HAL open science

Black hole entropy and $SU(2)$ Chern-Simons theory

Jonathan Engle, Karim Noui, Alejandro Perez

To cite this version:

Jonathan Engle, Karim Noui, Alejandro Perez. Black hole entropy and $SU(2)$ Chern-Simons theory. Physical Review Letters, 2010, 105 (3), pp.031302. 10.1103/PhysRevLett.105.031302 . hal-00391827

HAL Id: hal-00391827

https://hal.science/hal-00391827v1

Submitted on 7 Sep 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Distributed under a Creative Commons Attribution 4.0 International License

Black Hole Entropy and $SU(2)$ Chern-Simons Theory

Jonathan Engle,[1] Karim Noui,[2] and Alejandro Perez[1]
[1]*Centre de Physique Théorique, Campus de Luminy, 13288 Marseille, France*
[2]*Laboratoire de Mathématique et Physique Théorique, 37200 Tours, France*
(Received 14 September 2009; published 12 July 2010)

Black holes (BH's) in equilibrium can be defined locally in terms of the so-called isolated horizon boundary condition given on a null surface representing the event horizon. We show that this boundary condition can be treated in a manifestly $SU(2)$ invariant manner. Upon quantization, state counting is expressed in terms of the dimension of Chern-Simons Hilbert spaces on a sphere with punctures. Remarkably, when considering an ensemble of fixed horizon area a_H, the counting can be mapped to simply counting the number of $SU(2)$ intertwiners compatible with the spins labeling the punctures. The resulting BH entropy is proportional to a_H with logarithmic corrections $\Delta S = -\frac{3}{2}\log a_H$. Our treatment from first principles settles previous controversies concerning the counting of states.

DOI: 10.1103/PhysRevLett.105.031302 PACS numbers: 04.70.Dy, 04.60.Pp

Black holes are intriguing solutions of general relativity describing the physics of gravitational collapse. These fascinating systems—whose existence in our Universe is supported by a great amount of observational evidence—are remarkably simple. However, in the interior of the event horizon, the predictive power of classical general relativity breaks down due to the unavoidable appearance of unphysical divergences of the gravitational field (singularities). Dimensional arguments imply that quantum effects cannot be neglected near singularities. In this precise sense, black holes (BH) provide the most tantalizing theoretical evidence for the need of a more fundamental (quantum) description of the gravitational field.

Quantum effects are also important outside the horizon. Indeed the semiclassical calculations of Hawking [1] show that BH's radiate as perfect black bodies at temperature proportional to their surface gravity and have an entropy $S = a_H/4\ell_p^2$, where $\ell_p^2 = G\hbar/c^3$ is the Planck area. This entropy is expected to arise from the huge number of microstates of the underlying fundamental quantum theory describing the BH, and therefore its computation from basic principles is an important test of any candidate quantum theory of gravity. This Letter proposes a new and more fundamental framework for the computation of BH entropy in loop quantum gravity (LQG) and establishes a precise relationship between $SU(2)$ Chern-Simons (CS) theory and quantum black hole physics as first explored in [2].

Our treatment clarifies the description of both the classical as well as the quantum theory of black holes in LQG making the full picture more transparent. We show that, in contrast with prior results [3], the gauge symmetry of LQG need not be reduced from $SU(2)$ to $U(1)$ at the horizon. Even when the $U(1)$ reduction is perfectly viable at the classical level, it leads to imposition of certain components of the quantum constraints only in a weak sense. Our $SU(2)$ invariant formulation—equivalent to the $U(1)$ at the classical level—avoids this issue and allows the imposition of the constraints strongly in the Dirac sense. This leads to a drastic simplification of the quantum theory in which states of a black hole are now in one-to-one correspondence with the fundamental basic volume excitations of LQG given by single intertwiner states. This settles certain controversies concerning the relevant quantum numbers to be considered in the counting of states. The main quantitative result of our work is the correction of the value of the BH entropy.

The standard definition of a BH as a space-time region of no escape is a global definition. This notion of BH requires a complete knowledge of a space-time geometry and is therefore not suitable for describing local physics. This is solved by using instead the notion of Isolated horizons (IH), defined by extracting from the definition of a Killing horizon the minimum conditions necessary for the laws of BH mechanics to hold [4]. They may be thought of as "apparent horizons in equilibrium." Even though IH are very general, allowing rotation and distortion, for simplicity here we concentrate on the case in which the horizon geometry is spherically symmetric. In the vacuum case, the latter are easy to visualize in terms of the characteristic formulation of general relativity with initial data given on null surfaces: spacetimes with such IH are solutions to Einstein's equations where Schwarzschild data are given on the horizon and suitable free radiation is given at a transversal null surface [5] (see Fig. 1).

