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Proceeding: Perspectives in Mathematical Physics, International Conference in honor of Alex Grossmann, CIRM, Luminy, Marseille, France, 28th July - August 1st, 1997

Ginette Saracco, Matthias Holschneider

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PERSPECTIVES IN MATHEMATICAL PHYSICS: CONFERENCE IN THE HONOR OF ALEX GROSSMANN

MARSEILLE-LUMINY, JULY 28th - August 1st 1997 France

M. HOLSCHNEIDER & G. SARACCO Editors

To Alex... ***G. Saracco***

CPT-98/P.3748

Conference Supported by: Centre National de la Recherche Scientifique, CPT, Association du Grand Luminy, Conseil Général, NSF via Center for Theoretical Studies of Physical Systems, Clark Atlanta University, USA

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Perspective in Mathematical Physics: Conference in the honor of Alexander Grossmann

Proceeding of the International Conference

Marseille-Luminy, France - July 28th - August 1st 1997

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Section 3: Biology-Genetics.

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NSF via Center for Theoretical Studies of Physical Systems, Clark Atlanta University, USA

AVANT PROPOS

FOREWORD: report of the international conference:``Perspectives in Mathematical Physics": Conference in the honor of Alexander Grossmann*, Marseille, CFML-Luminy, July 28th to August 1st 1997. Organized by M. Holschneider (UPR-7061, Centre de Physique Théorique - Marseille) and G. Saracco (UPR-4661, Laboratoire de Géophysique - Rennes). Scientific committee: Section 1: JP Antoine, Y. Avron, R. Seiler. Section 2: I. Daubechies, Y. Meyer. Section 3: JL Risler.

La conférence internationale ``Perspectives in Mathematical Physics", en l'honneur d'Alexander Grossmann*, a eu lieu à Marseille - Luminy du 28 juillet 1997 au 1er août 1997.
Elle a réuni pendant une semaine, dans une atmosphère amicale et détendue, une soixantaine d'amis chercheurs qui ont tous connus Alex pendant des années de travail commun. La diversité des intérêts scientifiques d'Alex Grossmann se retrouvait naturellement dans le large spectre des origines scientifiques des personnes présentes, ainsi que dans leurs exposés. Ceux-ci se regroupaient en trois sections.

La première était la Physique Mathématique, champ d'activité le plus ancien d'Alex, et regroupait un fort potentiel d'amis et collaborateurs de longue date, ainsi qu'une nouvelle génération de scientifiques. Intervinrent, dans l'ordre: D. Kastler, J. Madore, H. Borchers, J. Marion, R. Stora, P. Hislop, S. Graffi, J. Zak, Y. Avron, Th. Paul, A. Tip, A. Martin et H. Bacry. Albeverio S. et dell Antonio G.F. n'ont pu être présents au dernier moment pour cause de santé. Les thèmes couvraient une grande part de la physique mathématique actuelle: allant de la géométrie non-commutative à la théorie des champs, de la théorie des milieux désordonnés à l'effet Hall quantique,...

La deuxième section fut consacrée à l'Analyse en Ondelettes: domaine où les contributions d'Alex Grossmann et de son école sont internationalement reconnues. Les intervenants ont montré l'expansion des activités dans ce domaine, touchant la turbulence, l'acoustique, la théorie de l'information, le traitement d'images, la résolution d'équations aux dérivées partielles, la géophysique,... Ainsi nous eurent respectivement des exposés de: Ph. Tchamitchian, G. Beylkin, R. Seiler, J.C. Risset, M. Farge, C. R. Handy, J-P. Antoine, I. Daubechies, P. Flandrin, M. V. Wickerhauser, F. Geshwind, R. Murenzi et P. Frick. Les thèmes très variés reflétaient ainsi la diversité des domaines d'applications de la transformation en ondelettes, ainsi que ses différents développements (bases d'ondelettes continues, orthogonales, bi-othogonales, frames, etc...). A noter en particulier, sur un sujet peu commun pour son auditoire, l'exposé de Jean-Claude Risset qui montra comment les travaux d'Alex Grossmann ont servi dans la recherche et composition musicale, ainsi que dans le domaine de la psychoacoustique.

La troisième et dernière section: la Biologie et, plus précisément la Génétique, actuel domaine d'intérêt d'Alex Grossmann et de ses collaborateurs, donna lieu à deux exposés fondamentaux. Jean-Loup Risler montra comment les travaux d'Alex ont permis, dans ce domaine, de mieux estimer les générateurs infinitésimaux de l'évolution protéique. Alain Arnéodo montra comment l'analyse multifractale basée sur la transformation en ondelettes continues permet de mettre en évidence des corrélations à longues portées dans des parties codantes d'un gène.

Mais ce qui caractérisa le plus cette conférence, lui rendant un charme particulier tout au long de cette semaine, fut son atmosphère non solennelle, chaleureuse et naturelle: Conférence à l'image même d'Alex, où discussions scientifiques, culturelles et rires florissaient, catalysés par sa propre personnalité généreuse et chaleureuse. Il y eut dès le premier soir, un vernissage des peintures de J. Mandelbrojt que l'on pouvait admirer durant la semaine. Le mardi, J. C. Risset nous donna en soirée, une démonstration musicale alliant piano acoustique, ordinateur, et synthèses numériques. Quelques oeuvres étaient justement obtenues grâce à la transformation en ondelettes dans sa forme développée par Alex Grossmann. Un banquet nous réunit le mercredi soir, où C. Korthals-Altes grâce à la complicité de Carlo Becchi, intervena délicieusement en relatant avec finesse et humour les parallèles de la vie d'Alex avec celle de la "Divine comédie" de A. Dante, sous les gravures visionnaires de G. Dorée.

Le point fort de cette conférence fut le non-traditionnel hommage dédié à la non moins traditionnelle carrière d'Alex, illustrée par I. Daubechies et si délicieusement "anecdotée" par Th. Paul. Il nous restera encore longtemps en mémoire leurs "surprise talk" sur ce que c'est que de travailler avec Alex.

Ce congrès a pu être organisé grâce aux aides financières du CNRS, du Conseil Général (PACA), de l'Association Grand Luminy, et du Center for Theoretical Studies of Physical Systems (Clark Atlanta University, Georgia, USA). Nous les en remercions vivement, ainsi que le personnel du Centre de PhysiqueThéorique (N. Catrain , N. Lambert, D. Roche, M. Rossignol, et A. Sueur) et celui du CFML. Un grand Merci à J.B. Erismann sans qui ce livre n'aurait pas vu le jour, ainsi qu'a P. Chiappetta (CPT) et merci encore au Laboratoire de Géophysique de Rennes I, notamment D. Gibert et A. Le Solliec.

G.S. & M. H.

* *Rappelons qu'Alexandre GROSSMANN est le premier Lauréat du Prix spécial créé en 1997 par la Société Française de Physique et a obtenu cette même année par les instances du CNRS les Eméritats.*

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* Photography p 6 : A. Grossmann in 1955, Havard University, USA

* Photagraphy p 174: J. Morlet, A. Grossmann, I. Daubechies, 28 july 1997 Marseille.

PERSPECTIVES IN MATHEMATICAL PHYSICS: CONFERENCE IN THE HONOR OF ALEX GROSSMANN

MARSEILLE-LUMINY, JULY 28th - August 1st 1997

France

PROGRAM

Organizing comittee: **Holschneider M. & Saracco G.**

Scientific comittee:

S1: **Antoine J.P., Avron Y., Seiler R.**
S2: **Daubechies I., Meyer Y.**
S3: **Risler J.L.**

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S1: **Mathematical Physics section**
S2: **Wavelet Analysis section**
S3: **Biology-Genetics section**

****

Monday, July 28th 1997

9h30 - 10h00 Welcome to CFML

Section 1

10h00 - 10h45 **Kastler D.**

"Quantification des mathématiques et interactions fondamentales: les efforts Marseillais"

10h45 - 11h30 **Madore J.**

"Non commutative geometry and the structure of space-time"

coffe break

11h45 - 12h30 **Borchers H.:**

"The modular theory as a tool in quantum field theory"

lunch

15h00 - 15h45 **dell Antonio G.F.**

"Point interaction"

15h45 - 16h40 **Stora R.**

"La symétrie de Slavnov des théories de jauge du point de vue de la cohomologie équivariante"

16h40 - 17h15 **Marion J.**

"Generalized Lie group structure associated in non-commutative geometry"

coffe break

17h30 - 18h15 **Hislop P.**

"Recent progress in random system"

18h15 - 19h00 **Graffi S.**

"Quantization of hyperbolic symplectomorphisms of the torus and deterministic spin models with glassy behavior"

****

Opening of the Exposition of J. Mandelbrojt's Paintings

****

Tuesday, July 29th 1997

Section 2

9h30 - 10h30 **Tchamitchian Ph.**

"The square root problem"

10h45 - 11h45 **Beylkin G.**

"A new class of stable discretisation schemes for the solution of non-linear PDES"

coffe break

11h45 - 12h30 **Seiler R.**

"Factorisation of loops, a full parametrisation of biorthogonal wavelets"

lunch

Section 2/1

14h45 - 15h45 **Risset J.C.**

"Music: A time-scale game?"

15h45 - 16h30 **Farge M.**

Are statistical and deterministic approaches compatible for fully-developed turbulence?"

coffe break

17h00 - 17h45 **Handy C.**

"Moment quantization methods to quantum mechanics"

17h45 - 18h30 **Antoine J.P.**

"Uncovering hidden symmetries with directional wavelets"

Dinner

****

21h15 Musical demonstration: Piano-Computer-Disklavier: J.C. Risset

****

Wednesday, July 30th 1997

Section 3

10h00 - 11h00 **Risler J.L.**

"On Darwinian evolution of proteins"

coffe break

11h30 - 12h30 **Arnéodo A.**

"What can we learn about DNA sequences from wavelet analysis"

lunch

****

14h30 "Surprise talk" in the Honor of Alex Grossmann :
I. Daubechies & Th. Paul

****

free Afternoon

****

Conference Dinner "L'abri Côtier"

****

Thursday, June 31st 1997

Section 1

9h30 - 10h30 **Zak J.**

"Balian-Low theorem for Landau Levels"

coffe break

11h00 - 11h45 **Avron Y.**

"The integral quantum Hall effect"

11h45 - 12h30 **Paul Th.**

"Semi-classical methods and coherent states"

lunch

15h00 - 15h45 **Albeverio S.**

"Some recent developements in infinite dimensional analysis and quantum fields"

15h45 - 16h30 **Tip A.**

"Canonical formalism and quantization for a class of classical field equation"

coffe break

17h00 - 17h45 **Martin A.**

"Regge poles 38 years later"

17h45 - 18h30 **Bacry H.**

"Arguments de symétrie en faveur d'une réinterprétation d'une gravure de Dürer"

****

Rock & Conference

****

Friday, August 1st 1997

Section 2

9h00 - 10h00 **Daubechies I.**

"Subdivision algorithms and smoothness"

coffe break

10h30 - 11h30 **Flandrin P.**

"Time-frequency localisation, symetries and generalized means"

11h30 - 12h15 **Wickerhauser V.**

"Designing a custom wavelet packet image compression scheme, with applications to fingerprints and seismic data"

lunch

14h00 - 14h50 **Geshwind F. & Coifman R.**

"Multiscale processing, non linear wavelet transform, noiselets and fast rotations"

coffe break

15h20 - 16h00 **Murenzi R.**

"Spatio-temporal wavelets: Application to tracking of moving targets in noisy"

16h00 - 16h40 **Frick P.**

"Wavelet analysis of observational data with gaps"

****

16h45: Drink - End of the Conference

****

Abstracts of the conference*

Perspectives in Mathematical Physics
Conference in the honor of Alex Grossmann
27. July - 1. August, 1997
Marseille, Luminy

* Supported by: Centre National de la Recherche Scientifique, Association du Grand Luminy, Conseil Général, NSF via Centre for Theoretical Studies of Physical Systems

Quantification des mathématiques et intéractions fondamentales: les efforts Marseillais

Daniel Kastler
Centre de Physique Théorique
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Abstract: Since 82 Alain Connes carries out a global recasting of Mathematics (by now largely accomplished!) amounting to a "quantization" analogous to the quantization of Physics half a century ago. Conceptually, this is a replacement of the "mathematics of spacees" by the "mathematics of non-commutative algebras", technically a combination of operator algebras and differential geometry-topology thereby becoming universal: relevant to manifolds (usual and conformal), foliations, groups (discrete and continuous), fractals, prime numbers. The emergence of this new paradigm closer in spirit to physics has begun since 81 to underspan the physics of fundamental interactions (lagrangian aspects of elementary particles and gravitation). We attempt to sketch this development with its Marseille episodes, showing how the successive discovery of two Pandora boxes (the "quantum Yang-Mills" and the "heat-equation expansion" - amazingly yielding the phenomenological lagrangians - the latter in combination with gravitation) in fact monitored thie construction of "non-commutative manifolds" for which the microworld has stood model! - the deep reason for that being that the standard model worked out by the high-energy phenologists is the only genuinly "quantal" man-made object. The historical corpus of mathematics is too "classical".

Non commutative geometry and the structure of space - time

John Madore
Université de Paris Sud
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Abstract: It is argued that space-time should be more properly described by a non-commutative $*$-algebra $\mathcal{A}_{\overline{k}}$ over the complex numbers with four hermitian generators q^λ. The parameter $\overline{k}$ is a fundamental area scale which is presumably of the order of the Planck area $G\hbar$. A review will be made of recent efforts to add a gravitational field to this noncommutative model of space-time. If there is a gravitational field then there must be some source, of characteristic mass μ. If $\mu^2\overline{k}$ tends to zero with $\overline{k}$ then the limiting 'classical' space-time will be without curvature. We are interested here in the case in which $\mu^2\overline{k}$ tends to some finite non-vanishing value as $\overline{k} \to 0$. It is claimed that there is then a rigid relation between the noncommutative structure of the space-time on the one hand and the nature of the gravitational field which remains as a 'shadow'

in the commutative limit on the other. This claim is entirely based on analogy with the results of simple models.

The modular theory as a tool in quantum field theory

Hans Borchers
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Abstract: These investigations will use the quantum field theory in the frame of Araki, Haag and Kastler (see the textbook of R. Haag for details). Due to the Reeh–Schlieder property of quantum field theory every algebra $\mathcal{M}(D)$ associated with a domain D, which has a spacelike complement with interior points, has cyclic and separating vectors. The vacuum vector is an example. Therefore, we obtain by the Tomita–Takesaki theory for every of these algebras a symmetry group

$$\sigma_D^t(\mathcal{M}(D)) = \Delta_D^{it}\mathcal{M}(D)\Delta_D^{-it} = \mathcal{M}(D).$$

Since all these local symmetry groups are present one should use them for the structure analysis of quantum field theory. In some cases the modular groups have been computed. Bisognano and Wichmann have calculated the action of the group for the wedge domain. This computation was done under the assumption that the algebras are generated by finitely many Poincaré covariant Wightman fields. In this situation the modular group coincides with the Lorentz boosts which map the wedge onto itself. For a massless Bose particle, not interacting with other massive particles, the vacuum vector is also cyclic and separating for the algebra $\mathcal{M}(V^+)$ associated with the forward lightcone V^+. For this situation Buchholz has shown that the modular group coincides with the dilatations of this theory. In conformal field theory one can use the result of Bisognano and Wichmann or that of Buchholz to compute the modular group for the algebra $\mathcal{M}(D)$ of a double cone D. This has been done by Hislop and Longo.

In all the examples one dealt with special field theories. For the general field theory our knowledge about the informations one can extract from the modular groups is limited. The situation will be better if also half-sided translations exist besides the modular group. These are unitary groups with positive generators which map the algebra into itself but only for positive arguments:

$$U(s)\mathcal{M}(D)U^*(s) \subset \mathcal{M}(D), \qquad \text{for} \quad s \geq 0.$$

These groups must also leave the vacuum vector unchanged. Also the relations of the half-sided translations with the modular group will be discussed. Half-sided translations appear for instance with the algebras associated with wedges. In addition we will investigate the two-parametric group defined by half-sided translations and the

modular group. This group is not unimodular. It will also be shown that half–sided translations can only occur if the algebra $\mathcal{M}(D)$ is of type III_1. Closely connected with the half–sided translations is the principle of half–sided modular inclusions which has been introduced by H.-W. Wiesbrock. This principle is interesting because of the natural order which comes with it. We will discuss the meaning of this order in terms of half–sided translations. At the end examples will be presented where half–sided translations have been used. Finally, indications of problems will be given which seem to be solvable with help of the half–sided translations.

