codimension one (a curve in the plane) in inviscid incompressible flow, across which the tangential component of the velocity has a jump discontinuity, while the normal component is continuous. The flow outside the sheet is irrotational. The evolution of the vortex sheet can be described by the Birkhoff-Rott (BR) equation [8, 67, 68]. This is a nonlinear singular integro-differential equation, which can be obtained formally from the Euler equations assuming that the evolution of a vortex sheet retains a curve-like structure:

$$\frac{\partial \bar{z}}{\partial t}(\Gamma, t) = \frac{1}{2\pi i} \ \text{p.v.} \int_{-\infty}^{\infty} \frac{d\Gamma'}{z(\Gamma, t) - z(\Gamma', t)},$$

here $z = x + iy$ is the complex position of the sheet and $\Gamma \in (-\infty, \infty)$ represents the circulation, that is $\gamma = 1/|z_\Gamma|$ is the vorticity density along the sheet. However, the initial data problem for the BR equation is ill-posed due to the Kelvin-Helmholtz instability [8,69]. Numerous results show that an initially real analytic vortex sheet (curve) can develop a finite time singularity in its curvature. This singularity formation was studied with asymptotic techniques in [23, 64] and numerically in [23, 46, 62]. Specific examples of solutions were constructed in [9, 29], where the development, in a finite time, of curvature singularity from initially analytic data was rigorously proved. After the appearance of the first singularity the solution becomes very irregular. This is a consequence of the elliptic nature of the Birkhoff-Rott equations: if solutions have a certain minimal regularity, then they are actually analytic ([50, 79, 80]). An open problem is the determination of this threshold of regularity that will imply analyticity. It was shown in [50] that any solution consisting of a closed chord arc vortex sheet that near a point belongs to $C^{1,\beta}$, $\beta > 0$ must be analytic. The conclusion is maintained if the vortex sheet is required to be a Lipschitz chord arc curve [79, 80].

The problem of the evolution of a vortex sheet can also be approached, in the general framework of weak solutions (in the distributional sense) of the Euler equations, as a problem of evolution of the vorticity, which is concentrated as a measure along a surface of codimension one. This approach was pioneered by DiPerna and Majda in [26–28]. The general problem of existence for mixed-sign vortex sheet initial data remains an open question. However, in 1991, Delort [25] proved a global in time existence of weak solutions of the 2D incompressible Euler equation for the vortex sheet initial data with initial vorticity being a Radon measure of a distinguished sign, see also [30, 53, 58, 59, 71, 72]. This result was later obtained as an inviscid limit of the Navier-Stokes regularizations of the Euler equations [58, 71], and as a limit of numerical vortex methods [53, 54, 72]. The Delort's result [25] was also extended to the case of mirror-symmetric flows with distinguished sign vorticity on each side of the mirror [57]. It is worth mentioning that uniqueness of solutions of the 2D Euler equation was obtained by Yudovich [81] for initially bounded vorticity, see, also, [76] for an improvement with vorticity in a class slightly larger than L^∞, and [75] for review of relevant two-dimensional results. This does not include vortex sheets, which admit measure-valued vorticity. There is also a non-uniqueness result for velocity in $C\left((0,T), L^2_{\text{weak}}\right)$ [24, 70, 73]. However, the problem of uniqueness of a weak solution with a fixed sign vortex sheet initial data is still unanswered, numerical evidences of non-uniqueness can be found, e.g., in [55, 66]. Furthermore, the structure of weak solutions given by Delort's theorem is not known, while the Birkhoff-Rott equations assume *a priori* that a vortex sheet remains a curve at a later time. A proposed criterion for the equivalence of a weak solution of the 2D Euler equations with vorticity being a Radon measure supported on a curve, and a weak solution of the Birkhoff-Rott equation can be found in [56]. Also, another definition of weak solutions of Birkhoff-Rott equation has been proposed in [79, 80]. For a recent survey of the subject, see [4].

The Euler-α model [15, 21, 37–39, 61] is an inviscid regularization of the Euler equations (1.1) given by

$$\begin{aligned} &\frac{\partial v}{\partial t} + (u \cdot \nabla)\, v + \sum_j v_j \nabla u_j + \nabla \pi = 0, \\ &v = \left(1 - \alpha^2 \Delta\right) u, \\ &\nabla \cdot u = \nabla \cdot v = 0, \\ &u(x,0) = u^{in}(x). \end{aligned} \tag{1.2}$$

Here u represents the "filtered" fluid velocity vector, π is the "filtered" pressure, and $\alpha > 0$ is a regularization lengthscale parameter representing the width of the filter.

The question of global existence of weak solutions for the three-dimensional Euler-α equations is still an open problem. On the other hand, the 2D Euler-α equations were studied in [65], where it has been shown that there exists a unique global weak solution to the Euler-α equations with initial vorticity in the space of Radon measures on $\mathbb{R}^2$, with a unique Lagrangian flow map describing the evolution of particles. In particular, it follows that the vorticity, initially supported on a curve, remains supported on a curve for all times.

In this paper we relate the weak solutions of Euler-α equations with a distinguished sign vortex sheet initial data to those of the 2D Euler equations, by proving their convergence, as the length scale $\alpha \to 0$. This produces a variant of the result of Delort [25], by obtaining a weak solution of Euler equations as a limit of an inviscid regularization of Euler equations, in addition to approximations obtained by smoothing the initial data, viscous regularization or numerical vortex methods [25, 53, 54, 58, 59, 71, 72]. Since a weak solution of Euler equations with vortex sheet is unlikely to be unique, a different regularization could produce a different weak solution.

We also present an analytical study of the α-analogue of the Birkhoff-Rott equation, the Birkhoff-Rott-α (BR-α) model, which is induced by the 2D Euler-α equations. The BR-α results that were reported in a short communication [3] are presented here with full details. The BR-α model was implemented computationally in [41], where a numerical comparison between the BR-α regularization and the existing regularizing methods, such as a vortex blob model [1, 19, 22, 45, 53] has been performed. In the BR-α case the singular kernel of the Biot-Savart law determining the velocity in terms of the vorticity is smoothed by a convolution with a smoothing function $G^\alpha = \frac{1}{\alpha^2}\frac{1}{2\pi} K_0\left(\frac{|x|}{\alpha}\right)$, which is the Green function associated with the Helmholtz operator $\left(I - \alpha^2 \Delta\right)$. The function K_0 is a modified Bessel function of the second kind of order zero. This is similar to vortex blob methods, however, unlike the standard vortex blob methods [1,6,19,22,45,47] (and, in particular, the proof of convergence of vortex blobs methods to a weak solution of 2D Euler equations [53]), the BR-alpha smoothing function G^α is unbounded at the origin. Also, unlike the vortex blob methods that regularize the singular Biot-Savart kernel, the Euler-α model regularizes the Euler equations themselves to obtain a smoother kernel.

Section 2 contains the preliminaries about the 2D Euler-α equations. In Section 3 we investigate the convergence of solutions of the Euler-α equations for vortex sheet initial data to those of the 2D Euler equations, as the regularization length scale α tends to zero. Specifically, we prove that for the vortex sheet initial data with initial vorticity of a distinguished sign Radon measure one can extract subsequences of weak solutions of the Euler-α equations which converge weak-$*$ in $L^\infty\left([0,T]; \mathcal{M}(\mathbb{R}^2)\right)$, as $\alpha \to 0$, to a weak solution of the 2D Euler equation. $\mathcal{M}(\mathbb{R}^2)$ denotes the space of finite Radon measures on $\mathbb{R}^2$.

In Section 4 we describe the BR-α equation. Section 5 studies the linear stability of a flat vortex sheet with uniform vorticity density for the 2D BR-α model. The linear stability analysis shows that the BR-α regularization controls the growth of high wave number perturbations, which is the reason for the well-posedness. This is unlike the case for the original BR problem for Euler equations that exhibits the Kelvin-Helmholtz instability, the main mechanism for its ill-posedness. In Section 6 we show global well-posedness of the 2D BR-α model in the space of Lipschitz functions and in the Hölder space $C^{n,\beta}$, $n \geq 1$, which is the space of n-times differentiable functions with Hölder continuous n^{th} derivative. Specifically, we show that (i) an initially Lipschitz chord arc vortex sheet (curve), evolving under the BR-α equation,

remains Lipschitz for all times, (ii) an initially Hölder $C^{1,\beta}$, $0 \leq \beta < 1$, chord arc curve remains in $C^{1,\beta}$ for all times, and finally, (iii) an initially Hölder $C^{n,\beta}$, $n \geq 1$, $0 < \beta < 1$, closed chord arc curve remains in $C^{n,\beta}$ for all times. Notice that for $n > 1$ we request β to be strictly larger than zero and the curve to be closed. In all these cases the weak Euler-α and the BR-α descriptions of the vortex sheet motion are equivalent. The convergence of BR-α solutions to the solutions of the original BR system on the short interval of existence of solutions will be reported in a forthcoming paper.

2 Euler-α equations

In two dimensions, the incompressible Euler equations in the vorticity form are obtained by taking the curl of (1.1) and are given by

$$\begin{aligned} &\frac{\partial q}{\partial t} + (v \cdot \nabla) q = 0, \\ &v = K * q, \\ &q(x,0) = q^{in}(x), \end{aligned} \tag{2.1}$$

where $K(x) = \frac{1}{2\pi}\nabla^{\perp} \log|x|$, v is the fluid velocity field, $q = \operatorname{curl} v$ is the vorticity, and q^{in} is the given initial vorticity. Delort [25] proved a global in time existence of weak solutions of the 2D Euler equations for the vortex sheet initial data with fixed sign initial vorticity in $\mathcal{M}(\mathbb{R}^2) \cap H^{-1}_{loc}\left(\mathbb{R}^2\right)$. The space $\mathcal{M}(\mathbb{R}^2)$ is the space of finite Radon measures on $\mathbb{R}^2$ with the norm

$$\|\mu\|_{\mathcal{M}} = \sup\left\{ \left| \int_{\mathbb{R}^2} \varphi d\mu \right| : \varphi \in C_0\left(\mathbb{R}^2\right), \|\varphi\|_{L^\infty} \leq 1 \right\},$$

$\mathcal{C}_0(\mathbb{R}^2)$ is the space of continuous functions vanishing at infinity. The space H^{-s} denotes the dual of the Sobolev space H^s. The localized Sobolev space $H^s_{loc}\left(\mathbb{R}^2\right)$, $s \in \mathbb{R}$ is the set of all distributions f such that $\rho f \in H^s(\mathbb{R}^N)$ for any $\rho \in C_c^\infty(\mathbb{R}^N)$, see, e.g., [33].

A vorticity $q \in L^\infty\left([0,T], \mathcal{M}(\mathbb{R}^2) \cap H^{-1}_{loc}\left(\mathbb{R}^2\right)\right) \cap Lip\left([0,T], H^{-L}_{loc}\left(\mathbb{R}^2\right)\right)$, $L > 1$, is called a weak solution of (2.1), if for every test function $\psi \in C_c^\infty\left(\mathbb{R}^2 \times (0,T)\right)$

$$W(q;\psi) \equiv \int_0^T \int_{\mathbb{R}^2} \partial_t \psi(x,t)\, dq(x,t)\, dt + \int_0^T \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} H_\psi(x,y,t)\, dq(y,t)\, dq(x,t)\, dt = 0, \tag{2.2}$$

where

$$H_\psi(x,y,t) = \frac{1}{4\pi} \frac{(x-y)^{\perp} \cdot (\nabla\psi(x,t) - \nabla\psi(y,t))}{|x-y|^2}.$$

The initial value is $q(x,0) = q^{in}(x)$ and it makes sense since $q \in Lip\left([0,T], H^{-L}_{loc}\left(\mathbb{R}^2\right)\right)$. The kernel H_ψ is bounded, continuous outside the diagonal $x = y$ and vanishes at infinity. This weak vorticity formulation is well-defined, since the H^{-1} vorticity has no discrete part (i.e., $q(\{x_0\}, t) = 0$ for all $x_0 \in \mathbb{R}^2$), which implies that the diagonal $x = y$ has $q(x,t)\, q(y,t)$-measure zero, see [25, 71]. Thorough discussions of Delort's theorem, its extension and different proofs of the result can be found in [12, 25, 30, 53, 58, 59, 71, 72].

Taking the curl of (1.2) yields the vorticity formulation of the 2D Euler-α model

$$\begin{aligned} &\frac{\partial q}{\partial t} + (u \cdot \nabla) q = 0, \\ &u = K^\alpha * q, \\ &q(x,0) = q^{in}(x). \end{aligned} \tag{2.3}$$

Here u represents the "filtered" fluid velocity, and $\alpha > 0$ is a regularization length scale parameter, which represents the width of the filter. At the limit $\alpha = 0$, we formally obtain the Euler equations (2.1). The

smoothed kernel is $K^\alpha = G^\alpha * K$, where G^α is the Green function associated with the Helmholtz operator $\left(I - \alpha^2\Delta\right)$, given by

$$G^\alpha(x) = \frac{1}{\alpha^2} G\left(\frac{x}{\alpha}\right) = \frac{1}{\alpha^2}\frac{1}{2\pi} K_0\left(\frac{|x|}{\alpha}\right), \tag{2.4}$$

here $x = (x_1, x_2) \in \mathbb{R}^2$ and K_0 is a modified Bessel function of the second kind of order zero [78]. To see this relationship in $\mathbb{R}^2$ one can take a Fourier transform of $v = \left(1 - \alpha^2\Delta\right) u$, and obtain G^α as the inverse Fourier transform of $\frac{1}{(1+\alpha^2|k|^2)}$. Notice that

$$K^\alpha(x) = \nabla^\perp \Psi^\alpha(|x|) = \frac{x^\perp}{|x|} D\Psi^\alpha(|x|), \tag{2.5}$$

where

$$\Psi^\alpha(r) = \frac{1}{2\pi}\left[K_0\left(\frac{r}{\alpha}\right) + \log r\right],$$
$$D\Psi^\alpha(r) = \frac{d\Psi^\alpha}{dr}(r) = \frac{1}{2\pi}\left[-\frac{1}{\alpha} K_1\left(\frac{r}{\alpha}\right) + \frac{1}{r}\right],$$

and K_1 denotes a modified Bessel functions of the second kind of order one. For details on Bessel functions, see, e.g., [78].

A weak solution of (2.3) is $q \in C\left([0,T]; \mathcal{M}(\mathbb{R}^2)\right)$ satisfying

$$W^\alpha(q;\psi) \equiv \int_0^T \int_{\mathbb{R}^2} \partial_t \psi(x,t)\, dq(x,t)\, dt + \int_0^T \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} H^\alpha_\psi(x,y,t)\, dq(x,t)\, dq(y,t)\, dt = 0, \tag{2.6}$$

for all test functions $\psi \in C_c^\infty\left(\mathbb{R}^2 \times (0,T)\right)$. The initial value is $q(x,0) = q^{in}(x)$ and it makes sense since $q \in C\left([0,T]; \mathcal{M}(\mathbb{R}^2)\right)$. The kernel H^α_ψ is a continuous vanishing at infinity function given by

$$H^\alpha_\psi(x,y,t) = \frac{1}{2} D\Psi^\alpha(|x-y|)\, \frac{(x-y)^\perp \cdot (\nabla\psi(x,t) - \nabla\psi(y,t))}{|x-y|}.$$

Oliver and Shkoller [65] showed global well-posedness of the Euler-α equations with initial vorticity in $\mathcal{M}(\mathbb{R}^2)$.

Theorem 2.1. (Oliver and Shkoller [65]) *For initial data $q^{in} \in \mathcal{M}(\mathbb{R}^2)$, there exists a unique global weak solution of Euler-α equations* (2.3) *in the sense of* (2.6).
Let $\mathcal{G}$ denote the group of all homeomorphism of $\mathbb{R}^2$, which preserve the Lebesgue measure and let $\eta_\alpha = \eta_\alpha(\cdot,t)$ denote the Lagrangian flow map induced by (2.3)*, i.e., which obeys the equation $\partial_t \eta_\alpha(x,t) = u(\eta_\alpha(x,t),t) := \int_{\mathbb{R}^2} K^\alpha(\eta_\alpha(x,t), \eta_\alpha(y,t))\, dq^{in}(y,t)$, $\eta_\alpha(x,0) = x$. Then the unique Lagrangian flow map $\eta_\alpha \in C^1([0,T];\mathcal{G})$ exists globally and the vorticity q_α is transported by the flow, i.e., $q_\alpha(x,t) = q^{in} \circ \eta_\alpha^{-1}(x,t)$.*

Notice that the original BR equations assume *a priori* that a vortex sheet remains a curve at a later time, however, in the 2D Euler-α case, it follows as a consequence of the existence of the unique Lagrangian flow map, that the vorticity that is initially supported on a curve remains supported on a curve for all times.

3 Convergence of a fixed sign Euler-α vortex sheet to an Euler vortex sheet

Let the initial vorticity $q^{in} \in \mathcal{M}(\mathbb{R}^2) \cap H^{-1}_{loc}\left(\mathbb{R}^2\right)$ be of a fixed sign, $q^{in} \geq 0$, and compactly supported. In this section we show that there is a subsequence of the solutions of 2D Euler-α model with initial data q^{in},

guaranteed by Theorem 2.1, that converge to a weak solution of 2D Euler equations in the sense of (2.2). This produces a variant of the result of Delort [25], by obtaining a weak solution of Euler equations as a limit of an solutions of inviscid regularization of Euler equations, namely, the Euler-α equations. This is in addition to different kind of regularizations obtained, for instance, by smoothing the initial data, viscous regularization or numerical vortex methods [25,53,54,58,59,71,72]. Since a weak solution of Euler equations with vortex sheet is unlikely to be unique, a different regularization could produce a different weak solution of Euler equations.

Our analysis is closely related to that of [25, 58, 59, 71], the facts that the solutions q_α of the Euler-α equations have uniform decay of the associated "filtered" vorticity ω_α, which defined by (3.1) below, in small circles and the contribution of $\int_{\mathbb{R}^2} d\left|\alpha^2\Delta\omega_\alpha\right| \to 0$, as $\alpha \to 0$, allow us to show the weak-$*$ convergence of q_α to a weak solution of Euler equations.

Theorem 3.1. *Let q_α be the solutions of the weak vorticity formulation of Euler-α equations* (2.6)*, guaranteed by Theorem 2.1, with initial data $q^{in} \in \mathcal{M}(\mathbb{R}^2) \cap H^{-1}_{loc}\left(\mathbb{R}^2\right)$, $q^{in} \geq 0$ and compactly supported and let $T > 0$. Then there exists a subsequence q_{α_j} that weak-$*$ converges to q in $L^\infty\left([0,T]; \mathcal{M}(\mathbb{R}^2)\right)$ and in $\mathcal{M}\left(\mathbb{R}^2\right)$ for each fixed t, as $\alpha_j \to 0$, and q is a weak solution of the Euler equations* (2.1) *in the sense of* (2.2) *with initial data q^{in}.*

The weak-$*$ convergence in $L^\infty\left([0,T]; \mathcal{M}(\mathbb{R}^2)\right)$ means that

$$\lim_{\alpha_j \to \infty} \int_0^T \int_{\mathbb{R}^2} \varphi(x,t)\, dq_{\alpha_j}(x,t)\, dt = \int_0^T \int_{\mathbb{R}^2} \varphi(x,t)\, dq(x,t)\, dt,$$

for all $\varphi \in L^1\left([0,T]; \mathcal{C}_0(\mathbb{R}^2)\right)$.

We denote, respectively, the velocity and the "filtered" velocity by v_α and u_α and their corresponding vorticities by $q_\alpha = \operatorname{curl} v_\alpha$ and $\omega_\alpha = \operatorname{curl} u_\alpha$.

