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# Generation, motion and thickness of transition layers for a nonlocal Allen-Cahn equation

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## Abstract

We investigate the behavior, as  $\varepsilon \rightarrow 0$ , of the nonlocal Allen-Cahn equation  $u_t = \Delta u + \frac{1}{\varepsilon^2} f(u, \varepsilon \int_{\Omega} u)$ , where  $f(u, 0)$  is of the bistable type. Given a rather general initial data  $u_0$  that is independent of  $\varepsilon$ , we perform a rigorous analysis of both the generation and the motion of interface, and obtain a new estimate for its thickness. More precisely we show that the solution develops a steep transition layer within the time scale of order  $\varepsilon^2 |\ln \varepsilon|$ , and that the layer obeys the law of motion that coincides with the limit problem within an error margin of order  $\varepsilon$ .

Key Words: reaction-diffusion equation, nonlocal PDE, singular perturbation, motion by mean curvature <sup>1</sup>.

## 1 Introduction

This paper is concerned with the singular limit, as  $\varepsilon \rightarrow 0$ , of the nonlocal Allen-Cahn equation

$$(P^\varepsilon) \quad \begin{cases} u_t = \Delta u + \frac{1}{\varepsilon^2} f\left(u, \varepsilon \int_{\Omega} u\right) & \text{in } \Omega \times (0, \infty) \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, \infty) \\ u(x, 0) = u_0(x) & \text{in } \Omega, \end{cases}$$

where  $\Omega$  is a smooth bounded domain in  $\mathbb{R}^N$  ( $N \geq 2$ ) and  $\nu$  the Euclidian unit normal vector exterior to  $\partial\Omega$ . We assume that the nonlinearity  $f(u, v)$  is smooth and that  $\tilde{f}(u) := f(u, 0)$  is given by  $\tilde{f}(u) := -W'(u)$ , where  $W(u)$

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<sup>1</sup>AMS Subject Classifications: 35K57, 45K05, 35B25, 35R35, 53C44.

is a double-well potential with equal well-depth, taking its global minimum value at  $u = \pm 1$ . More precisely we assume that  $\tilde{f}$  has exactly three zeros  $-1 < a < 1$  such that

$$\tilde{f}'(\pm 1) < 0, \quad \tilde{f}'(a) > 0 \quad (\text{bistable nonlinearity}), \quad (1.1)$$

and that

$$\int_{-1}^{+1} \tilde{f}(u) du = 0. \quad (1.2)$$

The condition (1.1) implies that the potential  $W(u)$  attains its local minima at  $u = \pm 1$ , and (1.2) implies that  $W(-1) = W(+1)$ . In other words, the two stable zeros of  $\tilde{f}$  have “balanced” stability.

Concerning the initial data  $u_0$ , we assume its smoothness and choose  $C_0 \geq 1$  such that

$$\|u_0\|_{C^0(\bar{\Omega})} + \|\nabla u_0\|_{C^0(\bar{\Omega})} + \|D^2 u_0\|_{C^0(\bar{\Omega})} \leq C_0. \quad (1.3)$$

Furthermore we define the “initial interface”  $\Gamma_0$  by

$$\Gamma_0 := \{x \in \Omega \mid u_0(x) = a\},$$

and suppose that  $\Gamma_0$  is a smooth closed hypersurface without boundary, such that,  $n$  being the Euclidian unit normal vector exterior to  $\Gamma_0$ ,

$$\Gamma_0 \subset\subset \Omega \quad \text{and} \quad \nabla u_0(x) \neq 0 \quad \text{if } x \in \Gamma_0, \quad (1.4)$$

$$u_0 > a \quad \text{in} \quad \Omega_0^+, \quad u_0 < a \quad \text{in} \quad \Omega_0^-, \quad (1.5)$$

where  $\Omega_0^-$  denotes the region enclosed by  $\Gamma_0$  and  $\Omega_0^+$  the region enclosed between  $\partial\Omega$  and  $\Gamma_0$ .

Before going into more details, let us recall known facts concerning the “usual” Allen-Cahn equation, namely

$$u_t = \Delta u + \frac{1}{\varepsilon^2} \tilde{f}(u).$$

The singular limit was first studied by Allen and Cahn [4] and by Kawasaki and Ohta [14]. By using formal asymptotic arguments, they show that the limit problem, as  $\varepsilon \rightarrow 0$ , is a free boundary problem: the motion of the limit interface is ruled by its mean curvature. More precisely, the solution  $u^\varepsilon$  of the Allen-Cahn equation tends to a step function taking the value  $+1$  on one side of an moving interface, and  $-1$  on the other side. This sharp interface, which we will denote by  $\Gamma_t$ , obeys the law of motion  $V_n = -\kappa$ , where  $V_n$  is the normal velocity of  $\Gamma_t$  in the exterior direction and  $\kappa$  the mean curvature at each point of  $\Gamma_t$ .

Then, some rigorous justification of this procedure were obtained. In the framework of classical solutions, let us mention the works of Bronsard and

Kohn [7], X. Chen [8, 9], and de Mottoni and Schatzman [18, 19]. Later, in [3], the authors prove an optimal estimate for this convergence for solutions with general initial data. By performing an analysis of both the generation and the motion of interface, they show that the solution develops a steep transition layer within a very short time, and that the layer obeys the law of motion that coincides with the formal asymptotic limit  $V_n = -\kappa$  within an error margin of order  $\varepsilon$  (previously, the best thickness estimate in the literature was of order  $\varepsilon |\ln \varepsilon|$ , [8]). For similar estimates of the thickness of the interface in related problems we refer to [1] (reaction-diffusion-convection system as a model for chemotaxis with growth), [15] (inhomogeneous Lotka-Volterra competition-diffusion system), [2] (fully anisotropic Allen-Cahn equation).

Since the classical motion by mean curvature may develop singularities in finite time (extinction, “pinch off” phenomena...), one has to define a generalized motion by mean curvature in order to study the singular limit of the Allen-Cahn equation for all time. One represents  $\Gamma_t$  as the level set of an auxiliary function which solves (in the viscosity sense) a nonlinear partial differential equation. This direct partial differential equation approach was developed by Evans and Spruck [13], Chen, Giga and Goto [11]. In this framework of viscosity solutions, we refer to Evans, Soner and Souganidis [12], Barles, Soner and Souganidis [5], Barles and Souganidis [6], Ilmanen [16] for the singular limit of reaction-diffusion equations, for all time.

We now turn back to the nonlocal Allen-Cahn equation. Problem  $(P^\varepsilon)$  was considered by Chen, Hilhorst and Logak [10]. In order to underline its relevance in population genetics and nervous transmission, they first show that  $(P^\varepsilon)$  can be seen as the limit, as  $\sigma \rightarrow 0$  and  $\tau \rightarrow 0$ , of the FitzHugh-Nagumo system

$$\begin{cases} u_t &= \Delta u + \frac{1}{\varepsilon^2} f(u, \varepsilon \frac{|\Omega|}{\gamma} v) \\ \tau v_t &= \frac{1}{\sigma} \Delta v + u - \frac{1}{\gamma} v. \end{cases}$$

Then, they study the motion of transition layers for the solutions  $u^\varepsilon$  of  $(P^\varepsilon)$ . More precisely, for “well-prepared” initial data, they prove that, as  $\varepsilon \rightarrow 0$ , the sharp interface limit, which we will denote by  $\Gamma_t$ , obeys the law of motion

$$(P^0) \quad \begin{cases} V_n = -\kappa + c_0(|\Omega_t^+| - |\Omega_t^-|) & \text{on } \Gamma_t \\ \Gamma_t|_{t=0} = \Gamma_0, \end{cases}$$

where  $V_n$  is the normal velocity of  $\Gamma_t$  in the exterior direction,  $\kappa$  the mean curvature at each point of  $\Gamma_t$ ,  $\Omega_t^-$  the region enclosed by  $\Gamma_t$ ,  $\Omega_t^+$  the region

enclosed between  $\partial\Omega$  and  $\Gamma_t$ ,  $c_0$  the constant defined by

$$c_0 = -\frac{\int_{-1}^{+1} \frac{\partial f}{\partial v}(u, 0) du}{\int_{-1}^{+1} [2(W(u) - W(-1))]^{1/2} du}, \quad (1.6)$$

and  $|A|$  the measure of the set  $A$ . As explained in [10], the Problem  $(P^0)$  possesses a unique smooth solution locally in time, say on some  $[0, T]$ . Moreover, in contrast with the “usual” motion by mean curvature which shrinks in finite time, the nonlocal effect allows the possibility of nontrivial stationary state (see [17] for a discussion in the radially symmetric case).