Point interactions

Gian Fausto dell'Antonio
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Abstract: I will present recent results in collaboration with R. Figari and A. Teta, on the problem of a quantum particle interacting with N point sources moving along preassigned trajectories (nd whose strength may depend on time). We prove that a solution exists and is unique in the sense of quadratic forms and the corresponding flow is unitary; we also give rather explicit solutions for smooth initial data. The solution of this problem may be regarded as a fist step in the analysis of non-linear coupled equations, in which the motion of the sources depends on the local behaviour of the wave function, or in problems of homogeneization. From the mathematical standpoint tha novelty of the problem consists in the fact that the form domain of the Schroedinger operator varies in time.

La symétrie de Slavnov des théories de jauge du point de vue de la cohomologie équivariante

Raymond Stora
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Abstract: The Faddeev Popov gauge fixing procedure and the ensuing Slavnov symmetry will be quickly reviewed. The geometrical set up will be reanalyzed and the cohomological character of the gauge fixing procedure will be established. The Slavnov symmetry will then emrge as a quotient of the equivariant cohomology construction. This geometry will be presented in a finite dimensional set up. A few remarks will be made about the infinite dimensional case.

Generalized Lie group structure associated with regular spectral triples in non-commutative geometry.

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Abstract: Let $(\mathcal{A}, (\mathcal{H}, \pi), \mathcal{D})$ be a spectral triple over a unital *-algebra such that $\pi(\mathcal{A})$ is contained in the domain of all positive powers of the derivation δ on $\mathcal{L}(\mathcal{H})$. For a large class of such objects $(\mathcal{A}, (\mathcal{H}, \pi), \mathcal{D})$, called regular spectral triples, we prove that A has a natural structure of unital multiplicatively convex Fréchet *-algebra, and that the group $\mathrm{Inv}(\mathcal{A})$ of its invertible elements in an open subset of $\mathcal{A}$ having a natural structure of generalized Lie group in the H. Omori's sense, with Lie algebra $\mathcal{A}$ and usual exponential mapping. This the case for instance of the algebra $C^\infty(M ; A)$ where M is a spin^c manifold and A a finite-dimensional multimatrix C-algebra, and of the algebra of the non-commtative 2k-dimensional tori. An important consequence is that $\mathcal{A}$, as well as some associated unital *-algebras such that the matrix algebras $M_n(\mathcal{A})$ and the path algebra $C(I ; \mathcal{A})$ have a holomorphic functional calculus

Recent progress in random system

Peter Hislop
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Abstract: This talk will present an overview of recent results of the speaker, in collaboration with J.-M. Barbaroux, J. M. Combes, E. Mourre, and A. Tip, on localization for electrons and for classical acoustic and electromagnetic waves propagating in disordered media. These results include localization near band edges of the spectrum of the unperturbed system, and the regularity of the integrated density of states at those energies. Some of the open problems in the field will be discussed.

Quantization of chaotic toral symplectomorphisms and deterministic spin models with glassy behaviour

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Abstract: There is a formal algebraic identity between two apparently unrelated systems, the quantized symplectomorphisms of the torus on one side and the deterministic spin models with glassy behaviour recently introduced by Parisi and coworkers on the other side. The coupling matrix $J_{l,k}^{(N)}$ defining the energy $H = -\sum_{l,k}^{N} \sigma_l \sigma_k$ of the N spins $\sigma_l : l = 1, \ldots, N$ does actually coincide with the (real part) of the unitary propagator quantizing the unit symplectic matrix on $\mathbf{T}^2$: this is a $N \times N$ matrix with $N = \frac{1}{\hbar}$, so that the thermodynamic and classical limits formally coincide. Some statistical mechanics results are reviewed here which are based on the arithmetic techniques employed to prove the convergence of the scalar products of the quantum observables among eigenfunctions to the corresponding classical ergodic means. One of them concerns the asymptotic degeneracy of the ground state of the Marinari-Parisi-Ritort model; another one deals with the introduction of a similar class of models based upon the quantization of a chaotic map instead of a periodic one. Unlike the first one, in this second class of models the critical temperature for the glassy transition can be computed by linearizing the TAP (mean field) equations, which are in turn obtained by explicit summation of the high-temperature expansion.

The square root problem

Philippe Tchamitchian
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Abstract: The square root problem was raised by T. Kato in the 60's, motivated by perturbation theory for some hyperbolic partial differential equations. The question is whether the domain of the square root of a divergence form operator is the natural Sobolev space or not. While it is trivial in the self-adjoint case, it turned out to be a profound problem in the general case, connected to abstract operator theory, modern harmonic analysis and the study of elliptic partial differential equations. The conference will describe a new approach to this problem (still unsolved in general), due to P. Auscher and the speaker, which takes advantage of recent developments in functional calculus and in harmonic analysis. This approach allows to unify previous results and to obtain new ones. In particular, it sheds light on relations between the study of the square root of an operator L and the properties of weak solutions to the inhomogeneous equation $Lu = 3Df$ with nice f.

A new class of stable discretization schemes for the solution of non-linear PDES

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Abstract: We describe a new class of time-discretization schemes for solving nonlinear evolution equations,

$$u_t = \mathcal{L}u + \mathcal{N}(u),$$

where $\mathcal{L}$ represents the linear and $\mathcal{N}(\cdot)$ the nonlinear terms, respectively. A distinctive feature of these new schemes is the exact evaluation of the contribution of the linear part (we call such schemes exact linear part (ELP) schemes). Namely, if the non-linear part is zero, then the scheme reduces to the evaluation of the exponential function of the operator (or matrix) $\mathcal{L}$ representing the linear part. We show that such schemes

have very good stability properties and, in fact, describe explicit schemes with stability regions similar to those of typical implicit schemes used in e.g. fluid dynamics applications.

Computing and applying exponential or other functions of operators in the usual manner typically requires evaluating dense matrices and is highly inefficient. An exception is the case where there is a fast transform that diagonalizes the operator. For example, if $\mathcal{L}$ is a convolution (or a circulant) matrix which is diagonalized by the Fourier Transform (FT) (or discrete FT), then computing functions of operators can be acomplished by a fast algorithm, e.g. the FFT. It turns out that the wavelet transform may be used in a similar manner for computing functions of operators from a wide class, in particular, elliptic operators with variable coefficients. In the wavelet system of coordinates computing exponentials of such operators always results in sparse matrices and, therefore, presents an efficient option.

We study the stability of ELP schemes and show that these schemes have distinctly different stability properties as compared with known mixed (implicit-explicit) schemes. For example, the stability properties of time-discretization schemes for advection-diffusion equations are controlled by the linear term and, therefore, require implicit treatment. Using an explicit ELP scheme, it is possible to achieve stability usually associated with implicit predictor-corrector schemes.

This is a joint work with J. Keiser and L. Vozovoi.

Factorisation of loops, a full parametrisation of biorthogonal wavelets

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Abstract: Parametrization of biorthogonal Wavelets is conveniently done in terms of Birkhoff factorization of loops in the group of 2x2 matrices with determinant of modulus one.

Connection to the lifting concept of Daubechies and Sweldons and to the theory of surfaces is discussed.

Musik: a time-scale game?

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Abstract: This presentation will be illustrated with sound examples.

Early lithophones may be the first instance of a log frequency scale. Music is "ars bene movandi", and it involves time. Western musical scores and musical automata seem to have examplified cartesian coordinates long before Oresmus and Descartes.

The Fourier representation of signals attempts to represent finite duration signals in terms of infinite duration. It does not allow a proper display of changes in the course of sound. The aural importance of changes can be demonstrated by synthesis. Ad hoc modifications of the Fourier representation have been proposed for some time. Clearly, the granular approach adopted by Gabor, with the Gabor representation, and by Morlet and Grossmann, with the wavelet transform, are in principle much more satisfactory. In fact, they are very effective for sound. One will hear ome of the earliest sound examples of wavelet reconstructions and intimate sound modifications using transformations of the time-scale image, produced by Richard Kronland-Martinet in the Equipe Informatique Musicale (IM) of the Laboratoire de Mécanique et d'Acoustique: Richard collaborated with Alex Grossmann in early explorations of the continuous wavelet transform. In fact, many early examples of wavelet trasforms were generated then, using a special digital audioprocessor, Syter.

The validity of representing sounds by grains, suggested by Gabor, was in fact first demonstrated by Bacry, Grossmann, and Zak in the context of quantum mechanics, five years before the work by Bastiaans in the field of signal processing. The first Gabor transforms were also realized at IM. Some transformations are easier to perform with a Gabor representation, as will be shown by examples of sound stretching realized by Daniel Arfib.

The wavelet transform provide useful representations. One can go beyond by stipulating specific types of processing. For instance, one can attempt to extract a model of a given sound. This is helped by the possibility of extracting ridges in a time-frequency or time-scale representation, investigated by the late Bernard Escudié, Grossmann and others: this permits to extract modulation parameters. Some examples of model extraction and sound reconstruction performed by Philippe Guillemain will be heard.

A musical demonstration will be given in the evening. One will hear music demonstrating movements of virtual sound sources in illusory space - the space here is not a priori,

it is generated by the music, or rather by the interaction of the signal with the auditory perceptual organization (perhaps similarly to models originating in the algebra of non-commutative operators). A short metaphoric event realized for the inauguration of the LEP in Genève will be presented - it includes the voice of Alex Grossmann. The following demonstration will then feature a piano duet with a single pianist, accompanied by a virtual-partner adding (on the same acoustic piano) a musical part that depends upon what the pianist plays and how he plays - with several types of relation between the pianist and his illusory partner. The presentation will be concluded by a tape where a soprano speaks and sing, in dialogue with sounds from an "invisible" partner, sounds produced by synthesis or processig using Gabor and wavelet transforms.

Are statistical and deterministic approaches compatible for fully-developed turbulence?

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Abstract: Fluid mechanics is governed by the Navier-Stokes equations, which are deterministic, but whose solutions are chaotic, namely sensitive to initial conditions. *Fully-developed turbulence* (corresponding to high Reynolds numbers) is the limit where the nonlinear advective term strongly dominates the linear dissipative term. In this limit we are unable to integrate the Navier-Stokes equations, even with the largest computers presently available or proposed for the future. Therefore in order to compute fully-developed turbulent flows we use a *combination of numerical integration and statistical modelling.*

The present state of the art, so called Large Eddy Simulation (LES), consists of integrating the dynamics of the large-scale motions (corresponding to the scales resolved by the computational grid) and of modelling statistically the behaviour of the small-scale motions (corresponding to the subgrid-scales). But this programme is not adequate for fully-developed turbulence because it requires a *spectral gap*, namely a decoupling between large-scale and small-scale dynamics. Such a decoupling may exist between the inertial range scales and the dissipative scales, which therefore allows the numerical simulation of weakly nonlinear flows (corresponding to low Reynolds numbers). However, if we want to compute fully-developed turbulent flows, we should find a another way to separate the modes to be deterministically computed and those which can be modelled statistically.

For the last ten years we have proposed wavelet representation to analyze, model and compute two-dimensional turbulence, which is relevant for geophysical flows due to the

combined effect of stable stratification and rotation of the reference frame. In 1988 we found that the strong wavelet coefficients correspond to the coherent structures, while the weak wavelet coefficients correspond to the background flow. We have also shown that both components are multiscale and therefore cannot be separated by Fourier filtering. We then proposed a new method to compute fully-developed two-dimensional turbulent flows based on wavelet phase-space segmentation. In this method we *compute the dynamics of the coherent structures* using a limited number of wavelet modes, keeping only the most excited ones and remapping the wavelet basis at each time step. We then *statistically model the background flow*, corresponding to the discarded wavelet modes, which are replaced by a stochastic forcing having the same statistical behaviour. The justification for this procedure is that the coherent structures are not numerous enough to reach a statistical equilibrium state, and therefore we have to compute their dynamics with a deterministic method. On the contrary, for the background flow, which is well mixed, we can assume ergodicity and thus define a statistical equilibrium which allows us to design an appropriate statistical model.

This work has been developed during the last ten years in collaboration with Eric Goirand, Matthias Holschneider, Nicholas Kevlahan, Kai Schneider, Thierry Philipovitch and Victor Wickerhauser. It is a direct consequence of the inspiring discusions I had since 1984 with Alex Grossmann, to whom I want to express here all my gratitude.

On the equivalence of moment quantization and continuous wavelet transform analysis for Schrödinger Hamiltonians

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Abstract: We have developed a new quantization formalism based on the use of appropriately rescaled and translated power moments. This approach defines a multiscale quantization procedure in which infinite scale information (i.e. the energy and certain moments) is used to systematically generate smaller scale quantities, including the recovery of the wavefunction. The latter results from solving a finite set of coupled, linear, first order differential equations in the inverse scale variable. This formalism enables the exact transformation and inversion of Schrödinger Hamiltonian problems into an equivalent continuous wavelet transform (CWT) representation, without the need for any Galerkin type of approximation. Application of the method to rational polynomial potential problems leads to excellent results. Our formalism is a direct consequence of CWT theory; although the particular representation adopted is unprecedented. The rationale for the simplicity of our formalism lies in the fact that the

space of polynomials (of degree no greater than N) maps into itself under affine transformations. As such, given the importance of affine transformations in CWT analysis, it is to be expected that the effective use of a moments representation should lead to a significant simplification in analyzing Sturm-Liouville quantum problems with rational polynomial potentials. This is confirmed by our theory.

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Uncovering hidden symmetries with directional wavelets

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Abstract: It is well-known that the two-dimensional continuous wavelet transform (CWT) is a powerful tool for detecting various features in a picture or a pattern, as opposed to the discrete WT, which is the prime choice for data compression. If the relevant features have a preferred direction, the tool analysis necessary for detecting them must fulfill two conditions: (i) One must use the full 2-D CWT, including the rotation parameter, in addition to the usual translations and dilations; (ii) One must choose a wavelet with some directional selectivity, such as a Morlet or a Cauchy wavelet. This is standard, for instance, in directional filtering or edge detection, two classical problems in image processing. In this talk, we will describe a novel application of such directional wavelets, namely the determination of the symmetries of a pattern, even approximate or local ones.

After a brief reminder of the 2-D CWT, we will first describe the class of Cauchy wavelets, which are particularly well-suited for the problem at hand, as shown by various calibration tests. Then we turn to the symmetry problem. Given a 2-D signal (an object, a pattern,...), let $S(a,\theta,\vec{b})$ be its wavelet transform with respect to a Cauchy wavelet. Our main tool for analyzing the symmetries of the signal is its *scale-angle measure*, defined as the positive-valued, bounded function:

$$\mu_S(a,\theta) = \int d^2\vec{b}\, |S(a,\theta,\vec{b})|^2,$$

which may also be viewed as a partial energy density in the scale and angle variables, that is, in spatial frequency space

We begin with a simplified version and eliminate the scale dependence by integrating over a, thus ending with a function α_S of the rotation angle only, called the *angular measure* of the object. In general, $\alpha_S(\theta)$ is 2π-periodic. But when the analyzed object

has rotational symmetry n, that is, it is invariant under a rotation of angle $\frac{2\pi}{n}$, then α_S is in fact $\frac{2\pi}{n}$-periodic. This is illustrated on simple geometrical figures, such as a square, a rectangle and a regular hexagon.

If the object has not only a rotational symmetry, but a combined rotation-dilation symmetry, one has to use the full scale-angle measure μ_S, which is then a doubly-periodic function of a and θ. This is exemplified on several types of pictures: a 'twisted snowflake', a Penrose tiling with local 10-fold symmetry, an octagonal dot pattern. The latter is particularly interesting. It has in fact two distinct combined rotation-dilation symmetries, one exact, the other one approximate. In each case, one gets a semigroup, with seemingly infinitely many orbits, and some defects in the second case. This example suggests a systematic wavelet analysis of 2-D lattices, which often show rich geometric and arithmetic properties. The technique could also be used for uncovering hidden symmetries of physical objects, such as quasicrystals or nanotubes, through their X-ray diffraction patterns. For instance, the helical structure of some nanotubes might be studied in this way.