The calculation of black hole entropy in LQG is done by quantizing the sector of the phase space of general relativity corresponding to solutions having an IH. At the technical level this sector is defined by postulating the existence of a null boundary $\Delta \subset \mathcal{M}$ with topology $S^2 \times \mathbb{R}$ with the pullback of the gravitational field to Δ satisfying the isolated horizon boundary conditions.

It is well known that the initial value formulation of general relativity can be characterized in terms of a triad field e_a^i through $\Sigma = e \wedge e$—encoding the intrinsic spatial metric of M as $q_{ab} = e_a^i e_b^j \delta_{ij}$—and certain components of the extrinsic curvature K_{ab} of M defined by $K_a^i = K_{ab} e_i^b$. It can be shown that the symplectic structure of gravity

 © 2010 The American Physical Society

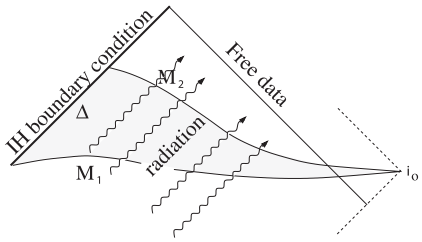

FIG. 1. The characteristic data for a (vacuum) spherically symmetric isolated horizon corresponds to Schwarzschild data on Δ, and free radiation data on the transversal null surface.

$$\Omega_M(\delta_1, \delta_2) = \frac{1}{8\pi G}\int_M [\delta_1\Sigma^i \wedge \delta_2 K_i - \delta_2\Sigma^i \wedge \delta_1 K_i] \quad (1)$$

is preserved in the presence of an IH. More precisely in the shaded space-time region in Fig. 1 one has

$$\Omega_{M_2}(\delta_1, \delta_2) = \Omega_{M_1}(\delta_1, \delta_2). \quad (2)$$

That is, the symplectic flux across the isolated horizon Δ vanishes due to the isolated horizon boundary condition [4,6]. One also has that, on shell, phase space tangent vectors δ_α, δ_v of the form

$$\delta_\alpha\Sigma = [\alpha, \Sigma], \qquad \delta_\alpha K = [\alpha, K];$$
$$\delta_v\Sigma = \mathcal{L}_v\Sigma, \qquad \delta_v K = \mathcal{L}_v K$$

for $\alpha : M \to \mathfrak{su}(2)$ and $v \in \mathrm{Vect}(M)$ tangent to the horizon, are degenerate directions of Ω_M from which one concludes that $SU(2)$ triad rotations and diffeomorphisms are gauge symmetries [7]. Hence, the IH boundary condition breaks neither the symmetry under these diffeomorphisms nor the $SU(2)$ internal gauge symmetry introduced by the use of triad variables.

Ashtekar-Barbero connection variables are necessary for the quantization in the manner of LQG. When there is no boundary the $SU(2)$ connection

$$A^i_a = \Gamma^i + \beta K^i_a \quad (3)$$

is canonically conjugate to $\epsilon^{abc}\beta^{-1}\Sigma^i_{bc}/2$ where β is the so-called Immirzi parameter. As shown below, in the presence of a boundary the situation is more subtle: the symplectic structure acquires a boundary term Ω_H. Because of the fact that, at the horizon, phase space tangent vectors δ are linear combinations of $SU(2)$ gauge transformations and diffeomorphisms tangent to the horizon $H = M \cap \Delta$, the boundary term Ω_H is completely fixed by the requirement of gauge invariance, i.e., the condition that local $SU(2)$ transformations, now taking the form

$$\delta_\alpha\Sigma = [\alpha, \Sigma], \qquad \delta_\alpha A = -d_A\alpha \quad (4)$$

as well as diffeomorphisms preserving H, continue to be degenerate directions of the symplectic structure. The symplectic structure, in the new variables, becomes [8]

$$\kappa\Omega_M = \int_M 2\delta_{[1}\Sigma^i \wedge \delta_{2]}A_i - \frac{a_H}{\pi(1-\beta^2)}\int_H \delta_1 A_i \wedge \delta_2 A^i, \quad (5)$$

where $\kappa = 8\pi G\beta$, a_H is the horizon area, and we have used the IH boundary condition which in terms of Ashtekar-Barbero variables is found [6] to take the form

$$\Sigma^i + \frac{a_H}{\pi(1-\beta^2)}F^i(A) = 0. \quad (6)$$

Note that the boundary contribution to the symplectic structure is given by an $SU(2)$ CS symplectic form. One can also show directly that the boundary term contribution is necessary for time evolution to preserve the symplectic form.