Given $q_\alpha \in \mathcal{M}(\mathbb{R}^2)$, we define a linear continuous functional $\omega_\alpha = \left(1-\alpha^2\Delta\right)^{-1} q_\alpha$ acting on every $\varphi \in C_0\left(\mathbb{R}^2\right)$ by

$$\langle \omega_\alpha, \varphi \rangle = \int_{\mathbb{R}^2} \left(\left(1-\alpha^2\Delta\right)^{-1}\varphi\right) dq_\alpha, \tag{3.1}$$

where $\psi = \left(1-\alpha^2\Delta\right)^{-1}\varphi$, the unique vanishing at infinity solution of $\varphi = \left(1-\alpha^2\Delta\right)\psi$, is given by

$$\left(1-\alpha^2\Delta\right)^{-1}\varphi = \int_{\mathbb{R}^2} \frac{1}{\alpha^2}\frac{1}{2\pi} K_0\left(\frac{|y|}{\alpha}\right)\varphi(x-y)\, dy,$$

the function K_0 is a modified Bessel function of the second kind of order zero, $K_0 > 0$, $\int_0^\infty K_0(r)\, rdr = 1$, see, e.g., [78]. From the above its follows that $\left\|\left(1-\alpha^2\Delta\right)^{-1}\varphi\right\|_{L^\infty} \leq \|\varphi\|_{L^\infty}$.

We observe that if $q_\alpha \geq 0$ then ω_α is a nonnegative linear functional. Indeed, let $\varphi \in C_0\left(\mathbb{R}^2\right)$, $\varphi \geq 0$, then

$$\left(1-\alpha^2\Delta\right)^{-1}\varphi = \int_{\mathbb{R}^2} \frac{1}{\alpha^2}\frac{1}{2\pi} K_0\left(\frac{|y|}{\alpha}\right)\varphi(x-y)\, dy \geq 0,$$

and hence by (3.1) $\langle \omega_\alpha, \varphi \rangle \geq 0$. Also,

$$|\langle \omega_\alpha, \varphi \rangle| \leq \|q_\alpha\|_{\mathcal{M}} \left\|\left(1-\alpha^2\Delta\right)^{-1}\varphi\right\|_{L^\infty} \leq \|q_\alpha\|_{\mathcal{M}} \|\varphi\|_{L^\infty}.$$

Therefore, by the Riesz representation theorem (see, e.g., [33, Chapter 7]) the functional ω_α can be represented by a unique nonnegative Radon measure, which we also denote by ω_α, and

$$\|\omega_\alpha\|_{\mathcal{M}} \leq \|q_\alpha\|_{\mathcal{M}}. \tag{3.2}$$

Again, by the Riesz representation theorem, a linear functional $\left(\alpha^2\Delta\omega_\alpha\right)$ defined by

$$\left\langle \alpha^2\Delta\omega_\alpha, \varphi\right\rangle = \int_{\mathbb{R}^2} \left(\alpha^2\Delta\left(1-\alpha^2\Delta\right)^{-1}\varphi\right) dq_\alpha, \tag{3.3}$$

for every $\varphi \in C_0\left(\mathbb{R}^2\right)$, can be identified with a Radon measure, which we also denote by $\alpha^2\Delta\omega_\alpha$. Observe that, since for every $\varphi \in C_0\left(\mathbb{R}^2\right)$

$$\alpha^2\Delta\left(1-\alpha^2\Delta\right)^{-1}\varphi = \left(1-\alpha^2\Delta\right)^{-1}\varphi - \varphi,$$

we have

$$\left|\left\langle \alpha^2\Delta\omega_\alpha, \varphi\right\rangle\right| \leq \|q_\alpha\|_{\mathcal{M}} \left\|\alpha^2\Delta\left(1-\alpha^2\Delta\right)^{-1}\varphi\right\|_{L^\infty} \leq 2\|q_\alpha\|_{\mathcal{M}}\|\varphi\|_{L^\infty},$$

that is, $\left\|\alpha^2\Delta\omega_\alpha\right\|_{\mathcal{M}(\mathbb{R}^2)} \leq 2\|q_\alpha\|_{\mathcal{M}(\mathbb{R}^2)}$.

We note that by Theorem 2.1 the solution q_α of Euler-α equations (2.6) is transported by the flow, that is $q_\alpha(x,t) = q^{in}\circ\eta_\alpha^{-1}(x,t)$, $\eta_\alpha \in C^1\left([0,T];\mathcal{G}\right)$, hence for all t

$$\|q_\alpha(\cdot,t)\|_{\mathcal{M}} = \left\|q^{in}\right\|_{\mathcal{M}}. \tag{3.4}$$

In addition, if $q^{in} \geq 0$, then $q_\alpha \geq 0$ for all times, and therefore also $\omega_\alpha \geq 0$ for all times.

The kernel H_ψ appearing in the non-linear term of (2.2) is discontinuous on the diagonal $x = y$, so, following [26, 59], to prove the convergence of the non-linear term we need the following estimate, which shows uniform decay of the "filtered" vorticity ω_α in small circles.

Lemma 3.2. *Let q_α be the solutions of (2.6) with initial data $q^{in} \in \mathcal{M}(\mathbb{R}^2) \cap H^{-1}_{loc}\left(\mathbb{R}^2\right)$, $q^{in} \geq 0$ and compactly supported. Then for $\omega_\alpha = \left(1-\alpha^2\Delta\right)^{-1}q_\alpha$ defined by (3.1), there exists a constant $C = C(T)$, such that for all $\alpha > 0$, $0 \leq t \leq T$, $0 < R < 1$ and $x_0 \in \mathbb{R}^2$ we have*

$$\int_{|x-x_0|<R} d\omega_\alpha(x,t) \leq C(T)\left|\log R\right|^{-1/2}. \tag{3.5}$$

Proof. Recall that $\omega_\alpha \geq 0$ for all times. The idea of the proof, which is shown in details below, is to convolve the initial data with a standard $C_c^\infty\left(\mathbb{R}^2\right)$ mollifier to obtain a sequence of solutions of the Euler-α equations that has a uniform decay of the circulation on small disks

$$\int_{|x-x_0|\leq R} \omega_{\alpha,\varepsilon}(x,t)\,dx \leq C(T)\left|\log R\right|^{-1/2},$$

$0 < \varepsilon \leq \varepsilon_0$, $0 \leq t \leq T$, $R < 1$, and then the weak-$*$ limit in $L^\infty\left([0,T],\mathcal{M}\left(\mathbb{R}^2\right)\right)$ of a subsequence $\omega_{\alpha,\varepsilon_j}$ when $\varepsilon_j \to 0$, which is the solution Euler-α equations with initial data q^{in}, satisfies a similar bound.

We observe that, similarly to the Euler equations, any smooth radially symmetric vanishing at infinity vorticity $\bar{q}(|x|)$ defines a stationary solution of Euler-α equation (2.3) with the corresponding velocity $\bar{v}(x) = \nabla^\perp\Delta^{-1}\bar{q}(|x|) = \frac{x^\perp}{|x|^2}\int_0^{|x|} s\bar{q}(s)\,ds$. This could be seen using the vorticity stream function formulation for Euler-α equations, which is

$$\begin{aligned} &q_t + J(\varphi, \Delta\psi) = 0,\\ &q = \Delta\psi, \end{aligned}$$

where ψ is the velocity stream function, $v = \nabla^\perp\psi$, and $\varphi = \left(1-\alpha^2\Delta\right)^{-1}\psi$ is the "filtered" stream function, $u = \nabla^\perp\varphi$. Since Δ and $\left(1-\alpha^2\Delta\right)$ are rotationally invariant, we have that the corresponding $\bar{\omega} = \left(1-\alpha^2\Delta\right)^{-1}\bar{q}$, $\bar{\psi} = \Delta^{-1}\bar{q}$ and $\bar{\varphi} = \left(1-\alpha^2\Delta\right)^{-1}\bar{\psi}$ are also radially symmetric, therefore $J\left(\bar{\varphi}, \Delta\bar{\psi}\right) = 0$ and hence $\bar{q}$ defines a stationary solution of Euler-α equation.

Let $\rho \in C_c^\infty\left(\mathbb{R}^2\right)$ be a standard mollifier, for example,

$$\rho\left(x\right)=\begin{cases} C\exp\left(1/\left(\left|x\right|^2-1\right)\right) & \text{if } \left|x\right|<1,\\ 0 & \text{if } \left|x\right|\geq 1,\end{cases}$$

$\int_{\mathbb{R}^2}\rho=1$, $\rho_\varepsilon\left(x\right)=\frac{1}{\varepsilon^2}\rho\left(\frac{x}{\varepsilon}\right)$. Smoothing the initial data by a mollification with ρ_ε, $q_\varepsilon^{in}=\rho_\varepsilon * q^{in}$, we have that for all $0<\varepsilon<\varepsilon_0$ the smoothed initial vorticities satisfy $q_\varepsilon^{in}\geq 0$, $\operatorname{supp} q_\varepsilon^{in}\subseteq\left\{x\right|\left|x\right|<R_0\}$ (since q^{in} is compactly supported), $\int_{\mathbb{R}^2}q_\varepsilon^{in}\left(x\right)dx=\int_{\mathbb{R}^2}dq^{in}\left(x\right)$. Following [26, 59] for the 2D Euler case we decompose the velocity into a combination of a stationary bounded velocity plus a time dependent velocity with finite total energy. Let $\bar{q}\left(\left|x\right|\right)$ be any smooth radially symmetric function with compact support, such that $\int_{\mathbb{R}^2}\bar{q}\left(\left|x\right|\right)dx=\int_{\mathbb{R}^2}dq^{in}\left(x\right)$. Define $\bar{v}=K*\bar{q}$, $\tilde{q}_\varepsilon^{in}=q_\varepsilon^{in}-\bar{q}$ and $\tilde{v}_\varepsilon^{in}=K*\tilde{q}_\varepsilon^{in}$. Notice, that by direct calculation $\operatorname{div}\bar{v}=0$ and $\bar{v},\nabla\bar{v},\partial^2\bar{v}\in L^\infty\left(\mathbb{R}^2\right)$. Since $\int_{\mathbb{R}^2}\tilde{q}_\varepsilon^{in}=0$, and $\tilde{q}_\varepsilon^{in}$ has compact support we have that $\tilde{v}_\varepsilon^{in}\in L^2\left(\mathbb{R}^2\right)$. Also, due to the fact that $q^{in}\in\mathcal{M}(\mathbb{R}^2)\cap H_{loc}^{-1}\left(\mathbb{R}^2\right)$ with compact support, and hence, for $\varepsilon\leq\varepsilon_0$, the smooth q_ε^{in} are uniformly bounded in L^1 with a common compact support and $v_\varepsilon^{in}=K*q_\varepsilon^{in}$ are uniformly bounded in L_{loc}^2, and since $\bar{q}$ is the same function for all ε, we have that $\tilde{v}_\varepsilon^{in}$ are uniformly bounded in $L^2\left(\mathbb{R}^2\right)$, for $\varepsilon\leq\varepsilon_0$.

Observe, that the stationary part

$$\bar{u}\left(x\right)=\left(1-\alpha^2\Delta\right)^{-1}\bar{v}\left(x\right)=\int_{\mathbb{R}^2}\frac{1}{\alpha^2}\frac{1}{2\pi}K_0\left(\frac{\left|y\right|}{\alpha}\right)\bar{v}\left(x-y\right)dy$$

satisfies[1]

$$\begin{aligned}&\left\|\bar{u}\right\|_{L^\infty}\leq\left\|\bar{v}\right\|_{L^\infty},\\ &\left\|\nabla\bar{u}\right\|_{L^\infty}\leq\left\|\nabla\bar{v}\right\|_{L^\infty},\\ &\left\|\partial^2\bar{u}\right\|_{L^\infty}\leq\frac{1}{2\pi}\frac{1}{\alpha}\left\|\nabla\bar{v}\right\|_{L^\infty}.\end{aligned}\tag{3.6}$$

[1]We show that for smooth vanishing at infinity $\psi=\left(1-\alpha^2\Delta\right)^{-1}\varphi$ one has that $\left\|\nabla\psi\right\|_{L^\infty}\leq\frac{1}{2\pi}\frac{1}{\alpha}\left\|\varphi\right\|_{L^\infty}$. We have

$$\psi\left(x\right)=\int_{\mathbb{R}^2}\frac{1}{2\pi}K_0\left(\left|y\right|\right)\varphi\left(x-y\alpha\right)dy.$$

Since $\left|\nabla\varphi\right|_{L^\infty}<\infty$ and $K_0\in L^1\left(\mathbb{R}^2\right)$, the differentiation under the $\int$ sign can be justified by Lebesgue dominated convergence theorem, and we obtain

$$\begin{aligned}\nabla\psi\left(x\right)&=\int_{\mathbb{R}^2}\frac{1}{2\pi}K_0\left(\left|y\right|\right)\nabla_x\varphi\left(x-y\alpha\right)dy\\ &=\int_{\mathbb{R}^2}\frac{1}{2\pi}\frac{1}{\alpha}K_0\left(\left|y\right|\right)\nabla_y\varphi\left(x-y\alpha\right)dy,\end{aligned}$$

now, by integration by parts and using that for large $r:K_0\left(r\right)\sim\left(\frac{\pi}{2}\right)^{1/2}\frac{1}{e^r\sqrt{r}}$, while $\nabla_y\varphi$ is bounded we have

$$\begin{aligned}\nabla\psi\left(x\right)&=\frac{1}{\alpha}\int_{\mathbb{R}^2}\frac{1}{2\pi}\nabla_yK_0\left(\left|y\right|\right)\varphi\left(x-y\alpha\right)dy\\ &=\int_{\mathbb{R}^2}\frac{1}{2\pi}\frac{1}{\alpha}K_1\left(\left|y\right|\right)\frac{y}{\left|y\right|}\varphi\left(x-y\alpha\right)dy,\end{aligned}$$

that is,

$$\begin{aligned}\left|\nabla\psi\left(x\right)\right|&\leq\frac{1}{\alpha}\left\|\varphi\right\|_{L^\infty}\int_{\mathbb{R}^2}\frac{1}{2\pi}K_1\left(\left|y\right|\right)dy\\ &=\frac{\pi}{2}\frac{1}{\alpha}\left\|\varphi\right\|_{L^\infty}.\end{aligned}$$

Consider the partial differential equation

$$\begin{aligned}\frac{\partial}{\partial t}\tilde{v}_{\alpha,\varepsilon}+\left(\tilde{u}_{\alpha,\varepsilon}\cdot\nabla\right)\tilde{v}_{\alpha,\varepsilon}+\sum_j\left(\tilde{v}_{\alpha,\varepsilon}\right)_j\nabla\left(\tilde{u}_{\alpha,\varepsilon}\right)_j \\ +\left(\tilde{u}_{\alpha,\varepsilon}\cdot\nabla\right)\bar{v}+\sum_j\bar{v}_j\nabla\left(\tilde{u}_{\alpha,\varepsilon}\right)_j \\ +\left(\bar{u}\cdot\nabla\right)\tilde{v}_{\alpha,\varepsilon}+\sum_j\left(\tilde{v}_{\alpha,\varepsilon}\right)_j\nabla\bar{u}_j+\nabla\tilde{\pi}_{\alpha,\varepsilon}=0, \\ \tilde{v}_{\alpha,\varepsilon}=\left(1-\alpha^2\Delta\right)\tilde{u}_{\alpha,\varepsilon}.\end{aligned} \tag{3.7}$$

This evolution equation is similar to the Euler-α equations. Moreover, if $\tilde{v}_{\alpha,\varepsilon}(x,t)$ is the solution of the equation(3.7) with initial data $\tilde{v}_\varepsilon^{in}$, then $v_{\alpha,\varepsilon}(x,t)=\tilde{v}_{\alpha,\varepsilon}(x,t)+\bar{v}(x)$ is the solution of the 2D Euler-α equations (1.2) with initial data $v_\varepsilon^{in}=K*q_\varepsilon^{in}$.

Similarly to the Euler case (see, e.g., [59]) this equation has a unique global infinitely smooth solution, since, as in 2D Euler case, we have an *a priori* uniform control over the L^∞ norm of the $\tilde{q}_{\alpha,\varepsilon}$, which implies the global existence, as in the proof of the Beale-Kato-Majda criterion [5]. The solution $\tilde{v}_{\alpha,\varepsilon}$ is in $C^1\left([0,\infty),H^s\left(\mathbb{R}^2\right)\right)$ for all $s>2$, and hence, by Sobolev embedding theorem, $\partial^k\tilde{v}_{\alpha,\varepsilon}$ and, consequently, $\partial^k\tilde{u}_{\alpha,\varepsilon}(x)=\int_{\mathbb{R}^2}\frac{1}{\alpha^2}\frac{1}{2\pi}K_0\left(\frac{|y|}{\alpha}\right)\tilde{v}_{\alpha,\varepsilon}(x-y)\,dy$ are also in $C_0\left(\mathbb{R}^2\right)$ for all k.

Moreover, the solution $\tilde{u}_{\alpha,\varepsilon}$ is in $L^\infty\left([0,\infty);H^1\left(\mathbb{R}^2\right)\right)$ due to the following *a priori* estimate. Taking the inner product of (3.7) with $\tilde{u}_{\alpha,\varepsilon}$ we have (omitting the subindices α and ε)

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(|\tilde{u}|_{L^2}^2+\alpha^2|\nabla\tilde{u}|_{L^2}^2\right)&=\alpha^2\left(\left(\bar{u}\cdot\nabla\right)\Delta\tilde{u},\tilde{u}\right)-\sum_j\left(\tilde{v}_j\nabla\bar{u}_j,\tilde{u}\right)\\&=I_1-I_2.\end{aligned}$$

Since $\operatorname{div}\bar{u}=0$, for I_1 we have

$$\begin{aligned}I_1&=-\alpha^2\sum_{i,j,k}\int\bar{u}_i\frac{\partial^2\tilde{u}_j}{\partial x_k^2}\frac{\partial}{\partial x_i}\tilde{u}_j\\&=\alpha^2\sum_{i,j,k}\int\frac{\partial\bar{u}_i}{\partial x_k}\frac{\partial\tilde{u}_j}{\partial x_k}\frac{\partial\tilde{u}_j}{\partial x_i}+\alpha^2\sum_{i,j,k}\int\bar{u}_i\left(\frac{\partial^2}{\partial x_i\partial x_k}\tilde{u}_j\right)\frac{\partial}{\partial x_k}\tilde{u}_j.\end{aligned}$$

Since the second term on the right is zero, we obtain that

$$|I_1|\le C\alpha^2\left\|\nabla\bar{u}\right\|_{L^\infty}\left\|\nabla\tilde{u}\right\|_{L^2}^2.$$

Now we estimate I_2

$$\begin{aligned}I_2&=\sum_{i,j,k}\int\tilde{u}_j\nabla\bar{u}_j\cdot\tilde{u}-\alpha^2\int\Delta\tilde{u}_j\nabla\bar{u}_j\cdot\tilde{u}\\&=I_{2_1}+I_{2_2}.\end{aligned}$$

We have

$$|I_{2_1}|\le C\left\|\tilde{u}\right\|_{L^2}^2\left\|\nabla\bar{u}\right\|_{L^\infty}$$

and

$$I_{2_2}=\alpha^2\sum_{i,j,k}\int\frac{\partial\tilde{u}_j}{\partial x_k}\frac{\partial\bar{u}_j}{\partial x_i}\frac{\partial\tilde{u}_i}{\partial x_k}+\alpha^2\sum_{i,j,k}\int\frac{\partial\tilde{u}_j}{\partial x_k}\frac{\partial^2\bar{u}_j}{\partial x_k\partial x_i}\tilde{u}_i,$$

hence

$$|I_{2_2}|\le C\alpha^2\left\|\nabla\tilde{u}\right\|_{L^2}^2\left\|\nabla\bar{u}\right\|_{L^\infty}+C\alpha^2\left\|\nabla\tilde{u}\right\|_{L^2}\left\|\tilde{u}\right\|_{L^2}\left\|\partial^2\bar{u}\right\|_{L^\infty}.$$