On Darwinian evolution of proteins

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Abstract: As (almost) everybody knows, all the information that is required for building a living cell is entirely contained into the chromosome(s), and is "coded" in the form of some "text" -called sequence- with an alphabet of only four "letters".

Other indispensable macromolecules in the cell are the proteins, that are translated from the sequences of the chromosomic genes into sequences of a twenty letters alphabet.

It is also well known that the chromosomes are prone to suffer from modifications in their sequence that are called "mutations". In many cases, these mutations will result in modifications of some protein sequence.

The proteins are highly structured macromolecules whose proper functioning requires a precise three-dimensional atomic arrangement. It is now clear that the sequence of a protein is severely constrained by the necessity of building the proper structure. Hence a mutation in the coding chromosomic gene, resulting in a modification of the protein sequence, can have different consequences : - the mutation is "neutral", that is to say, the three-dimensional structure of the protein can accomodate it. Such a mutation can be transmitted to the future generations without any modification of the protein

specificity. - the mutation is totally incompatible with the 3-D structure of the protein : it will often be lethal and, consequently, will be eliminated. - the mutation results in some modification of the structure, that will modify the properties of the protein : this is how a new function or specificity can appear.

During this talk, I shall discuss the structural constraints that severely limit the mutability of protein sequences, and will give some examples showing how the evolution cleverly gets round them to create the necessary variability of living organisms.

What can we learn about DNA sequences from wavelet analysis

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Abstract: The fractal scaling properties of DNA sequences are analyzed using the wavelet transform. Because the wavelet transform microscope can be made blind to the "patchiness" of genomic sequences, we demonstrate and quantify the existence of long-range correlations in both the (protein) coding and the noncoding regions of the human genome. Moreover, the fluctuations in the patchy landscapes of DNA walks are found to be homogeneous with Gaussian statistics. When looking at the introns, one notices some significant tendancy to the long-range correlation exponent H to increase with the percentage of GC content of the sequence. In particular, a few introns with a low percentage of GC content do not display long-range correlations (H=1/2) and therefore cannot be distinguished from actual exons. We show that long-range correlations also exist in exons when undersampling these coding sequences by retaining the third base of each codon only. This observation seems to corroborate the attractive biological conjecture that the correlations in coding DNA sequences could be attained through the degeneracy of the genetic code (most of the synonyms are due to change in the third base in the codon). Finally, we comment about the possible understanding of the origin of the observed long-range correlations in terms of the nonequilibrium dynamical processes that produce the "isochore structure" of the human genome.

Balian - Low theorem for Landau levels

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Abstract: The Balian-Low theorem relates to a set of orthogonal states on a von Neumann Lattice. This is a rectangular lattice in the phase plane with a unit cell of area h. Such a lattice was first introduced by von Neumann in the xy-plane and independently by Gabor in the time-frequency plane. The Balian-Low theorem can be formulated the following way. Given a square integrable function $\psi(x)$, one builds out of it the by translations in the phase plane a von-Neumann set $\psi_{m,n}(x)$, where the indices m, n label the number of the unit cell on the von Neumann lattice in the pahse plane $[\psi_{00}(x) = \psi(x)]$. m and n assume all integer values $0, \pm 1, \ldots$. The theorem then claims that if $\psi_{mn}(x)$ are orthogonal to $\psi_{m'n'}(x)$ for all $m, n \neq m', n'$ it follows that at least one of the two quantities $\langle x^2 \rangle$ or $\langle p^2 \rangle$ in the state $\psi(x)$ diverges (The triangular bracket denote the expectation value). Although discovered by physicists, the Balian-Low theorem has become of wide-spread interest in the Engineering literature of signal processing.

The Balian-Low theorem is applied the motion of an electron in xy-plane with a magnetic field B perpendicular this plane. The energy spectrum for this ploblem are the Landau levels. It is shown that the eigenfunctions for the Laundau levels cannot be chosen sufficiently localized in order to make both uncertainties Δx and Δy finite. A simila result holds for the coordinates of the orbit center. With this restriction on the localization, complete orthonormal sets are defined on von Neumann lattices.

The integral quantum Hall effect

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Abstract: In an experimental work that won a Nobel prize, von Klitzing discovered that the Hall conductance of certain two dimensional electronic systems in strong magnetic

fields is quantized to be an integer with great accuracy. This discovery was partially explained by Laughlin who related the quantization to gauge invariance. It gave birth to an extensive body of theoretical and mathematical understanding to which Thouless and Bellissard, among others, made seminal contributions. I shall review a geometric a geometric point of view which focusses on *families* of Schrödinger operators. I shall explain two relations for such families:

$$Conductance = Curvature$$

and

$$Conductance = Relative\ Index.$$

In some special but still important cases the family that enters is associated with direct integrals–something I first learned about from Alex.

The lecture is based on works done with Ruedi Seiler over a period of more than ten years and to a lesser extent on works with Barry Simon and Peter Zograf.

Semiclassical methods and coherent states

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Abstract: We will review some recent results concerning the semiclassical approximation of the Schrödinger equation obtained by using coherent states. Included will be the following topics: propagation of coherent states, trace formula "à la Gutzwiller" and a local version of it, and construction of quasi-modes associated to periodic trajectories. We will emphasise the situations where the underlying classical dynamics is chaotic and consider the "scaring phenomenon", that is the possibilty for unstable periodic trajectories to support bound states.

Some recent developements in infinite dimensional analysis and quantum fields

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Abstract: Recent developments in infinite dimensional analysis concern integration with respect to probability measures as well as infinite dimensional oscillatory integrals.

Applications are given to the construction of stochastic dynamics of various systems of classical and quantum statistical mechanics, of quantum fields models and of Chern-Simon's model of topological quantum fields.

Canonical formalism and quantization for a class of classical field equation

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Abstract: The research reported here is about physical problems related to photonic crystals; dielectric media with a periodic electric permeability $\varepsilon(\mathrm{x})$. Such systems have a band structure similar to that encountered in solid state physics. In particular bandgaps may occur and, if some randomness is introduced (Anderson) localisation intervals may occur within the original gap. Suppose an atom is introduced into such a system. If one of its transition frequencies ω_j falls into a gap one expects the corresponding atomic decay to be inhibited since there are no propagating field modes with frequency in the gap. But what happens if ω_j is in a localisation interval?

In order to treat such problems properly, the electromagnetic field must be quantised and a further complication is a possible frequency dependence of $\varepsilon(\mathrm{x})$. The latter leads in a natural way to consider the quantisation of evolution equations of the form

$$\partial_t F(t) = \mathrm{N}F(t) + \int_0^t ds \mathrm{M}(t-s)F(s),$$

acting in a separable, real, Hilbert space and where N is anti-selfadjoint. Thus we have a unitary time evolution for vanishing M . But also, if $\mathrm{M}(t)$ has a Fourier transform having suitable positivity properties we can embed the problem in a larger space where the time evolution is once more unitary. Applied to Maxwell's equations for an absorptive dielectric this gives rise to an interesting continuity equation.

The second step is to construct a suitable Lagrange-Hamilton formalism starting from a unitary time evolution. A fundamental point is here that if the generator has a non-empty null space, then we are led in a natural way to the gauge concept. In this setting the well known gauges (Coulomb, Lorentz) of electrodynamics have their counterparts. Quantisation then offers no further problems.

In the application to the quantised electromagnetic field coupled to material particles the usual divergencies turn up but the methods of non-relativistic renormalisation theory apply leading to a proper formalism. However. the renormalised mass is in general no longer a scalar.

A number of physical processes can already be described by replacing an atom by a two-level system and to truncate the field Fock space after the one-particle layer. This results in a precise mathematical model possessing a superselection law. Within this model it is directly clear that if the original atomic transition frequency ω_0 is in a gap of the classical dielectric, then the perturbed atomic eigenvalues remain real and isolated. In addition a new eigenvalue, not analytic in the atom-field coupling constant, appears. We end with a few notes upon the situation where ω_0 is in a localisation interval.

Regge poles 38 years later

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Abstract: We decribe the evolution of the use of Regge poles, which, in the mind of their inventor, Tullio Regge, were designed to study the scattering amplitude at fixed physical energy, in particular for t going to infinity, and became quickly used to describe high energy scattering near the forward direction. We show that the postulate that Regge trajectories are approximatively linear becomes natural if hadrons are made of confined quarks and can even explain the fact that baryons and meson trajectories are parallel.

Arguments de symétrie en faveur d'une réinterprétation d'une gravure de Dürer

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Abstract: L'une des plus célèbres gravures de Dürer, connue sous le nom de la *mélancolie* a fait l'objet de très nombreuses interprétations au cours d'études qui, bien que contradictoires, sont extrêmement précieuses car richement documentées.

Par tant d'arguments de symétrie nous sommes progressivement conduit vers une nouvelle interprétation, qui a l'avantage d'être confortée par des éléments tirés du contexte culturel de l'époque, faisant une large part au fait que Dürer fut l'un des plus grands humanistes de la Renaissance.

Subdivision algorithms and smoothness

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Abstract: Wavelet bases are usually defined so that their centers lie on regularly spaced grids. For some applications, including compression of irregularly spaced data, it is useful to have wavelet bases localized around uequally spaced points. These can still be defined in association with a multiresolution analysis, and in association with unequally spaced scaling functions, defined by subdivision scemes. The question then arises how to prove or derive smoothness for these scaling functions. This talk will show how communication rules can still be used in the irregular setting to discuss smoothness questions; in particular, for a generalization of the cubic Deslauriers-Dubuc scheme to unequally spaced grids, one can prove that the subdivision limit is as smooth as in the equally spaced case.

Time-frequency localization, symmetries and generalized means

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Abstract: (talk based on joint work with P. Goncalves, published in Appl. Comp. Harm. Anal., vol. 3, pp. 10–39, 1996)

In 1976, A. Grossmann first showed that the Wigner function is intimately connected to a symmetry operator in phase-space. This interpretation is indeed a key for understanding the localization of the Wigner function on lines in the plane, as well as for predicting its behavior when applied to any chirp in time-frequency analysis. The talk will discuss an extension of the original approach to the more general class of affine Bertrand distributions, which are known to perfectly localize on power-law curves in the plane. Emphasis will be put on geometrical interpretations and on a discussion of Stolarsky's generalized means, which happen to play a natural role in this context.

Designing a custom wavelet packet image compression scheme, with applications to fingerprints and seismic data

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Abstract: We present a meta-algorithm for designing a transform coding image compression algorithm specific to a given application. The goal is to select a decorrelating transform which performs best on a given collection of data. The method consists of conducting experimental trials with adapted wavelet transforms and the best basis algorithm, evaluating the basis choices made for a training set of images, then selecting a transform that, on average, delivers the best compression for the data set.

The meta-algorithm has been applied to two classes of data. A crude version of the method was used to design the WSQ fingerprint image compression algorithm. A more refined version was used with raw seismic data as the training set. The parameters of filter length, depth of decomposition, and quantization method were varied through 36 parameter settings and the rate-distortion relation was plotted and fitted with a line. The lines are compared to judge which parameter setting produces the highest quality for a given compression ratio on the sample data. It was found that long filters, moderate decomposition depths, and uniform variance-adjusted quantization yield the best results.

Multiscale processing, nonlinear wavelet transforms, noiselets and fast rotations

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Abstract: A general method to generate adaptively orthogonal transformations with a multiscae FFT structure will bescribed. In particular a nonlinear version of the Haar transform, as well as noiselet transforms will be constructed, these are connected to cellular automata and various paper folding mechanisms.

Spatio-temporel wavelets: application to tracking of moving targets in noisy environment

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Abstract: In this work we apply the spatio-temporal continuous wavelet transform to tracking moving targets in noisy environment. We focus our attention on handling more general classes of motion, such as acceleration. To accomplish this task the spatio-temporal wavelet transform is adapted to the motion parameters on a fram-by-frame basis. Three different energy densities, associated with velocity, location, and size, are used to determine motion parameters. Tracking results on synthethically generated images sequences demonstrate the capabilities of the proposed methods.

This work is partially supported by ONR (Office of Naval Research), Grant N00014-93-1-0561, BMDO (Ballistic Missile Defense Office), Grant DAAH04-95-1-0650 , and ARL (Army Research Laboratory), Grant DAAL01-96-2-0001 .

Wavelet analysis of observational data with gaps

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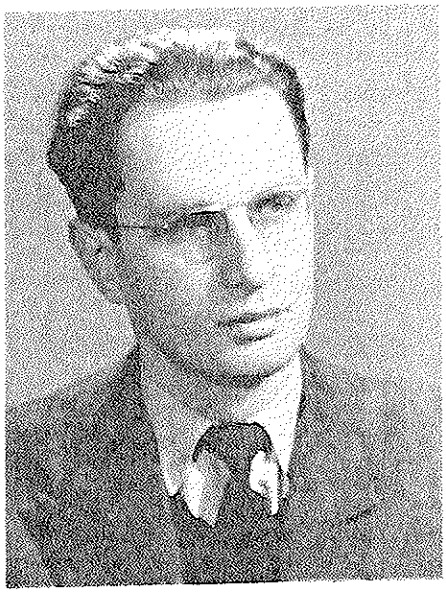
Abstract: In many domains of science, there are inevitable gaps in the time (or space) domains at which data can be recorded. When one applies spectral methods to such data, one is faced with the problem of separating the spectral properties of the signal from the spectral properies of the set of gaps. This situation is typical, e.g. for astronomy, where the gaps are caused by seasonal windows of observation, cloudy skies, telescope maintenance etc. In order to overcome these problems, different kinds of interpolation are currently used. However, extended gaps in time series are difficult to fill by interpolation (in some cases the total size of gaps is comparable with the duration of observations) and, secondly, any interpolation also introduces additional artifacts, leading at least to smoothing of the higher frequencies of the signal. These problems appear in Fourier analysis as well as in wavelet analysis. An extension of

Fourier transform to an uneven data set is known in astronomy as the Lomb-Scargle periodogram. The idea of this technique is to correct the basic functions $cos(\omega t)$ and $sin(\omega t)$ by a phase shift and a mean value subtraction to preserve their normalization conditions on a given set of observations.

We consider here a technique for the wavelet transform of signals with gaps. The main idea of this method, called the adaptive wavelet transform, is to considere the wavelet transform not as a convolution of signal with gaps with a given analysing wavelet, but as the convolution of a signal and of the wavelet with gaps. Transfering the gap problem from the unknown signal function to the known wavelet function, one tries to correct the broken wavelet in a way to provide at least the admissibility condition for any dilation and translation of analyzing wavelet. The properties of proposed method are illustrated by artificial examples and by the results of study of stellar chromospheric activity variations.



It is a joy, and a scaring honor, to be entrusted with opening the celebration of Alex Grossmann - a privilege, I think, bestowed upon me by my priority in meeting him. Our first encounter was in the summer 1954: we were both students of the Les Houches school, the then newly founded Abbaye de Thélème of theoretical physics, where Cecile Morette arranged for the best teachers to deliver crash courses to audiences destined to become the future elite of european theoretical physics. At first glance I would detect in Alex a profound thinker - he was evidently afflicted - or blessed, whatever you may say - by the heaviness of thougt - *die Schwere des Gedankens* - this was so-to-speak written on his forefront. Still more than by his superior american education in theoretical physics, Alex impressed the lonely autodidact I was by a unique style in his scientific statements - he spoke with a kind of slow penetration obviously stemming from profound independant thought - a sober seriousness high above my own frivolous fits of enthusiasm.

Our next encounter occurred a few years later in the second Abbaye de Thélème of theoretical physics: the I.H.I.S. where he was a guest, at a time when Visconti had already laid the foundations of our own wishfully delicious cloister. Right away I decided that we should get Alex to Marseille, proposing him to come (with Toni's telephonic assent) and simultaneously applying for a position - a kind of gambling which has allowed C.P.T. to emerge out of nothing in this republic of overcautious beaurocrats. As the time approached when Alex should rally Marseille, my application was rejected, throwing me into one of the worst squeezes of my otherwise quite bumpy gambling career: all I could possibly do was throw myself in desperation into the train for Paris, looking for a saver, whom I providentially found in the person of Prof. Maréchal, a nice optician who was in charge of an agency converting military money into funds for scientific research: Maréchal generously granted us a one-year position yielding the initial

condition for Alex's later stable settling to CPT. I was immensely relieved to be spared the extremity of telling my wife Lisl that we should halve my salary to share it with Alex, a proposal which might have put a temporary strain on the instinctive great sympathy she felt for him! By the way - and this might be interesting to hear for the fans who ignore it - Alex is genetically - as is also my wife Lisl - a grandchild of the double-headed eagle monarchy - the alas defunct great empire that was more liberal towards his subjects than our rationalistic republic towards her citizens. Alex is de facto a K-und-K aristocrat with a unique personal blend of the traditional K-und-K beautiful combination of noblesse and simplicity!