The goal of the present paper is to make a detailed study of the limiting behavior of the solution  $u^\varepsilon$  of Problem  $(P^\varepsilon)$ , without assuming that the initial datum already has a specific profile with a well-developed transition layer. In other words, we study the generation of interface from arbitrary initial data. Moreover, we obtain an improved error estimate, of  $O(\varepsilon)$ , between the solutions of  $(P^\varepsilon)$  and those of  $(P^0)$ .

Our main result, Theorem 1.1, describes the profile of the solution after a very short initial period. It asserts that: given a virtually arbitrary initial data  $u_0$ , the solution  $u^\varepsilon$  quickly becomes close to  $\pm 1$ , except in a small neighborhood of the initial interface  $\Gamma_0$ , creating a steep transition layer around  $\Gamma_0$  (*generation of interface*). The time needed to develop such a transition layer, which we will denote by  $t^\varepsilon$ , is of order  $\varepsilon^2 |\ln \varepsilon|$ . The theorem then states that the solution  $u^\varepsilon$  remains close to the step function  $\tilde{u}$  on the time interval  $[t^\varepsilon, T]$  (*motion of interface*), where  $\tilde{u}$  is defined by

$$\tilde{u}(x, t) = \begin{cases} -1 & \text{in } \Omega_t^- \\ +1 & \text{in } \Omega_t^+ \end{cases} \quad \text{for } t \in [0, T]. \quad (1.7)$$

In other words, the motion of the transition layer is well approximated by the limit interface equation  $(P^0)$ .

**Theorem 1.1** (Generation, motion and thickness of transition layers). *Let  $\eta$  be an arbitrary constant satisfying  $0 < \eta < \min(a + 1, 1 - a)$  and set*

$$\mu = \tilde{f}'(a).$$

*Then there exist positive constants  $\varepsilon_0$  and  $\mathcal{C}$  such that, for all  $\varepsilon \in (0, \varepsilon_0)$  and for all  $t^\varepsilon \leq t \leq T$ , where  $t^\varepsilon := \mu^{-1} \varepsilon^2 |\ln \varepsilon|$ , we have*

$$u^\varepsilon(x, t) \in \begin{cases} [-1 - \eta, +1 + \eta] & \text{if } x \in \mathcal{N}_{\mathcal{C}\varepsilon}(\Gamma_t) \\ [-1 - \eta, -1 + \eta] & \text{if } x \in \Omega_t^- \setminus \mathcal{N}_{\mathcal{C}\varepsilon}(\Gamma_t) \\ [+1 - \eta, +1 + \eta] & \text{if } x \in \Omega_t^+ \setminus \mathcal{N}_{\mathcal{C}\varepsilon}(\Gamma_t), \end{cases} \quad (1.8)$$

where  $\mathcal{N}_r(\Gamma_t) := \{x \in \Omega, \text{dist}(x, \Gamma_t) < r\}$  denotes the  $r$ -neighborhood of  $\Gamma_t$ .

The estimate (1.8) implies that, once a transition layer is formed, its thickness remains within order  $\varepsilon$  for the rest of time.

**Corollary 1.2** (Convergence). *As  $\varepsilon \rightarrow 0$ ,  $u^\varepsilon$  converges to  $\tilde{u}$  everywhere in  $\cup_{0 < t \leq T} (\Omega_t^\pm \times \{t\})$ .*

This paper is organized as follows. In Section 2 we study the generation of transition layers that takes place in a very short time range. Section 3 is devoted to the construction of a pair of sub- and super-solutions for the study of the motion of interface. In Section 4 by fitting the pair of sub- and super-solutions of Section 2 into the pair of Section 3, we prove our main result, Theorem 1.1. Since our arguments rely on a nonlocal comparison principle borrowed from [10], we recall it in a short appendix.

## 2 Generation of interface

In this section, we investigate the generation of interface, namely the rapid formation of internal layers that takes place in a neighborhood of  $\Gamma_0 = \{x \in \Omega \mid u_0(x) = a\}$  within the time span of order  $\varepsilon^2 |\ln \varepsilon|$ . In this earlier stage, the diffusion term is negligible and the partial differential equation is approximated by the nonlocal equation  $u_t = \frac{1}{\varepsilon^2} f(u, \varepsilon \int_\Omega u)$  and so, by the ordinary differential equation

$$u_t = \frac{1}{\varepsilon^2} (\tilde{f}(u) + O(\varepsilon)). \quad (2.1)$$

In the sequel,  $\eta_0$  will stand for the quantity

$$\eta_0 := \min(a + 1, 1 - a).$$

The main result of the present section is the following.

**Theorem 2.1** (Generation of interface). *Let  $\eta \in (0, \eta_0)$  be arbitrary and define  $\mu$  as the derivative of  $\tilde{f}(u)$  at the unstable zero  $u = a$ , that is*

$$\mu = \tilde{f}'(a). \quad (2.2)$$

*Then there exist positive constants  $\varepsilon_0$  and  $M_0$  such that, for all  $\varepsilon \in (0, \varepsilon_0)$ ,*

(i) *for all  $x \in \Omega$ ,*

$$-1 - \eta \leq u^\varepsilon(x, \mu^{-1} \varepsilon^2 |\ln \varepsilon|) \leq 1 + \eta, \quad (2.3)$$

(ii) *for all  $x \in \Omega$  such that  $|u_0(x) - a| \geq M_0 \varepsilon$ , we have that*

$$\text{if } u_0(x) \geq a + M_0 \varepsilon \text{ then } u^\varepsilon(x, \mu^{-1} \varepsilon^2 |\ln \varepsilon|) \geq 1 - \eta, \quad (2.4)$$

$$\text{if } u_0(x) \leq a - M_0 \varepsilon \text{ then } u^\varepsilon(x, \mu^{-1} \varepsilon^2 |\ln \varepsilon|) \leq -1 + \eta. \quad (2.5)$$

The above theorem will be proved by constructing a suitable pair of sub- and super-solutions based upon the ordinary differential equation (2.1). Note that the assumption of balanced nonlinearity (1.2) is useless for the proof of the generation of interface property.

## 2.1 The bistable ordinary differential equation

We first consider a slightly perturbed nonlinearity:

$$\tilde{f}_\delta(u) := \tilde{f}(u) + \delta,$$

where  $\delta$  is any constant. For  $|\delta|$  small enough, this function is still of the bistable type. More precisely, if  $\delta_0$  is small enough, then for any  $\delta \in (-\delta_0, \delta_0)$ ,  $\tilde{f}_\delta$  has exactly three zeros, namely  $\alpha_-(\delta) < a(\delta) < \alpha_+(\delta)$ , and there exists a positive constant  $C$  such that

$$|\alpha_-(\delta) + 1| + |a(\delta) - a| + |\alpha_+(\delta) - 1| \leq C|\delta|, \quad (2.6)$$

$$|\mu(\delta) - \mu| \leq C|\delta|, \quad (2.7)$$

where

$$\mu(\delta) := \tilde{f}'_\delta(a(\delta)) = \tilde{f}'(a(\delta)).$$

Now for each  $\delta \in (-\delta_0, \delta_0)$ , we define  $Y(\tau, \xi; \delta)$  as the solution of the ordinary differential equation

$$\begin{cases} Y_\tau(\tau, \xi; \delta) = \tilde{f}_\delta(Y(\tau, \xi; \delta)) & \text{for } \tau > 0 \\ Y(0, \xi; \delta) = \xi, \end{cases} \quad (2.8)$$

where  $\xi$  varies in  $(-2C_0, 2C_0)$ , with  $C_0$  being the constant defined in (1.3). We claim that  $Y(\tau, \xi; \delta)$  has the following properties.