To conclude, we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|\tilde{u}\|_{L^2}^2+\alpha^2\|\nabla\tilde{u}\|_{L^2}^2\right)\leq C\left(\alpha^2\|\nabla\bar{u}\|_{L^\infty}\|\nabla\tilde{u}\|_{L^2}^2+\|\tilde{u}\|_{L^2}^2\|\nabla\bar{u}\|_{L^\infty}+\alpha\|\nabla\tilde{u}\|_{L^2}\|\tilde{u}\|_{L^2}\alpha\left\|\partial^2\bar{u}\right\|_{L^\infty}\right).$$

Hence, thanks to (3.6),

$$\frac{1}{2}\frac{d}{dt}\left(\|\tilde{u}\|_{L^2}^2+\alpha^2\|\nabla\tilde{u}\|_{L^2}^2\right)\leq C\|\nabla\bar{v}\|_{L^\infty}\left(\alpha^2\|\nabla\tilde{u}\|_{L^2}^2+\|\tilde{u}\|_{L^2}^2\right),$$

and by Grönwall inequality

$$\begin{aligned}\|\tilde{u}(\cdot,t)\|_{L^2}^2+\alpha^2\|\nabla\tilde{u}(\cdot,t)\|_{L^2}^2&\leq e^{C\|\nabla\bar{v}\|_{L^\infty}t}\left(\|\tilde{u}(\cdot,0)\|_{L^2}^2+\alpha^2\|\nabla\tilde{u}(\cdot,0)\|_{L^2}^2\right)\\&\leq e^{C\|\nabla\bar{v}\|_{L^\infty}t}\|\tilde{v}(\cdot,0)\|_{L^2}^2.\end{aligned}$$

Hence we have that for all $0<\varepsilon\leq\varepsilon_0$, $0\leq t\leq T$, the solution of Euler-α equations with the smoothed initial data satisfies (we now put back the subindices α and ε)

$$\begin{aligned}\|u_{\alpha,\varepsilon}(\cdot,t)\|_{L^2(B(x_0,1))}&\leq\|\tilde{u}_{\alpha,\varepsilon}(\cdot,t)\|_{L^2(B(x_0,1))}+\|\bar{u}\|_{L^2(B(x_0,1))}\\&\leq\|\tilde{u}_{\alpha,\varepsilon}(\cdot,t)\|_{L^2(\mathbb{R}^2)}+\pi\|\bar{u}\|_{L^\infty(\mathbb{R}^2)}\\&\leq C(T),\end{aligned}$$

where $C(T)=C\left(\left\|q^{in}\right\|_{\mathcal{M}},\|\bar{q}\|_{L^\infty},\varepsilon_0,R_0\right)e^{C\|\nabla\bar{v}\|_{L^\infty}t}+\pi\|\bar{u}\|_{L^\infty(\mathbb{R}^2)}$. This is enough to show uniform decay of the vorticity $\omega_{\alpha,\varepsilon}$ in small circles (see [71], we remark that here the fixed sign of the vorticity comes in place[2]): for $R<1$

$$\int_{|x-x_0|\leq R}\omega_{\alpha,\varepsilon}(x,t)\,dx\leq C(T)\left|\log R\right|^{-1/2}.$$

By (3.2) $\|\omega_{\alpha,\varepsilon}(\cdot,t)\|_{\mathcal{M}}\leq\|q_{\alpha,\varepsilon}(\cdot,t)\|_{\mathcal{M}}=\left\|q_\varepsilon^{in}\right\|_{\mathcal{M}}=\left\|q^{in}\right\|_{\mathcal{M}}$, hence there exists a subsequence $\omega_{\alpha,\varepsilon_j}$ which converges weak-$*$ in $\mathcal{M}\left(\mathbb{R}^2\right)$ for each fixed t and also in $L^\infty\left([0,T],\mathcal{M}\left(\mathbb{R}^2\right)\right)$ to the limit ω_α. This limit has a similar decay

$$\int_{|x-x_0|<R}d\omega_\alpha(x,t)\leq\liminf_{\varepsilon_j\to0}\int_{|x-x_0|<R}\omega_{\alpha,\varepsilon_j}(x,t)\,dx\leq C(T)\left|\log R\right|^{-1/2}.$$

Furthermore, $q_\alpha=\left(1-\alpha^2\Delta\right)\omega_\alpha$ is the solution of the Euler-α equation (2.3), the passing to the limit in $\lim_{\varepsilon_j\to0}W^\alpha\left(q_{\alpha,\varepsilon_j};\psi\right)=W^\alpha\left(q_\alpha;\psi\right)$ is straightforward since $H_\psi^\alpha\in C\left([0,T],\left(C_0\left(\mathbb{R}^2\right)\right)^2\right)$ and q_{α,ε_j} are

[2]In [71] to prove the uniform decay of the vorticity in small circles one defines for $R<1$

$$\delta_R(x)=\begin{cases}1 & |x|\leq R,\\ \frac{\log(\sqrt{R}/|x|)}{\log(1/\sqrt{R})} & R\leq|x|\leq\sqrt{R},\\ 0 & |x|\geq\sqrt{R}.\end{cases}$$

Then $|\nabla\delta_R|_{L^2}\leq C\left|\log R\right|^{-1/2}$. We have

$$\begin{aligned}\int_{|x-x_0|\leq R}\omega_{\alpha,\varepsilon}(x,t)\,dx&\leq\int_{\mathbb{R}^2}\delta_R(x-x_0)\,\omega_{\alpha,\varepsilon}(x,t)\,dx\\&\leq\left|\int_{\mathbb{R}^2}\nabla^\perp\delta_R(x-x_0)\,u_{\alpha,\varepsilon}(x,t)\,dx\right|\\&\leq|\nabla\delta_R|_{L^2}\,|u_{\alpha,\varepsilon}(\cdot,t)|_{L^2(B(x_0,1))}\\&\leq C(T)\left|\log R\right|^{-1/2}.\end{aligned}$$

Here in the second transaction we used the fact that $\omega_{\alpha,\varepsilon}\geq0$.

equicontinuous in time with values in a negative Sobolev space $W^{-2,1}$ (which, together with $q_{\alpha,\varepsilon_j} \overset{*}{\rightharpoonup} q_\alpha$ in $L^\infty\left([0,T],\mathcal{M}\left(\mathbb{R}^2\right)\right)$, implies $q_{\alpha,\varepsilon_j}(x,t)\, q_{\alpha,\varepsilon_j}(y,t) \overset{*}{\rightharpoonup} q_\alpha(x,t)\, q_\alpha(y,t)$ in $L^\infty\left([0,T],\mathcal{M}\left(\mathbb{R}^2\right)\right)$, see [71, Lemma 3.2]). The equicontinuity follows from the fact that $|x| D\Psi^\alpha(|x|)$ is bounded (in fact, it is bounded independent of α) and hence we have for all $\psi \in C_c^\infty\left(\mathbb{R}^2 \times (0,T)\right)$

$$
\begin{aligned}
&\left|\int_0^T \int_{\mathbb{R}^2} \partial_t \psi(x,t)\, q_{\alpha,\varepsilon_j}(x,t)\, dxdt\right| = \qquad (3.8)\\
&= \left|\frac{1}{2}\int_0^T \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} D\Psi^\alpha(|x-y|)\, \frac{(x-y)^\perp \cdot (\nabla\psi(x,t) - \nabla\psi(y,t))}{|x-y|} q_{\alpha,\varepsilon_j}(x,t)\, q_{\alpha,\varepsilon_j}(y,t)\, dxdydt\right|\\
&\le \frac{1}{2}\left\||x-y| D\Psi^\alpha(x-y)\right\|_{L^\infty} \int_0^T \left\|D^2\psi(\cdot,t)\right\|_{L^\infty(\mathbb{R}^2)} \int_{\mathbb{R}^2} q_{\alpha,\varepsilon_j}(x,t)\, dx \int_{\mathbb{R}^2} q_{\alpha,\varepsilon_j}(y,t)\, dydt\\
&\le C\left\|q^{in}\right\|_{\mathcal{M}}^2 \|\psi\|_{L^1([0,T],W^{2,\infty}(\mathbb{R}^2))}\\
&\le C\left\|q^{in}\right\|_{\mathcal{M}}^2 \|\psi\|_{L^1([0,T],H^4(\mathbb{R}^2))},
\end{aligned}
$$

where in the last inequality we used the Sobolev embedding theorem. Hence $\partial_t q_{\alpha,\varepsilon_j}$ uniformly bounded in $L^\infty\left([0,T],H^{-4}\left(\mathbb{R}^2\right)\right)$, and hence q_{α,ε_j} are uniformly bounded in $Lip\left([0,T];H^{-4}\left(\mathbb{R}^2\right)\right)$. □

We also need the following result

Lemma 3.3. *Let q be a finite Radon measure, $q = \left(1-\alpha^2\Delta\right)\omega$, as defined in (3.1)-(3.3), then*

$$\int_{\mathbb{R}^2} d\left|\alpha^2\Delta\omega\right| \le C\alpha \|q\|_{\mathcal{M}}.$$

Proof. For the theory of Radon measures, see, e.g., [33]. First, we show that for all compact $K \subset \mathbb{R}^2$

$$\left|\alpha^2\Delta\omega\right|(K) \le C\alpha \|q\|_{\mathcal{M}}.$$

By Riesz representation theorem (see, e.g., [33, Chapter 7])

$$\left|\alpha^2\Delta\omega\right|(K) = \inf\left\{\int_{\mathbb{R}^2} f d\left|\alpha^2\Delta\omega\right| : f \in C_c\left(\mathbb{R}^2\right), f \ge \chi_K\right\}.$$

Let R be such that $K \subset B(0,R)$, take $\theta \in C_c^\infty(\mathbb{R}^2)$ with $0 \le \theta(x) \le 1$ for all x, with $\theta(x) = 1$ if $|x| \le R$, $\theta(x) = 0$ if $|x| \ge R+1$. For example, $\theta = \chi_{B(0,R+1/2)} * \rho^{\varepsilon=1/4}$. Then using that $\left\|\alpha\Delta\left(1-\alpha^2\Delta\right)^{-1}\theta\right\|_{L^\infty} \le C\|\nabla\theta\|_{L^\infty}$ (see, e.g., (3.6)), we have

$$
\begin{aligned}
\left|\alpha^2\Delta\omega\right|(K) &\le \int_{\mathbb{R}^2} \theta d\left|\alpha^2\Delta\omega\right|\\
&\le \int_{\mathbb{R}^2} \left|\alpha^2\Delta\left(1-\alpha^2\Delta\right)^{-1}\theta\right| d|q|\\
&\le C\alpha\|q\|_{\mathcal{M}}\|\nabla\theta\|_{L^\infty}\\
&\le C\alpha\|q\|_{\mathcal{M}}.
\end{aligned}
$$

Now, since a Radon measure is inner regular we have

$$
\begin{aligned}
\left|\alpha^2\Delta\omega\right|\left(\mathbb{R}^2\right) &= \sup\left\{\left|\alpha^2\Delta\omega\right|(K) : K \subset \mathbb{R}^2, K \text{ compact}\right\}\\
&\le C\alpha\|q\|_{\mathcal{M}}.
\end{aligned}
$$

□

Now we are ready to prove Theorem 3.1. Due to (3.4) there exists a subsequence, that we relabel as q_α, such that $q_\alpha \rightharpoonup q$ weak-$*$ in $L^\infty\left([0,T],\mathcal{M}\left(\mathbb{R}^2\right)\right)$ and in $\mathcal{M}\left(\mathbb{R}^2\right)$ for each fixed t, as $\alpha \to 0$. We notice that (3.8) implies that $q_\alpha \in Lip\left([0,T];H^{-4}\left(\mathbb{R}^2\right)\right)$, and hence we also have that $q_\alpha(x,t)\,q_\alpha(y,t) \rightharpoonup q(x,t)\,q(y,t)$ weak-$*$ both in $L^\infty\left([0,T],\mathcal{M}\left(\mathbb{R}^2\right)\right)$ and in $\mathcal{M}\left(\mathbb{R}^2\right)$ for each fixed $t \in [0,T]$, as $\alpha \to 0$ (see [71, Lemma 3.2]).

Since q_α is uniformly bounded in $\mathcal{M}(\mathbb{R}^2)$ and $Lip\left([0,T];H^{-4}\left(\mathbb{R}^2\right)\right)$ (by (3.8)), $\mathcal{M}(\mathbb{R}^2) \hookrightarrow H^{-s}_{loc}\left(\mathbb{R}^2\right) \overset{comp}{\hookrightarrow} H^{-4}_{loc}\left(\mathbb{R}^2\right)$ for $1 < s < 4$, then by Arzela-Ascoli theorem there is a subsequence of q_α that converges to some $\bar{q}$ in $C\left([0,T];H^{-4}_{loc}\right)$, and hence $\bar{q}$ is also in $Lip\left([0,T];H^{-4}_{loc}\right)$. Applying both types of convergence of the q_α to the integral $\int_0^T \int_{\mathbb{R}^2} \psi(t)\,\varphi(x)\,dq_\alpha(x,t)$ for every $\psi \in C_c([0,T]), \varphi \in C_c\left(\mathbb{R}^2\right)$ shows that $\bar{q} = q$, and hence the limit q belongs to $Lip\left([0,T],H^{-4}_{loc}\left(\mathbb{R}^2\right)\right)$ as well.

We observe that $\omega_\alpha(t,\cdot)$ also weak-$*$ converges to q in $\mathcal{M}\left(\mathbb{R}^2\right)$ for every $t \in [0,T]$, as $\alpha \to 0$. Indeed, let $\varphi \in C_c\left(\mathbb{R}^2\right)$ then

$$\begin{aligned}\left|\int_{\mathbb{R}^2} \varphi(x)\,dq(x,t) - \int_{\mathbb{R}^2} \varphi(x)\,d\omega_\alpha(x,t)\right| \leq & \left|\int_{\mathbb{R}^2} \varphi(x)\,dq(x,t) - \int_{\mathbb{R}^2} \varphi(x)\,dq_\alpha(x,t)\right| \\ & + \left|\int_{\mathbb{R}^2} \varphi(x)\,dq_\alpha(x,t) - \int_{\mathbb{R}^2} \varphi(x)\,d\omega_\alpha(x,t)\right|\end{aligned}$$

the first term on the right-hand side converges to 0, since $q_\alpha \overset{*}{\rightharpoonup} q$ in $\mathcal{M}\left(\mathbb{R}^2\right)$, as $\alpha \to 0$, and the second term is equal to $\left|\int_{\mathbb{R}^2} \varphi d\left(\alpha^2 \Delta \omega_\alpha\right)\right| \leq \|\varphi\|_{L^\infty} \int_{\mathbb{R}^2} d\left|\alpha^2 \Delta \omega\right| \to 0$, as $\alpha \to 0$, due to Lemma 3.3. Hence also q decays in small disks, that is,for all $0 \leq t \leq T$, $0 < R < 1$ and $x_0 \in \mathbb{R}^2$

$$\int_{|x-x_0|<R} dq(x,t) \leq \liminf_{\alpha \to 0} \int_{|x-x_0|<R} d\omega_\alpha(x,t) \leq C(T)\,|\log R|^{-1/2}. \tag{3.9}$$

Next we show that q is a weak solution of the Euler equations (2.2), namely, for every test function $\psi \in C_c^\infty\left(\mathbb{R}^2 \times (0,T)\right)$

$$W(q;\psi) = \lim_{\alpha \to 0} W^\alpha(q_\alpha;\psi) = 0.$$

The convergence of the linear term is obvious from the weak-$*$ convergence $q_\alpha \rightharpoonup q$ in $L^\infty\left([0,T];\mathcal{M}(\mathbb{R}^2)\right)$, as $\alpha \to 0$. Hence we need to show the convergence for the non-linear term

$$\lim_{\alpha \to 0} W^\alpha_{NL}(q_\alpha;\psi) = W_{NL}(q;\psi).$$

We rewrite $W_{NL}(q;\psi) - W^\alpha_{NL}(q_\alpha;\psi)$ as

$$\begin{aligned}W_{NL}(q;\psi) - W^\alpha_{NL}(q_\alpha;\psi) &= \int_0^T \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} H_\psi(x,y,t)\left[dq(x,t)\,dq(y,t) - dq_\alpha(x,t)\,dq_\alpha(y,t)\right]dt \\ &\quad + \int_0^T \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left(H_\psi(x,y,t) - H^\alpha_\psi(x,y,t)\right) dq_\alpha(x,t)\,dq_\alpha(y,t)\,dt \\ &= I_1 + I_2.\end{aligned}$$

We recall that the kernel H_ψ is bounded by a constant times $\left\|D^2\psi\right\|_{L^\infty}$, tends to zero at infinity, and it is discontinuous on the diagonal $x = y$ (see [71]).

Let $\theta(|x|) \in C_c^\infty(\mathbb{R}^2)$ be a fixed cutoff function $0 \leq \theta \leq 1$ with $\theta = 1$ for $|x| \leq 1$ and $\theta = 0$ for $|x| \geq 2$. Let $0 < \delta < 1$. Write I_1 as

$$\begin{aligned}I_1 &= \int_0^T \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left[1 - \theta\left(\frac{|x-y|}{\delta}\right)\right] H_\psi(x,y,t)\left(dq(x,t)\,dq(y,t) - dq_\alpha(x,t)\,dq_\alpha(y,t)\right)dt \\ &\quad + \int_0^T \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \theta\left(\frac{|x-y|}{\delta}\right) H_\psi(x,y,t)\left(dq(x,t)\,dq(y,t) - dq_\alpha(x,t)\,dq_\alpha(y,t)\right)dt \\ &= I_{11} + I_{12}.\end{aligned}$$

Since $\left[1-\theta\left(\frac{|x-y|}{\delta}\right)\right]H_\psi \in C\left([0,T],\left(C_0\left(\mathbb{R}^2\right)\right)^2\right)$ and $q_\alpha(x,t)\,q_\alpha(y,t) \rightharpoonup q(x,t)\,q(y,t)$ weak-$*$ in $L^\infty\left([0,T],\mathcal{M}\left(\mathbb{R}^2\right)\right)$ as $\alpha\to 0$, then $\lim_{\alpha\to 0} I_{11}=0$. Now we estimate I_{12}

$$\begin{aligned}|I_{12}| &\le \int_0^T \int\int_{|x-y|<2\delta} |H_\psi(x,y,t)|\,dq(x,t)\,dq(y,t)\,dt\\ &+\int_0^T \int\int_{|x-y|<2\delta} |H_\psi(x,y,t)|\,dq_\alpha(x,t)\,dq_\alpha(y,t)\,dt\\ &= I_{121}+I_{122}.\end{aligned}$$

For I_{121}, due to uniform decay of the vorticity q in small circles (3.9), we have for $2\delta<1$

$$\begin{aligned}I_{121} &\le |H_\psi|_{L^\infty}\int_0^T\int_{\mathbb{R}^2}\int_{B(y,2\delta)} dq(x,t)\,dq(y,t)\,dt\\ &\le C(T)\,|\log 2\delta|^{-1/2}\left\|q^{in}\right\|_{\mathcal{M}}.\end{aligned}$$

To estimate I_{122} we use (3.5) (for $2\delta<1$) and Lemma 3.3.

$$\begin{aligned}I_{122} &\le |H_\psi|_{L^\infty}\int_0^T\int\int_{|x-y|<2\delta} d\left(\left(1-\alpha^2\Delta\right)\omega_\alpha\right)(x,t)\,d\left(\left(1-\alpha^2\Delta\right)\omega_\alpha\right)(y,t))dt\\ &= |H_\psi|_{L^\infty}\int_0^T\int\int_{|x-y|<2\delta} d\omega_\alpha(x,t)\,d\omega_\alpha(y,t))dt\\ &+|H_\psi|_{L^\infty}\int_0^T\left(2\int_{\mathbb{R}^2} d\omega_\alpha(x,t)\int_{\mathbb{R}^2} d\left|\alpha^2\Delta\omega_\alpha(x,t)\right|+\left(\int_{\mathbb{R}^2} d\left|\alpha^2\Delta\omega_\alpha(x,t)\right|\right)^2\right)dt\\ &\le C(T)\,|\log 2\delta|^{-1/2}\left\|q^{in}\right\|_{\mathcal{M}}^2+\alpha(1+\alpha)CT\left\|q^{in}\right\|_{\mathcal{M}}^2.\end{aligned}$$

Thus, $I_{12}\to 0$, as δ and α converge to zero.