Dear Alex! - now time has passed! Although still stacked with juvenile projects for the future, we old men reached the age of critical weighing of our life-experience. Looking down at my early gambling for acquiring you, I am rejoicing in self-gratifying contentment! What a brilliant idea I had to ask you to join us - what a poetical anticipation of the glory you would win for Marseille. Your profound thought and untiring labour has procured a major drive of our development: besides a bulk of versatile contributions ranging from gravitation to solids (a germinal work of yours was e.g. a starting point for Jean Bélissard) your audacious departing from crowded roads for launching the wawelets with your friend Morlet was epoch-making! You have made C.P.T. the mother-house of what is now a world-wide multinationale in science, owning two house-periodicals and a rainbow of departments ranging between extremes like mathematical music and mathematical genetics! You propulsed the reputation of C.P.T to all four corners of the world on the crest of your wawelets! What an accressment of our reputation! - we, your friends, are happily splattered by this glory!

Mon cher Alex, permets-moi au nom de tous de t'exprimer notre profonde reconnaissance - je te dis cela avec la gravité de l'admiration et le sourire de l'amitié.

Daniel Kastler.

NONCOMMUTATIVE GEOMETRY AND FUNDAMENTAL PHYSICAL INTERACTIONS.

(HISTORICAL SKETCH AND DESCRIPTION OF THE PRESENT SITUATION).

These notes consist of two heterogeneous parts. The descriptive section [I] - a (dispensable) frontispiece to Thomas Schücker's lectures - is a tale of the history of the subject written by a witness of the first hour, showing how physics and mathematics interwove to develop a mathematical concept of quantum manifold relevant to elementary particle theory. The second, more technical, section [II] provides independant access to the present doctrine of the "universal spectral action".

Alain Connes successively proposed two Pandora boxes: "quantum Yang-Mills" and "spectral action", both miraculously spilling (granted the required computational work) all the complicated terms of the bosonic lagrangian of the standard model of elementary particles - the spectral action even in combination with the lagrangian of gravitation (Einstein-Hilbert-Glashow-Salam-Weinberg - with some admixture of Weyl - in a thimble!). These lagrangian-creating paradigms are so far both confined to the classical ("tree"-) level of the theory, stopping short of field quantization, hence without firm physical predictions - however seriously hinting at a potential capacity to compute masses, e. g. the ratio of the Higgs to the top mass.

The relationship between both schemes is for the moment unclear. The question of wether and to which extent the second (carrying the "primal matter" philosophy, with renormalization-group "descent" to accelerator energies) relegates the first (mainly studied in view of (hints of) realistic predictions), is open.

We conclude these notes with a short evocation of perspectives.

[I] NONCOMMUTATIVE GEOMETRY AND BASIC PHYSICS (SKETCH AND GLIMPSE TROUGH HISTORY).

After a study, now appearing as a detour, of the noncommutative geometry of C*-dynamical systems based on their (plentiful) derivations [1], Alain Connes realized the inadequacy for noncommutative geometry at large of the Koszul-complex-context and its wedge-product (Grassmann algebra of a Lie algebra of derivations) [1]. Inspired by Atiyah's "Global elliptic operators" [2] and Kasparov's work [3] (see also Mishenko and Fomenko [4]), he discovered cyclic cohomology as the right conceptual backbone of noncommutativity - the habitat, in duality with K-theory, of the index theorems [5a][5]. In his hands the abstract equipment of cyclic cohomology and K-theory has stragegic importance for checking the fundamentality of supposedly basic physical concepts. However physics requires concreter ground for its computational practice (differential operators rather than homology classes).

The equipment for physics comes with the "metric geometry" [6][7] [0,Chapter VI] originating in the crucial recognition that all the information of a (compact closed) spin manifold **M** lies in its Dirac operator $\tilde{D}$ [2] - a profound truth borne out by the fact that $\tilde{D}$ probes the geodesic distance between points $x,y\in\mathbf{M}$ as the sup of $|a(x)-a(y)|$ for all lipshitzian functions fulfilling $\|[D,a]\|\leq 1$. [3] Since the constitutive properties of the Dirac operator do not mention commutativity (cf. footnote 2), they still make sense in a non-commutative (or, for that matter, discrete) context. This suggests to *define* the differential geometry of noncommutative and/or discrete algebras as the existence of a generalized Dirac operator formalized as follows: an *(even, d-summable) K-cycle* (**H**,D) of the *-algebra **A** is the data of a $\mathbb{Z}/2$-graded Hilbert space **H** carrying a (faithful) *-representation of **A** by even bounded operators, moreover endowed with an odd self-adjoint operator D such that: all commutators [D,a], $a\in\mathbf{A}$, are bounded, the resolvent $(D+i)^{-1}$ is compact; and one has $\mu_n = O(n^{-1/d})$ for the nth characteristic value of D. We call *(even, d-dimensional) spectral triples* the "metric quantum spaces" (**A**,**H**,D) obtained in this way, defining the *gauge group* of the triple (**A**,**H**,D) as the group of unitaries of **A** (note that the relevant algebras are *-subalgebras of C*-algebras "small" in the sense that their "cohomological dimension" d is finite). As an important consequence of "d-

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[1] I happened to be at IHES at the very moment of discovery: he would pop out of his office like an out-of-the-box-devil, exclaiming: "The wedge product is no good! the wedge product is no good!". It took me months to realize why!

[2] We denote by $\tilde{D}$ the *Atiyah-Singer-Lichnérowicz-Dirac operator* $\tilde{D}=\gamma^{\mu}\tilde{\nabla}_{\mu}$ acting on L^2 of the spin bundle $\mathbb{S}_{\mathbf{M}}$ (we affect with $\sim$ the items pertaining to the spin-structure) Identifying the (smooth) functions on **M** with their multiplicative action, we recall that $[\tilde{D},a]=\gamma^{\mu}\partial_{\mu}a$ is a bounded operator for each function a, that the resolvent of $\tilde{D}$ is compact and that the characteristic values μ_n are $O(n^{-1/d})$, d the dimension of **M**.

[3] This fact was known to Caratheodory, and rediscovered by Connes who realized its huge implications.

dimensionality" [7], one has the existence of a *quantum integration* $⨍(D^{-d}\cdot)$ [4] stemming from a *quantum volume-form* $⨍$, (constructible) common value on "measurable operators" of all *Dixmier traces* Tr_ω, (non-constructive) ω-limits of the renormalized logarithmically divergent traces of discrete-spectrum operators with characteristic values $\mu_n = O(n^{-1})$ ($(D+i)^{-d}$ is an example, more generally any PDO of order -d, on which $⨍$ reduces to the Wodzicki residue).

This notion of spectral triple evidently encompasses the "classical case" $(C^\infty(\mathbf{M})=\mathbf{A},\mathbf{H}=L^2(\mathbb{S}_M),D=\tilde{D})$ which served as a model, and whose features should now proceed from the (archetipical) "Dirac K-cycle" as a first test of the theory. We now show how one actually constructs the DeRham complex $\Omega(\mathbf{M})$ of $\mathbf{M}$ and the Yang-Mills action (physically $\approx$ classical electrodynamics) by means of the Dirac operator $\tilde{D}$:

– (i): DeRham complex: we proceed as follows: first construct a differential algebra $(\Omega\mathbf{A},d)$ of "formal forms" by symbols a, da, $a\in\mathbf{A}$ and relations stating whatever should be the case: linearity of a and da w.r.t. a, and product ab in $\Omega\mathbf{A}$, reducing to those in $\mathbf{A}$, $d\mathbb{1}=0$, and the Leibniz rule for d(ab) - this allows to write any "word" as a linear combination of words of the type $\omega=a_0da_1...da_n$, $a_0,..,a_n\in\mathbf{A}$. The (coherent) requirement $d\omega=a_0da_1...da_n$ then yields a differential algebra $(\Omega\mathbf{A},d)$ which by itself carries little information (e.g.it is acyclic), however becomes richly informative when represented on $\mathbf{H}$ as follows:

$$\pi_D(a_0da_1...da_n)=a_0[D,a_1]...[D,a_n]. \tag{1}$$

(π_D is a representation of the algebra $\Omega\mathbf{A}$ because the substitution $d\to[D,\cdot]$ replaces a derivation by another one, the multiplication algorithm in $\Omega\mathbf{A}$ essentially proceeding from the derivation rule). Note that, since $([D,\cdot])^2\neq 0$, $\pi_D(\Omega\mathbf{A})$ is not a differential algebra (the kernel K of π_D is not a differential ideal), however one easily checks that K+dK is a differential ideal, making d to pass to the quotient $\Omega_D\mathbf{A}=\Omega\mathbf{A}/(K+dK)$ to yield a differential algebra $(\Omega_D\mathbf{A},d)$. It turns out now that *in the case* $\mathbf{A}=C^\infty(\mathbf{M})$ $(\Omega_D\mathbf{A},d)$ *is isomorphic to* $(\Omega(\mathbf{M}),d)$: we thus reconstructed the DeRham complex of $\mathbf{M}$ via the Dirac operator! - this moreover by means of a construction valid for general spectral triples $(\mathbf{A},\mathbf{H},D)$, thus producing their "quantum DeRham complex" $(\Omega_D,\mathbf{A},d)$.

– (ii): Yang-Mills action [6]: given a connexion ∇ of a smooth bundle over $\mathbf{A}$ (acting on its $\mathbf{A}$-module $\mathbf{E}$ of smooth sections) [5] we should "integrate the square of the curvature". Looking at connexions ∇ in the guise of their covariant exterior derivatives, these are grade-one d-derivations of the $\mathbf{A}$-module $\mathbf{E}\otimes_{\mathbf{A}}\Omega_D\mathbf{A}=\mathbf{E}\otimes_{C^\infty(M)}\Omega(\mathbf{M})$ (of $\mathbf{E}$-valued differential forms), the

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[4] Interpret D^{-d} as any $(D+i\alpha)^{-d}$, $\alpha\in\mathbb{R}$ (choice of α indifferent).

[5] Note that the "quantum Yang -Mills" needs as data a spectral triple (A,H,D) plus a projective finite module E as the receptacle of connexions. In most of our cases however E will be A itself taken as a right A-module (as is the case for electrodynamics).

corresponding curvature being the endomorphism (=two-form) Θ representing the square ∇^2. Note that these items exclusively use the differential algebra $\Omega_D\mathbf{A}$ constructed via D, thus persist in the non-commutative and/or discrete case. The "quantum integration" then allows to define the "quantum Yang-Mills action" as $⨍ (D^{-d}\,\Theta^2)$ (valid for general spectral triples $(\mathbf{A},\mathbf{H},D)$, and merging as it should in the classical case $\mathbf{E}=\mathbf{A}=C^\infty(\mathbf{M})$ into the usual classical Yang-Mills action). Note that, to achieve manifest gauge-invariance, D^{-d} has to be replaced in the integration by the equivalent $(D+A)^{-d}$ - the difference is absorbed by the Dixmier trace).

Where are we now? We have constructed generic "quantum versions" (using the spectral triples $(\mathbf{A},\mathbf{H},D)$ embodying our "quantum geometries") of both the DeRham complex and the Yang-Mills action, these quantum objects merging into the customary items in the case of a classical even-dimensional spin manifold.[6] We now look for a non-classical example to inaugurate our "metric quantum geometry". At this point something remarkable happens: the simplest possible choice (derisively simple! - still commutative, but *discrete*) turns out to be (germinally) of the highest significance for elementary particle physics! This example is as follows: the algebra is the "two-point algebra" $\mathbf{A}=\mathbb{C}\oplus\mathbb{C}$; the $\mathbb{Z}$:2-graded Hilbert space $\mathbf{H}=\mathbb{C}^N\oplus\mathbb{C}^N$, $N\in\mathbb{N}$ (with grading involution $1\oplus-1$) is acted upon by $(p,q)\in\mathbb{C}\oplus\mathbb{C}$ as $p\mathbb{1}\oplus q\mathbb{1}$; the Dirac operator is $D=\begin{pmatrix}0 & M^*\\ M & 0\end{pmatrix}$, M a N×N matrix (note that, as should be the case, the algebra is even and the Dirac operator is odd). Working out the DeRham complex, the hermitean connexions of the (right) $\mathbf{A}$-module $\mathbf{A}$, their curvatures, and the Yang-Mills action is an easy exercise. The connexions are parametrized by a complex number ϕ in terms of which the Yang-Mills action turns out as $(|\phi|^2-1)^2$ - an expression which a physicist recognizes with gleaming eyes as *the typical expression of the Higgs potential!* [7] We find the Higgs in a nutshell, the "embryonal Higgs"! This at once enlightens our physical picture: the world is two-sheeted, the mysterious Higgs is nothing but a gauge boson, however needing noncommutative geometry to be recognized as such because the corresponding potential is not a connection within the realm of classical differential geometry, but a *discrete connection* (so-to-speak with parallel transport jumping from one world-sheet to the other).

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[6] We somewhat wronged history by presenting the quantum DeRham complex prior to Yang-Mills. The order of discovery was first Yang-Mills, performed with the formal $(\Omega\mathbf{A},d)$ instead of $(\Omega_D\mathbf{A},d)$, thus dragging along the "junk" (K+KI in second order) as "adynamical fields" (not figuring in the action) to be eliminated by minimization of the action, a procedure subsequently streamlined by $\Omega_D\mathbf{A}$.

[7] The standard model of elementary particles is a puzzling object in that its phenomen(ologic)al success - seeking deviations from it is a thoroughy frustrating sport! - seems in total contrast with the completely ad hoc way in which it was historically constructed: namely through heuristical "symmetry-breaking" of the esthetical U(1)×SU(2)×SU(3) gauge theory (unphysical because leading to massless particles) by a mysterious "Higgs boson" scalar field subject to a double-well potential. This model gloriously survives in contrast with the waste-basket-destiny of hundreds of other models! Fishing out the right fish was a long and strenuous process!

This tempting insight provided by the "embryonal Higgs" (inner degrees of freedom) made us already extrapolate towards a multisheeted space-time combining space-time with the "inner space". We now need for describing the space-time Higgs to perform the actual construction of such a compound object At this point mathematics plays into our hands. Alain Connes' "quantum Yang-Mills" indeed allows a natural tensoring of two spectral triples $(\mathbf{A}',\mathbf{H}',D')$ and $(\mathbf{A}''\mathbf{H}'',D'')$ yielding a compound triple $(\mathbf{A},\mathbf{H},D)$ as follows: one has $\mathbf{A}=\mathbf{A}'\otimes\mathbf{A}''$; $\mathbf{H}=\mathbf{H}'\otimes\mathbf{H}''$; $\gamma=\gamma'\otimes\gamma''$; $a=a'\otimes a'',a\in\mathbf{A},a'\in\mathbf{A}',a''\in\mathbf{A}''$; and $D=D'\otimes\mathbb{1}''+\gamma'\otimes D''$: [8] This general procedure applied to the classical space-time Dirac-K-cycle as the primed K-cycle and the embryonal Higgs as the double-primed now produces a sensible toy-model: the electroweak sector of the (simplified) standard model of a fictitious word with N generations of a single fermion. We are thus on the right track, but need a more elaborate spectral triple to become realistic. In their original paper [9] [9] Connes and Lott first considered the case of the bipoint algebra $\mathbb{C}\oplus\mathbb{C}$ endowed with a K-cycle featuring a projective-finite module generated by a projection tailored to feature the leptons- a rather complicated formalism which they then traded, whilst appending the quarks, for a simpler and more natural scheme based on the algebra $\mathbb{C}\oplus\mathbb{H}$ (dictated by the electroweak gauge-group $U(1)\times SU(2)$) serving also as a right module over itself for accomodating connections (as was above the case for the pure electrodynamics in the quantum Yang-Mills guise). The inner Hilbert space H is then chosen so as to fit phenomenology (with a basis indexed by the fermions - leptons and quarks - as we know them; with the action of the algebra dictated by the gauge-group behaviour of the fermions; and the inner Dirac operator D a couple of finite matrices, the "Cabibbo-Kobayashi-Maskawa" matrix and its hermitean conjugate acting in an odd way (for details which would encumber this historical sketch we refer to Schücker's lectures or to **[II]C** below). The involved computation of the quantum Yang-Mills action (elimination of the junk somewhat nightmarish!) then *yields the bosonic electroweak sector of the standard model (Glashow-Salam-Weinberg)* - a hardly fortuitous "miracle"!. There follows another striking situation: whilst we got above the correct action, the fermions come out with wrong hypercharges. On the other hand, to describe physics (full standard model), we have to append the chromodynamics sector to the previously constructed electroweak sector, knowing that the gluon-field is a "vectorial field" producing no Higgs particles. We achieve this by tensoring the previous "inner electroweak spectral triple" by a "inner chromodynamics spectral triple" made up of the algebra $\mathbb{C}\oplus M_3(\mathbb{C})$ and a vanishing Dirac (the compound Dirac of the tensor-product triple is thus entirely on the electroweak side). Now, providentially, the "modular adjustment" collapsing the three U(1)'s into a single one effected to obtain the right gauge group (mathematically coherent

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[8] This procedure is in fact the exterior multiplication of Kasparov K-cycles.