**Lemma 2.2.** *There exist positive constants  $\delta_0$  and  $C$  such that, for all  $(\tau, \xi, \delta) \in (0, \infty) \times [-2C_0, 2C_0] \times [-\delta_0, \delta_0]$ ,*

- (i)  $|Y(\tau, \xi; \delta)| \leq 2C_0$
- (ii)  $0 < Y_\xi(\tau, \xi; \delta)$
- (iii)  $|\frac{Y_{\xi\xi}}{Y_\xi}(\tau, \xi; \delta)| \leq C(e^{\mu(\delta)\tau} - 1)$ .

Property (i) is a direct consequence of the profile of  $\tilde{f}_\delta$  and so of the qualitative properties of the solution of the bistable ordinary differential equation (2.8); for proofs of (ii) and (iii) we refer to [3], subsection 4.1.  $\square$

## 2.2 Construction of sub- and super-solutions

We are now ready to construct a pair of sub- and super-solutions in order to prove the generation of interface property. By using some cut-off initial data (see [3], subsection 3.2) we can modify slightly  $u_0$  near the boundary  $\partial\Omega$  and make, without loss of generality, the additional assumption

$$\frac{\partial u_0}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \quad (2.9)$$

We set

$$w_\varepsilon^\pm(x, t) = Y\left(\frac{t}{\varepsilon^2}, u_0(x) \pm \varepsilon^2 r(\pm\varepsilon\mathcal{G}, \frac{t}{\varepsilon^2}); \pm\varepsilon\mathcal{G}\right),$$

where the function  $r(\delta, \tau)$  is given by

$$r(\delta, \tau) = C_\star(e^{\mu(\delta)\tau} - 1),$$

and the constant  $\mathcal{G}$  by

$$\mathcal{G} = 2C_0|\Omega| \max_{(u,v) \in [-2C_0, 2C_0] \times [-1, 1]} \left| \frac{\partial f}{\partial v}(u, v) \right|.$$

**Lemma 2.3.** *There exist positive constants  $\varepsilon_0$  and  $C_\star$  such that, for all  $\varepsilon \in (0, \varepsilon_0)$ ,  $(w_\varepsilon^-, w_\varepsilon^+)$  is a pair of sub- and super-solutions for Problem  $(P^\varepsilon)$ , in the domain  $\Omega \times (0, \mu^{-1}\varepsilon^2 |\ln \varepsilon|)$ .*

Before proving the lemma, we remark that  $w_\varepsilon^-(x, 0) = w_\varepsilon^+(x, 0) = u_0(x)$ . Consequently, by the comparison principle, we obtain

$$w_\varepsilon^-(x, t) \leq u^\varepsilon(x, t) \leq w_\varepsilon^+(x, t) \quad \text{for all } \bar{\Omega} \times [0, \mu^{-1}\varepsilon^2 |\ln \varepsilon|]. \quad (2.10)$$

**Proof.** First, the inequality  $w_\varepsilon^- \leq w_\varepsilon^+$  follows from the fact that  $Y(\tau, \xi; \delta)$  increases with both  $\xi$  (see (ii) Lemma 2.2) and  $\delta$  (as easily seen from the ordinary differential equation). Next, (2.9) implies that both  $w_\varepsilon^+$  and  $w_\varepsilon^-$  satisfy the Neumann homogeneous boundary conditions. Hence, it remains to prove the inequalities  $\mathcal{L}_+ w_\varepsilon^+ \geq 0$  and  $\mathcal{L}_- w_\varepsilon^- \leq 0$  (see Definition A.1), provided that the constants  $\varepsilon_0$  and  $C_\star$  are appropriately chosen.

If  $\varepsilon_0$  is sufficiently small, we note that  $\pm\varepsilon\mathcal{G} \in (-\delta_0, \delta_0)$  and that, in the range  $0 \leq t \leq \mu^{-1}\varepsilon^2 |\ln \varepsilon|$ ,

$$|\varepsilon^2 C_\star (e^{\mu(\pm\varepsilon\mathcal{G})t/\varepsilon^2} - 1)| \leq \varepsilon^2 C_\star (\varepsilon^{-\mu(\pm\varepsilon\mathcal{G})/\mu} - 1) \leq C_0,$$

using (2.7). The above inequality implies

$$u_0(x) \pm \varepsilon^2 r(\pm\varepsilon\mathcal{G}, \frac{t}{\varepsilon^2}) \in [-2C_0, 2C_0].$$



These observations allow us to use the results of Lemma 2.2 with the choices  $\tau := t/\varepsilon^2$ ,  $\xi := u_0(x) \pm \varepsilon^2 r(\pm \varepsilon \mathcal{G}, t/\varepsilon^2)$  and  $\delta := \pm \varepsilon \mathcal{G}$ . In particular, it follows from property (i) that  $|\int_{\Omega} w_{\varepsilon}^{\pm}(x, t) dx| \leq 2C_0|\Omega|$  which in turn implies (thanks to the choice of  $\mathcal{G}$ )

$$\max_{\int_{\Omega} w_{\varepsilon}^{-} \leq s \leq \int_{\Omega} w_{\varepsilon}^{+}} f(w_{\varepsilon}^{+}, \varepsilon s) \leq \tilde{f}(w_{\varepsilon}^{+}) + \varepsilon \mathcal{G}. \quad (2.11)$$

In view of the above inequality, some straightforward calculations yield

$$\mathcal{L}_{+} w_{\varepsilon}^{+} \geq \frac{1}{\varepsilon^2} Y_{\tau} + C_{\star} \mu(\varepsilon \mathcal{G}) e^{\mu(\varepsilon \mathcal{G})t/\varepsilon^2} Y_{\xi} - |\nabla u_0|^2 Y_{\xi\xi} - \Delta u_0 Y_{\xi} - \frac{1}{\varepsilon^2} \tilde{f}(Y) - \frac{1}{\varepsilon} \mathcal{G},$$

where the argument  $\left(\frac{t}{\varepsilon^2}, u_0(x) + \varepsilon^2 C_{\star} (e^{\mu(\varepsilon \mathcal{G})t/\varepsilon^2} - 1); \varepsilon \mathcal{G}\right)$  of the function  $Y$  and its derivatives is omitted. Noticing that the ordinary differential equation (2.8) writes as  $Y_{\tau} = \tilde{f}(Y) + \varepsilon \mathcal{G}$ , we get

$$\mathcal{L}_{+} w_{\varepsilon}^{+} \geq Y_{\xi} \left[ C_{\star} \mu(\varepsilon \mathcal{G}) e^{\mu(\varepsilon \mathcal{G})t/\varepsilon^2} - \Delta u_0 - \frac{Y_{\xi\xi}}{Y_{\xi}} |\nabla u_0|^2 \right].$$

Using the estimate (iii) in Lemma 2.2, we obtain

$$\begin{aligned} \mathcal{L}_{+} w_{\varepsilon}^{+} &\geq Y_{\xi} \left[ C_{\star} \mu(\varepsilon \mathcal{G}) e^{\mu(\varepsilon \mathcal{G})t/\varepsilon^2} - |\Delta u_0| - C(e^{\mu(\varepsilon \mathcal{G})t/\varepsilon^2} - 1) |\nabla u_0|^2 \right] \\ &\geq Y_{\xi} \left[ (C_{\star} \mu(\varepsilon \mathcal{G}) - C |\nabla u_0|^2) e^{\mu(\varepsilon \mathcal{G})t/\varepsilon^2} - |\Delta u_0| + C |\nabla u_0|^2 \right]. \end{aligned}$$

In view of (2.7), this inequality implies that, for  $\varepsilon \in (0, \varepsilon_0)$ , with  $\varepsilon_0$  small enough,

$$\mathcal{L}_{+} w_{\varepsilon}^{+} \geq Y_{\xi} \left[ C_{\star} \frac{1}{2} \mu - C C_0^2 - C_0 \right] \geq 0,$$

by choosing  $C_{\star}$  large enough.