It remains to estimate I_2

$$I_2=\frac{1}{4\pi}\int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{1}{\alpha}K_1\left(\frac{|x-y|}{\alpha}\right)(x-y)^\perp\frac{\nabla(\psi(x,t)-\psi(y,t))}{|x-y|}dq_\alpha(x,t)\,dq_\alpha(y,t)\,dt.$$

Now, for $\frac{r}{\alpha}\to\infty$ $\frac{r}{\alpha}K_1\left(\frac{r}{\alpha}\right)\le C\left(\frac{\pi}{2}\right)^{1/2}\frac{\sqrt{\frac{r}{\alpha}}}{e^{\frac{r}{\alpha}}}\to 0$ [78]. Hence, for each $\varepsilon>0$, there is an L large enough, depending on ε, such that $\frac{r}{\alpha}K_1\left(\frac{r}{\alpha}\right)<\varepsilon$, whenever $\frac{r}{\alpha}\ge L$. Write I_2 as

$$\begin{aligned}I_2 &= \frac{1}{4\pi}\int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left[1-\theta\left(\frac{|x-y|}{\alpha L}\right)\right]\frac{1}{\alpha}K_1\left(\frac{|x-y|}{\alpha}\right)(x-y)^\perp\frac{\nabla(\psi(x,t)-\psi(y,t))}{|x-y|}dq_\alpha(x,t)\,dq_\alpha(y,t)\,dt\\ &+\frac{1}{4\pi}\int_0^T\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\theta\left(\frac{|x-y|}{\alpha L}\right)\frac{1}{\alpha}K_1\left(\frac{|x-y|}{\alpha}\right)(x-y)^\perp\frac{\nabla(\psi(x,t)-\psi(y,t))}{|x-y|}dq_\alpha(x,t)\,dq_\alpha(y,t)\,dt\\ &= I_{21}+I_{22}.\end{aligned}$$

We have

$$\begin{aligned}I_{21} &\le \frac{1}{4\pi}\int_0^T\int\int_{\frac{|x-y|}{\alpha}>L}\left[1-\theta\left(\frac{|x-y|}{\alpha L}\right)\right]\frac{|x-y|}{\alpha}K_1\left(\frac{|x-y|}{\alpha}\right)\frac{|\nabla(\psi(x,t)-\psi(y,t))|}{|x-y|}dq_\alpha(x,t)\,dq_\alpha(y,t)\,dt\\ &\le \varepsilon\left\|D^2\psi\right\|_{L^\infty}\frac{1}{4\pi}\int_0^T\int\int_{\frac{|x-y|}{\alpha}>L} dq_\alpha(x,t)\,dq_\alpha(y,t)\,dt\\ &\le \varepsilon\left\|D^2\psi\right\|_{L^\infty}\frac{1}{4\pi}\int_0^T\left(\int_{\mathbb{R}^2} dq_\alpha(x,t)\right)^2\\ &\le \varepsilon\left\|D^2\psi\right\|_{L^\infty}\frac{1}{4\pi}T\left\|q^{in}\right\|_{\mathcal{M}}^2.\end{aligned}$$

Since $\frac{r}{\alpha}K_1\left(\frac{r}{\alpha}\right) \leq C$ for all r (independent of α), then similarly to the bound on I_{122}, we have that for $\alpha < \frac{1}{2L}$

$$I_{22} \leq C(T)\left|\log 2\alpha L\right|^{-1/2}\left\|D^2\psi\right\|_{L^\infty}\left\|q^{in}\right\|_{\mathcal{M}} + \alpha(1+\alpha)CT\left\|D^2\psi\right\|_{L^\infty}\left\|q^{in}\right\|_{\mathcal{M}}^2.$$

Hence for each $\varepsilon > 0$, there is an L large enough, depending on ε, such that (for $\alpha < \frac{1}{2L}$)

$$I_2 \leq C\left(T, \psi, \left\|q^{in}\right\|_{\mathcal{M}}\right)(\varepsilon + \left|\log 2\alpha L\right|^{-1/2} + \alpha(1+\alpha)).$$

For each $\varepsilon > 0$, there is δ^* such that $\left|\log r\right|^{-1/2} < \varepsilon$, whenever $r < \delta^*$. Hence, for $\alpha < \min\left\{\frac{\delta^*}{2L}, \frac{1}{2L}, \varepsilon\right\}$

$$I_2 \leq \varepsilon C\left(T, \psi, \left\|q^{in}\right\|_{\mathcal{M}}\right).$$

Therefore, $\lim_{\alpha\to 0} I_2 = 0$. This concludes the proof that q is a weak solution of the Euler equations (2.2) with initial data q^{in}.

4 Birkhoff-Rott-α equation

The Birkhoff-Rott-α equation, based on the Euler-α equations (2.3) is derived similarly to the original Birkhoff-Rott equation. Detailed descriptions of the Birkhoff-Rott equation as a model for the evolution of the vortex sheet can be found, e.g., in [59, 60, 68]. We remark, however, that while the BR equations assume *a priori* that a vortex sheet remains a curve at a later time, in the 2D Euler-α case, if the vorticity is initially supported on a curve, then due to the existence of the unique Lagrangian flow map $\partial_t\eta(x,t) = \int_{\mathbb{R}^2} K^\alpha(x,y)\,dq(y,t)$, $\eta(x,0) = x$, $q(x,t) = q^{in}\circ\eta^{-1}(x,t)$, given by Theorem 2.1 of Oliver and Shkoller [65], it remains supported on a curve for all times. Existence of the unique Lagrangian flow map implies that the BR-α equation gives an equivalent description of the vortex sheet evolution, as the weak solution of 2D Euler-α equations. It is described in the following proposition.

Proposition 4.1. *Let $q^{in} \in \mathcal{M}(\mathbb{R}^2)$ supported on the sheet (curve) $\Sigma^{in} = \left\{x = x(\sigma) \in \mathbb{R}^2 | \sigma_0^{in} \leq \sigma \leq \sigma_1^{in}\right\}$, with a density $\gamma^{in}(\sigma)$, that is, the vorticity q^{in} satisfies*

$$\int_{\mathbb{R}^2}\varphi(x)dq^{in}(x) = \int_{\sigma_0^{in}}^{\sigma_1^{in}}\varphi(x(\sigma))\,\gamma^{in}(\sigma)|x_\sigma(\sigma)|d\sigma,$$

for every $\varphi \in C_c^\infty\left(\mathbb{R}^2\right)$, $\gamma^{in} \in L^1(|x_\sigma|\,d\sigma)$[3]. Let q be the solution of (2.3) in the sense of the Theorem 2.1. Then, for as long as the curve $\Sigma(t) = \left\{x = x(\sigma,t) \in \mathbb{R}^2 |\, \sigma_0(t) \leq \sigma \leq \sigma_1(t)\right\}$ remains nice enough

[3]Let Σ be a curve parametrized by $x(\sigma) : [\sigma_0, \sigma_1] \to \mathbb{R}^2$, such that $x_\sigma \in L^1([\sigma_0, \sigma_1])$, and let $q \in \mathcal{M}(\mathbb{R}^2)$ be supported on the curve Σ, with a density γ. Then $\gamma \in L^1(|x_\sigma|\,d\sigma)$ (and vice versa).

Proof. First, assume $q \geq 0$, and let θ_n be a truncating sequence, $\theta_n \in C_c^\infty(\mathbb{R}^2)$, $\theta_n(x) = \theta_1\left(\frac{x}{n}\right)$, $\theta_1 \in C_c^\infty(\mathbb{R}^2)$, $0 \leq \theta_1 \leq 1$ with $\theta_1 = 1$ for $|x| \leq 1$ and $\theta_1 = 0$ for $|x| \geq 2$. Then, on the one hand,

$$\int_{\sigma_0}^{\sigma_1}\theta_n(x(\sigma))\,\gamma(\sigma)|x_\sigma(\sigma)|d\sigma \geq \int_{\{\sigma:|x(\sigma)|\leq n\}\cap[\sigma_0,\sigma_1]}\gamma(\sigma)|x_\sigma(\sigma)|d\sigma,$$

on the other hand

$$\int_{\sigma_0}^{\sigma_1}\theta_n(x(\sigma))\,\gamma(\sigma)|x_\sigma(\sigma)|d\sigma = \int_{\mathbb{R}^2}\theta_n(x)dq^{in}(x) \leq \|\theta_n\|_{L^\infty}\|q\|_{\mathcal{M}} \leq \|q\|_{\mathcal{M}},$$

hence

$$\int_{\{\sigma:|x(\sigma)|\leq n\}\cap[\sigma_0,\sigma_1]}\gamma(\sigma)|x_\sigma(\sigma)|d\sigma \leq \|q\|_{\mathcal{M}}.$$

Since n can be taken arbitrary large this implies that $\int_{\sigma_0}^{\sigma_1}\gamma(\sigma)|x_\sigma(\sigma)|d\sigma < \infty$. Now, for signed measure we apply the previous result to each of the nonnegative measures q^+, q^-, given by the Jordan Decomposition of q, $q = q^+ - q^-$, which is defined by

$$\int_{\mathbb{R}^2}\varphi(x)dq^\pm(x) = \int_{\sigma_0}^{\sigma_1}\varphi(x(\sigma))\,\gamma^\pm(\sigma)|x_\sigma(\sigma)|d\sigma.$$

□

so that x_σ makes sense a.e., q has a density $\gamma(\sigma,t)$ supported on the sheet $\Sigma(t)$, $\gamma(\cdot,t) \in L^1(|x_\sigma|\,d\sigma)$, $\gamma(\sigma,t)\,|x_\sigma(\sigma,t)|\,d\sigma = \gamma(\sigma,0)\,|x_\sigma(\sigma,0)|\,d\sigma$ and the sheet evolves according to the equation

$$\frac{\partial}{\partial t}x(\sigma,t) = \int_{\sigma_0(t)}^{\sigma_1(t)} K^\alpha(x(\sigma,t) - x(\sigma',t))\,\gamma(\sigma',t)\,|x_\sigma(\sigma',t)|\,d\sigma',$$

where K^α is given by (2.5). Additionally, if $\Gamma(\sigma,t) = \int_{\sigma^*}^{\sigma} \gamma(\sigma',t)\,|x_\sigma(\sigma',t)|\,d\sigma'$, where $x(\sigma^*,t)$ is some fixed reference point on $\Sigma(t)$, defines a strictly increasing function of σ (e.g., as in the case of positive vorticity), then the evolution equation is given by the Birkhoff-Rott-α (BR-α) equation

$$\frac{\partial}{\partial t}x(\Gamma,t) = \int_{\Gamma_0}^{\Gamma_1} K^\alpha(x(\Gamma,t) - x(\Gamma',t))\,d\Gamma' \tag{4.1}$$

with $\gamma = 1/|x_\Gamma|$ being the vorticity density along the curve and $-\infty < \Gamma_0 < \Gamma_1 < \infty$.

In Section 6 we show the global well-posedness of the Birkhoff-Rott-α (4.1) in the space of Lipschitz functions and in the Hölder space $C^{n,\beta}$, $n \geq 1$, which is the space of n-times differentiable functions with Hölder continuous n^{th} derivative. Thus the solutions the Birkhoff-Rott-α and of the Euler-α are equivalent for the initial data being a finite positive Radon measure supported on Lipschitz or Hölder $C^{1,\beta}((\Gamma_0,\Gamma_1))$, $0 \leq \beta < 1$, chord arc curve, or supported on $C^{n,\beta}((\Gamma_0,\Gamma_1))$, $n \geq 1$, $0 < \beta < 1$, closed chord arc curve.

Here σ_0, σ_1 can represent either a finite length curve, or an infinite one. We remark that the smoothed kernel $K^\alpha(x)$ is a bounded continuous function, that for $\frac{|x|}{\alpha} \to 0$ behaves asymptotically as $K^\alpha(x) = -\frac{1}{4\pi}\frac{1}{\alpha^2}x^\perp \log\frac{|x|}{\alpha} + O\left(\frac{|x|}{\alpha^2}\right)$, i.e., it is non-singular kernel at the origin. For the case where $\gamma(\cdot,t) \in L^1(|x_\sigma|\,d\sigma)$ we can show the integrability of the relevant terms, even though $|K^\alpha(x)|$ is decaying like $|x|^{-1}$ at infinity.

5 Linear stability of a flat vortex sheet with uniform vorticity density for 2D BR-α model

The initial data problem for the BR equation is highly unstable due to an ill-posed response to small perturbations called Kelvin-Helmholtz instability [8, 69]. The linear stability analysis of the BR-α equation shows that the ill-posedness of the original problem is mollified, and the Kelvin-Helmholtz instability of the original system now disappears. We assume that the vortex sheet can be parameterized as a graph $x_2 = x_2(x_1,t)$, the proof can be easily adapted to establish the result in general. The flat sheet $x_2^0 \equiv 0$, with uniformly concentrated intensity γ_0 (notice that this density is not integrable on the curve), is a stationary solution of the following general BR-α system

$$\begin{aligned} \frac{\partial x_2}{\partial t} &= -\frac{\partial x_2}{\partial x_1}u_1 + u_2, \\ \frac{\partial \gamma}{\partial t} &= -\frac{\partial}{\partial x_1}(\gamma u_1), \end{aligned} \tag{5.1}$$

with velocity $u = (u_1,u_2)^t$ given by

$$u(x_1,t) = \text{p.v.}\int_{\mathbb{R}} K^\alpha(x(x_1,t) - x(x_1',t))\,\gamma(x_1',t)\,dx_1',$$

where $x(x_1,t) = (x_1, x_2(x_1,t))^t$ and p.v. is taken at infinity. By linearization about the flat sheet with uniformly concentrated intensity γ_0 we obtain the following linear system

$$\begin{aligned} \frac{\partial \tilde{x}_2}{\partial t} &= \tilde{u}_2, \\ \frac{\partial \tilde{\gamma}}{\partial t} &= -\gamma_0\frac{\partial \tilde{u}_1}{\partial x_1}, \end{aligned}$$

where

$$\tilde{u}_1(x_1,t) = -\gamma_0 \left(\operatorname{sgn}(x_1)\, D\Psi^\alpha(|x_1|)\right) * \frac{\partial \tilde{x}_2}{\partial x_1},$$
$$\tilde{u}_2(x_1,t) = \left(\operatorname{sgn}(x_1)\, D\Psi^\alpha(|x_1|)\right) * \tilde{\gamma},$$

and $(\tilde{x}_2, \tilde{\gamma})$ is a small perturbation about the flat sheet $x_2 \equiv 0$ with the constant density $\gamma = \gamma_0$.

Consequently, the equation for the Fourier modes (transform) of the above system is given by

$$\frac{d}{dt}\begin{pmatrix} \widehat{\tilde{x}_2} \\ \widehat{\tilde{\gamma}} \end{pmatrix} = \begin{pmatrix} 0 & \frac{i}{2}\operatorname{sgn}(k) d(k) \\ -i\frac{\gamma_0^2}{2} k^2 \operatorname{sgn}(k) d(k) & 0 \end{pmatrix} \begin{pmatrix} \widehat{\tilde{x}_2} \\ \widehat{\tilde{\gamma}} \end{pmatrix}, \tag{5.2}$$

where

$$d(k) = \left(1 + \frac{1}{\alpha^2 k^2}\right)^{-1/2} - 1.$$

Observe that in order to calculate the Fourier transform

$$\mathcal{F}\left(\operatorname{sgn}(x_1)\, D\Psi^\alpha(|x_1|)\right)(k) = \frac{i}{2}\operatorname{sgn}(k) d(k),$$

we used here the integral representation of the modified Bessel function of the second kind $K_1(x_1) = x_1 \int_1^\infty e^{-x_1 t}\left(t^2-1\right)^{1/2} dt$, (see, e.g., [78]). The eigenvalues of the coefficient matrix, given in (5.2), are

$$\lambda(k) = \pm\frac{1}{2}|\gamma_0|\,|k|\left(1 - \left(1 + \frac{1}{\alpha^2 k^2}\right)^{-1/2}\right). \tag{5.3}$$

We observe that while the linear system for the original Birkhoff-Rott equation is elliptic (in space and time)

$$\frac{\partial^2 \widehat{\tilde{x}_2}}{\partial t^2} - \frac{1}{4}\gamma_0^2 k^2 \widehat{\tilde{x}_2} = 0,$$

for a Birkhoff-Rott-α equation it is no longer an elliptic system

$$\frac{\partial^2 \widehat{\tilde{x}_2}}{\partial t^2} - \frac{1}{4}\gamma_0^2 d^2(k) k^2 \widehat{\tilde{x}_2} = 0,$$

since $\left|d^2(k)k^2\right|$ is bounded and behaves like $\frac{1}{\alpha^4 k^2}$, as $k \to \infty$ (for α fixed).

To conclude, the α-regularization mollifies the Kelvin-Helmholtz instability as follows: we have an algebraic decay of the eigenvalues to zero of order $\frac{1}{\alpha^2|k|}$, as $k \to \infty$ (for α fixed). While, for $\alpha \to 0$, for fixed k, we recover the eigenvalues of the original BR equations $\pm\frac{1}{2}|\gamma_0|\,|k|$ (see, e.g., [74]).

For the sake of comparison, we note that for the vortex blob regularization of Krasny [46], where the singular BR kernel, $K(x)$, was replaced with the smoothed kernel

$$K_\delta(x) = K(x)\frac{|x|^2}{|x|^2 + \delta^2} = \frac{1}{2\pi}\frac{x^\perp}{|x|^2 + \delta^2},$$

the eigenvalues are

$$\lambda(k) = \pm\frac{1}{2}e^{-\delta k}|\gamma_0|\,|k|$$

with an exponential decay to zero, as $k \to \infty$ ($\delta > 0$ is fixed). As $\delta \to 0$, for fixed k, one recovers again the eigenvalues of the original BR equations.

The behavior of the eigenvalues of the linearized system (5.2) indicates that high wave number perturbations grow exponentially in time with a rate that decays to zero, as $k \to \infty$, which is the reason for well-posedness of the α-regularized model. This is unlike the original BR problem that exhibits the Kelvin-Helmholtz instability. It is worth mentioning that the α-regularization is "closer" to the original system than the vortex-blob method at the high wave numbers, due to the algebraic decay instead of exponential one in the vortex blob method. This result was also evaluated computationally in [41].

6 Global regularity for BR-α equation

In this section we present the global existence and uniqueness of solutions of the BR-α equation (4.1) in the space of Lipschitz functions and in the Hölder space $C^{n,\beta}$, $n \geq 1$, which is the space of n-times differentiable functions with Hölder continuous n^{th} derivative.

Let us first describe the Hölder space $C^{n,\beta}\left(J \subset \mathbb{R}; \mathbb{R}^2\right)$, $0 < \beta \leq 1$, which is the space of functions $x : J \subset \mathbb{R} \to \mathbb{R}^2$, with a finite norm

$$\|x\|_{C^{n,\beta}(J)} = \sum_{k=0}^{n} \left\| \frac{d^k}{d\Gamma^k} x \right\|_{C^0(J)} + \left| \frac{d^n}{d\Gamma^n} x \right|_{\beta(J)},$$

where

$$\|x\|_{C^0(J)} = \sup_{\Gamma \in J} |x(\Gamma)|$$

and $|\cdot|_\beta$ is the Hölder semi-norm

$$|x|_{\beta(J)} = \sup_{\substack{\Gamma, \Gamma' \in J \\ \Gamma \neq \Gamma'}} \frac{|x(\Gamma) - x(\Gamma')|}{|\Gamma - \Gamma'|^\beta}.$$

The Lipschitz space $\text{Lip}(J)$ is the $C^{0,1}$space, that is, with the finite norm $\|x\|_{\text{Lip}(J)} = \|x\|_{C^0(J)} + |x|_{1(J)}$.