Another consequence of the fact that $SU(2)$ transformations and diffeomorphisms preserving H are gauge is that (in the canonical formulation) they are Hamiltonian vector fields generated by first class constraints. More precisely one has that

$$\begin{aligned}\Omega(\delta_\alpha, \delta) + \delta G[\alpha, A, \Sigma] = 0,\\ \Omega(\delta_v, \delta) + \delta V[v, A, \Sigma] = 0,\end{aligned} \quad (7)$$

where G and V are the Gauss and diffeo constraints, respectively. They take the form

$$\begin{aligned}G[\alpha, A, \Sigma] = &\int_M \alpha_i[d_A\Sigma^i/(\kappa\beta)]\\ &+ \int_H \alpha_i\left[\frac{a_H}{\pi\kappa\beta(1-\beta^2)}F^i + \frac{1}{\kappa\beta}\Sigma^i\right] \approx 0,\end{aligned}$$

for all $\alpha : M \to \mathfrak{su}(2)$, and

$$\begin{aligned}V[v, A, \Sigma] = &\int_M \frac{1}{\kappa\beta}[\Sigma_i \wedge v\lrcorner F^i - v\lrcorner A_i d_A\Sigma^i]\\ &- \int_H v\lrcorner A_i\left[\frac{a_H}{\pi\kappa\beta(1-\beta^2)}F^i + \frac{1}{\kappa\beta}\Sigma^i\right]\\ &\approx 0,\end{aligned}$$

for all $v \in \mathrm{Vect}(M)$ that is tangent to H at the horizon. Notice that the previous constraints have the usual Gauss and diffeo constraint bulk terms, plus boundary terms given by smearings of (6) on H. Their Poisson algebra is

$$\begin{aligned}\{G[\alpha, A, \Sigma], G[\beta, A, \Sigma]\} &= G([\alpha, \beta], A, \Sigma)\\ \{G[\alpha, A, \Sigma], V[v, A, \Sigma]\} &= G(\mathcal{L}_v\alpha, A, \Sigma)\\ \{V[v, A, \Sigma], V[w, A, \Sigma]\} &= V([v, w], A, \Sigma),\end{aligned} \quad (8)$$

where we have ignored the Poisson brackets involving the scalar constraint as its smearing must vanish on H; i.e., it does not affect the first class nature of the previous constraints. Thus (6) are first class constraints and can be implemented in the manner of Dirac in the quantum theory.

This fact and the form of the symplectic structure motivates one to handle the quantization of the bulk and horizon degrees of freedom (d.o.f.) separately. As in standard LQG [9] one first considers (bulk) Hilbert spaces $\mathcal{H}^B_\gamma$ defined on a graph $\gamma \subset M$ with end points on H, denoted $\gamma \cap H$. The quantum operator associated with Σ in (6) is

$$\epsilon^{ab}\hat{\Sigma}^i_{ab}(x) = 8\pi G\beta \sum_{p\in\gamma\cap H} \delta(x, x_p)\hat{J}^i(p), \quad (9)$$

where $[\hat{J}^i(p), \hat{J}^j(p)] = \epsilon^{ij}{}_k\hat{J}^k(p)$ at each $p \in \gamma \cap H$.