Since one can prove  $\mathcal{L}_{-} w_{\varepsilon}^{-} \leq 0$  by similar arguments, this completes the proof of Lemma 2.3.  $\square$

### 2.3 Proof of the generation of interface property

In order to prove Theorem 2.1 we first quote a lemma from [3]; it makes more precise the bistable behavior of the ordinary differential equation by giving basic estimates of the function  $Y(\tau, \xi; \pm \varepsilon \mathcal{G})$  at time  $\tau = \mu^{-1} |\ln \varepsilon|$ .

**Lemma 2.4.** *Let  $\eta \in (0, \eta_0)$  be arbitrary; there exist positive constants  $\varepsilon_0$  and  $M_{\star}$  such that, for all  $\varepsilon \in (0, \varepsilon_0)$ ,*

(i) *for all  $\xi \in (-2C_0, 2C_0)$ ,*

$$-1 - \eta \leq Y(\mu^{-1} |\ln \varepsilon|, \xi; \pm \varepsilon \mathcal{G}) \leq 1 + \eta; \quad (2.12)$$

(ii) for all  $\xi \in (-2C_0, 2C_0)$  such that  $|\xi - a| \geq M_*\varepsilon$ , we have that

$$\text{if } \xi \geq a + M_*\varepsilon \text{ then } Y(\mu^{-1}|\ln \varepsilon|, \xi; \pm\varepsilon\mathcal{G}) \geq 1 - \eta, \quad (2.13)$$

$$\text{if } \xi \leq a - M_*\varepsilon \text{ then } Y(\mu^{-1}|\ln \varepsilon|, \xi; \pm\varepsilon\mathcal{G}) \leq -1 + \eta. \quad (2.14)$$

**Proof of Theorem 2.1.** By setting  $t = \mu^{-1}\varepsilon^2|\ln \varepsilon|$  in (2.10), we get

$$\begin{aligned} & Y\left(\mu^{-1}|\ln \varepsilon|, u_0(x) - \varepsilon^2 r(-\varepsilon\mathcal{G}, \mu^{-1}|\ln \varepsilon|); -\varepsilon\mathcal{G}\right) \\ & \leq u^\varepsilon(x, \mu^{-1}\varepsilon^2|\ln \varepsilon|) \leq Y\left(\mu^{-1}|\ln \varepsilon|, u_0(x) + \varepsilon^2 r(\varepsilon\mathcal{G}, \mu^{-1}|\ln \varepsilon|); +\varepsilon\mathcal{G}\right). \end{aligned} \quad (2.15)$$

We note that (2.7) implies

$$\lim_{\varepsilon \rightarrow 0} \frac{\mu - \mu(\pm\varepsilon\mathcal{G})}{\mu} \ln \varepsilon = 0, \quad (2.16)$$

so that, if  $\varepsilon_0$  is sufficiently small,

$$\varepsilon^2 r(\pm\varepsilon\mathcal{G}, \mu^{-1}|\ln \varepsilon|) = C_*\varepsilon(\varepsilon^{(\mu - \mu(\pm\varepsilon\mathcal{G}))/\mu} - \varepsilon) \in \left(\frac{1}{2}C_*\varepsilon, \frac{3}{2}C_*\varepsilon\right),$$

and, for all  $x \in \Omega$ , it holds that  $u_0(x) \pm \varepsilon^2 r(\pm\varepsilon\mathcal{G}, \mu^{-1}|\ln \varepsilon|) \in (-2C_0, 2C_0)$ . Hence, the result (2.3) of Theorem 2.1 is a direct consequence of (2.12) and (2.15).

Next we prove (2.4). We take  $x \in \Omega$  such that  $u_0(x) \geq a + M_0\varepsilon$ ; then

$$\begin{aligned} u_0(x) - \varepsilon^2 r(-\varepsilon\mathcal{G}, \mu^{-1}|\ln \varepsilon|) & \geq a + M_0\varepsilon - \frac{3}{2}C_*\varepsilon \\ & \geq a + M_*\varepsilon, \end{aligned}$$

if we choose  $M_0$  large enough. Using (2.15) and (2.13) we see that inequality (2.4) is true. The inequality (2.5) can be shown the same way. This completes the proof of Theorem 2.1.  $\square$

### 3 Motion of interface

In Section 2, we have proved that the solution  $u^\varepsilon$  of Problem  $(P^\varepsilon)$  develops a clear transition layer within a very short time. The aim of the present section is to show that, once such a clear transition layer is formed, it persists for the rest of time and that its law of motion is well approximated by the interface equation  $(P^0)$ . In order to study this latter time range where the motion of interface occurs, we will construct another pair of sub- and super-solutions  $(u_\varepsilon^-, u_\varepsilon^+)$  for Problem  $(P^\varepsilon)$ . To begin with we present mathematical tools which are essential for this construction.

### 3.1 Preliminaries

**The “cut-off signed distance function”.** Let  $\Gamma = \cup_{0 < t \leq T} (\Gamma_t \times \{t\})$  be the solution of the limit geometric motion problem  $(P^0)$  and let  $\tilde{d}$  be the signed distance function to  $\Gamma$  defined by:

$$\tilde{d}(x, t) = \begin{cases} -\text{dist}(x, \Gamma_t) & \text{for } x \in \Omega_t^- \\ \text{dist}(x, \Gamma_t) & \text{for } x \in \Omega_t^+, \end{cases} \quad (3.1)$$

where  $\text{dist}(x, \Gamma_t)$  is the distance from  $x$  to the hypersurface  $\Gamma_t$  in  $\Omega$ . The “cut-off signed distance function”  $d$  is defined as follows. First, choose  $d_0 > 0$  small enough so that the signed distance function  $\tilde{d}$  defined in (3.1) is smooth in the following tubular neighborhood of  $\Gamma$ :

$$\{(x, t) \in \bar{\Omega} \times [0, T] \mid |\tilde{d}(x, t)| < 3d_0\},$$

and that

$$\text{dist}(\Gamma_t, \partial\Omega) \geq 3d_0 \quad \text{for all } t \in [0, T]. \quad (3.2)$$

Next let  $\zeta(s)$  be a smooth increasing function on  $\mathbb{R}$  such that

$$\zeta(s) = \begin{cases} s & \text{if } |s| \leq d_0 \\ -2d_0 & \text{if } s \leq -2d_0 \\ 2d_0 & \text{if } s \geq 2d_0. \end{cases}$$

We then define the cut-off signed distance function  $d$  by

$$d(x, t) = \zeta(\tilde{d}(x, t)). \quad (3.3)$$

Note that, in view of (3.2) and the definition of  $d$ , the equality  $\nabla d = 0$  holds in a neighborhood of  $\partial\Omega$ . Note also that the equality  $|\nabla d| = 1$  holds in the region  $\{(x, t) \in \bar{\Omega} \times [0, T] \mid |d(x, t)| < d_0\}$ . Moreover, since  $\nabla d$  coincides with the outward normal unit vector to the hypersurface  $\Gamma_t$ , we have  $d_t(x, t) = -V_n$ , where  $V_n$  is the normal velocity of the interface  $\Gamma_t$  in the exterior direction. It is also known that the mean curvature  $\kappa$  of the interface is equal to  $\Delta d$ . Hence, since the moving interface  $\Gamma$  satisfies Problem  $(P^0)$ , an alternative equation for  $\Gamma$  is given by

$$d_t = \Delta d - c_0 \gamma(t) \quad \text{on } \Gamma_t, \quad (3.4)$$

where  $\gamma(t) := |\Omega_t^+| - |\Omega_t^-|$ .