We also use the notation

$$|x|_* = \inf \frac{|x(\Gamma) - x(\Gamma')|}{|\Gamma - \Gamma'|},$$

where the infimum is taken over all $\Gamma, \Gamma' \in J$ such that $\Gamma \neq \Gamma'$, or, in the case of a closed curve (without loss of generality, over S^1), the infimum is taken over all $\Gamma, \Gamma' \in S^1$ such that $\Gamma \neq \Gamma' \mod 2\pi$.

We consider the BR-α equation as an evolution functional equation in the Banach spaces Lip, C^1 or $C^{n,\beta}$, $n \geq 1$, $0 < \beta < 1$,

$$\begin{aligned} &\frac{\partial x}{\partial t}(\Gamma, t) = \int_{\Gamma_0}^{\Gamma_1} K^\alpha (x(\Gamma, t) - x(\Gamma', t))\, d\Gamma', \\ &x(\Gamma, 0) = x_0(\Gamma) \end{aligned} \tag{6.1}$$

with $\gamma = 1/|x_\Gamma|$ being the vorticity density along the sheet and $-\infty < \Gamma_0 < \Gamma_1 < \infty$. Notice that the density $\gamma(\Gamma)$ is in $C^{n-1,\beta}((\Gamma_0, \Gamma_1))$ for the subset $\{|x|_* > 0\}$ of $C^{n,\beta}((\Gamma_0, \Gamma_1))$, and $\gamma(\Gamma) \in L^\infty((\Gamma_0, \Gamma_1))$ for the subset $\{|x|_* > 0\}$ of $\text{Lip}((\Gamma_0, \Gamma_1))$.

Theorem 6.1. *Let $-\infty < \Gamma_0 < \Gamma_1 < \infty$, let V be either the space $C^{1,\beta}((\Gamma_0, \Gamma_1))$, $0 \leq \beta < 1$ or the space $\text{Lip}((\Gamma_0, \Gamma_1))$ and let $x_0 \in V \cap \{|x|_* > 0\}$, then for any $T > 0$ there is a unique solution $x \in C^1([-T, T]; V \cap \{|x|_* > 0\})$ of (6.1) with initial value $x(\Gamma, 0) = x_0(\Gamma)$.*

Furthermore, let x_0 be a closed curve and without restriction of generality, we assume $x_0(\Gamma) \in C^{n,\beta}(S^1) \cap \{|x|_* > 0\}$, then for all $n \geq 1$, $0 < \beta < 1$, $T > 0$ there is a unique solution $x \in C^1\left([-T, T]; C^{n,\beta}(S^1) \cap \{|x|_* > 0\}\right)$ of (4.1) with initial value $x(\Gamma, 0) = x_0(\Gamma)$. In particular, if $x_0 \in C^\infty(S^1) \cap \{|x|_* > 0\}$ then $x \in C^1\left([-T, T]; C^\infty(S^1) \cap \{|x|_* > 0\}\right)$.

Notice that for $n > 1$ we request β to be strictly larger than zero and the curve to be closed.

We remark that, although the kernel K^α is a continuous bounded function, its derivatives are unbounded near the origin, and the chord arc condition $|x|_* > 0$, which implies simple curves, allows us to show the integrability of the relevant terms.

The following are the main steps involved in the proof of Theorem 6.1. In the first step, we apply the Contraction Mapping Principle to the BR-α equation (4.1) to prove the short time existence and uniqueness of solutions in the appropriate space of functions. Specifically, we show that an initially Lipschitz or $C^{1,\beta}, 0 \leq \beta < 1$ smooth solutions of (4.1) remain, respectively, Lipschitz or $C^{1,\beta}$ smooth for a finite short time. Next, we derive an *a priori* bound for the controlling quantity for continuing the solution for all time. At step three we extend the $C^{1,\beta}, 0 < \beta < 1$ result for higher derivatives for the case of a closed curve.

6.1 Step 1. Local well-posedness.

First we show the local existence and uniqueness of solutions. To apply the Contraction Mapping Principle to the BR-α equation (6.1) we first prove the following result

Proposition 6.2. *Let $-\infty < \Gamma_0 < \Gamma_1 < \infty$, $1 < M < \infty$, V be either the space $C^{1,\beta}((\Gamma_0,\Gamma_1))$, $0 \le \beta < 1$ or the space $\mathrm{Lip}((\Gamma_0,\Gamma_1))$, and let K^M be the set*

$$K^M = \left\{ x \in V : |x|_1 < M, |x|_* > \frac{1}{M} \right\}.$$

Then the mapping

$$x(\Gamma) \mapsto u(x(\Gamma)) = \int_{\Gamma_0}^{\Gamma_1} K^\alpha(x(\Gamma) - x(\Gamma'))\, d\Gamma' \tag{6.2}$$

defines a locally Lipschitz continuous map from K^M, equipped with the topology induced by the $\|\cdot\|_V$ norm, into V.

Proof. Notice that K^M is an open set in V. We recall that $K^\alpha(x) = \nabla^\perp \Psi^\alpha(|x|) = \frac{x^\perp}{|x|} D\Psi^\alpha(|x|)$, where $\Psi^\alpha(r) = \frac{1}{2\pi}\left[K_0\left(\frac{r}{\alpha}\right) + \log r\right]$ and $D\Psi^\alpha(r) = \frac{d\Psi^\alpha}{dr}(r) = \frac{1}{2\pi}\left[-\frac{1}{\alpha}K_1\left(\frac{r}{\alpha}\right) + \frac{1}{r}\right]$. The functions K_0 and K_1 denote the modified Bessel functions of the second kind of orders zero and one, respectively. For details on Bessel functions, see, e.g., [78]. We observe that $D\Psi^\alpha$ is bounded

$$D\Psi^\alpha(r) \le \frac{C}{\alpha}, \tag{6.3}$$

derivatives of Ψ^α decay to zero as $\frac{r}{\alpha} \to \infty$, and as $\frac{r}{\alpha} \to 0$ satisfy

$$\begin{aligned} D\Psi^\alpha(r) &= -\frac{1}{4\pi}\frac{r}{\alpha^2}\log\frac{r}{\alpha} + O\left(\frac{r}{\alpha^2}\right), \\ D^2\Psi^\alpha(r) &= -\frac{1}{4\pi}\frac{1}{\alpha^2}\log\frac{r}{\alpha} + O\left(\frac{1}{\alpha^2}\right), \\ D^3\Psi^\alpha(r) &= -\frac{1}{4\pi}\frac{1}{r\alpha^2} + O\left(\frac{r}{\alpha^4}\log\frac{r}{\alpha}\right). \end{aligned} \tag{6.4}$$

The constant C will denote a generic constant independent of the parameters, while, $C(\diamond)$ denotes a constant which depends on $\diamond$.

First we show the local existence and uniqueness of solutions in $C^{1,\beta}$, $0 < \beta < 1$.

We start by showing that $u(x(\Gamma))$ maps K^M into $C^{1,\beta}$. Let $x \in K^M$. Using the boundness of $D\Psi^\alpha$ (6.3) we have

$$|u(x(\Gamma))| \le \int_{\Gamma_0}^{\Gamma_1} D\Psi^\alpha(|x(\Gamma) - x(\Gamma')|)\, d\Gamma' \le \frac{C}{\alpha}(\Gamma_1 - \Gamma_0). \tag{6.5}$$

Using that

$$\frac{du}{d\Gamma}(x(\Gamma)) = \int_{\Gamma_0}^{\Gamma_1} \nabla K^\alpha(x(\Gamma) - x(\Gamma'))\, \frac{dx}{d\Gamma}(\Gamma)\, d\Gamma',$$

(which can be justified by applying Lebesgue dominated convergence theorem) and the fact that $\left|\frac{dx}{d\Gamma}(\Gamma)\right| < M$, we obtain

$$\begin{aligned} \left|\frac{du}{d\Gamma}(x(\Gamma))\right| &\le M \int_{\Gamma_0}^{\Gamma_1} |\nabla K^\alpha(x(\Gamma) - x(\Gamma'))|\, d\Gamma', \\ &= M\left(\int_{(\Gamma_0,\Gamma_1)\cap\left\{\frac{|\Gamma-\Gamma'|}{\alpha}<\varepsilon\right\}} + \int_{(\Gamma_0,\Gamma_1)\setminus\left\{\frac{|\Gamma-\Gamma'|}{\alpha}<\varepsilon\right\}}\right), \\ &= M(I_1 + I_2), \end{aligned}$$

where ε is to be defined later. Due to (2.5), (6.4) and

$$\frac{1}{M}\frac{|\Gamma-\Gamma'|}{\alpha}<|x|_*\frac{|\Gamma-\Gamma'|}{\alpha}\le\frac{|x(\Gamma)-x(\Gamma')|}{\alpha}\le\left\|\frac{dx}{d\Gamma}\right\|_{C^0}\frac{|\Gamma-\Gamma'|}{\alpha}\le M\varepsilon, \tag{6.6}$$

we have that for a fixed small ε

$$\begin{aligned} I_1 &\le \int_{\frac{|\Gamma-\Gamma'|}{\alpha}<\varepsilon}\left(\frac{1}{2\pi\alpha^2}\left|\log\left(C(M)\frac{|\Gamma-\Gamma'|}{\alpha}\right)\right|+C(M)\frac{1}{\alpha^2}\right)d\Gamma' \\ &= C(M)\frac{1}{\alpha}. \end{aligned}$$

For I_2 due to the boundness of $|\nabla K^\alpha|$ in $(\Gamma_0,\Gamma_1)\setminus\left\{\frac{|\Gamma-\Gamma'|}{\alpha}<\varepsilon\right\}$ we obtain

$$I_2\le\frac{C(M)}{\alpha^2}(\Gamma_1-\Gamma_0).$$

Summing up,

$$\int_{\Gamma_0}^{\Gamma_1}|\nabla K^\alpha(x(\Gamma)-x(\Gamma'))|\,d\Gamma'\le C\left(M,\Gamma_1,\Gamma_0,\frac{1}{\alpha}\right) \tag{6.7}$$

and hence

$$\left|\frac{du}{d\Gamma}(x(\Gamma))\right|\le C\left(M,\Gamma_1,\Gamma_0,\frac{1}{\alpha}\right). \tag{6.8}$$

To show the Hölder continuity of $\frac{du}{d\Gamma}(x(\Gamma))$ we write

$$\begin{aligned} \left|\frac{du}{d\Gamma}(x(\Gamma))-\frac{du}{d\Gamma}\left(x\left(\bar\Gamma\right)\right)\right| &\le \int_{\Gamma_0}^{\Gamma_1}|\nabla K^\alpha(x(\Gamma)-x(\Gamma'))|\left|\frac{dx}{d\Gamma}(\Gamma)-\frac{dx}{d\Gamma}\left(\bar\Gamma\right)\right|d\Gamma' \\ &+\int_{\Gamma_0}^{\Gamma_1}\left|\nabla K^\alpha(x(\Gamma)-x(\Gamma'))-\nabla K^\alpha\left(x\left(\bar\Gamma\right)-x(\Gamma')\right)\right|\left|\frac{dx}{d\Gamma}\left(\bar\Gamma\right)\right|d\Gamma' \\ &\le\left|\frac{dx}{d\Gamma}\right|_\beta|\Gamma-\Gamma'|^\beta\int_{\Gamma_0}^{\Gamma_1}|\nabla K^\alpha(x(\Gamma)-x(\Gamma'))|\,d\Gamma'+ \\ &+M\int_{\Gamma_0}^{\Gamma_1}\left|\nabla K^\alpha(x(\Gamma)-x(\Gamma'))-\nabla K^\alpha\left(x\left(\bar\Gamma\right)-x(\Gamma')\right)\right|d\Gamma'. \end{aligned}$$

The first term on the right-hand side is bounded by $C\left(M,\Gamma_1,\Gamma_0,\frac{1}{\alpha}\right)\left\|\frac{dx}{d\Gamma}\right\|_\beta\left|\Gamma-\bar\Gamma\right|^\beta$due to (6.7), as for the second one, let $r=\frac{|\Gamma-\bar\Gamma|}{\alpha}$, and write

$$\begin{aligned} I&=\int_{\Gamma_0}^{\Gamma_1}\left|\nabla K^\alpha(x(\Gamma)-x(\Gamma'))-\nabla K^\alpha\left(x\left(\bar\Gamma\right)-x(\Gamma')\right)\right|d\Gamma' \\ &=\int_{(\Gamma_0,\Gamma_1)\cap\left\{\frac{|\Gamma-\Gamma'|}{\alpha}<2r\right\}}+\int_{(\Gamma_0,\Gamma_1)\cap\left\{\frac{|\Gamma-\Gamma'|}{\alpha}\ge2r\right\}} \\ &=I_1+I_2. \end{aligned}$$

For I_1 we have $\frac{|\Gamma-\Gamma'|}{\alpha}<2r$, and hence $\frac{|\bar\Gamma-\Gamma'|}{\alpha}<3r$. Due to (2.5), the fact that

$$\left|D^2\Psi^\alpha(s)\right|\le\frac{1}{4\pi}\frac{1}{\alpha^2}\left|\log\frac{s}{\alpha}\right|+\frac{C}{\alpha^2} \tag{6.9}$$

and (6.6) we obtain

$$\begin{aligned}
I_1 &\leq \int_{(\Gamma_0,\Gamma_1)\cap\left\{\frac{|\Gamma-\Gamma'|}{\alpha}<2r\right\}} \left|\nabla K^\alpha\left(x\left(\Gamma\right)-x\left(\Gamma'\right)\right)\right| + \left|\nabla K^\alpha\left(x\left(\bar{\Gamma}\right)-x\left(\Gamma'\right)\right)\right| d\Gamma' \\
&\leq \frac{C}{\alpha^2}\left(\int_{\frac{|\Gamma-\Gamma'|}{\alpha}<2r}\left|\log\frac{\left|x\left(\Gamma\right)-x\left(\Gamma'\right)\right|}{\alpha}\right| d\Gamma' + \int_{\frac{|\bar{\Gamma}-\Gamma'|}{\alpha}<3r}\left|\log\frac{\left|x\left(\bar{\Gamma}\right)-x\left(\Gamma'\right)\right|}{\alpha}\right| d\Gamma' + r\alpha\right) \\
&\leq \frac{C}{\alpha^2}\left(\int_{\frac{|\Gamma-\Gamma'|}{\alpha}<2r}\left|\log C\left(M\right)\frac{\left|\Gamma-\Gamma'\right|}{\alpha}\right| d\Gamma' + \int_{\frac{|\bar{\Gamma}-\Gamma'|}{\alpha}<3r}\left|\log C\left(M\right)\frac{\left|\bar{\Gamma}-\Gamma'\right|}{\alpha}\right| d\Gamma' + r\alpha\right) \\
&\leq C\left(M,\frac{1}{\alpha}\right) r\left(\left|\log r\right|+1\right).
\end{aligned}$$

For I_2 we have $\frac{|\Gamma-\Gamma'|}{\alpha}\geq 2r$, and hence $\frac{|\bar{\Gamma}-\Gamma'|}{\alpha}\geq r$. By the mean value theorem (MVT), (2.5) and the fact that

$$D^3\Psi^\alpha\left(s\right)\leq\frac{1}{4\pi\alpha^2}\frac{1}{s}+\frac{C}{\alpha^3} \tag{6.10}$$

we have that for $\Gamma''\in\left[\Gamma,\bar{\Gamma}\right]$

$$\begin{aligned}
\left|\nabla K^\alpha\left(x\left(\Gamma\right)-x\left(\Gamma'\right)\right)-\nabla K^\alpha\left(x\left(\bar{\Gamma}\right)-x\left(\Gamma'\right)\right)\right| &\leq r\left\|\frac{dx}{d\Gamma}\right\|_{C^0}\left(\frac{C}{\alpha^2\left|x\left(\Gamma''\right)-x\left(\Gamma'\right)\right|}+\frac{C}{\alpha^3}\right) \\
&\leq C\left(M,\frac{1}{\alpha}\right) r\left(\frac{1}{\left|\Gamma''-\Gamma'\right|}+1\right),
\end{aligned}$$

we also have that $\frac{|\Gamma''-\Gamma'|}{\alpha}\geq r$ and $\Gamma_0\leq\Gamma''\leq\Gamma_1$. Hence

$$\begin{aligned}
\left|I_2\right| &\leq r\frac{C\left(M\right)}{\alpha^3}\int_{(\Gamma_0,\Gamma_1)\cap\left\{\frac{|\Gamma''-\Gamma'|}{\alpha}\geq r\right\}}\left(\frac{\alpha}{\left|\Gamma''-\Gamma'\right|}+1\right)d\Gamma' \\
&\leq C\left(M,\frac{1}{\alpha},\Gamma_1,\Gamma_0\right) r\left(1+\left|\log r\right|\right).
\end{aligned}$$

Summing up we obtain

$$\left|I\right|\leq C\left(M,\frac{1}{\alpha},\Gamma_1,\Gamma_0\right)\left|\bar{\Gamma}-\Gamma\right|\left(\left|\log\left|\bar{\Gamma}-\Gamma\right|\right|+1\right), \tag{6.11}$$

which implies the Hölder continuity of $\frac{du}{d\Gamma}\left(x\left(\Gamma\right)\right)$ for $0<\beta<1$.

It remains to show that $u\left(x\right)$ is locally Lipschitz continuous on K^M. We will show that for $x\in K^M$, $y\in C^{1,\beta}\left(\left(\Gamma_0,\Gamma_1\right)\right)$

$$\left\|D_x u\left(x\right)y\right\|_{1,\beta}\leq C\left(\frac{1}{\alpha},M,\left\|x\right\|_{1,\beta},\Gamma_1,\Gamma_0,\beta\right)\left\|y\right\|_{1,\beta}.$$

Hence for any $x\in K^M$, let δ be such that $B\left(x,\delta\right)\subset K^M$, then for every $\bar{x},\tilde{x}\in B\left(x,\delta\right)$ by the fundamental theorem of calculus

$$\begin{aligned}
\left\|u\left(\bar{x}\right)-u\left(\tilde{x}\right)\right\|_{1,\beta} &= \left\|\int_0^1\frac{d}{d\varepsilon}u\left(\bar{x}+\varepsilon\left(\tilde{x}-\bar{x}\right)\right)d\varepsilon\right\|_{1,\beta} \\
&= \left\|\int_0^1 D_x u\left(\bar{x}+\varepsilon\left(\tilde{x}-\bar{x}\right)\right)\left(\tilde{x}-\bar{x}\right)d\varepsilon\right\|_{1,\beta} \\
&\leq \left\|\tilde{x}-\bar{x}\right\|_{1,\beta}\int_0^1 C\left(\frac{1}{\alpha},M,\left\|\bar{x}+\varepsilon\left(\tilde{x}-\bar{x}\right)\right\|_{1,\beta},\Gamma_1,\Gamma_0,\beta\right)d\varepsilon \\
&\leq C\left(\frac{1}{\alpha},M,\left\|x\right\|_{1,\beta},\delta,\Gamma_1,\Gamma_0,\beta\right)\left\|\tilde{x}-\bar{x}\right\|_{1,\beta},
\end{aligned}$$

that is, the map is locally Lipschitz. Here we used the fact that the Banach space $C^{1,\beta}$ is an algebra.