Consider a basis of $\mathcal{H}^B_\gamma$ of eigenstates of both $J_p \cdot J_p$ as well as J^3_p for all $p \in \gamma \cap H$ with eigenvalues $\hbar^2 j_p(j_p+1)$ and $\hbar m_p$, respectively. These states are spin network states, here denoted $|\{j_p, m_p\}_1^n; \cdots\rangle$, where j_p and m_p are the spins and magnetic numbers labeling n edges puncturing the horizon at points x_p (other labels are left implicit). They are also eigenstates of the horizon area operator $\hat{a}_H$

$$\hat{a}_H|\{j_p, m_p\}_1^n; \cdots\rangle = 8\pi\beta\ell_p^2 \sum_{p=1}^{n} \sqrt{j_p(j_p+1)} \times |\{j_p, m_p\}_1^n; \cdots\rangle.$$

We can decompose $\mathcal{H}^B_\gamma$ according to

$$\mathcal{H}^B_\gamma = \bigoplus_{\{j_p\}_{p\in\gamma\cap H}} \mathcal{H}^B_\gamma(\{j_p\}) \tag{10}$$

for spaces $\mathcal{H}^B_\gamma(\{j_p\})$ spanned by states $|\{j_p, m_p\}_1^n; \cdots\rangle$ for a given n-tuple $\{j_p\}$.

Substituting the expression (9) into (6) we get

$$-\frac{a_H}{\pi(1-\beta^2)}\epsilon^{ab}\hat{F}^i_{ab} = 8\pi G\beta \sum_{p\in\gamma\cap H} \delta(x, x_p)\hat{J}^i(p). \tag{11}$$

This equation tells us that the surface Hilbert space $\mathcal{H}^H_{\gamma\cap H}$ is precisely the one corresponding to (the well-studied [9]) $SU(2)$ CS theory in the presence of particles with CS level $k = a_H/[2\pi\beta(1-\beta^2)\ell_p^2]$. The curvature of the (quantum) CS connection vanishes everywhere on H except at the position of the defects where we find conical singularities of strength encoded in the quantum operators $\hat{J}^i_p$.

The solutions of (11) restricted to the graph γ are found to be elements of the Hilbert space [6]

$$\mathcal{H}_\gamma = \bigoplus_{\{j_p\}_{p\in\gamma\cap H}} \mathcal{H}^{\rm inv}_\gamma(\{j_p\}) \otimes \mathcal{H}^{\rm CS}_k(\{j_p\}), \tag{12}$$

where $\mathcal{H}^{\rm inv}_\gamma(\{j_p\})$ is a proper subspace of $\mathcal{H}^B_\gamma(\{j_p\})$ spanned by area eigenstates, and $\mathcal{H}^{\rm CS}_k(\{j_p\})$ are the CS Hilbert spaces which turn out to be completely determined by the total spin of punctures $\{j_p\}$ [9]. The full Hilbert space of solutions of (11) is obtained as the projective limit of the spaces $\mathcal{H}_\gamma$. The IH boundary condition implies that lapse must be zero at the horizon so that the scalar constraint is only imposed in the bulk.

The entropy of the IH is computed by the formula $S = {\rm tr}(\rho_{\rm IH}\log\rho_{\rm IH})$ where the density matrix $\rho_{\rm IH}$ is obtained by tracing over the bulk d.o.f., while restricting to horizon states that are compatible with the macroscopic area parameter a_H. Assuming that there exist at least one solution of the bulk constraints for every admissible state on the boundary, the entropy is given by $S = \log(\mathcal{N})$ where $\mathcal{N}$ is the number of horizon states compatible with the given macroscopic horizon area a_H. After a moment of reflection one sees that

$$\mathcal{N} = \sum_{n;(j)_1^n} {\rm dim}[\mathcal{H}^{\rm CS}_k(j_1\cdots j_n)], \tag{13}$$

where the labels $j_1\cdots j_p$ of the punctures are constrained by the condition

$$a_H - \epsilon \le 8\pi\beta\ell_p^2 \sum_{p=1}^{n} \sqrt{j_p(j_p+1)} \le a_H + \epsilon. \tag{14}$$

Similar formulas were first used in [10].

It turns out that due to (14) we can compute the entropy for $a_H \gg B\ell_p^2$ (not necessarily infinite). The reason is that the representation theory of $U_q[SU(2)]$—describing $\mathcal{H}^{\rm CS}_k$ for finite k—implies

$${\rm dim}[\mathcal{H}^{\rm CS}_k(j_1\cdots j_n)] = {\rm dim}[{\rm Inv}(\otimes_p j_p)], \tag{15}$$

as long as all the j_p as well as the interwining internal spins are less than $k/2$. But for Immirzi parameter in the range $|\beta| \le \sqrt{3}$ this is precisely granted by (14) [6]. All this simplifies the entropy formula considerably. The previous dimension corresponds to the number of independent states one has if one models the black hole by a single $SU(2)$ intertwiner.

Let us conclude with a few remarks.