**The one dimensional standing wave  $U_0$ .** Let  $U_0(z)$  be the unique solution of the stationary problem

$$\begin{cases} U_0'' + \tilde{f}(U_0) = 0 \\ U_0(-\infty) = -1, \quad U_0(0) = a, \quad U_0(+\infty) = +1. \end{cases} \quad (3.5)$$

This solution represents the first approximation of the profile of a transition layer around the interface observed in the stretched coordinates; it naturally arises when performing a formal asymptotic expansion of the solution  $u^\varepsilon$  (see [3], Section 2). Note that the “balanced stability assumption”, i.e. the integral condition  $\int_{-1}^{+1} \tilde{f}(u) du = 0$ , guarantees the existence of such a standing wave. In the simple case where  $\tilde{f}(u) = u(1 - u^2)$ , we know that  $U_0(z) = \tanh(z/\sqrt{2})$ . In the general case, the following standard estimates hold.

**Lemma 3.1.** *There exist positive constants  $C$  and  $\lambda$  such that*

$$\begin{aligned} 0 < 1 - U_0(z) &\leq C e^{-\lambda|z|} && \text{for } z \geq 0 \\ 0 < U_0(z) + 1 &\leq C e^{-\lambda|z|} && \text{for } z \leq 0. \end{aligned}$$

In addition,  $U_0$  is a strictly increasing function and, for  $j = 1, 2$ ,

$$|D^j U_0(z)| \leq C e^{-\lambda|z|} \quad \text{for } z \in \mathbb{R}. \quad (3.6)$$

**The solution  $U_1$  of a linearized problem.** Let  $U_1(z, t)$  be the solution of the problem

$$\begin{cases} U_{1zz} + \tilde{f}'(U_0(z))U_1 = \gamma(t) \left( -\frac{\partial f}{\partial v}(U_0(z), 0) - c_0 U_0'(z) \right), \\ U_1(0, t) = 0, \quad U_1(\cdot, t) \in L^\infty(\mathbb{R}), \end{cases} \quad (3.7)$$

where

$$c_0 := -\frac{\int_{\mathbb{R}} \frac{\partial f}{\partial v}(U_0(z), 0) U_0'(z) dz}{\int_{\mathbb{R}} (U_0')^2(z) dz}. \quad (3.8)$$

Again, the above problem arises when performing a formal asymptotic expansion of the solution  $u^\varepsilon$ . Since (3.7) can be seen as a linearized problem for (3.5), its solvability follows from a Fredholm alternative: thanks to the definition of  $c_0$ ,  $U_0'$  turns out to be orthogonal to the right-hand side member of (3.7). Moreover, there exist constants  $M > 0$  and  $C > 0$  such that

$$|U_1(z, t)| \leq M, \quad (3.9)$$

$$|U_{1t}(z, t)| \leq C, \quad (3.10)$$

$$|U_{1z}(z, t)| + |U_{1zz}(z, t)| \leq C e^{-\lambda|z|}, \quad (3.11)$$

for all  $(z, t) \in \mathbb{R} \times [0, T]$ . We omit the details and refer the reader to [3], Section 2.

Note that, by multiplying equation (3.5) by  $U_0'$  and integrating from  $-\infty$  to  $z$ , we obtain  $U_0'(z) = [2(W(U_0(z)) - W(-1))]^{1/2}$ . Using this, it is now a matter of routine to deduce from (3.8) the more intrinsic expression (1.6).

### 3.2 Construction of sub- and super-solutions

We look for a pair of sub- and super-solutions  $u_\varepsilon^\pm$  for  $(P^\varepsilon)$  of the form

$$u_\varepsilon^\pm(x, t) = U_0 \left( \frac{d(x, t) \pm \varepsilon p(t)}{\varepsilon} \right) + \varepsilon U_1 \left( \frac{d(x, t) \pm \varepsilon p(t)}{\varepsilon}, t \right) \pm q(t), \quad (3.12)$$

where

$$\begin{aligned} p(t) &= -e^{-\beta t/\varepsilon^2} + e^{Lt} + K, \\ q(t) &= \sigma(\beta e^{-\beta t/\varepsilon^2} + \varepsilon^2 L e^{Lt}). \end{aligned}$$

Note that  $q = \sigma \varepsilon^2 p_t$ . It is clear from the definition of  $u_\varepsilon^\pm$  that

$$\lim_{\varepsilon \rightarrow 0} u_\varepsilon^\pm(x, t) = \begin{cases} +1 & \text{for all } (x, t) \in \cup_{0 \leq t \leq T} (\Omega_t^+ \times \{t\}) \\ -1 & \text{for all } (x, t) \in \cup_{0 \leq t \leq T} (\Omega_t^- \times \{t\}). \end{cases} \quad (3.13)$$

The main result of this section is the following:

**Lemma 3.2.** *Choose  $\beta > 0$  and  $\sigma > 0$  appropriately. Then for any  $K > 1$ , there exist positive constants  $\varepsilon_0$  and  $L$  such that, for any  $\varepsilon \in (0, \varepsilon_0)$ ,  $(u_\varepsilon^-, u_\varepsilon^+)$  is a pair of sub- and super-solutions for  $(P^\varepsilon)$  in the domain  $\bar{\Omega} \times [0, T]$ .*

**Proof.** First, we claim that (A.14) and (A.16) hold as a consequence of (4.5). Then, since  $d$  is constant in a neighborhood of  $\partial\Omega$ , both  $u_\varepsilon^+$  and  $u_\varepsilon^-$  satisfy the Neumann homogeneous boundary conditions. Hence it remains to prove the inequalities  $\mathcal{L}_+ u_\varepsilon^+ \geq 0$  and  $\mathcal{L}_- u_\varepsilon^- \leq 0$ , provided that the various constants are appropriately chosen.

We start with some useful inequalities. On the one hand, by assumption (1.1), there exist positive constants  $b, m$  such that

$$\tilde{f}'(U_0(z)) \leq -m \quad \text{if } U_0(z) \in [-1, -1+b] \cup [1-b, 1]. \quad (3.14)$$

On the other hand, since the region  $\{z \in \mathbb{R} \mid U_0(z) \in [-1+b, 1-b]\}$  is compact and since  $U_0' > 0$  on  $\mathbb{R}$ , there exists a constant  $a_1 > 0$  such that

$$U_0'(z) \geq a_1 \quad \text{if } U_0(z) \in [-1+b, 1-b]. \quad (3.15)$$

We set

$$\beta = \frac{m}{4}, \quad (3.16)$$

and choose  $\sigma$  that satisfies

$$0 < \sigma \leq \min(\sigma_0, \sigma_1, \sigma_2), \quad (3.17)$$

where

$$\sigma_0 := \frac{a_1}{m + F_1}, \quad \sigma_1 := \frac{1}{\beta + 1}, \quad \sigma_2 := \frac{4\beta}{H(\beta + 1)},$$

the constants  $F_1$  and  $H$  being given by

$$F_1 := \|\tilde{f}'\|_{L^\infty(-1,1)}, \quad H := \max_{(u,v) \in [-3,3] \times [-1,1]} \|Hess_{(u,v)} f\|, \quad (3.18)$$

where  $\|A\| := \max_{i,j} |a_{ij}|$ . Combining (3.14) and (3.15), and considering that  $\sigma \leq \sigma_0$ , we obtain

$$U_0'(z) - \sigma \tilde{f}'(U_0(z)) \geq \sigma m \quad \text{for } -\infty < z < \infty. \quad (3.19)$$