Let $x \in K^M$, $y \in C^{1,\beta}((\Gamma_0,\Gamma_1))$, we now compute

$$\begin{aligned} D_x u(x(\Gamma))\, y(\Gamma) &= \left.\frac{d}{d\varepsilon} u(x(\Gamma)+\varepsilon y(\Gamma))\right|_{\varepsilon=0} \\ &= \left.\frac{d}{d\varepsilon}\int_{\Gamma_0}^{\Gamma_1} K^\alpha(x(\Gamma)+\varepsilon y(\Gamma)-x(\Gamma')-\varepsilon y(\Gamma'))\, d\Gamma'\right|_{\varepsilon=0} \\ &= \int_{\Gamma_0}^{\Gamma_1} \nabla K^\alpha(x(\Gamma)-x(\Gamma'))\,(y(\Gamma)-y(\Gamma'))\, d\Gamma'. \end{aligned}$$

Now, we show that

$$\|D_x u(x(\cdot))\, y(\cdot)\|_{1,\beta} \le C\left(\frac{1}{\alpha}, M, \|x\|_{1,\beta}, \Gamma_1, \Gamma_0, \beta\right)\|y\|_{1,\beta}.$$

To estimate $\|D_x u(x)\, y\|_{C^0}$ we use (6.7)

$$|D_x u(x)\, y| \le C\left(M, \Gamma_1, \Gamma_0, \frac{1}{\alpha}\right)\|y\|_{C^0}. \tag{6.12}$$

Next we estimate $\left\|\frac{d}{d\Gamma} D_x u(x)\, y\right\|_{C^0}$. For $\Gamma' \neq \Gamma$, $\nabla K^\alpha(x(\Gamma)-x(\Gamma'))(y(\Gamma)-y(\Gamma'))$ is differentiable in Γ, hence (which can be justified by using Lebesgue dominated convergence theorem)

$$\begin{aligned} \frac{d}{d\Gamma} D_x u(x(\Gamma))\, y(\Gamma) &= \int_{\Gamma_0}^{\Gamma_1} \nabla K^\alpha(x(\Gamma)-x(\Gamma'))\,\frac{dy}{d\Gamma}(\Gamma)\, d\Gamma' \\ &\quad + \int_{\Gamma_0}^{\Gamma_1} \sum_{i,j=1}^{2} \partial_{x_i}\partial_{x_j} K^\alpha(x(\Gamma)-x(\Gamma'))\,\frac{dx_i}{d\Gamma}(\Gamma)\,(y_j(\Gamma)-y_j(\Gamma'))\, d\Gamma'. \end{aligned}$$

Notice, that although, for Γ' close to Γ, $\left|D^2 K^\alpha(x(\Gamma)-x(\Gamma'))\right| = O\left(\frac{1}{\alpha^2|x(\Gamma)-x(\Gamma')|}\right)$ (see (2.5) and (6.4)), the term $(y(\Gamma)-y(\Gamma'))$ cancels the singularity in $\frac{1}{x(\Gamma)-x(\Gamma')}$, so this is not a singular integral.

$$\begin{aligned} \left|\frac{d}{d\Gamma} D_x u(x(\Gamma))\, y(\Gamma)\right| &\le \left\|\frac{dy}{d\Gamma}\right\|_{C^0}\int_{\Gamma_0}^{\Gamma_1} |\nabla K^\alpha(x(\Gamma)-x(\Gamma'))|\, d\Gamma' \\ &\quad + \left\|\frac{dx}{d\Gamma}\right\|_{C^0}\int_{\Gamma_0}^{\Gamma_1} \left|D^2 K^\alpha(x(\Gamma)-x(\Gamma'))\right| |y(\Gamma)-y(\Gamma')|\, d\Gamma'. \end{aligned}$$

Write the second integral on the right-hand side as

$$\begin{aligned} \int_{\Gamma_0}^{\Gamma_1} \left|D^2 K^\alpha(x(\Gamma)-x(\Gamma'))\right| |y(\Gamma)-y(\Gamma')|\, d\Gamma' &= \int_{(\Gamma_0,\Gamma_1)\cap\left\{\frac{|\Gamma-\Gamma'|}{\alpha}<\varepsilon\right\}} + \int_{(\Gamma_0,\Gamma_1)\setminus\left\{\frac{|\Gamma-\Gamma'|}{\alpha}<\varepsilon\right\}} \\ &= I_1 + I_2. \end{aligned}$$

Then due to (2.5), (6.4) and (6.6), we obtain

$$\begin{aligned} I_1 &\le C\frac{1}{\alpha^2}\left\|\frac{dy}{d\Gamma}\right\|_{C^0}\int_{\frac{|\Gamma-\Gamma'|}{\alpha}<\varepsilon} \frac{M}{|\Gamma-\Gamma'|}|\Gamma-\Gamma'|\, d\Gamma' \\ &\le CM\frac{1}{\alpha}\left\|\frac{dy}{d\Gamma}\right\|_{C^0}. \end{aligned}$$

For I_2 we have

$$\begin{aligned} I_2 &\leq C\left(M, \frac{1}{\alpha}\right)\left\|\frac{dy}{d\Gamma}\right\|_{C^0}\int_{(\Gamma_0,\Gamma_1)\cap\{|\Gamma-\Gamma'|\geq\varepsilon\alpha\}}|\Gamma-\Gamma'|\,d\Gamma' \\ &\leq C\left(M,\Gamma_1,\Gamma_0,\frac{1}{\alpha}\right)\left\|\frac{dy}{d\Gamma}\right\|_{C^0}. \end{aligned}$$

Hence

$$\left|\frac{d}{d\Gamma}D_x u\left(x\left(\Gamma\right)\right)y\left(\Gamma\right)\right| \leq \|y\|_{C^1} C\left(M,\Gamma_1,\Gamma_0,\frac{1}{\alpha}\right). \tag{6.13}$$

It remains to estimate $\left|\frac{d}{d\Gamma}D_x u\left(x\right)y\right|_\beta$.

$$\begin{aligned} \frac{d}{d\Gamma}D_x u\left(x\left(\Gamma\right)\right)y\left(\Gamma\right) - \frac{d}{d\Gamma}D_x u\left(x\left(\bar{\Gamma}\right)\right)y\left(\bar{\Gamma}\right) &= \int_{\Gamma_0}^{\Gamma_1}\left(\nabla K^\alpha\left(x\left(\Gamma\right)-x\left(\Gamma'\right)\right)\frac{dy}{d\Gamma}\left(\Gamma\right) - \nabla K^\alpha\left(x\left(\bar{\Gamma}\right)-x\left(\Gamma'\right)\right)\frac{dy}{d\Gamma}\left(\bar{\Gamma}\right)\right)d\Gamma' \\ &+ \int_{\Gamma_0}^{\Gamma_1}\left[\sum_{i,j}\partial_{x_i}\partial_{x_j}K^\alpha\left(x\left(\Gamma\right)-x\left(\Gamma'\right)\right)\frac{dx_i}{d\Gamma}\left(\Gamma\right)\left(y_j\left(\Gamma\right)-y_j\left(\Gamma'\right)\right) - \right. \\ &\left. - \sum_{i,j}\partial_{x_i}\partial_{x_j}K^\alpha\left(x\left(\bar{\Gamma}\right)-x\left(\Gamma'\right)\right)\frac{dx_i}{d\Gamma}\left(\bar{\Gamma}\right)\left(y_j\left(\bar{\Gamma}\right)-y_j\left(\Gamma'\right)\right)\right]d\Gamma' \\ &= I_1 + I_2. \end{aligned}$$

We write I_1 as

$$\begin{aligned} |I_1| &\leq \int_{\Gamma_0}^{\Gamma_1}\left|\nabla K^\alpha\left(x\left(\Gamma\right)-x\left(\Gamma'\right)\right)\right|\left(\frac{dy}{d\Gamma}\left(\Gamma\right)-\frac{dy}{d\Gamma}\left(\bar{\Gamma}\right)\right|d\Gamma' \\ &+ \int_{\Gamma_0}^{\Gamma_1}\left|\nabla K^\alpha\left(x\left(\Gamma\right)-x\left(\Gamma'\right)\right)-\nabla K^\alpha\left(x\left(\bar{\Gamma}\right)-x\left(\Gamma'\right)\right)\right|\left|\frac{dy}{d\Gamma}\left(\bar{\Gamma}\right)\right|d\Gamma' \\ &\leq I_{11}+I_{12}. \end{aligned}$$

Using (6.7) to bound I_{11} and (6.11) to bound I_{12} we obtain that

$$|I_1| \leq C\left(M,\frac{1}{\alpha},\Gamma_1,\Gamma_0\right)\left|\bar{\Gamma}-\Gamma\right|^\beta\|y\|_{1,\beta}.$$

Now, we estimate I_2. Let $r=\frac{|\Gamma-\bar{\Gamma}|}{\alpha}$, write I_2 as

$$\begin{aligned} I_2 &= \int_{(\Gamma_0,\Gamma_1)\cap\left\{\frac{|\Gamma-\Gamma'|}{\alpha}<2r\right\}} + \int_{(\Gamma_0,\Gamma_1)\cap\left\{\frac{|\Gamma-\Gamma'|}{\alpha}\geq 2r\right\}} \\ &= I_{21}+I_{22}. \end{aligned}$$

Using (2.5) and (6.10) for I_{21} we have

$$\begin{aligned} |I_{21}| &\leq \int_{(\Gamma_0,\Gamma_1)\cap\frac{|\Gamma-\Gamma'|}{\alpha}<2r}\left|D^2K^\alpha\left(x\left(\Gamma\right)-x\left(\Gamma'\right)\right)\right|\left|\frac{dx}{d\Gamma}\left(\Gamma\right)\right|\left|y\left(\Gamma\right)-y\left(\Gamma'\right)\right|d\Gamma' \\ &+ \int_{(\Gamma_0,\Gamma_1)\cap\frac{|\bar{\Gamma}-\Gamma'|}{\alpha}<3r}\left|D^2K^\alpha\left(x\left(\bar{\Gamma}\right)-x\left(\Gamma'\right)\right)\right|\left|\frac{dx}{d\Gamma}\left(\bar{\Gamma}\right)\right|\left|y\left(\bar{\Gamma}\right)-y\left(\Gamma'\right)\right|d\Gamma' \\ &\leq C\left\|\frac{dx}{d\Gamma}\right\|_{C^0}\left\|\frac{dy}{d\Gamma}\right\|_{C^0}\int_{\frac{|\Gamma-\Gamma'|}{\alpha}<3r}\left(\frac{C}{\alpha^2}\frac{M}{|\Gamma-\Gamma'|}+\frac{C}{\alpha^3}\right)|\Gamma-\Gamma'|\,d\Gamma' \\ &\leq C\left(\frac{1}{\alpha},M,\Gamma_1,\Gamma_0\right)\left\|\frac{dx}{d\Gamma}\right\|_{C^0}\|y\|_{1,\beta}\left|\Gamma-\bar{\Gamma}\right|^\beta \end{aligned}$$

We write I_{22} as

$$\begin{aligned}
I_{22} &= \int_{(\Gamma_0,\Gamma_1)\cap\left\{\frac{|\Gamma-\Gamma'|}{\alpha}\geq 2r\right\}} \sum_{i,j} \partial_{x_i}\partial_{x_j} K^\alpha\left(x\left(\Gamma\right)-x\left(\Gamma'\right)\right)\frac{dx_i}{d\Gamma}\left(\Gamma\right)\left(y_j\left(\Gamma\right)-y_j\left(\bar{\Gamma}\right)\right)d\Gamma' \\
&+ \int_{(\Gamma_0,\Gamma_1)\cap\left\{\frac{|\Gamma-\Gamma'|}{\alpha}\geq 2r\right\}} \sum_{i,j} \partial_{x_i}\partial_{x_j} K^\alpha\left(x\left(\Gamma\right)-x\left(\Gamma'\right)\right)\left(\frac{dx_i}{d\Gamma}\left(\Gamma\right)-\frac{dx_i}{d\Gamma}\left(\bar{\Gamma}\right)\right)\left(y_j\left(\bar{\Gamma}\right)-y_j\left(\Gamma'\right)\right)d\Gamma' \\
&+ \int \sum_{i,j} \left(\partial_{x_i}\partial_{x_j} K^\alpha\left(x\left(\Gamma\right)-x\left(\Gamma'\right)\right)-\partial_{x_i}\partial_{x_j} K^\alpha\left(x\left(\bar{\Gamma}\right)-x\left(\Gamma'\right)\right)\right)x_{i\Gamma}\left(\bar{\Gamma}\right)\left(y_j\left(\bar{\Gamma}\right)-y_j\left(\Gamma'\right)\right)d\Gamma' \\
&= I_{221}+I_{222}+I_{223}.
\end{aligned}$$

For I_{221} :

$$\begin{aligned}
|I_{221}| &\leq \left\|\frac{dx}{d\Gamma}\right\|_{C^0}\left\|\frac{dy}{d\Gamma}\right\|_{C^0}\left|\Gamma-\bar{\Gamma}\right|\int_{(\Gamma_0,\Gamma_1)\cap\left\{\frac{|\Gamma-\Gamma'|}{\alpha}\geq 2r\right\}}\left|D^2K^\alpha\left(x\left(\Gamma\right)-x\left(\Gamma'\right)\right)\right|d\Gamma' \\
&\leq \left\|\frac{dx}{d\Gamma}\right\|_{C^0}\left\|\frac{dy}{d\Gamma}\right\|_{C^0}\left|\Gamma-\bar{\Gamma}\right|\int_{(\Gamma_0,\Gamma_1)\cap\left\{\frac{|\Gamma-\Gamma'|}{\alpha}\geq 2r\right\}}\left(\frac{1}{\alpha^2}\frac{M}{|\Gamma-\Gamma'|}+\frac{C}{\alpha^3}\right)d\Gamma' \\
&\leq C\left(\frac{1}{\alpha},M,\Gamma_1,\Gamma_0,\left\|\frac{dx}{d\Gamma}\right\|_{C^0}\right)\|y\|_{1,\beta}\left|\Gamma-\bar{\Gamma}\right|\left[1+\left|\log\left|\Gamma-\bar{\Gamma}\right|\right|\right],
\end{aligned}$$

which implies Hölder continuity for $0<\beta<1$.

For I_{222} :

$$\begin{aligned}
|I_{222}| &\leq \int_{(\Gamma_0,\Gamma_1)\cap\left\{\frac{|\Gamma-\Gamma'|}{\alpha}\geq 2r\right\}}\left|D^2K^\alpha\left(x\left(\Gamma\right)-x\left(\Gamma'\right)\right)\right|\left|\frac{dx}{d\Gamma}\left(\Gamma\right)-\frac{dx}{d\Gamma}\left(\bar{\Gamma}\right)\right|\left|y\left(\bar{\Gamma}\right)-y\left(\Gamma'\right)\right|d\Gamma' \\
&\leq C\left|\frac{dx}{d\Gamma}\right|_\beta\left|\Gamma-\bar{\Gamma}\right|^\beta\left\|\frac{dy}{d\Gamma}\right\|_{C^0}\int_{(\Gamma_0,\Gamma_1)\cap\left\{\frac{|\Gamma-\Gamma'|}{\alpha}\geq 2r\right\}}\left(\frac{1}{\alpha^2}\frac{M}{|\Gamma-\Gamma'|}+\frac{C}{\alpha^3}\right)\left|\bar{\Gamma}-\Gamma'\right|d\Gamma' \\
&\leq C\left(\frac{1}{\alpha},M,\left|\frac{dx}{d\Gamma}\right|_\beta,\Gamma_1,\Gamma_0\right)\left|\Gamma-\bar{\Gamma}\right|^\beta\|y\|_{1,\beta},
\end{aligned}$$

here we also used that

$$\left|\bar{\Gamma}-\Gamma'\right|\leq\left|\bar{\Gamma}-\Gamma\right|+\left|\Gamma-\Gamma'\right|.$$

For I_{223} :

$$|I_{223}|\leq\left\|\frac{dx}{d\Gamma}\right\|_{C^0}\left\|\frac{dy}{d\Gamma}\right\|_{C^0}\int_{(\Gamma_0,\Gamma_1)\cap\left\{\frac{|\Gamma-\Gamma'|}{\alpha}\geq 2r\right\}}\left|D^2K^\alpha\left(x\left(\Gamma\right)-x\left(\Gamma'\right)\right)-D^2K^\alpha\left(x\left(\bar{\Gamma}\right)-x\left(\Gamma'\right)\right)\right|\left|\bar{\Gamma}-\Gamma'\right|d\Gamma'$$

Since $\frac{|\Gamma-\Gamma'|}{\alpha}\geq 2r$ and $\frac{|\bar{\Gamma}-\Gamma'|}{\alpha}\geq r$, i.e., $D^2K^\alpha\left(x\left(\Gamma\right),x\left(\Gamma'\right)\right)-D^2K^\alpha\left(x\left(\bar{\Gamma}\right),x\left(\Gamma'\right)\right)$ is differentiable in $\left[\Gamma,\bar{\Gamma}\right]$, we can apply the MVT to obtain that for $\Gamma''\in\left[\Gamma,\bar{\Gamma}\right]$

$$\left|D^2K^\alpha\left(x\left(\Gamma\right),x\left(\Gamma'\right)\right)-D^2K^\alpha\left(x\left(\bar{\Gamma}\right),x\left(\Gamma'\right)\right)\right|=r\alpha\frac{C\left(M\right)}{\alpha^3}\left(\frac{\alpha}{\left|\Gamma''-\Gamma'\right|^2}+1\right).$$

We also have that $\frac{|\Gamma''-\Gamma'|}{\alpha}\geq r$. Hence

$$\begin{aligned}
|I_{223}| &\leq \left(\frac{1}{\alpha},M\right)\left\|\frac{dx}{d\Gamma}\right\|_{C^0}\left\|\frac{dy}{d\Gamma}\right\|_{C^0}r\alpha\int_{(\Gamma_0,\Gamma_1)\cap\frac{|\Gamma''-\Gamma'|}{\alpha}\geq r}\left(\frac{\alpha}{\left|\Gamma''-\Gamma'\right|^2}+1\right)\left|\bar{\Gamma}-\Gamma'\right|d\Gamma' \\
&\leq C\left(\frac{1}{\alpha},M,\left\|\frac{dx}{d\Gamma}\right\|_{C^0},\Gamma_1,\Gamma_0\right)\|y\|_{1,\beta}\left|\bar{\Gamma}-\Gamma\right|\left(1+\left|\log\left|\bar{\Gamma}-\Gamma\right|\right|\right),
\end{aligned}$$

where we have also used that

$$\left|\bar{\Gamma}-\Gamma'\right| \leq \left|\bar{\Gamma}-\Gamma''\right| + \left|\Gamma''-\Gamma'\right| \leq \left|\bar{\Gamma}-\Gamma\right| + \left|\Gamma''-\Gamma'\right|.$$

This implies Hölder continuity for $0<\beta<1$.

Summing up we have

$$\left|\frac{d}{d\Gamma}D_x u\left(x\right)y\right|_{\beta} \leq C\left(\frac{1}{\alpha}, M, \left\|x\right\|_{1,\beta}, \Gamma_1, \Gamma_0, \beta\right)\left\|y\right\|_{1,\beta}.$$

Now, the local existence and uniqueness of solutions in C^1 is a particular case of the above proof (see bounds (6.5),(6.8),(6.12),(6.13)).