[9] The program was annonced by Connes verbally in Sept 1989 at the Cortona meeting on Quantum groups and in writing at the 1990 Oxford Conference [8].

using the DeLaHarpe-Skandalis determinant [10] - but physically heuristic) [10] *can be achieved in such a way as to correct for the wrong hypercharges.* Non-commutative geometry refuses to treat the electroweak sector in isolation (wrong hypercharges), and so-to-speak asks (even exactly! cf. [12]) for tensorization by the gluons for their hypercharge-correcting virtue! There is more to this: this electroweak-chromodynamics "duality" (structureless in the usual theory) is recognized by Alain Connes as *a manifestation of a mathematically canonical noncommutative Poincaré duality* : a novel concept of pure mathematics monitored by the inner space of the full standard model of elementary particles! [11]. It seems extremely rewarding that the combination of the electroweak and chromodynamics sectors be required by non-commutative geometry for a basic mathematical reason. We formalize generically the concept we needed for the Connes-Lott inner space: our electroweak, resp. chromodynamic inner algebras are the archetypes of pairs **A**,**B** of the following type: A *metric dual pair* [12] is a spectral triple (**A**⊗B,**H**,D) fulfilling the *first-order condition*:

(2) $$[[D,a],b]=0 \qquad , a\in\mathbf{A}, b\in\mathbf{B}.$$

(symmetric in **A** and **B** owing to the Jacobi identity). Moreover the metric dual pair *fulfills Poincaré dualiy* whenever its intersection form is nondegenerate [13]

Up to now we mentioned only the bosonic part of the action. The fermionic part of the standard-model action is, on the other hand, compactly expressed (third "miracle"!) as

(3) $$(\psi, (D+A)\psi),$$

ψ in the compound Hilbert space (indexed by the fermions, asked to be chiral), D+A the compound covariant generalized Dirac opeator (A is the potential).

The Connes-Lott model based on the metric dual pair $((\mathbb{C}\oplus\mathbb{H})\otimes(\mathbb{C}\oplus M_3(\mathbb{C}),H,D)$ accumulates an impressive amount of coherence! Contact with the traditional standard-model lagrangian of elementary-particle (both bosonic and fermionic) is perfect. The generic "Connes-

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[10] Modular adjustment corrects for the fact that the SU(3) gauge group of the gluons is imperfectly represented by the algebra $M_3(\mathbb{C})$ with unitary group U(3), whence the necessity to assign to the gluons the algebra $\mathbb{C}\oplus M_3(\mathbb{C})$ and then collapse the three ambient U(1)'s into one. Interestingly this "modular adjustment" is precisely required by anomaly-freeness [18].

[11] This generalization features a couple of Poincaré dual algebras sitting in a bimodule (K-homology of the one mapped on the K-cohomology of the other). In our case the bimodule is the tensor-product spectral triple where one of the algebras is replaced by its opposite.

[12] personnal terminology: this concept will later turn up to coïncide with the S_0-real specialization of *real spectral triples* -see below.

[13] The first definition of noncommutative Poincaré duality [0] incorporated the first-order condition. The more recent usage (cf. definition below of "spectral geometries") defines Poincaré duality as the nondegeracy of the intersection form, mapping K-homology of A on K-cohology of B.

Lott (=quantum Yang-Mills) models" are in fact much more constraining than the comparatively loose traditional Yang-Mills models [11], thus entail more information. The above Connes-Lott model describes the Higgs in a much tighter way than the traditional standard model which features the Higgs mass as a totally loose parameter. Noncommutative geometry implies constraints allowing a tree-approximation computation of the Higgs mass in terms of the top mass (for what such calculations are worth in absence of renormalization, but any indication of a possible all-important computation of masses is worth investigating). This aspect has been intensively studied. Early investigations using a one-parameter coupling constant had revealed for the most symmetric choice a situation reminiscent of grand unification [13][14]. Alain Connes later advocated [9a] an oppositely maximally uncommitted choice of the coupling constant (matricial, within the commutant of algebra and Dirac)[14], a prescription which was noticed by Thomas Schücker [16][16a][17] to lead to very coherent tree-approximation-constraints practically fixing the tree-level Higgs mass irrespective of the possible choices [15].

Interesting as it is, the Connes-Lott theory has however imperfections:

- the notion of metric dual pair pertains only to inner space (since one does not wish to tensorially double space-time).
- the relevant connections are not all the connections, but the "biconnections" ("remembering" the tensorial splitting $(\mathbb{C}\oplus\mathbb{H}) \otimes (\mathbb{C}\oplus M_3(\mathbb{C}))$.[16]
- the theory understresses particle-antiparticle (=charge-conjugation) symmetry.
- "modular correction" is heuristic, not conceptual, thus insufficiently understood despite the link with anomaly-freeness [18].
- the elimination of the "junk" may appear unesthetically complicated.

In May 1994 in Trež Alain Connes proposed his notion of "real spectral triple" [20] which removes the two first drawbacks and halves the third. The concept originates in a combination of the Tomita-Takesaki-theory [21], KO-theory [22] and the idea of charge-(particle-antiparticle-) conjugation. Two distinct things have to be considered:

(1) - the general mathematical notion of real spectral triple - along with its S_0-real specialization.

(2) - the S_0-real "spectral triple of the standard model" replacing (and constructed from) the

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[14] This attitude is opposite to the one lately advocated by Chamseddine-Connes, claiming that the universal action (as can also be done for the Yang-Mills action, cf. [13]) pertains to "primal matter" (replacing the traditional grand unification).

[15] Once again, these results are dubious because they are non-renormalized and upset [18] - but slowly - by the renormalization group. Such dubious exploratory computations are worth doing owing to the tremendous physical importance of the Higgs mass, but the results must be considered with due care. The universal action proposes a different scheme for calculating the Higgs mass (see below, and Schücker's course).

[16] Connes motivated the use of biconnections by a gauge-group argument [14]. We attempted a more basic justification [19]. The problem evaporates whilst passing to the real spectral triples.

above dual metric pair. [17]

(1) - A *d-dimensional real spectral triple* (**A**,**H**,D,J) is a d-dimensional even spectral triple (**A**,**H**,D) with a *real structure*, i.e. an antilinear operator J of **H** fulfilling the following properties: $J^*J=\mathbb{1}$ (antiunitarity); J commutes with D; $J\mathbf{A}J^*$ commutes with both **A** and [D,**A**] ("order one"); $J^2=\varepsilon\mathbb{1}$; $J\chi=\varepsilon'\chi J$ (χ the grading involution of **H**) with the signs ε and ε' as follows:

(4)

d mod8	0	2	4	6
ε	1	-1	-1	1
ε'	1	-1	1	-1

Definitions:

- we set:

(5) $$a\xi b=aJb^*J^{-1}\xi \ (=Jb^*J^{-1}a\xi\ ) \qquad , a,b\in A,\ \xi\in H.$$

- the elements u of the gauge group $\mathcal{G}$ of the spectal triple (unitaries of **A**) act on **H** by the *adjoint representation*::

(6) $$\xi\rightarrow {}^u\xi=uJu^*J^{-1}\xi \ (=u\xi u^*),$$

- the *covariant Dirac operator* D_A, indexed by:$A\in\Omega_D(\mathbf{A})$, is defined as;

(7) $$D_A=D+A+JAJ^{-1}.$$

- the *fermionic action* is defined as:

(8) $$I_F(A,\psi)=(\psi,D_A\psi) \qquad , \psi\in H,\ A\in\Omega_D(\mathbf{A}).$$

Results:

- H acquires by (5) the structure of a **A**-bimodule such that $JaJ^{-1}=\cdot a^*$, $a\in\mathbf{A}$, where $\cdot a^*$ denotes multiplicaton in **H** by a* from the right (as in (5)).
- (6) defines a unitary representation of $\mathcal{G}$ on **H**, such that $J({}^u\xi)={}^u(J\xi)$, $\xi\in H$, $u\in\mathcal{G}$.
- for each $A\in\Omega_D(\mathbf{A})$, $(\mathbf{A},\mathbf{H},D_A,J)$ is a d-dimensional even spectral triple.
- the fermionic action is gauge-invariant: one has

(9) $$I_F({}^uA,{}^u\psi)=I_F(A,\psi) \qquad , \psi\in H,\ A\in\Omega_D(\mathbf{A}),\ u\in\mathcal{G},$$

where ${}^uA=A+u^*du$.

- The (euclidean) "charge conjugation C (such that $C^*=C^{-1}=-C$, $C\gamma^5=\gamma^5C$, $C\gamma^\mu=\gamma^\mu C$, $\mu=1,2,3,4$) is a real structure of the classical spectral triple $(C^\infty(\mathbf{M}),L^2(\mathbb{S}_M),\tilde{D})$.
- Tensoring as above as spectral triples the d'- , resp. d"-dimensional real spectral triples $(\mathbf{A}',\mathbf{H}',D',J')$ and $(\mathbf{A}'',\mathbf{H}'',D'',J'')$ yields a (d'+d")-dimensional real spectral triple with real structure $J'\otimes J''$.

The following specialization of real spectral triples arises in a one-to-one fashion from

[17] The general notion arose as a generalization of both the "classical example" and the "spectral triple of the standard model"

"metric dual pairs": A *S_0-real spectral triple* $(\mathcal{A},\mathcal{H},\mathcal{D},J,P)$ with grading involution χ) is a real spectral triple $(\mathcal{A},\mathcal{H},\mathcal{D},J)$ (with grading involution χ) together with an hermitean projection $P(=\mathbb{1}-\bar{P})$ of $\mathbf{H}$ fulfilling: $PJ+JP=J(\Leftrightarrow PJ=J\bar{P}\Leftrightarrow \bar{P}J=JP)$; $P\chi=\chi P(=P\chi P)$; $P\mathcal{D}=\mathcal{D}P(=P\mathcal{D}P)$; $Pa=aP(=PaP)$, $a\in\mathbf{A}$.

Proposition: (i): S_0-real spectral triples $(\mathcal{A},\mathcal{H},\mathcal{D},J,P)$ are one-to-one with metric dual pairs $(\mathbf{A}'\otimes\mathbf{A}'',\mathbf{H},D,J)$, bijection specified in both ways as follows:

(10)

$$\begin{cases} \mathcal{A}=\mathbf{A}'\oplus\mathbf{A}'', \\ \mathcal{H}=H\oplus\bar{H}, \\ J(\xi,\bar{\eta})=(\eta,\varepsilon\bar{\xi}),\ \xi,\eta\in H, \\ \chi=(\chi\oplus\varepsilon'\chi), \\ \mathcal{D}=(D\oplus D), \\ (a',a'')=(a'\oplus a'') \end{cases} , \quad \begin{cases} H=P\mathcal{H} \\ \mathbf{A}'=P\mathcal{A}P \\ \mathbf{A}''=\bar{P}\mathcal{A}\bar{P} \\ \chi=\text{restr. to } H \text{ of } P\chi P=P\chi=\chi P, \\ D=\text{restr. to } H \text{ of } P\mathcal{D}P=P\mathcal{D}=\mathcal{D}P, \\ a'=\text{restr. to } H \text{ of } (a',0)P=P(a',0)=(a',0)P, \\ a''=\text{restr. to } H \text{ of } PJ(0,a'')^*J^*P=PJ(0,a'')^*=(0,a'')^*J^*P \end{cases}$$

(here $\bar{H}$ is the conjugate Hilbert space of H, and the shorthand $\alpha S\oplus\beta T$, $S,T\in\mathrm{End}H$, $\alpha,\beta\in\mathbb{R}$, means the prescription $(\alpha S\oplus\beta T)(\xi,\bar{\eta})=(\alpha S\xi,\beta\overline{T^*\eta})$, $\xi,\eta\in H$).

(ii): A S_0-real spectral triple $(\mathbf{A}_0\oplus\mathbf{A}'\oplus\mathbf{A}'_0\oplus\mathbf{A}''\ \mathcal{H},\mathcal{D},J,P)$ with $\mathbf{A}'_0\approx\mathbf{A}_0$ can be "cut down" to $(\mathbf{A}_0\oplus\mathbf{A}'\oplus\mathbf{A}''\ \mathcal{H},\mathcal{D},J,P)$ by identifying $\mathbf{A}'_0$ with $\mathbf{A}_0$.

(2) - [18] The S_0-real "inner spectral triple of the Connes model" $(\mathbb{C}\oplus\mathbb{H}\oplus M_3(\mathbb{C}),\mathcal{H},\mathcal{D},J)$ then arises by applying the above procedure (i), and (ii) (with $\mathbf{A}_0=\mathbb{C}$), to the "inner dual metric pair $((\mathbb{C}\oplus\mathbb{H})\otimes(\mathbb{C}\oplus M_3(\mathbb{C}),H,D)$ of the Connes-Lott model". The S_0-real "(full) spectral triple of the Connes model" is then the compound real spectral triple $(\mathcal{A},(\mathcal{H}=\underline{\mathbb{H}}\oplus\bar{\underline{\mathbb{H}}}\ ,\chi,\mathcal{D}),J)$ obtained by tensorizing the space-time real tensor triple $(C^\infty(\mathbf{M}),(L^2(\mathbb{S}(\mathbf{M})),\gamma^5,\tilde{D}),C)$ by the inner real spectral triple $(\mathbf{A},\mathcal{H}=H\oplus\bar{H}\ ,\chi,\mathcal{D},J)$:

The passage from the Connes-Lott model based on a metric dual pair to the Connes real spectral triple standard model achieves a threefold progress: (i): particle-antiparticle symmetry is now duly emphasised, (ii): the "biconnections" come automatically with one-forms the charge conjugation-symmetrized A+JAJ, (iii): half of the modular adjustment is automatic (reduction by 1 of the number of plethoric U(1)'s). One shows that the passage from the definition-representation to the adjoint representation of the gauge-group does not alter the hypercharges of the fermions.

For a time we believed that trading the Connes-Lott model based on a metric dual pair for the Connes real spectral triple standard model would not change the action. That this was false

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[18] described in detail in [II]C below - see also Thomas Schücker's lectures. There should be no confusion between the particle- , resp. anti particle-Hilbert space $\underline{\mathbb{H}}$, resp $\bar{\underline{\mathbb{H}}}$, and the quaternion algebra $\mathbb{H}$.

was recognized by Lionel Carminati (the point is that for the junk-computation the simplifying use of the tensor structure [25] is no longer available): this slightly upsets the conclusions on tree-level mass constraints mentioned above - the problem was reexamined in [23] [24] - for details see Thomas Schücker's lectures.