Now let  $K > 1$  be arbitrary. In what follows we will show that  $\mathcal{L}_+ u_\varepsilon^+ \geq 0$  provided that the constants  $\varepsilon_0$  and  $L$  are appropriately chosen. We recall that  $-1 < U_0 < 1$  and that  $|U_1| \leq M$ . We go on under the following assumption

$$\varepsilon_0 M \leq 1, \quad \varepsilon_0^2 L e^{LT} \leq 1. \quad (3.20)$$

Then, given any  $\varepsilon \in (0, \varepsilon_0)$ , since  $\sigma \leq \sigma_1$  we have  $0 \leq q(t) \leq 1$ , so that

$$-3 \leq u_\varepsilon^\pm(x, t) \leq +3. \quad (3.21)$$

In order to evaluate the “nonlocal part” of  $\mathcal{L}_+ u_\varepsilon^+$ , we need bounds for the quantities  $\int_\Omega u_\varepsilon^\pm(x, t) dx$ . For the sake of clarity, the arguments of most of the functions are omitted in the following. We write  $\int_\Omega u_\varepsilon^+ dx = \int_{\Omega_t^+} (U_0 - 1) dx + |\Omega_t^+| + \int_{\Omega_t^-} (U_0 + 1) dx - |\Omega_t^-| + \int_\Omega (\varepsilon U_1 + q) dx$ , which yields

$$\begin{aligned} \int_\Omega u_\varepsilon^+ dx - \gamma(t) &= |\Omega|q(t) + \int_\Omega \varepsilon U_1 dx \\ &\quad + \int_{\Omega_t^+} (U_0 - 1) dx + \int_{\Omega_t^-} (U_0 + 1) dx \\ &=: |\Omega|q(t) + I_\varepsilon(t) + I_+(t) + I_-(t). \end{aligned}$$

In the following we will denote by  $C$  various positive constants that are independent of  $\varepsilon \in (0, \varepsilon_0)$ . Since  $U_1$  is bounded, we have  $|I_\varepsilon(t)| \leq C\varepsilon$ .

In order to estimate  $I_+(t)$ , we use the partition

$$\Omega_t^+ = \{x \mid d(x, t) \geq d_0\} \cup \{x \mid 0 < d(x, t) < d_0\}.$$

First assume  $d(x, t) \geq d_0$ . From Lemma 3.1, we deduce that

$$0 \leq 1 - U_0\left(\frac{d(x, t) + \varepsilon p(t)}{\varepsilon}\right) \leq C e^{-\lambda|d(x, t) + \varepsilon p(t)|/\varepsilon} \leq C e^{-\lambda d_0/\varepsilon},$$

from which we infer that

$$0 \leq \int_{d(x,t) \geq d_0} \left( 1 - U_0\left(\frac{d(x,t) + \varepsilon p(t)}{\varepsilon}\right) \right) dx \leq C e^{-\lambda d_0/\varepsilon}.$$

In order to estimate the integral on  $\{x \mid 0 < d(x,t) < d_0\}$ , we use arguments similar to those used in [10]. We denote by  $J(s,d)$  the Jacobi of the transformation  $x \mapsto (s,d)$ , where  $s(x,t)$  is the projection of  $x$  on  $\Gamma_t$  along the normal of  $\Gamma_t$  and  $d(x,t)(= \tilde{d}(x,t))$  is the signed distance defined above; we define  $C_J := \max_{0 \leq t \leq T} \|J(\cdot, \cdot)\|_{L^\infty(\Gamma_t \times [-d_0, d_0])}$ . This yields

$$\begin{aligned} 0 &\leq \int_{0 < d(x,t) < d_0} \left( 1 - U_0\left(\frac{d(x,t) + \varepsilon p(t)}{\varepsilon}\right) \right) dx \\ &= \int_{\Gamma_t} \int_0^{d_0} \left( 1 - U_0\left(\frac{r + \varepsilon p(t)}{\varepsilon}\right) \right) J(s,r) dr ds \\ &\leq C_J \varepsilon \int_{\Gamma_t} \int_0^{d_0/\varepsilon} (1 - U_0(z + p(t))) dz ds \\ &\leq C_J \varepsilon \int_{\Gamma_t} \int_0^{+\infty} (1 - U_0(u)) du ds \leq C \varepsilon. \end{aligned}$$

As far as  $I_-(t)$  is concerned, we first assume that  $d(x,t) \leq -d_0$ . Note that  $0 < K - 1 \leq p \leq e^{LT} + K$ . Consequently, if we assume

$$e^{LT} + K \leq \frac{d_0}{2\varepsilon_0}, \quad (3.22)$$

then  $\frac{d_0}{\varepsilon} - |p| \geq \frac{d_0}{2\varepsilon}$ . By using similar arguments as the ones above, we obtain

$$0 \leq \int_{d(x,t) \leq -d_0} \left( U_0\left(\frac{d(x,t) + \varepsilon p(t)}{\varepsilon}\right) + 1 \right) dx \leq C e^{-\lambda d_0/(2\varepsilon)}.$$

Concerning the region  $\{x \mid -d_0 < d(x,t) < 0\}$ , we get

$$\begin{aligned} 0 &\leq \int_{-d_0 < d(x,t) < 0} \left( U_0\left(\frac{d(x,t) + \varepsilon p(t)}{\varepsilon}\right) + 1 \right) dx \\ &= \int_{\Gamma_t} \int_{-d_0}^0 \left( U_0\left(\frac{r + \varepsilon p(t)}{\varepsilon}\right) + 1 \right) J(s,r) dr ds \\ &\leq C_J \varepsilon \int_{\Gamma_t} \int_{-d_0/\varepsilon}^0 (U_0(z + p(t)) + 1) dz ds \\ &\leq C_J \varepsilon \int_{\Gamma_t} \int_{-\infty}^{d_0/(2\varepsilon_0)} (U_0(u) + 1) du ds \leq C \varepsilon. \end{aligned}$$

Since one would obtain similar estimates with  $u_\varepsilon^+$  replaced by  $u_\varepsilon^-$ , the above estimates yield

$$\left| \int_{\Omega} u_\varepsilon^\pm dx - \gamma(t) \right| \leq C\varepsilon + Cq(t), \quad (3.23)$$

which, in turn, implies

$$\max_{\int_{\Omega} u_\varepsilon^- \leq s \leq \int_{\Omega} u_\varepsilon^+} f(u_\varepsilon^+, \varepsilon s) \leq f(u_\varepsilon^+, \varepsilon \gamma(t)) + C\varepsilon^2 + C\varepsilon q(t) \quad (3.24)$$

and

$$\min_{\int_{\Omega} u_\varepsilon^- \leq s \leq \int_{\Omega} u_\varepsilon^+} f(u_\varepsilon^-, \varepsilon s) \geq f(u_\varepsilon^-, \varepsilon \gamma(t)) - C\varepsilon^2 - C\varepsilon q(t). \quad (3.25)$$

Now, we can turn back to the proof of  $\mathcal{L}_+ u_\varepsilon^+ \geq 0$ . From the above inequality, we get

$$\mathcal{L}_+ u_\varepsilon^+ \geq (u_\varepsilon^+)_t - \Delta u_\varepsilon^+ - \frac{1}{\varepsilon^2} f(u_\varepsilon^+, \varepsilon \gamma(t)) - C - C \frac{1}{\varepsilon} q(t). \quad (3.26)$$

Straightforward computations yield

$$\begin{aligned} (u_\varepsilon^+)_t &= U_0' \left( \frac{dt}{\varepsilon} + p_t \right) + \varepsilon U_{1t} + U_{1z} (dt + \varepsilon p_t) + q_t \\ \Delta u_\varepsilon^+ &= U_0'' \frac{|\nabla d|^2}{\varepsilon^2} + U_0' \frac{\Delta d}{\varepsilon} + U_{1zz} \frac{|\nabla d|^2}{\varepsilon} + U_{1z} \Delta d, \end{aligned}$$

where the function  $U_0$ , as well as its derivatives, are evaluated at  $z = (d(x, t) + \varepsilon p(t))/\varepsilon$ , whereas the function  $U_1$ , as well as its derivatives, are evaluated at  $\left( (d(x, t) + \varepsilon p(t))/\varepsilon, t \right)$ . We also have

$$\begin{aligned} f(u_\varepsilon^+, \varepsilon \gamma(t)) &\leq \tilde{f}(U_0) + (\varepsilon U_1 + q) \tilde{f}'(U_0) + \varepsilon \gamma(t) \frac{\partial f}{\partial v}(U_0, 0) \\ &\quad + H \left( \frac{1}{2} (\varepsilon U_1 + q)^2 + \frac{1}{2} \varepsilon^2 \gamma^2(t) + (\varepsilon U_1 + q) \varepsilon \gamma(t) \right), \end{aligned}$$