The proof of the Lipschitz case is similar to the proof of the $C^{1,\beta}$ case, for example, to show Lipschitz continuity of $u\left(x\left(\Gamma\right)\right)$ for $x \in \mathrm{Lip}\left(\left(\Gamma_0,\Gamma_1\right)\right)$, denote $r = \frac{\left|\Gamma-\bar{\Gamma}\right|}{\alpha}$ and write

$$\begin{aligned}\left|u\left(x\left(\Gamma\right)\right)-u\left(x\left(\bar{\Gamma}\right)\right)\right| &\leq \int_{\Gamma_0}^{\Gamma_1}\left|K^{\alpha}\left(x\left(\Gamma\right)-x\left(\Gamma'\right)\right)d\Gamma' - K^{\alpha}\left(x\left(\bar{\Gamma}\right)-x\left(\Gamma'\right)\right)\right|d\Gamma' \\ &= \int_{\left(\Gamma_0,\Gamma_1\right)\cap E_r} + \int_{\left(\Gamma_0,\Gamma_1\right)\setminus E_r} = I_1 + I_2,\end{aligned}$$

where

$$E_r = \left\{\Gamma' \in \left(\Gamma_0,\Gamma_1\right) : \frac{\left|x\left(\Gamma\right)-x\left(\Gamma'\right)\right|}{\alpha} < 2rM\right\}.$$

For I_1, due to $\frac{1}{M}\frac{\left|\Gamma-\Gamma'\right|}{\alpha} < \left|x\right|_*\frac{\left|\Gamma-\Gamma'\right|}{\alpha} \leq \frac{\left|x(\Gamma)-x\left(\Gamma'\right)\right|}{\alpha}$, we have that $\frac{\left|\Gamma-\Gamma'\right|}{\alpha} < 2rM^2$ and hence $\frac{\left|\bar{\Gamma}-\Gamma'\right|}{\alpha} < r\left(1+2M^2\right)$. Thus by (2.5) and (6.3) we obtain

$$\begin{aligned}I_1 &\leq \int_{\left(\Gamma_0,\Gamma_1\right)\cap\left\{\frac{\left|\Gamma-\Gamma'\right|}{\alpha}<2r\right\}}\left|K^{\alpha}\left(x\left(\Gamma\right)-x\left(\Gamma'\right)\right)\right| + \left|K^{\alpha}\left(x\left(\bar{\Gamma}\right)-x\left(\Gamma'\right)\right)\right|d\Gamma' \\ &\leq \frac{C}{\alpha}\left(\int_{\frac{\left|\Gamma-\Gamma'\right|}{\alpha}<2r} d\Gamma' + \int_{\frac{\left|\bar{\Gamma}-\Gamma'\right|}{\alpha}<r\left(1+2M^2\right)} d\Gamma'\right) \leq \frac{C\left(M\right)}{\alpha}r.\end{aligned}$$

For I_2 due to $M\frac{\left|\Gamma-\Gamma'\right|}{\alpha} \geq \frac{\left|x(\Gamma)-x\left(\Gamma'\right)\right|}{\alpha} \geq 2rM$, we have that $\frac{\left|\Gamma-\Gamma'\right|}{\alpha} \geq 2r$, and hence $\frac{\left|\bar{\Gamma}-\Gamma'\right|}{\alpha} \geq r$, which in turn implies that $\frac{\left|x\left(\bar{\Gamma}\right)-x\left(\Gamma'\right)\right|}{\alpha} > \frac{1}{M}\frac{\left|\bar{\Gamma}-\Gamma'\right|}{\alpha} \geq \frac{r}{M}$. Also, due to $\left|x\left(\Gamma\right)-x\left(\bar{\Gamma}\right)\right| \leq M\left|\Gamma-\bar{\Gamma}\right| = Mr\alpha$ we have for every $x\left(\Gamma''\right) \in B\left(x\left(\Gamma\right), \left|x\left(\Gamma\right)-x\left(\bar{\Gamma}\right)\right|\right)$ that $\frac{\left|x\left(\Gamma''\right)-x\left(\Gamma'\right)\right|}{\alpha} \geq Mr$. Hence by the mean value theorem and (6.9)

$$\begin{aligned}\left|K^{\alpha}\left(x\left(\Gamma\right)-x\left(\Gamma'\right)\right)-K^{\alpha}\left(x\left(\bar{\Gamma}\right)-x\left(\Gamma'\right)\right)\right| &\leq \sup_{x\left(\Gamma''\right)\in B\left(x\left(\Gamma\right),\left|x\left(\Gamma\right)-x\left(\bar{\Gamma}\right)\right|\right)}\left|\nabla K^{\alpha}\left(x\left(\Gamma''\right)-x\left(\Gamma'\right)\right)\right|\left|x\left(\Gamma\right)-x\left(\bar{\Gamma}\right)\right| \\ &\leq \left(\frac{1}{4\pi}\frac{1}{\alpha^2}\left|\log\frac{\left|x\left(\Gamma''\right)-x\left(\Gamma'\right)\right|}{\alpha}\right| + \frac{C}{\alpha^2}\right)\left|x\left(\Gamma\right)-x\left(\bar{\Gamma}\right)\right| \\ &\leq C\left(M,\frac{1}{\alpha}\right)\left|\Gamma-\bar{\Gamma}\right|\left(\log\left(\frac{\left|\Gamma''-\Gamma'\right|}{\alpha}\right)+1\right)\end{aligned}$$

Hence

$$\begin{aligned}\left|I_2\right| &\leq rC\left(M,\frac{1}{\alpha}\right)\int_{\left(\Gamma_0,\Gamma_1\right)\cap\left\{\frac{\left|\Gamma''-\Gamma'\right|}{\alpha}\geq r\right\}}\left(\log\left(\frac{\left|\Gamma''-\Gamma'\right|}{\alpha}\right)+1\right)d\Gamma' \\ &\leq C\left(M,\frac{1}{\alpha},\Gamma_1,\Gamma_0\right)r\left(1+r\left|\log r\right|\right).\end{aligned}$$

Hence $u(x(\Gamma))$ is Lipschitz continuous. We remark, that in the proof of the $C^{1,\beta}$ part we used partitions using the fact that $x(\Gamma)$ is a differentiable, however, given the fact that differentiable functions are Lipschitz, one could have used the partitioning introduced in the proof of Lipschitz case on subsets of $x(\Gamma)$ also for $C^{1,\beta}$ results. □

Proposition 6.2 implies the local existence and uniqueness of solutions:

Proposition 6.3. *Let $-\infty < \Gamma_0 < \Gamma_1 < \infty$, let V be either the space $C^{1,\beta}((\Gamma_0,\Gamma_1))$, $0 \le \beta < 1$ or the space $\mathrm{Lip}((\Gamma_0,\Gamma_1))$, let $K^M = \left\{x \in V : \|x\|_1 < M, |x|_* > \frac{1}{M}\right\}$ and let $x_0 \in V \cap \{|x|_* > 0\}$, then for any M, $1 < M < \infty$, such that $x_0 \in K^M$, there exists a time $T(M)$, such that the system (6.1) has a unique local solution $x \in C^1((-T(M),T(M)); K^M)$.*

6.2 Step 2. Global existence.

To show the global existence, we assume by contradiction, that $T_{\max} < \infty$, where $[0, T_{\max})$ is the maximal interval of existence, and hence the solution leaves in a finite time the open set K^M, for all $M > 1$, that is, $\limsup_{t\to T_{\max}^-} \|x\|_V = \infty$ or $\limsup_{t\to T_{\max}^-} \frac{1}{|x(\cdot,t)|_*} = \infty$. Therefore, if we show global bounds on $\frac{1}{|x(\cdot,t)|_*}$ and $\|x(\cdot,t)\|_V$ in $[0, T_{\max})$, we obtain a contradiction to the blow-up and thus the obtained local solutions can be continued for all time. The result extends to negative times as well.

To control the quantities $\frac{1}{|x(\cdot,t)|_*}$ and $\|x(\cdot,t)\|_V$ we need to bound $\int_0^{T_{\max}} \|\nabla_x u(x(\cdot,t),t)\|_{L^\infty((\Gamma_0,\Gamma_1))}\, dt$. The next proposition provides the bound on the gradient of the velocity .

Proposition 6.4. *Let $x_0 \in \mathrm{Lip}((\Gamma_0,\Gamma_1))$ and $|x_0|_* > 0$. Suppose the solution exists on $[0, T_{\max})$, then for $t \in [0, T_{\max})$ we have*

$$|\nabla_x u(x(\Gamma,t),t)| \le \frac{1}{\alpha} C(|x_0|_*)\, e^{e^{tC_1}} + C_1, \tag{6.14}$$

where $C_1 = C\frac{1}{\alpha^2}(\Gamma_1 - \Gamma_0)$.

Proof. We write $\nabla_x u(x(\Gamma,t),t)$ as

$$\begin{aligned}\nabla_x u(x(\Gamma,t),t) &= \int_{\Gamma_0}^{\Gamma_1} \nabla_x K^\alpha(x(\Gamma,t) - x(\Gamma',t))\, d\Gamma' \\ &= \int_{(\Gamma_0,\Gamma_1)\cap E_\varepsilon} + \int_{(\Gamma_0,\Gamma_1)\setminus E_\varepsilon} = I_1 + I_2,\end{aligned}$$

where

$$E_\varepsilon = \left\{\Gamma' \in (\Gamma_0,\Gamma_1) : \frac{|x(\Gamma,t) - x(\Gamma',t)|}{\alpha} < \varepsilon\right\},$$

for a fixed small $0 < \varepsilon < 1$, to be further refined later.

Let the vorticity $q(x,t)$ be supported on the curve $\{x(\Gamma,t) : \Gamma_0 \le \Gamma \le \Gamma_1\}$, with a density $\gamma(\Gamma,t) = 1/|x_\Gamma(\Gamma,t)|$ (due to the Lipschitz continuity of $x(\Gamma,t)$ its derivative exists almost everywhere and is essentially bounded, and also due to $\{|x|_* > 0\}$, the vorticity density $\gamma(\Gamma,t) \in L^\infty((\Gamma_0,\Gamma_1)))$, that is for every $\varphi \in C_c^\infty(\mathbb{R}^2)$

$$\int_{\mathbb{R}^2} \varphi(x) dq(x,t) = \int_{\Gamma_0}^{\Gamma_1} \varphi(x(\Gamma,t))\, d\Gamma.$$

Observe that the vorticity $q(x,t)$ is a finite Radon measure which is the unique weak solution of the Euler equations given by Theorem 2.1 of Oliver and Shkoller [65]. Also, $\|q\|_{\mathcal{M}} = \Gamma_1 - \Gamma_0$.

Let η denote the unique Lagrangian flow map $\partial_t \eta(y,t) = \int_{\mathbb{R}^2} K^\alpha(y,z)\, dq(z,t)$, $\eta(y,0) = y$, $q = q^{in} \circ \eta^{-1}$, $y \in \mathbb{R}^2$ given by Theorem 2.1. We remark that in the formulation of BR-α model, we assumed the positivity of the vorticity q, see Proposition 4.1. Denote the distance between two points $\eta(y,t)$ and $\eta(y',t)$ by $r(t) = |\eta(y,t) - \eta(y',t)|$, where $r(0) = |y - y'|$.

Then, using the estimate (2.14) of [65], we have

$$\begin{aligned}\left|\frac{d}{dt}r\left(t\right)\right| &\leq \int_{\mathbb{R}^2}\left|K^{\alpha}\left(y,z\right)-K^{\alpha}\left(y',z\right)\right|dq\left(z,t\right)\\ &\leq C\frac{1}{\alpha}\varphi\left(\frac{r\left(t\right)}{\alpha}\right)\left\|q\right\|_{\mathcal{M}}\\ &= C\frac{1}{\alpha}\varphi\left(\frac{r\left(t\right)}{\alpha}\right)\left\|q^{in}\right\|_{\mathcal{M}},\end{aligned}$$

where

$$\varphi\left(r\right)=\begin{cases}0, & r=0,\\ r\left(1-\log r\right), & 0<r<1,\\ 1, & r\geq 1.\end{cases}$$

By comparison with the solution of the differential equation[4]

$$\begin{aligned}\frac{d}{dt}r\left(t\right) &= -C\frac{1}{\alpha}\varphi\left(\frac{r\left(t\right)}{\alpha}\right)\left\|q^{in}\right\|_{\mathcal{M}},\\ r\left(0\right) &= \left|x\left(\Gamma,0\right)-x\left(\Gamma',0\right)\right|,\end{aligned}$$

we can choose ε small enough, $\varepsilon<e^{1-e^{tC_1}}$, where $C_1=C\frac{1}{\alpha^2}\left\|q^{in}\right\|_M=\Gamma_1-\Gamma_0$, e.g., $\varepsilon=e^{-e^{tC_1}}$, such that, for $\frac{\left|x(\Gamma,t)-x\left(\Gamma',t\right)\right|}{\alpha}<\varepsilon$, we have that $\frac{\left|x(\Gamma,0)-x\left(\Gamma',0\right)\right|}{\alpha}=\frac{r(0)}{\alpha}<1$.[5] Hence

$$\begin{aligned}\frac{\left|x\left(\Gamma,t\right)-x\left(\Gamma',t\right)\right|}{\alpha}\geq\frac{r\left(t\right)}{\alpha} &= \left(\frac{r\left(0\right)}{\alpha}\right)^{e^{tC_1}}e^{1-e^{tC_1}}\\ &= \left(\frac{\left|x\left(\Gamma,0\right)-x\left(\Gamma',0\right)\right|}{\alpha}\right)^{e^{tC_1}}e^{1-e^{tC_1}}.\end{aligned}\tag{6.15}$$

Now, using also that $\left|x_0\right|_*$ is bounded away from zero, we can bound $\frac{\left|x(\Gamma,t)-x\left(\Gamma',t\right)\right|}{\alpha}$ from below, using (6.15),

$$1>\varepsilon>\frac{\left|x\left(\Gamma,t\right)-x\left(\Gamma',t\right)\right|}{\alpha}\geq\left(\frac{\left|x\left(\Gamma,0\right)-x\left(\Gamma',0\right)\right|}{\alpha}\right)^{e^{tC_1}}e^{1-e^{tC_1}}\geq\left|x_0\right|_*^{e^{tC_1}}\left(\frac{\left|\Gamma-\Gamma'\right|}{\alpha}\right)^{e^{tC_1}}e^{1-e^{tC_1}},$$

[4]

$$\frac{r\left(t\right)}{\alpha}=\begin{cases}\left(\frac{r(0)}{\alpha}\right)^{e^{tC_1}}e^{1-e^{tC_1}}, & \frac{r(0)}{\alpha}<1,\\ \frac{r(0)}{\alpha}-C_1t, & \frac{r(0)}{\alpha}\geq 1, \quad t<t^{**},\\ e^{1-e^{tC_1}-\frac{r(0)}{\alpha}+1}, & \frac{r(0)}{\alpha}\geq 1, \quad t\geq t^{**},\end{cases}$$

where

$$\begin{aligned}C_1 &= C\frac{1}{\alpha^2}\left\|q^{in}\right\|_M,\\ t^{**} &= \frac{1}{C_1}\left(\frac{r\left(0\right)}{\alpha}-1\right).\end{aligned}$$

[5]Otherwise, $\frac{\left|x(\Gamma,0)-x\left(\Gamma',0\right)\right|}{\alpha}=\frac{r(0)}{\alpha}\geq 1$, hence

$$\varepsilon>\frac{\left|x\left(\Gamma,t\right)-x\left(\Gamma',t\right)\right|}{\alpha}\geq\frac{r\left(t\right)}{\alpha}\geq\begin{cases}\frac{r(0)}{\alpha}-C_1t, & t<t^{**},\\ e^{1-e^{tC_1}-\frac{r(0)}{\alpha}+1}, & t\geq t^{**}.\end{cases}$$

Let $t\geq t^{**}$, then we have $\varepsilon\geq e^{1-e^{tC_1}}$, a contradiction. Otherwise $t<t^{**}=\frac{1}{C_1}\left(\frac{r(0)}{\alpha}-1\right)$, hence $C_1t<\left(\frac{r(0)}{\alpha}-1\right)$

$$\varepsilon>\frac{r\left(t\right)}{\alpha}\geq\frac{r\left(0\right)}{\alpha}-C_1t>\frac{r\left(0\right)}{\alpha}-\frac{r\left(0\right)}{\alpha}+1=1,$$

a contradiction.

which in turn implies the bound (using also (2.5) and (6.9))

$$\begin{aligned} I_1 &\leq \int_{(\Gamma_0,\Gamma_1)\cap\frac{|x(\Gamma,t)-x(\Gamma',t)|}{\alpha}<\varepsilon} \left(\frac{1}{2\pi}\frac{1}{\alpha^2}\log\frac{|x(\Gamma,t)-x(\Gamma',t)|}{\alpha}+\frac{C}{\alpha^2}\right)d\Gamma' \\ &\leq \frac{1}{\alpha}C\left(|x_0|_*\right)e^{e^{tC_1}}. \end{aligned}$$

While to bound I_2, we use the boundness of $|\nabla_x K^\alpha(x(\Gamma,t)-x(\Gamma',t))|$ in $\{\Gamma'\in(\Gamma_0,\Gamma_1):\frac{|x(\Gamma,t)-x(\Gamma',t)|}{\alpha}\geq\varepsilon\}$.

$$\begin{aligned} I_2 &\leq \sup_{\frac{|x(\Gamma,t)-x(\Gamma',t)|}{\alpha}\geq\varepsilon}|\nabla_x K^a(x(\Gamma,t),x(\Gamma',t))|\int_{\Gamma_0}^{\Gamma_1}d\Gamma' \\ &\leq C\frac{1}{\alpha^2}\left(|\log\varepsilon|+1\right)(\Gamma_1-\Gamma_0) \\ &= C_1\left(e^{tC_1}+1\right). \end{aligned}$$

□

Now, the bound on $\|x(\cdot,t)\|_{C^0}$ on $[0,T_{\max})$ follows from $\frac{dx}{dt}(\Gamma,t)=u(x(\Gamma,t),t)$ and the fact that

$$|u(x(\Gamma,t),t)|\leq\int_{\Gamma_0}^{\Gamma_1}|K^a(x(\Gamma,t)-x(\Gamma',t))|\,d\Gamma'\leq\frac{C}{\alpha}(\Gamma_1-\Gamma_0),$$

due to the boundness of K^a (see (2.5),(6.3)). Also, by Grönwall inequality the bound (6.14) provides bounds on $\frac{1}{|x(\cdot,t)|_*}$ and $|x(\cdot,t)|_1$ on $[0,T_{\max})$.

Finally, for the initial data in $C^{1,\beta}((\Gamma_0,\Gamma_1))$, the bound (6.14) provides bound on $\left\|\frac{dx}{d\Gamma}(\cdot,t)\right\|_{C^0}$ on $[0,T_{\max})$ by Grönwall inequality. While the bound on $\left|\frac{dx}{d\Gamma}(\cdot,t)\right|_\beta$ on $[0,T_{\max})$ is a consequence of

$$\frac{d}{dt}x_\Gamma(\Gamma,t)=\nabla_x u(x(\Gamma,t),t)\cdot x_\Gamma(\Gamma,t),$$

the bound (which is shown in a local existence part, see (6.11))

$$|\nabla_x u(x(\cdot,t),t)|_\beta\leq C\left(\frac{1}{\alpha},\|x_\Gamma(\cdot,t)\|_{L^\infty},|x(\cdot,t)|_*,\Gamma_1,\Gamma_0\right),$$

(6.14) and the Grönwall inequality.

This yields global in time existence and uniqueness of Lip and $C^{1,\beta}, 0\leq\beta<1$, solutions of (6.1).

6.3 Step 3. Higher regularity for closed curves.

Now we show the higher regularity for an initially closed curve $x_0(\Gamma)\in C^{n,\beta}(S^1)\cap\{|x|_*>0\}$, $n\geq1$, $0<\beta<1$. We remark that the high derivatives of the kernel $K^\alpha(x)$ are singular at the origin, thus the condition on closedness of the curve.

To provide an *a priori* bound for higher derivatives in terms of lower ones, we show that for $x\in K^M\cap C^{n,\beta}(S^1)$, the map u defined by (6.2) satisfies

$$\|u(x)\|_{n,\beta}\leq C\left(\frac{1}{\alpha},M,\|x\|_{n-1,\beta},\frac{1}{\beta}\right)\|x\|_{n,\beta},$$

hence by Grönwall inequality and the induction argument, it is enough to control $|x|_*$ and $\|x\|_{1,\beta}$, to guarantee that $x(\Gamma)\in C^{n,\beta}(S^1)$, for all $n\geq1$, (and consequently in $C^\infty(S^1)$, whenever $x_0\in C^\infty(S^1)\cap\{|x|_*>0\}$).