The notion of real spectral triple leads to the following notion of "noncommutative spin manifolds". A *spectral geometry* is the specification of a d-dimensional real spectral triple $(\mathbf{A},\mathbf{H},D,J)$, even or odd, thus (in both cases) fulfilling:

- ("reality"): existence of an antiunitary J as above in the even case, for the odd case we refer to [26])
- ("dimension"): $(D+i)^{-1}$ is compact; and $\mu_n=O(n^{-1/d})$ for the nth characteristic value of D.
- ("first order"): $J\mathbf{A}J$ commutes with both $\mathbf{A}$ and $[D,\mathbf{A}]$.

moreover endowed with the following properties:

- ("regulariry") for each $a\in A$, a and $[D,a]$ belong to $\cap_{n\in\mathbb{N}}$ Domain δ^n, $\delta=[|D|,.]$.
- ("finitude"): the $\mathbf{A}$-module $\mathcal{E}=\cap_{n\in\mathbb{N}}\mathrm{Domain}D^n$ is projective-finite with hermitean structure:

(11) $$(a\xi,\eta)=\mathrm{Tr}_\omega(D^{-d}a(\xi,\eta)) \qquad , \xi,\eta\in\mathcal{E},\ a\in\mathbf{A}.$$

- ("orientability": there is a Hochschild cocycle $c=a_0da_1...da_n$ such that $\pi_D(c)=\gamma$ (even case) or $\pi_D(c)=\mathbb{1}$ (odd case) where $\pi_D(c)=a_0[D,a_1]...[D,a_n]$.
- ("Poincaré-duality"): the intersection form: $K_*(\mathbf{A})\times K_*(\mathbf{A})\to\mathbb{Z}$ composition of the Fredholm index of D and the diagonal $K_*(\mathbf{A})\times K_*(\mathbf{A})\to K_*(\mathbf{A}\otimes\mathbf{A})\to K_*(\mathbf{A})$ is invertible.

These axioms are comforted by the remarkable fact that their abelian version (where the algebra $\mathbf{A}$ is abelian and the bimodule $\mathbf{H}$ has the natural real structure $JaJ=a^*$, $a\in A$) goes far towards an axiomatic characterization of classical spin manifolds [26] (the gap in the proof seems essentially technical). In spite of this undubitable progress over a traditional definition (assembling flat neighbourhoods) which falls regrettably short of the ideal persuit of axiomatic definitions as clearcut frames for a subsequent search of invariants, a mathematician ignorant of physics is likely to be puzzeled by a system of axioms seeming strange if one does not know their genesis as features shared by the "classical example" and the (derisively minute? - however revealing) inner spectral triple of the Connes standard model of elementary particles: the latter is apparently the only man-made object "quantal" enough to monitor the search for "noncommutative manifolds" (the rest of mathematics is too "classical")! Here mathematics and physics beautifully interwove their strides towards the exploration of the right (noncommutative) space! [19]

An exhaustive classification of the finite Poincaré-dual real spectral triples has been performed [27] - see also [28].

[19] Alain Connes claims that the role of the CERN accelerator is "exploring the right space".

[II] THE SPECTRAL ACTION.

A - PREHISTORY. I may evoque personal recollections illustrating the course of discovery. In the spring 92, whilst bringing me to the airport in his car on his way home from a Collège de France lecture, Alain Connes asked: "did you notice that the Dirac operator contains the information of gravitation?" - stating that Einstein-Hilbert sticks in the Wodzicki residue of $(\text{Dirac})^{-2}$. I was fascinated: so the wondrous Dirac operator, in addition to producing electrodynamics (even, if decorated, the standard model) generated also general relativity! Wishing to know the details, I performed the computation (pretty nasty, one has to dive by two levels of PDO-indices!), actually finding: cnst × scalar curvature [29]. Was there more to it? - unfortunately not: replacement of the nacked Dirac by the Dirac plus $\gamma^\mu A_\mu$ ends in frustration: one gets no coupling with electrodynamics, the six additional terms cancel each other. Since, conversely, the spin connection drops out of the Connes-Lott computation, I sadly concluded that " the two theories seem to repell each other at that level". Wolfgang Kalau and Markus Walze reported about identical results in a Marseille talk [30]. On the other hand I knew that $\text{res}_W(\tilde{D}^{-2})$ equals the second coefficient $a_2(\tilde{D}^2)$ of the heat expansion up to a constant - but alas I neglected this aspect, that dispensed me of looking at the third coefficient $a_4(\tilde{D}^2)$. The shock came with a telephone call of Alain in the early spring of 96 reporting that he had found a formalism yielding jointly gravitation and the standard model and synthetizing diffeomorphisms with the gauge group - he was then alarmed to get the right relative signs, particularly of a term which he hoped coherently tractable by the renormalization group as prescribed by Wilson. Shortly afterwards, we learnt more from Ali Chamseddine who was spending three weeks in Marseille before going to IHES for the final writing of their spectral action paper [32]. I decided to perform independently the computation of the spectral action, asking my friends to do the same for the renormalization group corrections: this resulted in the papers [33] [34] continuing our Marseille tradition of performing computations independently for self-education and expounding them in detail. [20]

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[20] Ironically: I was asked to referee the Chamseddine-Connes paper which they submitted to a supposedly serious physics magazine in an attempt to break out of the noncommutative club. My enthusiastic recommendation - competent to the extent that I had just independently checked the action computation - was willingly ignored by the editor of the magazine who rejected a "paper without experimental confirmations" (in contrast with the rest of the mathematical physics literature, including the strings abounding in this magazine!?). Regunt sacerdotes ad majorem ecclesiae gloriam!.

B - DEFINITION AND RESULTS. The *spectral action* [21] is the following functional of the metric g of **M** and the one-form $\mathbb{A}$:

$$(12) \qquad I(g,\mathbb{A},\psi)=(4\pi)^{-2}\mathrm{Tr}\ F(\frac{1}{\Lambda^2}\mathcal{D}_{\mathbb{A}}^2),$$

where $\mathcal{D}_{\mathbb{A}}=\mathcal{D}+\mathbb{A}+\mathbb{J}\mathbb{A}\mathbb{J}$ is the "covariant Dirac operator" of the standard model in the spectral formalism, and ψ an element of $\mathbb{E}=\mathbb{S}(\mathbf{M})\otimes\mathcal{H}$. F is a function: $\mathbb{R}_+^*\to\mathbb{R}$ such that $F(\frac{1}{\Lambda}\mathcal{D}_{\mathbb{A}}^2)$ is trace-class, where Λ is a cut-off parameter $\cong$ the inverse of the Plank length: $\Lambda=l_p^{-1}$.

The required trace-class property, as well as positivity of the bosonic action are achieved by choosing for F: $[0,1]\to[0,F(0)]$ a positive smooth function decreasing from a positive value F(0) to the value 0 at 1 mimicking the characteristic function $\chi_{[0,1]}$ of the interval [0,1].

Remark: Call an endomorphism X of $\mathcal{H}$ an *observable* whenever X has discrete spectrum, leaves $\underline{\mathbb{H}}$ and $\bar{\mathbb{H}}$ stable, and fulfills $X=\mathbb{J}X\mathbb{J}^{-1}$. Functions of observables are observables. The operators χ, $\mathcal{D}+\mathbb{A}+\mathbb{J}\mathbb{A}\mathbb{J}$, $\mathbb{A}\in\Omega(\mathcal{A})^1$, and $F(\frac{1}{\Lambda^2}\mathcal{D}_{\mathbb{A}}^2)$ are observables. One sees immediately that, for X an observable, one has $\mathrm{Tr}_{\underline{\mathbb{H}}}X=\mathrm{Tr}_{\bar{\mathbb{H}}}X=1/2\mathrm{Tr}_{\mathcal{H}}X$: thus restricting the trace in (12) to either the particle or the antiparticle Hilbert space just looses a factor 1/2. Further, since the euclidean $\chi=\chi^+-\chi^-$ commutes with $\mathbb{J}$, $\mathrm{Tr}_{\underline{\mathbb{H}}}X=\mathrm{Tr}_{\underline{\mathbb{H}}}[(\chi^++\chi^-)X]$ equals $\mathrm{Tr}_{\underline{\mathbb{H}}}[(\chi^+X)+\mathrm{Tr}_{\bar{\mathbb{H}}}(\chi^-X)$, a welcome sum over chiral particles and antichiral antiparticles refuting the a priori objection that one "doubles the fermions" by defining $\underline{\mathbb{H}}$ as the tensor product of a spinor space with $\mathbb{Z}/2$-grading and an inner space with R-L $\mathbb{Z}/2$-grading

Results:

(i): The three first terms of the asymptotic development in Λ^{-2} of the spectral action-density read: [22]

$$(13) \quad I_B(x,g,A)=90(2f_0)\Lambda^4-f_2\Lambda^2\{15s-8A|\Phi|^2\}$$
$$+f_4\{\frac{80}{9}N\mathbf{f}_{\mu\nu}\mathbf{f}^{\mu\nu}+\frac{4}{3}N\mathbf{h}^s{}_{\mu\nu}\mathbf{h}_s{}^{\mu\nu}+\frac{4}{3}N\mathbf{g}^a{}_{\mu\nu}\mathbf{g}_a{}^{\mu\nu}+4A|\mathbf{D}\Phi|^2+\frac{2}{3}As|\Phi|^2+4B|\Phi|^4-\frac{9}{4}C^2\}$$
$$+\text{ the surface terms } f_4\{11\pi^2\chi_4+\frac{8}{3}\Delta s+\frac{4}{3}A\Delta(|\Phi|^2)\}.$$

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[21] bosonic part of the action. The *fermionic action* is $(\psi,\mathcal{D}_{\mathbb{A}}\psi)$. $\mathcal{D}_{\mathbb{A}}$ acts on the Hilbert space $\underline{\mathbb{H}}\oplus\bar{\mathbb{H}}$ direct sum of a particle- and an antiparticle Hibert space (no confusion with the symbol $\mathbb{H}$ denoting the quatrenions).

[22] after modular adjustment. This is an asymptotic development, up to negligible terms (negative powers of Λ). This developement is obtained from the heat-kernel expansion via Laplace transform of F, (23) stemming from

$$\mathrm{Tr}F(\frac{1}{\Lambda^2}\mathcal{D}_{\mathbb{A}}^2)=\Lambda^4f_0a_0(\mathcal{D}_{\mathbb{A}}^2)+\Lambda^2f_2a_2(\mathcal{D}_{\mathbb{A}}^2)+f_4a_4(\mathcal{D}_{\mathbb{A}}^2) \text{ where } a_j(\mathcal{D}_{\mathbb{A}}^2)=\int_M a_j(x,\mathcal{D}_{\mathbb{A}}^2)dv.$$

where $f_0=\int F(u)udu$, $f_2=\int F(u)du$, $f_4=F(0)$ (note that for $F=\chi_{[0,1]}$ one has $2f_0=f_2=f_4$), $\chi_4 dv$ is the Euler form, and:

$$(14)\quad \begin{cases} A=tr_N[3(M_u{}^*M_u+M_d{}^*M_d)+M_e{}^*M_e], \\ B=tr_N[3(M_u{}^*M_uM_u{}^*M_u+M_d{}^*M_dM_d{}^*M_d)+M_e{}^*M_eM_e{}^*M_e]. \end{cases}$$

(ii): (tree-level computation): Identification of the corresponding terms in (23) with the bosonic action-density of the standard model:

$$(15)\quad \mathcal{L}_{stand}=\frac{1}{4}G^a{}_{\mu\nu}G_a{}^{\mu\nu}+\frac{1}{4}W^s{}_{\mu\nu}W_s{}^{\mu\nu}+\frac{1}{4}B_{\mu\nu}B^{\mu\nu}+(D_\mu\phi)^*(D^\mu\phi)-\frac{\mu^2}{v^2}(\phi^*\phi)^2+\mu^2\phi^*\phi,$$

where

$$(16)\quad \begin{cases} G^a{}_{\mu\nu}=\partial_\mu G^a{}_\nu-\partial_\nu G^a{}_\mu+g_3 f_{abc}G^b{}_\mu G^c{}_\nu, \\ W_s{}^{\mu\nu}=\partial_\mu W^s{}_\nu-\partial_\nu W^s{}_\mu+g_2\varepsilon_{srt}W^r{}_\mu W^t{}_\nu, \\ B_{\mu\nu}=\partial_\mu B_\nu-\partial_\nu B_\mu. \end{cases} \qquad \begin{cases} M_H=\sqrt{2}\mu \\ M_Z=\frac{1}{2}vg \\ M_W=M_Z\cos\theta_W=\frac{1}{2}vg_2 \\ tg\theta_W=\frac{g_1}{g_2} \end{cases},$$

yields $\Phi=2A^{-1/2}g_2\phi$, $\mu^2=2\Lambda, v=\frac{A}{2B^{1/2}}\mu$, thus the relations: [23]

$$(17)\quad \begin{cases} g_2=g_3 \\ \sin^2\theta_W=\frac{3}{8} \\ M_H=2\Lambda \\ M_W=\frac{1}{2}vg_2=\frac{A}{2\sqrt{(2B)}}\Lambda \text{ , under top dominance } M_W=\frac{\sqrt{3}}{2\sqrt{2}}\Lambda \end{cases}$$

These results show that the spectral action pertains to very high energies - or early times (primal matter). Realistic results are expected to result from a renormalization-group treatment. for which we refer to [32][33][34], and to Thomas Schücker lectures.

The spectral action depends only upon the eigenvalues of the covariant Dirac operator $\mathcal{D}_A$ (*spectral invariance principle* , a strengthening of the Einstein equivalence principle, implying invariance under the automorphism group of the tensor-product algebra $\mathcal{A}=C^\infty(\mathbf{M})\otimes\mathbf{A}$ semi-direct product of the dffeomorphisms and the gauge transformations).

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[23] dominance of the top mass implies $A\sqrt{B}=\sqrt{3}$.

The rest of this section, devoted to a description of the spectral action computation, is organized as follows: **C** describes in detail the Connes spectral standard model outlined in **[1]**. **D** displays the covariant (generalizes) Dirac operator $\mathcal{D}_A$ as a classical Dirac operator of a twisted Clifford bundle **E** displays the canonical decomposition of the generalized laplacian $\mathcal{D}_A{}^2$ as the sum of connection-laplacian Δ^{∇} and an endomorphism E. The remaining sections **F**, **G** treat the routine evaluation of the heat-expansion coefficients as given [35] in terms of ∇ and E.

C - THE CONNES SPECTRAL MODEL.

The input of the spectral action formalism is the same as that of the Connes-Yang-Mills model evoqued earlier, viz. the S_0-real inner spectral triple $(\mathbb{C}\oplus\mathbb{H}\oplus M_3(\mathbb{C}),\mathcal{H},\mathcal{D},J)$, derivate of the inner dual metric pair $((\mathbb{C}\oplus\mathbb{H})\otimes(\mathbb{C}\oplus M_3(\mathbb{C}),H,D)$ of the Connes-Lott model. We now need a detailed description of these objects. We recall our euclidean frame: **M** is a 4-dimensional smooth compact oriented spin manifold without boundary, with algebra of smooth functions $C^\infty(\mathbf{M})$, volume element dv, and Levi-Civita connexion ∇^M.[24] $\mathbb{S}_M$ denotes the spin-bundle of **M**, $\mathbb{Z}/2$-graded by γ^5, with module of smooth sections $\mathbb{S}(\mathbf{M})$, the latter a Clifford-module under the action γ of the (archetypical) Clifford module $\mathbb{C}l(\mathbf{M})$ of **M**. The $C^\infty(\mathbf{M})$-module $\mathbb{S}(\mathbf{M})$ is acted upon by the spin connexion $\tilde{\nabla}^M=\tilde{\nabla}$, the Atiyah-Singer-Lichnérowicz- Dirac operator $\tilde{D}$ and the (euclidean) charge-conjugation $C=-C^{-1}$. These data are subsumed by the "space-time real spectral triple" $(C^\infty(\mathbf{M}),(L^2(\mathbb{S}_M), \gamma^5, \tilde{D}),C)$. We recall that in our euclidean setting $\mathbb{S}(\mathbf{M})$ is endowed with a hermitean scalar product for wich the γ^μ and γ^5 are hermitean and commute with C.

The metric dual pair $((\mathbb{C}\oplus\mathbb{H})\otimes(\mathbb{C}\oplus M_3(\mathbb{C}),H,D)$. We recall that there are 45 types of fermions coming in 3 analogous generations (the e-, μ-, and τ-generation) and 2 chiralities (right and left): the *leptons* e_R ν_L e_L and the *quarks*: u_R d_R u_L d_L (u stands for: upper, d for: lower) [25]. The fermions are acted upon by SU(2), the right-handed particles as singlets (e_R) (u_R) (d_R) (acted upon by the representation D_0), the left-handed particles as doublets $(\nu_L e_L)$ $(u_L d_L)$ (acted upon by $D_{1/2}$). The (inner) Hilbert space $H=H_q\oplus H_l$ is thus the direct sum of a quark summand:

$$(18q) \qquad H_q=(\underbrace{\mathbb{C}^2_R}_{u_R\,d_R}\oplus \underbrace{\mathbb{C}^2_L}_{u_L\,d_L})\otimes \underbrace{\mathbb{C}^N}_{\text{generations}}\otimes \underbrace{\mathbb{C}^3}_{\text{color}} \qquad (\dim H_q=12N=36),$$

and a lepton summand:

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[24] with scalar curvature and Ricci-tensor denoted s, resp. R.