where  $H$  was defined in (3.18). Combining the above expressions with the equations (3.5) and (3.7) for  $U_0$  and  $U_1$ , we obtain

$$\mathcal{L}_+ u_\varepsilon^+ \geq E_1 + \dots + E_6,$$

where:

$$\begin{aligned} E_1 &= -\frac{1}{\varepsilon^2} q \left( \tilde{f}'(U_0) + \frac{1}{2} H q \right) + U_0' p_t + q_t \\ E_2 &= \left( \frac{U_0''}{\varepsilon^2} + \frac{U_{1zz}}{\varepsilon} \right) (1 - |\nabla d|^2) \end{aligned}$$



$$\begin{aligned}
E_3 &= \left( \frac{U_0'}{\varepsilon} + U_{1z} \right) (d_t - \Delta d + c_0 \gamma(t)) \\
E_4 &= \varepsilon U_{1z} p_t + \frac{1}{\varepsilon} q (-H\gamma(t) - HU_1 - C) \\
E_5 &= -c_0 \gamma(t) U_{1z} - \frac{1}{2} H U_1^2 - \frac{1}{2} H \gamma^2(t) - H\gamma(t) U_1 - C \\
E_6 &= \varepsilon U_{1t}.
\end{aligned}$$

In the sequel, we estimate the terms  $E_1$ — $E_6$  and denote by  $C_i$  various positive constants that are independent of  $\varepsilon$ .

### 3.2.1 The term $E_1$

Direct computation gives

$$E_1 = \frac{\beta}{\varepsilon^2} e^{-\beta t/\varepsilon^2} (I - \sigma\beta) + L e^{Lt} (I + \varepsilon^2 \sigma L),$$

where

$$I = U_0' - \sigma \tilde{f}'(U_0) - \frac{\sigma^2}{2} H (\beta e^{-\beta t/\varepsilon^2} + \varepsilon^2 L e^{Lt}).$$

In virtue of (3.19), we have

$$I \geq \sigma m - \frac{\sigma^2}{2} H (\beta + \varepsilon^2 L e^{LT}).$$

Combining this, (3.16), (3.20) and the inequality  $\sigma \leq \sigma_2$ , we obtain  $I \geq 2\sigma\beta$ . Consequently, we have

$$E_1 \geq \frac{\sigma\beta^2}{\varepsilon^2} e^{-\beta t/\varepsilon^2} + 2\sigma\beta L e^{Lt}.$$

### 3.2.2 The term $E_2$

First, in the region where  $|d| < d_0$ , we have  $|\nabla d| = 1$ , hence  $E_2 = 0$ . Next we consider the region where  $|d| \geq d_0$ . We deduce from Lemma 3.1 and from (3.11) that

$$|E_2| \leq C \left( \frac{1}{\varepsilon^2} + \frac{1}{\varepsilon} \right) e^{-\lambda|d+\varepsilon p|/\varepsilon} \leq \frac{2C}{\varepsilon^2} e^{-\lambda(d_0/\varepsilon - |p|)}.$$

In view of (3.22) we have  $0 < p \leq \frac{d_0}{2\varepsilon}$  so that

$$|E_2| \leq \frac{2C}{\varepsilon^2} e^{-\lambda d_0/(2\varepsilon)} \leq C_2.$$

### 3.2.3 The term $E_3$

Recall that

$$(d_t - \Delta d)(x, t) + c_0 \gamma(t) = 0 \quad \text{on } \Gamma_t = \{x \in \Omega, d(x, t) = 0\}.$$

Since the interface  $\Gamma_t$  is smooth, both  $\Delta d$  and  $d_t$  are Lipschitz continuous near  $\Gamma_t$ . It follows from the mean value theorem applied on both sides of  $\Gamma_t$  that there exists a constant  $N > 0$  such that:

$$|(d_t - \Delta d)(x, t) + c_0 \gamma(t)| \leq N|d(x, t)| \quad \text{for all } (x, t) \in \Omega \times (0, T).$$

Applying Lemma 3.1 and the estimate (3.11) we deduce that

$$\begin{aligned} |E_3| &\leq 2NC \frac{|d|}{\varepsilon} e^{-\lambda|d/\varepsilon+p|} \\ &\leq 2NC \max_{\xi \in \mathbb{R}} |\xi| e^{-\lambda|\xi+p|} \\ &\leq 2NC \max(|p|, \frac{1}{\lambda}). \end{aligned}$$

Thus, recalling that  $|p| \leq e^{Lt} + K$ , we obtain

$$|E_3| \leq C_3(e^{Lt} + K) + C_3',$$

where  $C_3 := 2NC$  and  $C_3' := 2NC/\lambda$ .

### 3.2.4 The terms $E_4, E_5$ and $E_6$

Since  $U_1, U_{1z}, U_{1t}$  and  $\gamma$  are bounded, it is a matter of routine to see that

$$|E_4| \leq C_4 \left( \frac{1}{\varepsilon} \beta e^{-\beta t/\varepsilon^2} + \varepsilon L e^{Lt} \right), \quad |E_5| \leq C_5, \quad |E_6| \leq \varepsilon C_6.$$

### 3.2.5 Completion of the proof

Collecting all these estimates gives

$$\mathcal{L}_+ u_\varepsilon^+ \geq \left( \frac{\sigma \beta^2}{\varepsilon^2} - \frac{C_4 \beta}{\varepsilon} \right) e^{-\beta t/\varepsilon^2} + (2\sigma \beta L - C_3 - \varepsilon C_4 L) e^{Lt} - C_7, \quad (3.27)$$

where  $C_7 := C_2 + KC_3 + C_3' + C_5 + C_6$ . Now we set

$$L := \frac{1}{T} \ln \frac{d_0}{4\varepsilon_0},$$

which, for  $\varepsilon_0$  small enough, validates assumptions (3.20) and (3.22). For  $\varepsilon_0$  small enough, the first term of the right-hand side of (3.27) is positive, and

$$\mathcal{L}_+ u_\varepsilon^+ \geq [\sigma \beta L - C_3] e^{Lt} - C_7 \geq \frac{1}{2} \sigma \beta L - C_7 \geq 0.$$

The proof of (A.12) is now complete, with the choice of the constants  $\beta, \sigma$  as in (3.16), (3.17). Since one can prove (A.13) by similar arguments, this completes the proof of Lemma 3.2.  $\square$

## 4 Proof of Theorem 1.1

Let  $\eta \in (0, \eta_0)$  be arbitrary. Choose  $\beta$  and  $\sigma$  that satisfy (3.16), (3.17) and

$$\sigma\beta \leq \frac{\eta}{3}. \quad (4.1)$$

By the generation of interface property, Theorem 2.1, there exist positive constants  $\varepsilon_0$  and  $M_0$  such that (2.3), (2.4) and (2.5) hold with the constant  $\eta$  replaced by  $\sigma\beta/2$ . Since  $\nabla u_0 \neq 0$  everywhere on  $\Gamma_0 = \{x \in \Omega \mid u_0(x) = a\}$  and since  $\Gamma_0$  is a compact hypersurface, we can find a positive constant  $M_1$  such that

$$\begin{aligned} \text{if } d_0(x) \geq M_1\varepsilon & \quad \text{then } u_0(x) \geq a + M_0\varepsilon \\ \text{if } d_0(x) \leq -M_1\varepsilon & \quad \text{then } u_0(x) \leq a - M_0\varepsilon. \end{aligned} \quad (4.2)$$