Lemma 6.5. *Let u and K^M be as defined in Proposition 6.2 and $x \in K^M \cap C^{n,\beta}\left(S^1\right)$, $n \geq 1$, $0 < \beta < 1$ then*

$$\|u(x)\|_{n,\beta} \leq C\left(\frac{1}{\alpha}, M, \|x\|_{n-1,\beta}, \frac{1}{\beta}\right)\|x\|_{n,\beta}$$

Proof. We show the proof for $n = 2$, the proof for general n is similar. The derivative of u with respect to Γ (in the sense of distributions) satisfies (see Appendix, Lemma A.1)

$$\begin{aligned}\frac{d^2}{d\Gamma^2}u(x(\Gamma)) &= \int_0^{2\pi} \nabla K^\alpha(x(\Gamma) - x(\Gamma'))\frac{d^2x}{d\Gamma^2}(\Gamma)\,d\Gamma' \\ &\quad + \text{p.v.}\int_0^{2\pi}\sum_{i,j=1}^{2}\partial_{x_i}\partial_{x_j}K^\alpha(x(\Gamma) - x(\Gamma'))\frac{dx_i}{d\Gamma}(\Gamma)\frac{dx_j}{d\Gamma}(\Gamma)\,d\Gamma' \\ &= I_1 + I_2.\end{aligned}$$

I_1 can be bounded using similar arguments as for (6.7),

$$|I_1| \leq \frac{1}{\alpha^2}C\left(\frac{1}{\alpha}, M\right)\|x\|_{2,\beta}.$$

We write I_2 as

$$\begin{aligned}I_2 &= \text{p.v.}\left(\int_{\frac{|\Gamma-\Gamma'|}{\alpha}<\varepsilon} + \int_{(0,2\pi)\setminus\left\{\frac{|\Gamma-\Gamma'|}{\alpha}<\varepsilon\right\}}\right)\sum_{i,j=1}^{2}\partial_{x_i}\partial_{x_j}K^\alpha(x(\Gamma) - x(\Gamma'))\frac{dx_i}{d\Gamma}(\Gamma)\frac{dx_j}{d\Gamma}(\Gamma)\,d\Gamma' \\ &= I_{21} + I_{22},\end{aligned}$$

where, due to the closedness of the curve, we can fix a small $\varepsilon < \pi/2$ independent of Γ, by taking $I_{22} = \int_{D\setminus\left\{\frac{|\Gamma-\Gamma'|}{\alpha}<\varepsilon\right\}}$, where $D = (0, 2\pi)$ if $\varepsilon\alpha < \Gamma < 2\pi - \varepsilon\alpha$, $D = (-\pi, \pi)$ if $0 \leq \Gamma \leq \varepsilon\alpha$, or $D = (\pi, 3\pi)$ if $2\pi - \varepsilon\alpha \leq \Gamma \leq 2\pi$. Treating I_{22} as in the local existence proof, we have

$$\begin{aligned}|I_{22}| &\leq C\left(M, \frac{1}{\alpha}\right)\|x\|_{1,\beta}^3\int_{D\cap\frac{|\Gamma-\Gamma'|}{\alpha}\geq\varepsilon}\left(\frac{\alpha}{|\Gamma-\Gamma'|} + 1\right)d\Gamma' \\ &\leq C\left(\frac{1}{\alpha}, M, \|x\|_{1,\beta}\right)|\log\varepsilon|.\end{aligned}$$

For I_{21} we have that

$$\begin{aligned}I_{21} &= \frac{1}{4\pi\alpha^2}\text{p.v.}\int_{\frac{|\Gamma-\Gamma'|}{\alpha}<\varepsilon}\sum_{i,j}\frac{\sigma_{ij}(x(\Gamma) - x(\Gamma'))\frac{dx_i}{d\Gamma}(\Gamma)}{|x(\Gamma) - x(\Gamma')|}\frac{dx_j}{d\Gamma}(\Gamma)\,d\Gamma' \\ &\quad + \frac{1}{4\pi\alpha^2}\int_{\frac{|\Gamma-\Gamma'|}{\alpha}<\varepsilon}\sum_{i,j}O\left(\left|\frac{|x(\Gamma) - x(\Gamma')|}{\alpha^2}\log\frac{|x(\Gamma) - x(\Gamma')|}{\alpha}\right|\right)\frac{dx_i}{d\Gamma}(\Gamma)\frac{dx_j}{d\Gamma}(\Gamma)\,d\Gamma' \\ &= I_{211} + I_{212},\end{aligned}$$

where

$$\sigma_{11}(x) = \frac{1}{|x|^3}\begin{pmatrix}-x_2\left(x_1^2 - x_2^2\right) \\ -x_1(x_1^2 + 3x_2^2)\end{pmatrix},\ \sigma_{12}(x) = \sigma_{21}(x) = \frac{1}{|x|^3}\begin{pmatrix}x_1\left(x_1^2 - x_2^2\right) \\ x_2\left(x_1^2 - x_2^2\right)\end{pmatrix}, \sigma_{22}(x) = \frac{1}{|x|^3}\begin{pmatrix}x_2\left(3x_1^2 + x_2^2\right) \\ -x_1\left(x_1^2 - x_2^2\right)\end{pmatrix}. \tag{6.16}$$

I_{212} is not a singular integral and due to

$$|x(\Gamma) - x(\Gamma')| \leq \|x_\Gamma\|_{C^0}|\Gamma - \Gamma'|,$$

we obtain that

$$|I_{212}| \leq C\frac{1}{\alpha^2}\left\|x\right\|_{1,\beta}^3.$$

We use the observation

$$|f(x)-f(y)-(x-y)f'(x)| \leq |x-y|^{1+\beta}\left|f'\right|_\beta \tag{6.17}$$

to desingularize the I_{211}. We rewrite

$$\begin{aligned}
\sum_{i,j}\frac{\sigma_{ij}\left(x(\Gamma)-x(\Gamma')\right)\frac{dx_i}{d\Gamma}(\Gamma)}{|x(\Gamma)-x(\Gamma')|}\frac{dx_j}{d\Gamma}(\Gamma) &= \sum_{i,j}\frac{\sigma_{ij}\left(x(\Gamma)-x(\Gamma')\right)\frac{dx_i}{d\Gamma}(\Gamma)(\Gamma-\Gamma')}{|x(\Gamma)-x(\Gamma')|(\Gamma-\Gamma')}\frac{dx_j}{d\Gamma}(\Gamma)\\
&= \sum_{i,j}\frac{\sigma_{ij}\left(x(\Gamma)-x(\Gamma')\right)\left(\frac{dx_i}{d\Gamma}(\Gamma)(\Gamma-\Gamma')-x_i(\Gamma)+x_i(\Gamma')\right)}{|x(\Gamma)-x(\Gamma')|(\Gamma-\Gamma')}\frac{dx_j}{d\Gamma}(\Gamma)\\
&\quad+\sum_{i,j}\frac{\sigma_{ij}\left(x(\Gamma)-x(\Gamma')\right)\left(x_i(\Gamma)-x_i(\Gamma')\right)}{|x(\Gamma)-x(\Gamma')|(\Gamma-\Gamma')}\frac{dx_j}{d\Gamma}(\Gamma)\\
&= J_1+J_2.
\end{aligned}$$

Observe that $J_2=\frac{1}{(\Gamma-\Gamma')}\binom{1}{-1}$ and $J_1\leq C(M)\left\|x\right\|_{1,\beta}^2|\Gamma-\Gamma'|^{-1+\beta}$ due to $|\sigma_{ij}|\leq 1$ (see (6.16)) and (6.17). Hence

$$\begin{aligned}
|I_{211}| &= \left|\frac{1}{4\pi\alpha^2}\int_{\frac{|\Gamma-\Gamma'|}{\alpha}<\varepsilon}J_1 d\Gamma'+\frac{1}{4\pi\alpha^2}\text{p.v.}\int_{\frac{|\Gamma-\Gamma'|}{\alpha}<\varepsilon}\binom{1}{-1}\frac{1}{(\Gamma-\Gamma')}d\Gamma'\right|\\
&\leq C(M)\frac{1}{\alpha^{2-\beta}}\left\|x\right\|_{1,\beta}^2\frac{1}{\beta}\varepsilon^\beta.
\end{aligned}$$

Summing up, we have that

$$\left|\frac{d^2}{d\Gamma^2}u(x(\Gamma))\right|\leq C\left(\frac{1}{\alpha},M,\left\|x\right\|_{1,\beta},\left|x\right|_*^{-1},\frac{1}{\beta}\right)\left\|x\right\|_{2,\beta}.$$

Using the same ideas we also bound $\left|\frac{d^2}{d\Gamma^2}u(x(\Gamma))\right|_\beta$. □

Appendix

Lemma A.1. *Let $x\in C^{2.\beta}((\Gamma_0,\Gamma_1))\cap\{|x|_*>0\}$ then*

$$\begin{aligned}
\frac{d^2}{d\Gamma^2}(x(\Gamma)) &= \int_{\Gamma_0}^{\Gamma_1}\nabla K^\alpha(x(\Gamma)-x(\Gamma'))\frac{d^2x}{d\Gamma^2}(\Gamma)\,d\Gamma'\\
&\quad+\text{p.v.}\int_{\Gamma_0}^{\Gamma_1}\sum_{i,j=1}^{2}\partial_{x_i}\partial_{x_j}K^\alpha(x(\Gamma)-x(\Gamma'))\frac{dx_i}{d\Gamma}(\Gamma)\frac{dx_j}{d\Gamma}(\Gamma)\,d\Gamma'
\end{aligned}$$

(in the sense of distributions).

Proof. By the definition of the distribution derivative, for all $\varphi\in C_c^\infty\left((\Gamma_0,\Gamma_1);\mathbb{R}^2\right)$

$$\begin{aligned}
\left\langle\frac{d^2}{d\Gamma^2}(x(\Gamma),),\varphi(\Gamma)\right\rangle &= -\left\langle\frac{du}{d\Gamma}(x(\Gamma)),\frac{d\varphi}{d\Gamma}(\Gamma)\right\rangle\\
&= -\left\langle\int_{\Gamma_0}^{\Gamma_1}\nabla K^\alpha(x(\Gamma)-x(\Gamma'))\frac{dx}{d\Gamma}(\Gamma)\,d\Gamma',\frac{d\varphi}{d\Gamma}(\Gamma)\right\rangle\\
&= -\lim_{\varepsilon\to 0}\int_{\Gamma_0}^{\Gamma_1}\int_{(\Gamma_0,\Gamma_1)\cap\frac{|\Gamma-\Gamma'|}{\alpha}>\varepsilon}\frac{d\varphi}{d\Gamma}(\Gamma)\nabla K^\alpha(x(\Gamma)-x(\Gamma'))\frac{dx}{d\Gamma}(\Gamma)\,d\Gamma d\Gamma'
\end{aligned}$$

where for a fixed Γ we take $\varepsilon < \min\left\{\frac{\Gamma-\Gamma_0}{\alpha}, \frac{\Gamma_1-\Gamma}{\alpha}\right\}$. Denote $D = (\Gamma_0, \Gamma_1) \cap \frac{|\Gamma-\Gamma'|}{\alpha} > \varepsilon$, by integration by parts we get

$$\begin{aligned} &= \lim_{\varepsilon\to 0}\int_{\Gamma_0}^{\Gamma_1}\left(-\left[\varphi(\Gamma)\nabla K^\alpha(x(\Gamma)-x(\Gamma'))\frac{dx}{d\Gamma}(\Gamma)\right]_{\partial D}+\int_D\varphi(\Gamma)\frac{d}{d\Gamma}\left[\nabla K^\alpha(x(\Gamma)-x(\Gamma'))\frac{dx}{d\Gamma}(\Gamma)\right]d\Gamma\right)d\Gamma' \\ &= \lim_{\varepsilon\to 0}\int_{\Gamma_0}^{\Gamma_1}(A+B)\,d\Gamma' \end{aligned}$$

For A we have

$$\begin{aligned} A &= -\varphi(\Gamma'-\varepsilon\alpha)\nabla K^\alpha(x(\Gamma'-\varepsilon\alpha)-x(\Gamma'))\frac{dx}{d\Gamma}(\Gamma'-\varepsilon\alpha) \\ &\quad+\varphi(\Gamma'+\varepsilon\alpha)\nabla K^\alpha(x(\Gamma'+\varepsilon\alpha)-x(\Gamma'))\frac{dx}{d\Gamma}(\Gamma'+\varepsilon\alpha) \\ &= I_1+I_2+I_3, \end{aligned}$$

where

$$\begin{aligned} I_1 &= [\varphi(\Gamma'+\varepsilon\alpha)-\varphi(\Gamma'-\varepsilon\alpha)]\nabla K^\alpha(x(\Gamma'+\varepsilon\alpha)-x(\Gamma'))\frac{dx}{d\Gamma}(\Gamma'+\varepsilon\alpha), \\ I_2 &= \varphi(\Gamma'-\varepsilon\alpha)[\nabla K^\alpha(x(\Gamma'+\varepsilon\alpha)-x(\Gamma'))-\nabla K^\alpha(x(\Gamma'-\varepsilon\alpha)-x(\Gamma'))]\frac{dx}{d\Gamma}(\Gamma'+\varepsilon\alpha), \\ I_3 &= \varphi(\Gamma'-\varepsilon\alpha)\nabla K^\alpha(x(\Gamma'-\varepsilon\alpha)-x(\Gamma'))\left[\frac{dx}{d\Gamma}(\Gamma'+\varepsilon\alpha)-\frac{dx}{d\Gamma}(\Gamma'-\varepsilon\alpha)\right]. \end{aligned}$$

Now, since for $y\in\mathbb{R}^2$, $\frac{|y|}{\alpha}\to 0 : |\nabla K^\alpha(y)| \le -\frac{1}{2\pi}\frac{1}{\alpha^2}\log\frac{|y|}{\alpha}+O\left(\frac{1}{\alpha^2}\right)$ and $\left|\frac{dx}{d\Gamma}\right|\varepsilon \ge \frac{|x(\Gamma'+\varepsilon\alpha)-x(\Gamma')|}{\alpha} \ge |x|_*\varepsilon$ we have

$$\begin{aligned} |\nabla K^\alpha(x(\Gamma'+\varepsilon\alpha)-x(\Gamma'))| &\le -\frac{1}{2\pi}\frac{1}{\alpha^2}\log\frac{|x(\Gamma'+\varepsilon\alpha)-x(\Gamma')|}{\alpha}+\frac{C}{\alpha^2} \\ &\le C\left(|x|_*,\frac{1}{\alpha}\right)(\log\varepsilon+1), \end{aligned}$$

Hence

$$|I_1| \le C\left(|x|_*,\frac{1}{\alpha}\right)\left\|\frac{d\varphi}{d\Gamma}\right\|_{C^0}\left\|\frac{dx}{d\Gamma}\right\|_{C^0}\varepsilon(\log\varepsilon+1)\to 0, \text{ as } \varepsilon\to 0.$$

Similarly,

$$|I_3| \le C\left(|x|_*,\frac{1}{\alpha}\right)\|\varphi\|_{C^0}\left\|\frac{d^2x}{d\Gamma^2}\right\|_{C^0}\varepsilon(\log\varepsilon+1)\to 0, \text{ as } \varepsilon\to 0.$$

For I_2 we have

$$|I_2| \le \|\varphi\|_{C^0}\left\|\frac{dx}{d\Gamma}\right\|_{C^0}|\nabla K^\alpha(x(\Gamma'+\varepsilon\alpha)-x(\Gamma'))-\nabla K^\alpha(x(\Gamma'-\varepsilon\alpha)-x(\Gamma'))|,$$

using that for $y\in\mathbb{R}^2$

$$\nabla K^\alpha(y) = \frac{1}{|y|}D\Psi^\alpha(|y|)(\sigma(y)+J)-\sigma(y)D^2\Psi^\alpha(|y|),$$

where

$$\sigma(y) = \frac{1}{|y|^2}\begin{pmatrix} y_1y_2 & y_2^2 \\ -y_1^2 & -y_1y_2\end{pmatrix}, J = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},$$

for $\frac{|y|}{\alpha}, \frac{|y'|}{\alpha} \to 0$, we obtain

$$|\nabla K^\alpha(y) - \nabla K^\alpha(y')| \le \frac{1}{4\pi}\frac{1}{\alpha^2}|\sigma(y) - \sigma(y')| + C\frac{|y|^2}{\alpha^4}\left|\log\frac{|y|}{\alpha}\right| + C\frac{|y'|^2}{\alpha^4}\left|\log\frac{|y'|}{\alpha}\right| + \frac{1}{4\pi}\frac{1}{\alpha^2}|\log|y'| - \log|y||.$$

Now, since

$$|\log|x(\Gamma' - \varepsilon\alpha) - x(\Gamma')| - \log|x(\Gamma' + \varepsilon\alpha) - x(\Gamma')|| \le C\frac{\alpha^\beta\varepsilon^\beta\left|\frac{dx}{d\Gamma}\right|_\beta}{|x|_*},$$

and since $\sigma(x)$ and x are continuous functions, we obtain

$$|\nabla_x K^\alpha(x(\Gamma' + \varepsilon\alpha) - x(\Gamma')) - \nabla_x K^\alpha(x(\Gamma' - \varepsilon\alpha) - x(\Gamma'))| \to 0, \text{ as } \varepsilon \to 0.$$

We also have that $|\nabla_x K^\alpha(x(\Gamma' + \varepsilon\alpha) - x(\Gamma')) - \nabla_x K^\alpha(x(\Gamma' - \varepsilon\alpha) - x(\Gamma'))|$ is bounded by

$$\begin{aligned}|\nabla_x K^\alpha(x(\Gamma' + \varepsilon\alpha) - x(\Gamma')) - \nabla_x K^\alpha(x(\Gamma' - \varepsilon\alpha) - x(\Gamma'))| &\le \frac{1}{4\pi}\frac{1}{\alpha^2} + \frac{C}{\alpha^2}\left\|\frac{dx}{d\Gamma}\right\|_{C^0}^2\varepsilon^2\left|\log\left\|\frac{dx}{d\Gamma}\right\|_{C^0}\varepsilon\right| + C\frac{\alpha^\beta\varepsilon^\beta\left|\frac{dx}{d\Gamma}\right|_\beta}{|x|_*} \\ &\le C\left(\alpha, \left\|\frac{dx}{d\Gamma}\right\|_{1,\beta}, |x|_*\right),\end{aligned}$$

hence by the Lebesgue's dominated convergence theorem

$$\lim_{\varepsilon\to 0}\int_{\Gamma_0}^{\Gamma_1} A d\Gamma' = 0.$$

For B we have

$$\begin{aligned}\frac{d}{d\Gamma}\left[\nabla K^\alpha(x(\Gamma) - x(\Gamma'))\frac{dx}{d\Gamma}(\Gamma)\right] &= \nabla K^\alpha(x(\Gamma) - x(\Gamma'))\frac{d^2x}{d\Gamma^2}(\Gamma) \\ &\quad + \sum_{i,j}\partial_{x_i}\partial_{x_j}K^\alpha(x(\Gamma) - x(\Gamma'))\frac{dx_i}{d\Gamma}(\Gamma)\frac{dx_j}{d\Gamma}(\Gamma).\end{aligned}$$

Hence

$$\begin{aligned}&\left\langle\frac{d^2}{d\Gamma^2}(x(\Gamma,t),t), \varphi(\Gamma)\right\rangle = \int_{\Gamma_0}^{\Gamma_1} d\Gamma\varphi(\Gamma) \\ &\cdot\left(\int_{\Gamma_0}^{\Gamma_1}\nabla K^\alpha(x(\Gamma) - x(\Gamma'))\frac{d^2x}{d\Gamma^2}(\Gamma)d\Gamma' + \text{p.v.}\int_{\Gamma_0}^{\Gamma_1}\sum_{i,j}\partial_{x_i}\partial_{x_j}K^\alpha(x(\Gamma) - x(\Gamma'))\frac{dx_i}{d\Gamma}(\Gamma)\frac{dx_j}{d\Gamma}(\Gamma)d\Gamma'\right),\end{aligned}$$

which concludes the proof. □

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