[25] the subscripts q and l stand for quark, resp. lepton, the subscripts R and L stand for right, resp. left, N is the number of generations (experimentally N=3), the quarks have a threefold colour whilst the leptons are colourless. There are no right-handed neutrinos (right-left asymmetry - parity-breaking!).

(18l) $$H_l=(\underset{e_R}{\mathbb{C}^1_R}\oplus\underset{\nu_L\, e_L}{\mathbb{C}^2_L})\otimes\underset{\text{generations}}{\mathbb{C}^N}\otimes\underset{\text{no color}}{\mathbb{C}^1}\qquad(\dim H_l=3N=9),$$

themselves direct sums of a right-handed and a left-handed part yielding the $\mathbb{Z}/2$-grading of (right-left) parity χ (+ at right, - at left). Denoting the endomorphisms of H_q, resp H_l, by 4×4 matrices with entries in $M_N(\mathbb{C})$ tensorized by $M(\mathbb{C}^3_{colour})$, resp by 3×3 matrices with entries in $M_N(\mathbb{C})$, we then have the algebra $\mathbb{C}\oplus\mathbb{H}$ acting as $(p,q)=(p,q)_q\oplus(p,q)_l$, the algebra $\mathbb{C}\oplus M_3(\mathbb{C})$ acting as $(p,m)=(p,m)_q\oplus(p,m)_l$, and the Dirac operator $D=D_q\oplus D_l$ given as follows:

(19q) $$(p,q)_q=\begin{array}{c} \begin{array}{cccc} u_R & d_R & u_L & d_L \end{array} \\ \begin{pmatrix} \bar{p} & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & -\bar{b} & \bar{a} \end{pmatrix} \end{array} \begin{array}{c} u_R \\ d_R \\ u_L \\ d_L \end{array} \otimes 1_3 , \qquad \left(q=\begin{pmatrix} a & -\bar{b} \\ 0 & \bar{a} \end{pmatrix} \right),$$

(19l) $$(p,q)_l=\begin{array}{c} \begin{array}{ccc} e_R & \nu_L & e_L \end{array} \\ \begin{pmatrix} p & 0 & 0 \\ 0 & a & b \\ 0 & -\bar{b} & \bar{a} \end{pmatrix} \end{array} \begin{array}{c} e_R \\ \nu_L \\ e_L \end{array} ,$$

(20q) $$(p,m)_q=\begin{array}{c} \begin{array}{cccc} u_R & d_R & u_L & d_L \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \begin{array}{c} u_R \\ d_R \\ u_L \\ d_L \end{array} \otimes m$$

(20l) $$(p,m)_l=p\begin{array}{c} \begin{array}{ccc} e_R & \nu_L & e_L \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array} \begin{array}{c} e_R \\ \nu_L \\ e_L \end{array}$$

(21q) $$D_q=\begin{array}{c} \begin{array}{cccc} u_R & d_R & u_L & d_L \end{array} \\ \begin{pmatrix} 0 & 0 & M_u^* & 0 \\ 0 & 0 & 0 & M_d^* \\ M_u & 0 & 0 & 0 \\ 0 & M_d & 0 & 0 \end{pmatrix} \end{array} \begin{array}{c} u_R \\ d_R \\ u_L \\ d_L \end{array} \otimes 1_3$$

(21l) $$D_l=\begin{array}{c} \begin{array}{ccc} e_R & \nu_L & e_L \end{array} \\ \begin{pmatrix} 0 & 0 & M_e \\ 0 & 0 & 0 \\ M_e^* & 0 & 0 \end{pmatrix} \end{array} \begin{array}{c} e_R \\ \nu_L \\ e_L \end{array} ,$$

with $M_u=M_u^*$, M_d, $M_e=M_e^*\in \mathrm{End}\mathbb{C}^N$ the fermion mass-matrices. [26]

Inner-space one-forms. Since the inner Dirac operator D (21q,l) commutes with the chromodynamics part (20q,l) of the inner algebra **A**, the latter does not contribute to the inner one-forms which thus exclusively stem from the electroweak part (19q,l) of **A** (as the linear combinations of the commutators of D with the latter). They are thus of type $A=A_q+A_l$ with:

$$\text{(22q)}\qquad A_q=\begin{array}{c} \begin{array}{cccc} u_R & u_R & u_L & d_L \end{array} \\ \begin{pmatrix} 0 & 0 & h'_2\otimes M_u^* & h'^1\otimes M_u^* \\ 0 & 0 & -h'_1\otimes M_d^* & h'^2\otimes M_d^* \\ h_2\otimes M_u & h^1\otimes M_d & 0 & 0 \\ -h_1\otimes M_d & h^2\otimes M_d & 0 & 0 \end{pmatrix} \end{array} \begin{array}{c} u_R \\ d_R \\ u_L \\ d_L \end{array} \otimes \mathbb{1}_3 ,$$

$$\text{(22l)}\qquad A_l=\begin{array}{c} \begin{array}{ccc} e_R & \nu_L & e_L \end{array} \\ \begin{pmatrix} 0 & -h'_1\otimes M_d^* & h'^2\otimes M_d^* \\ h_1\otimes M_d & 0 & b \\ 0 & h_2\otimes M_d & 0 \end{pmatrix} \end{array} \begin{array}{c} e_R \\ \nu_L \\ e_L \end{array} ,$$

expressions indexed by a pair of quaternions $\begin{pmatrix} h'_2 & h'^1 \\ -h'_1 & h'^2 \end{pmatrix}$, $\begin{pmatrix} h'_2 & h'^1 \\ -h'_1 & h'^2 \end{pmatrix}$ which should be taken hermitean-conjugate of each other to get hermitean one-forms.

The S_0-real inner spectral triple $(\mathbb{C}\oplus\mathbb{H}\oplus M_3(\mathbb{C}),\mathcal{H},\mathcal{D},J)$ is then obtained as in (2) above by the procedure (i) with $\mathbf{A}'=\mathbb{C}\oplus\mathbb{H}$, $\mathbf{A}''=\mathbb{C}\oplus M_3(\mathbb{C})$, cf. (10), followed by the compression (ii) where $\mathbf{A}_0=\mathbb{C}$. Specifically the inner real spectral triple is as follows: the "inner Hilbert space", direct sum $\mathcal{H}=H\oplus\bar{H}$ of the "particle inner Hilbert space" H and the "antiparticle inner Hilbert space" $\bar{H}$ (taken as the conjugate Hilbert space of H) [27] is acted upon as follows by the chirality χ, the element (p,q,m) of **A**, the Dirac operator $\mathcal{D}$, and the charge-conjugation $J=J^{-1}$: [28]

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[26] We comply to the common usage of choosing our fermion mass-matrices such that M_e *and* M_u *are diagonal positive matrices, whilst* $M_d=C|M_d|$, *with* C *unitary and* $|M_d|$ *strictly positive.* Furthermore we assume that *all fermion masses are different* (the eigenvalues of M_e, M_u and $|M_d|$ consists of positive numbers (the masses of leptons and quarks) all different from one another - experiment!). We further assume that *no eigenstate of* $|M_d|$ *is an eigenstate of* C (experiment!).

[27] with conjugation denoted $\xi\to\bar{\xi}$, and mutiplication by $\alpha\in\mathbb{C}$: $\xi\to\bar{\alpha}.\bar{\xi}$, whence $\overline{M\xi}=\bar{M}.\bar{\xi}$, M a matrix, $\bar{M}$ the complex-conjugate matrix.

[28] χ, (p,q,m), and $\mathcal{D}$ preserve H and $\bar{H}$, whilst J exchanges them

$$(23)\qquad \begin{cases} \chi(\xi,\bar{\eta})=(\chi\xi,\overline{\chi\eta}), \\ (p,q,m)(\xi,\bar{\eta})=((p,q)\xi,\overline{(p,m)\eta}), \\ \mathcal{D}(\xi,\bar{\eta})=(D\xi,\overline{D\eta}), \\ J(\xi,\bar{\eta})=(\eta,\bar{\xi}), \end{cases} \quad , \quad \begin{cases} (p,q,m)\in\mathbf{A}, \\ (\xi,\bar{\eta})\in\mathcal{H} \end{cases} .$$

The compound real spectral triple $(\mathcal{A},\ (\mathcal{H}=\underline{\mathbb{H}}\oplus\bar{\mathbb{H}},\ \chi,\mathcal{D}),\mathbb{J})$. We now tensorize as follows the space time real tensor triple $(C^\infty(\mathbf{M}),\ (L^2(\mathbb{S}(\mathbf{M})),\gamma^5,\ \tilde{D}),C)$ by the inner real spectral triple $(\mathbf{A},\ (\mathcal{H}=H\oplus\bar{H}\ ,\chi,\mathcal{D},J)$:

$$(24)\qquad \begin{cases} \mathcal{A}=C^\infty(\mathbf{M})\otimes\mathbf{A} \\ \mathcal{H}=L^2(\mathbb{S}(\mathbf{M}))\otimes\mathcal{H} \qquad (\ \underline{\mathbb{H}}=L^2(\mathbb{S}(\mathbf{M}))\otimes H\ ) \\ \chi=\gamma^5\otimes\chi \\ \mathcal{D}=\tilde{D}\otimes id_E+\gamma^5+\otimes D \end{cases} .$$

Note that whilst $\mathbb{J}$ exchanges the "particle space" $\underline{\mathbb{H}}$ and the "antiparticle space" $\bar{\mathbb{H}}$, all other operators X of the theory leave $\underline{\mathbb{H}}$ and $\bar{\mathbb{H}}$ stable. Denoting the endomorphisms of $\underline{\mathbb{H}}_q$, resp $\underline{\mathbb{H}}_l$, by 4×4 matrices with entries in $\mathrm{End}\mathbb{S}(\mathbf{M})\otimes M_N(\mathbb{C})$ *tensorized by* $M(\mathbb{C}^3_{colour})$, resp by 3×3 matrices with entries in $\mathrm{End}\mathbb{S}(\mathbf{M})\otimes M_N(\mathbb{C})$, the compound Dirac operator $\mathcal{D}=\mathcal{D}_q\oplus\mathcal{D}_l$ is then specified by the matrices:

$$(25q)\qquad \mathcal{D}_q=\begin{array}{c} \begin{array}{cccc} u_R & d_R & u_L & d_L \end{array} \\ \begin{pmatrix} D\otimes 1_N & 0 & \gamma^5\otimes M_u^* & 0 \\ 0 & D\otimes 1_N & 0 & \gamma^5\otimes M_d^* \\ \gamma^5\otimes M_u & 0 & D\otimes 1_N & 0 \\ 0 & \gamma^5\otimes M_d & 0 & D\otimes 1_N \end{pmatrix} \end{array} \begin{array}{c} \\ u_R \\ d_R \\ u_L \\ d_L \end{array} \otimes 1_3,$$

resp.

$$(25l)\qquad \mathcal{D}_l=\begin{array}{c} \begin{array}{ccc} e_R & \nu_L & e_L \end{array} \\ \begin{pmatrix} D\otimes 1_N & 0 & \gamma^5\otimes M_e^* \\ 0 & D\otimes 1_N & 0 \\ \gamma^5\otimes M_e & 0 & D\otimes 1_N \end{pmatrix} \end{array} \begin{array}{c} \\ e_R \\ \nu_L \\ e_L \end{array}$$

The compound one-forms have the tensorial decomposition [25]:

$$(26)\qquad \Omega(\mathcal{A})^1=\Omega(\mathbf{M})^1\otimes A\oplus(\alpha\gamma^5+\beta\mathbb{1})C^\infty(\mathbf{M})\otimes\Omega(\mathbf{A})^1,$$

yielding in combination with (16q,l) the form $\mathbb{A}=\mathbb{A}_q+\mathbb{A}_l$ with:

$$
(27q)\qquad \mathbb{A}_q=\begin{array}{c} \begin{array}{cccc} u_R & d_R & u_L & d_L \end{array} \\ \begin{pmatrix} -\gamma(a)\otimes\mathbb{1}_N & 0 & H^2\gamma^5\otimes M_u^* & -H^1\gamma^5\otimes M_u^* \\ 0 & \gamma(a)\otimes\mathbb{1}_N & H_1\gamma^5\otimes M_d^* & H_2\gamma^5\otimes M_d^* \\ H_2\gamma^5\otimes M_u & H^1\gamma^5\otimes M_d & \gamma(b^1{}_1)\otimes\mathbb{1}_N & \gamma(b^1{}_2)\otimes\mathbb{1}_N \\ -H_1\gamma^5\otimes M_u & H^2\gamma^5\otimes M_d & \gamma(b^2{}_1)\otimes\mathbb{1}_N & \gamma(b^2{}_2)\otimes\mathbb{1}_N \end{pmatrix} \end{array} \begin{array}{c} u_R \\ d_R \\ u_L \\ d_L \end{array} \otimes\mathbb{1}_3
$$

$$
(27l)\qquad \mathbb{A}_l=\begin{array}{c} \begin{array}{ccc} e_R & \nu_L & e_L \end{array} \\ \begin{pmatrix} \gamma(a)\otimes\mathbb{1}_N & H_1\gamma^5\otimes M_e^* & H_2\gamma^5\otimes M_e^* \\ H^1\gamma^5\otimes M_e & \gamma(b^1{}_1)\otimes\mathbb{1}_N & \gamma(b^1{}_2)\otimes\mathbb{1}_N \\ H^2\gamma^5\otimes M_e & \gamma(b^2{}_1)\otimes\mathbb{1}_N & \gamma(b^2{}_2)\otimes\mathbb{1}_N \end{pmatrix} \end{array} \begin{array}{c} e_R \\ \nu_L \\ e_L \end{array} ,
$$

where H^i, $H_i=\bar{H}^i\in C^\infty(\mathbf{M},\mathbb{C})$; [29]

$$
(28q)\qquad (\mathbb{JAJ})_q=\begin{array}{c} \begin{array}{cccc} u_R & d_R & u_L & d_L \end{array} \\ \begin{pmatrix} \gamma(c_0)\otimes\mathbb{1}_N & 0 & 0 & 0 \\ 0 & \gamma(c_0)\otimes\mathbb{1}_N & 0 & 0 \\ 0 & 0 & \gamma(c_0)\otimes\mathbb{1}_N & 0 \\ 0 & 0 & 0 & \gamma(c_0)\otimes\mathbb{1}_N \end{pmatrix} \end{array} \begin{array}{c} u_R \\ d_R \\ u_L \\ d_L \end{array} \otimes\mathbb{1}_3
$$

$$
+\begin{array}{c} \begin{array}{cccc} u_R & d_R & u_L & d_L \end{array} \\ \begin{pmatrix} \gamma(c_0)\otimes\mathbb{1}_N & 0 & 0 & 0 \\ 0 & \gamma(c_0)\otimes\mathbb{1}_N & 0 & 0 \\ 0 & 0 & \gamma(c_0)\otimes\mathbb{1}_N & 0 \\ 0 & 0 & 0 & \gamma(c_0)\otimes\mathbb{1}_N \end{pmatrix} \end{array} \begin{array}{c} u_R \\ d_R \\ u_L \\ d_L \end{array} \otimes\frac{\lambda}{2} ,
$$

$$
(28l)\qquad (\mathbb{JAJ})_l=\begin{array}{c} \begin{array}{ccc} e_R & \nu_L & e_L \end{array} \\ \begin{pmatrix} \gamma(a) & 0 & 0 \\ 0 & \gamma(a) & 0 \\ 0 & 0 & \gamma(a) \end{pmatrix} \end{array} \begin{array}{c} e_R \\ \nu_L \\ e_L \end{array} ,
$$

Remark: Note that the operators pertaining to the leptons are obtained from the corresponding operators pertaining to the quarks by the following process of *leptonic reduction:* (i): suppress the first row and the first column of the matrix (ii): effect the changes: $M_u\rightarrow 0$, $M_e\rightarrow M_e$.