Here  $d_0(x) := d(x, 0)$  denotes the signed distance function associated with the hypersurface  $\Gamma_0$ . Now we define functions  $H^+(x), H^-(x)$  by

$$\begin{aligned} H^+(x) &= \begin{cases} +1 + \sigma\beta/2 & \text{if } d_0(x) > -M_1\varepsilon \\ -1 + \sigma\beta/2 & \text{if } d_0(x) \leq -M_1\varepsilon, \end{cases} \\ H^-(x) &= \begin{cases} +1 - \sigma\beta/2 & \text{if } d_0(x) \geq M_1\varepsilon \\ -1 - \sigma\beta/2 & \text{if } d_0(x) < M_1\varepsilon. \end{cases} \end{aligned}$$

Then from (2.3), (2.4), (2.5) (with  $\eta$  replaced by  $\sigma\beta/2$ ) and (4.2), we see that

$$H^-(x) \leq u^\varepsilon(x, \mu^{-1}\varepsilon^2|\ln \varepsilon|) \leq H^+(x) \quad \text{for } x \in \Omega. \quad (4.3)$$

Next we fix a sufficiently large constant  $K > 1$  such that

$$U_0(-M_1 + K) \geq 1 - \frac{\sigma\beta}{3} \quad \text{and} \quad U_0(M_1 - K) \leq -1 + \frac{\sigma\beta}{3}. \quad (4.4)$$

For this  $K$ , we choose  $\varepsilon_0$  and  $L$  as in Lemma 3.2. We claim that

$$u_\varepsilon^-(x, 0) \leq H^-(x), \quad H^+(x) \leq u_\varepsilon^+(x, 0) \quad \text{for } x \in \Omega, \quad (4.5)$$

with  $(u_\varepsilon^-, u_\varepsilon^+)$  the pair of sub- and super-solutions defined in (3.12) for the study of the motion of interface. We shall only prove the former inequality, as the proof of the latter is virtually the same. Then it amounts to showing that

$$u_\varepsilon^-(x, 0) = U_0\left(\frac{d_0(x)}{\varepsilon} - K\right) + \varepsilon U_1\left(\frac{d_0(x)}{\varepsilon} - K, 0\right) - \sigma(\beta + \varepsilon^2 L) \leq H^-(x). \quad (4.6)$$

Recall that  $|U_1| \leq M$ . Therefore, by choosing  $\varepsilon_0$  small enough so that  $\varepsilon_0 M \leq \sigma\beta/6$ , we see that

$$u_\varepsilon^-(x, 0) \leq U_0\left(\frac{d_0(x)}{\varepsilon} - K\right) - \frac{5}{6}\sigma\beta.$$

In the range where  $d_0(x) < M_1\varepsilon$ , the fact that  $U_0$  is an increasing function and the second inequality in (4.4) imply

$$U_0\left(\frac{d_0(x)}{\varepsilon} - K\right) - \frac{5}{6}\sigma\beta \leq U_0(M_1 - K) - \frac{5}{6}\sigma\beta \leq -1 - \frac{\sigma\beta}{2} = H^-(x).$$

On the other hand, in the range where  $d_0(x) \geq M_1\varepsilon$ , we have

$$U_0\left(\frac{d_0(x)}{\varepsilon} - K\right) - \frac{5}{6}\sigma\beta \leq 1 - \frac{5}{6}\sigma\beta \leq H^-(x).$$

This proves (4.6), hence (4.5) is established.

Combining (4.3) and (4.5), we obtain

$$u_\varepsilon^-(x, 0) \leq u^\varepsilon(x, \mu^{-1}\varepsilon^2|\ln\varepsilon|) \leq u_\varepsilon^+(x, 0).$$

Since  $(u_\varepsilon^-, u_\varepsilon^+)$  is a pair of sub- and super-solutions for Problem  $(P^\varepsilon)$ , the comparison principle yields

$$u_\varepsilon^-(x, t) \leq u^\varepsilon(x, t + t^\varepsilon) \leq u_\varepsilon^+(x, t) \quad \text{for } 0 \leq t \leq T - t^\varepsilon, \quad (4.7)$$

where  $t^\varepsilon = \mu^{-1}\varepsilon^2|\ln\varepsilon|$ . Note that, in view of (3.13), this is enough to prove Corollary 1.2. Now let  $\mathcal{C}$  be a positive constant such that

$$U_0(\mathcal{C} - e^{LT} - K) \geq 1 - \frac{\eta}{2} \quad \text{and} \quad U_0(-\mathcal{C} + e^{LT} + K) \leq -1 + \frac{\eta}{2}. \quad (4.8)$$

One then easily checks, using (4.7) and (4.1), that, for  $\varepsilon_0$  small enough, for  $0 \leq t \leq T - t^\varepsilon$ , we have

$$\begin{aligned} \text{if } d(x, t) \geq \mathcal{C}\varepsilon & \quad \text{then } u^\varepsilon(x, t + t^\varepsilon) \geq 1 - \eta \\ \text{if } d(x, t) \leq -\mathcal{C}\varepsilon & \quad \text{then } u^\varepsilon(x, t + t^\varepsilon) \leq -1 + \eta, \end{aligned} \quad (4.9)$$

and

$$u^\varepsilon(x, t + t^\varepsilon) \in [-1 - \eta, 1 + \eta],$$

which completes the proof of Theorem 1.1.  $\square$

## Appendix - Comparison principle

The definition of sub- and super-solutions is the one proposed by Chen, Hilhorst and Logak [10]. It involves simultaneously a super- and a sub-solution.

**Definition A.1.** Let  $(u_\varepsilon^-, u_\varepsilon^+)$  be a pair of smooth functions defined on  $\bar{\Omega} \times [0, T]$  and satisfying

$$u_\varepsilon^- \leq u_\varepsilon^+ \quad \text{in } \bar{\Omega} \times [0, T], \quad (\text{A.10})$$

and

$$\frac{\partial u_\varepsilon^-}{\partial \nu} \leq 0 \leq \frac{\partial u_\varepsilon^+}{\partial \nu} \quad \text{on } \partial\Omega \times (0, T). \quad (\text{A.11})$$

We say that  $(u_\varepsilon^-, u_\varepsilon^+)$  is a pair of sub- and super-solutions for Problem  $(P^\varepsilon)$  if

$$\mathcal{L}_+ u_\varepsilon^+ := (u_\varepsilon^+)_t - \Delta u_\varepsilon^+ - \frac{1}{\varepsilon^2} \max_{\int_\Omega u_\varepsilon^- \leq s \leq \int_\Omega u_\varepsilon^+} f(u_\varepsilon^+, \varepsilon s) \geq 0 \quad \text{in } \Omega \times (0, T), \quad (\text{A.12})$$

$$\mathcal{L}_- u_\varepsilon^- := (u_\varepsilon^-)_t - \Delta u_\varepsilon^- - \frac{1}{\varepsilon^2} \min_{\int_\Omega u_\varepsilon^- \leq s \leq \int_\Omega u_\varepsilon^+} f(u_\varepsilon^-, \varepsilon s) \leq 0 \quad \text{in } \Omega \times (0, T). \quad (\text{A.13})$$

As proved in [10], the following comparison principle holds.

**Proposition A.2.** *Let a pair of sub- and super-solutions be given. Assume that, for all  $x \in \Omega$ ,*

$$u_\varepsilon^-(x, 0) \leq u_0(x) \leq u_\varepsilon^+(x, 0). \quad (\text{A.14})$$

*Then, if we denote by  $u^\varepsilon$  the solution of Problem  $(P^\varepsilon)$ , the function  $u^\varepsilon$  satisfies*

$$u_\varepsilon^-(x, t) \leq u^\varepsilon(x, t) \leq u_\varepsilon^+(x, t), \quad (\text{A.15})$$

*for all  $(x, t) \in \Omega \times [0, T]$ .*

As easily seen from the proof in [10], one could replace the assumption (A.10) by the assumption (A.14) together with the condition that

$$\int_\Omega u_\varepsilon^-(x, t) dx \leq \int_\Omega u_\varepsilon^+(x, t) dx, \quad (\text{A.16})$$

for all  $t \in [0, t_0]$  with  $t_0 > 0$ . More precisely if (A.14), (A.16), (A.11), (A.12) and (A.13) hold, then the conclusion (A.15) follows.

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