We have shown that the spherically symmetric isolated horizon is described by a symplectic form Ω_M that, when written in the (connection) variables suitable for quantization, acquires a horizon contribution corresponding to an $SU(2)$ CS theory. Our derivation of the (conserved) symplectic structure is straightforward. We first observe that $SU(2)$ and diffeomorphism gauge invariance is not broken by the IH boundary condition: they continue to be degenerate directions of Ω_M on shell. This by itself is then sufficient for deriving the boundary term that arises when writing the symplectic structure in terms of Ashtekar-Barbero connection variables (see also [6]).

Note that no d.o.f. is available at the horizon in the classical theory as the IH boundary condition completely fixes the geometry at Δ [the IH condition allows a single (characteristic) initial data once a_H is fixed (see Fig. 1)]. Nevertheless, nontrivial d.o.f. arise as would be gauge d.o.f. upon quantization. These are described by $SU(2)$ CS theory with (an arbitrary number of) defects which couple to gauge d.o.f. through the dimensionless parameter $16\pi^2\beta(1-\beta^2)\ell_p^2\sqrt{j(j+1)}/a_H$, i.e., the ratio of a basic quantum of area carried by the defect to the total area of the horizon. These would-be gauge excitations are entirely responsible for the entropy.

We obtain a remarkably simple formula for the horizon entropy: the number of states of the horizon is simply given in terms of the (well-studied) dimension of the Hilbert spaces of CS theory with punctures labeled by spins, which—due to the area constraint (14), and for the range $|\beta| \le \sqrt{3}$ including the physical value of β [11]—is just the dimension of the singlet component in the tensor product of the representations carried by punctures. The black hole density matrix $\rho_{\rm IH}$ is the identity on ${\rm Inv}(\otimes_p j_p)$ for admissible j_p. Similar counting formulas have been proposed in the literature [12] by means of heuristic argu-

ments. Our derivation from first principles, in particular, clarifies previous proposals.

General arguments and simple estimates indicate that the entropy will turn out to be $S_{\rm BH} = \beta_0 a_H/(4\beta\ell_p^2)$, where β_0 is a constant to be determined. The new counting techniques of [13] are expected to be very useful for this. Thus the result to leading order remains unchanged. However, subleading corrections will have the form $\Delta S = -\frac{3}{2}\log a_H$ [instead of $\Delta S = -\frac{1}{2}\log a_H$ in the $U(1)$ treatment] matching other approaches [14]. This is due to the full $SU(2)$ nature of the IH quantum constraints imposed here, and this is a clear-cut indication that the $U(1)$ treatment overcounts states. The value β_0 and the log correction has been recently computed for $|\beta| < \sqrt{3}$ [11]. The range $|\beta| \geq \sqrt{3}$ is unphysical as the quantum group structure imposes additional constraints driving the entropy below the physical value $a_H/4$.

In Ref. [4] the classical description of the IH was first done in terms of the null tetrad formalism. In this case the null surface defining the Horizon provides the natural structure for a partial gauge fixing from the internal gauge $\mathrm{SL}(2,\mathbb{C})$ to $U(1)$. In this setting one fixes an internal direction $r^i \in \mathfrak{su}(2)$ and the IH boundary condition (6) becomes

$$dV + \frac{2\pi}{a_H}\Sigma^i r_i = 0, \qquad \Sigma^i x_i = 0, \qquad \Sigma^i y_i = 0, \quad (16)$$

where x^i, $y^i \in \mathfrak{su}(2)$ are arbitrary vectors completing an internal triad. In the quantum theory [3] only the first of the previous constraints is imposed strongly, while—due to the noncommutativity of Σ^i in LQG—the other two can only be imposed weakly, namely, in [3] one has $\langle\Sigma^i x_i\rangle = \langle\Sigma^i x_i\rangle = 0$. However, this leads to a larger set of admissible states (over counting). To solve this problem, within the $U(1)$ model, one would have to solve the two constraints $\Sigma^i x_i = 0 = \Sigma^i y_i$ at the classical level first, implementing the reduction also on the pullback of two forms Σ^i on H. However, this would introduce formidable complications for the quantization of the bulk degrees of freedom in terms of LQG techniques. Our $SU(2)$ treatment resolves this problem as now the three components of (6) are first class constraints. Dirac implementation leads to a smaller subset of admissible surface states that are relevant in the entropy calculation.

We thank A. Ashtekar, M. Knecht, M. Montesinos, D. Pranzetti, M. Reisenberger, and C. Rovelli for discussions, and an anonymous referee for exchanges that have considerably improved the presentation of our results. This work was supported in part by the Agence Nationale de la Recherche; Grant No. ANR-06-BLAN-0050. J. E. was supported by NSF Grant No. OISE-0601844, and thanks Thomas Thiemann and the Albert-Einstein-Institut for hospitality. A. P. is a member l'Institut Universitaire de France. Support was received from Unité Mixte de Recherche (UMR 6207) du CNRS et des Universités Aix-Marseille I, Aix-Marseille II, et du Sud Toulon-Var; laboratoire afilié à la FRUMAM (FR 2291) and Fédération Denis Poisson Orléans-Tours, CNRS/UMR 6083.

[1] S. W. Hawking, Commun. Math. Phys. **43**, 199 (1975).
[2] K. Krasnov, Gen. Relativ. Gravit. **30**, 53 (1998); L. Smolin, J. Math. Phys. (N.Y.) **36**, 6417 (1995).
[3] A. Ashtekar, J. Baez, A. Corichi, and K. Krasnov, Phys. Rev. Lett. **80**, 904 (1998); A. Ashtekar, J. C. Baez, and K. Krasnov, Adv. Theor. Math. Phys. **4**, 1 (2000).
[4] A. Ashtekar, A. Corichi, and K. Krasnov, Adv. Theor. Math. Phys. **3**, 419 (2000).
[5] J. Lewandowski, Classical Quantum Gravity **17**, L53 (2000).
[6] J. Engle, K. Noui, A. Perez, and D. Pranzetti, arXiv:1006.0634.
[7] C. Crnkovic and E. Witten, in *Three Hundred Years of Gravitation*, edited by S. Hawking and W. Israel (Cambridge University Press, Cambridge, 1987); J. Lee and R. M. Wald, J. Math. Phys. (N.Y.) **31**, 725 (1990); A. Ashtekar, L. Bombelli, and O. Reula, in *200 Years After Lagrange*, edited by M. Francaviglia and D. Holm (North-Holland, Amsterdam, 1991).
[8] This follows first from $\Omega_M(\delta_\alpha, \delta) = 0$ implying

$$\begin{aligned} -\kappa\Omega_H &= \int_M \delta_\alpha\Sigma_i \wedge \delta A^i - \delta\Sigma_i \wedge \delta_\alpha A^i \\ &= \int_M [\alpha, \Sigma]_i \wedge \delta A^i + \delta\Sigma_i \wedge d_A\alpha^i \\ &= \int_M d(\alpha_i\delta\Sigma^i) - \alpha_i\delta(d_A\Sigma^i) \\ &= -\frac{a_H}{\pi(1-\beta^2)}\int_H \alpha_i\delta F^i(A) \\ &= \frac{a_H}{\pi(1-\beta^2)}\int_H \delta_\alpha A_i \wedge \delta A^i. \end{aligned}$$

where we used the Gauss law $\delta(d_A\Sigma) = 0$, condition (6), and that boundary terms at infinity vanish. A similar calculation for diffeos completes the proof (see [6] for all details).
[9] E. Witten, Commun. Math. Phys. **121**, 351 (1989).
[10] R. K. Kaul and P. Majumdar, Phys. Lett. B **439**, 267 (1998); Phys. Rev. Lett. **84**, 5255 (2000).
[11] I. Agulló, J. Fernando Barbero G., E. F. Borja, J. Díaz-Polo, and E. J. S. Villaseñor, Phys. Rev. D **80**, 084006 (2009).
[12] C. Rovelli, Phys. Rev. Lett. **77**, 3288 (1996); E. Livine and D. Terno, Nucl. Phys. **B741**, 131 (2006).
[13] I. Agulló, J. Fernando Barbero G., J. Díaz-Polo, E. Fernández-Borja, and E. J. S. Villaseñor, Phys. Rev. Lett. **100**, 211301 (2008); J. Fernando Barbero G. and E. J. S. Villaseñor, Phys. Rev. D **77**, 121502 (2008); Classical Quantum Gravity **26**, 035017 (2009).
[14] A. Ghosh and P. Mitra, Phys. Rev. D **71**, 027502 (2005); G. Gour, Phys. Rev. D **66**, 104022 (2002); S. Carlip, Classical Quantum Gravity **17**, 4175 (2000).