---

[29] Note that $H^\cdot=\begin{pmatrix} H_2 & H^1 \\ -H_1 & H^2 \end{pmatrix}\in C^\infty(\mathbf{M},\mathbb{H})$.

C - CONVERSION INTO CLASSICAL OBJECTS: THE COVARIANT DIRAC OPERATOR $\mathcal{D}_A=\mathcal{D}+A+JAJ$ AS A DIFFERENTIAL OPERATOR. From now on we shall use instead of the particle Hilbert space $\underline{\mathbb{H}}$ [30] its smooth dense sub-$C^\infty(\mathbf{M})$-module $\mathbb{E}=\mathbb{S}(\mathbf{M})\otimes\mathbb{H}$, left invariant by all the operators under consideration. We have the following situation:

(i): $\mathbb{E}$ is a finite-projective $C^\infty(\mathbf{M})$-module, expressible as the tensor product

(29) $$\mathbb{E}=\mathbb{S}(\mathbf{M})\otimes_{C^\infty(\mathbf{M})}\mathbf{E} \qquad \text{with} \qquad \mathbf{E}=C^\infty(\mathbf{M})\otimes H,$$

becoming a Clifford module $(\mathbb{E},c)$ under the $\mathbb{Z}/2$-grading χ and the Clifford action

(30) $$c=\gamma\otimes id_E,$$

and split in a direct sum of a quarkonic and the leptonic $C^\infty(\mathbf{M})$-module according to the decomposition $\mathbf{E}=\mathbf{E}_q\oplus\mathbf{E}_l$, *where* $\mathbf{E}_q=C^\infty(\mathbf{M})\otimes H_q$ *and* $\mathbf{E}_l=C^\infty(\mathbf{M})\otimes H_l$

(ii): We have $\mathcal{D}_A=D^\nabla+\Phi=ic^\mu\nabla_\mu+\Phi$, direct sum $(\mathcal{D}_A)_q\oplus(\mathcal{D}_A)_l$ of the quark and the lepton parts:

(31) $$\begin{cases}(\mathcal{D}_A)_q=(D^\nabla)_q+\Phi_q\\(\mathcal{D}_A)_l=(D^\nabla)_l+\Phi_l\end{cases} \qquad \text{with} \qquad \begin{cases}(D^\nabla)_q=ic^\mu\nabla_{q\mu}\\(D^\nabla)_l=ic^\mu\nabla_{l\mu}\end{cases},$$

where:

- the endomorphisms Φ_q, Φ_l of $\mathbb{E}$ respectively act on the quark and lepton subspaces as the matrices: [31]

(32q) $$\Phi_q=\begin{array}{c} \begin{array}{cccc} u_R & d_R & u_L & d_L \end{array} \\ \begin{pmatrix} 0 & 0 & \Phi^2\gamma^5\otimes M_u^* & -\Phi^1\gamma^5\otimes M_u^* \\ 0 & 0 & \Phi_1\gamma^5\otimes M_d^* & \Phi_2\gamma^5\otimes M_d^* \\ \Phi_2\gamma^5\otimes M_u & \Phi^1\gamma^5\otimes M_d & 0 & 0 \\ -\Phi_1\gamma^5\otimes M_u & \Phi^2\gamma^5\otimes M_d & 0 & 0 \end{pmatrix} \end{array} \begin{array}{c} u_R \\ d_R \\ u_L \\ d_L \end{array} \otimes\mathbb{1}_3$$

and

(32l) $$\Phi_l=\begin{array}{c} \begin{array}{ccc} e_R & \nu_L & e_L \end{array} \\ \begin{pmatrix} 0 & \Phi_1\gamma^5\otimes M_e^* & \Phi_2\gamma^5\otimes M_e^* \\ \Phi^1\gamma^5\otimes M_e & 0 & 0 \\ \Phi^2\gamma^5\otimes M_e & 0 & 0 \end{pmatrix} \end{array} \begin{array}{c} e_R \\ \nu_L \\ e_L \end{array},$$

where $\Phi_l=\bar{\Phi}^i\in C^\infty(\mathbf{M},\mathbb{C})$: $\Phi^\cdot_\cdot=H^\cdot_\cdot+1$ [32]

- the connection ∇ of $\mathbb{E}$ is the tensor-product:

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[30] which it suffices to consider by charge-conjugation symmetry.

[31] Note that Φ anticommutes with the c^μ, whilst Φ commmutes with the c^μ.

[32] Note that $\Phi^\cdot_\cdot=\begin{pmatrix}\Phi_2 & \Phi^1\\ -\Phi_1 & \Phi^2\end{pmatrix}\in C^\infty(\mathbf{M},\mathbb{E})$.

(33) $$\mathbb{V}=\tilde{\nabla}\otimes id_E+id_{\mathbb{S}(M)}\otimes\nabla^E,$$

of the spin connexion $\tilde{\nabla}$ *of* $\mathbb{S}(\mathbf{M})$ by the connexion ∇^E *of* $\mathbf{E}$ specified as follows: ∇^E is the direct sum $\nabla^E{}_q\oplus\nabla^E{}_l$ of a quark and a lepton connexion acting respectively on the quark and lepton subspaces as the sum of the exterior derivative and the matrices:

(34q) $$id_{\mathbb{S}(M)}\otimes(\nabla^E{}_q-\partial)_\mu=-i\begin{array}{c} \begin{array}{cccc} u_R & d_R & u_L & d_L \end{array} \\ \begin{pmatrix} (-\mathbf{a}_\mu+\mathbf{c}^0{}_\mu)\otimes\mathbb{1}_N & 0 & 0 & 0 \\ 0 & (\mathbf{a}_\mu+\mathbf{c}^0{}_\mu)\otimes\mathbb{1}_N & 0 & 0 \\ 0 & 0 & \mathbf{b}^1{}_{1\mu}\otimes\mathbb{1}_N & \mathbf{b}^1{}_{2\mu}\otimes\mathbb{1}_N \\ 0 & 0 & \mathbf{b}^2{}_{1\mu}\otimes\mathbb{1}_N & \mathbf{b}^2{}_{2\mu}\otimes\mathbb{1}_N \end{pmatrix} \end{array}\begin{array}{c} u_R \\ d_R \\ u_L \\ d_L \end{array}\otimes\mathbb{1}_3$$

$$-i\begin{array}{c} \begin{array}{cccc} u_R & d_R & u_L & d_L \end{array} \\ \begin{pmatrix} \mathbf{c}^0{}_\mu\otimes\mathbb{1}_N & 0 & 0 & 0 \\ 0 & \mathbf{c}^0{}_\mu\otimes\mathbb{1}_N & 0 & 0 \\ 0 & 0 & \mathbf{c}^0{}_\mu\otimes\mathbb{1}_N & 0 \\ 0 & 0 & 0 & \mathbf{c}^0{}_\mu\otimes\mathbb{1}_N \end{pmatrix} \end{array}\begin{array}{c} u_R \\ d_R \\ u_L \\ d_L \end{array}\otimes\frac{\lambda_a}{2},$$

and

(34l) $$id_{\mathbb{S}(M)}\otimes(\nabla^E{}_l-\partial)_\mu=-i\begin{array}{c} \begin{array}{ccc} e_R & \nu_L & e_L \end{array} \\ \begin{pmatrix} 2\mathbf{a}_\mu\otimes\mathbb{1}_N & 0 & 0 \\ 0 & \mathbf{b}^1{}_{1\mu}\otimes\mathbb{1}_N & \mathbf{b}^1{}_{2\mu}\otimes\mathbb{1}_N \\ 0 & \mathbf{b}^2{}_{1\mu}\otimes\mathbb{1}_N & \mathbf{b}^2{}_{2\mu}\otimes\mathbb{1}_N \end{pmatrix} \end{array}\begin{array}{c} e_R \\ \nu_L \\ e_L \end{array}.$$

Here: [33]

- $\mathbf{a}$ and $\mathbf{c}_0$ are classical U(1)-vector-potentials: $\bar{\mathbf{a}}=\mathbf{a},\in\Omega(\mathbf{M},\mathbb{C})^1$,
- $\mathbf{b}^{\cdot}$ is a classical U(2)-vector-potentials: $\mathbf{b}^{\cdot}=\begin{pmatrix} \mathbf{b}^1{}_1=\bar{\mathbf{b}}^1{}_1 & \mathbf{b}^1{}_2=\bar{\mathbf{b}}^2{}_1 \\ \mathbf{b}^2{}_1 & \mathbf{b}^2{}_2=-\mathbf{b}^1{}_1 \end{pmatrix}\in\Omega(\mathbf{M},i\mathbb{H}_{traceless})^1$,
- $\mathbf{c}^{\cdot}=(\mathbf{c}^a)_{a=1,\dots,8}$ is a SU(3)-vector-potentials (the λ_a are the eight Gell-Mann matrices).

Note that $\mathbb{V}$ is a Clifford connection, thus fulfills:

(35) $$[\mathbb{V}_\mu,c(\lambda)]=c(\nabla^M_\mu\lambda) \qquad , \lambda\in\Omega(\mathbf{M})^1.$$

Proof: $\mathbb{E}=\mathbb{S}(\mathbf{M})\otimes H$ is the finite-projective $C^\infty(\mathbf{M})$-module pull-back of the $\mathbb{C}$-module H by the $\mathbb{A}$-$\mathbb{C}$-bimodule $\mathbb{S}(\mathbf{M})$, thus obviously expressible as the tensor product of $C^\infty(\mathbf{M})$-modules $\mathbb{E}=\mathbb{S}(\mathbf{M})_A\otimes\mathbf{E}$. The action (27) of $\mathbb{C}l(\mathbf{M})$ on $\mathbb{E}$ then makes it a Clifford module $(\mathbb{E},c)$: indeed $\mathbb{E}$ is a $\mathbb{Z}/2$-graded $\mathbb{C}l(\mathbf{M})$-module owing to the Clifford relations $c^\mu c^\nu+c^\nu c^\mu=g^{\mu\nu}$. The remaining claims follow from the matrix form of the Dirac operators $\mathbb{D}_q$, $\mathbb{D}_l$ and of the vector-potentials $\mathbb{A}_q$, $\mathbb{A}_l$, cf. (18q,l), (20q,l) and (21q,l).

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[33] Following the physicists' usage we multiply by i our connexion one-forms to make them self-adjoint ("vector potentials"). Note that a quaternion is antihermitean iff it is traceless.

E - CANONICAL DECOMPOSITION OF $\mathcal{D}_A{}^2$. The generalized Laplacian $\mathcal{D}_A{}^2$ has the canonical spltting $\mathcal{D}_A{}^2 = \Delta^{\nabla} + E$ into the connection-Laplacian $\Delta^{\nabla} = -g^{\mu\nu}(\nabla_\mu \nabla_\nu - \Gamma^{\alpha}_{\mu\nu}\nabla_\alpha)$ plus the endomorphism: [34]

(36) $$E = \frac{1}{4}s\mathbb{1} - \frac{1}{2}c(R^E) + ic^\mu[\nabla_\mu, \Phi] + \Phi^2 \quad \text{with} \quad c(R^E) = -\gamma^\mu\gamma^\nu \otimes R^E(e_\mu, e_\nu),$$

where s is the scalar curvature of **M**, and R^E is the curvature of ∇^E, and we have the following matrix expressions:

(37q) $$c^\mu[\nabla_{q\mu}, \Phi_q] =$$

$$= \begin{pmatrix} 0 & 0 & \gamma(\mathbf{D}\Phi^2)\gamma^5 \otimes M_u{}^* & -\gamma(\mathbf{D}\Phi^1)\gamma^5 \otimes M_u{}^* \\ 0 & 0 & \gamma(\mathbf{D}\Phi_1)\gamma^5 \otimes M_d{}^* & \gamma(\mathbf{D}\Phi_2)\gamma^5 \otimes M_d{}^* \\ \gamma(\mathbf{D}\Phi_2)\gamma^5 \otimes M_u & \gamma(\mathbf{D}\Phi^1)\gamma^5 \otimes M_d & 0 & 0 \\ -\gamma(\mathbf{D}\Phi_1)\gamma^5 \otimes M_u & \gamma(\mathbf{D}\Phi^2)\gamma^5 \otimes M_d & 0 & 0 \end{pmatrix} \otimes \mathbb{1}_3$$

(37l) $$c^\mu[\nabla_{l\mu}, \Phi_l] = \begin{pmatrix} 0 & \gamma(\mathbf{D}\Phi_1)\gamma^5 \otimes M_e{}^* & \gamma(\mathbf{D}\Phi_2)\gamma^5 \otimes M_e{}^* \\ \gamma(\mathbf{D}\Phi^1)\gamma^5 \otimes M_e & 0 & 0 \\ \gamma(\mathbf{D}\Phi^2)\gamma^5 \otimes M_e{}^* & 0 & 0 \end{pmatrix}$$

where

(38) $$\begin{cases} \mathbf{D}\Phi^j = \mathbf{d}\Phi^j + i(a\Phi^j - b^j{}_k\Phi^k) \\ \mathbf{D}\Phi_j = \mathbf{d}\Phi_j - i(a\Phi_j - b^k{}_j\Phi_k) \end{cases}, \quad i=1,2, \quad \left(\text{i.e.} \begin{cases} \mathbf{D}\Phi^{\cdot} = \mathbf{d}\Phi^{\cdot} + i(a - b^a\frac{\tau_a}{2})\Phi^{\cdot} \\ \mathbf{D}\Phi_{\cdot} = \mathbf{d}\Phi_{\cdot} - i\Phi_{\cdot}(a - b^a\frac{\tau_a}{2}) \end{cases} \right)$$

Proof: We have

(39) $$\mathcal{D}_A{}^2 = (ic^\mu\nabla_\mu + \Phi)(ic^\nu\nabla_\nu + \Phi) = -c^\mu\nabla_\mu c^\nu\nabla_\nu + ic^\mu\nabla_\mu\Phi + i\Phi c^\mu\nabla_\mu + \Phi^2$$
$$= \mathbb{D}^2 + ic^\mu[\nabla_\mu, \Phi] + \Phi^2 = \Delta^{\nabla} + \frac{1}{4}s\mathbb{1} - \frac{1}{2}c(R^E) + ic^\mu[\nabla_\mu, \Phi] + \Phi^2,$$

where we plugged in the Lichnérowicz formula for the square of $\mathbb{D}$, cf. **App. 1**:

(40) $$\mathbb{D}^2 = \Delta^{\nabla} + \frac{1}{4}s\mathbb{1} - \frac{1}{2}c(R^E).$$

The expressions (37ql) are computed using (32ql) and (34ql) (details in [33] - observe that Φ commutes with the spin-connection one-form since the latter commutes with γ^5. It also commutes with the gluon-connection one-forms whose matrices are diagonal with entries Clifford scalars. Thus it suffices to compute $[\mathrm{id}_{\mathbb{S}(M)} \otimes (\nabla'^E{}_q - \partial)_\mu, \Phi]$, with ∇'^E obtained from ∇^E by deleting the gluon-connection).

---

[34] Cf. **App. 1**. Note that $[\nabla_\mu, \Phi]$ lies in $\mathrm{End}_A(\mathcal{E})$ as the commutator of a ∂_μ-derivation and a 0-derivation.

We now have all the ingredients to compute the heat-expansion coefficients:

$$(41)\quad \begin{cases} a_0(x,\mathbb{D}_A{}^2)=(4\pi)^{-2}tr_x(\mathbb{1}) \\ a_2(x,\mathbb{D}_A{}^2)=(4\pi)^{-2}tr_x(\frac{1}{6}s\mathbb{1}-E) \\ a_4(x,\mathbb{D}_A{}^2)=(4\pi)^{-2}\frac{1}{360}tr_x\{5s^2\mathbb{1}-2\mathbf{r}^2\mathbb{1}+2\mathbf{R}^2\mathbb{1}-60sE+180E^2+30\mathbb{R}_{\mu\nu}\mathbb{R}^{\mu\nu}\} \end{cases}$$

where s, **r**, **R** are the Levi-Civita scalar curvature, Ricci tensor, and Riemann curvature tensor of **M**, with $\mathbf{r}^2=\mathbf{r}^{\mu\nu}\mathbf{r}_{\mu\nu}$, $\mathbf{R}^2=\mathbf{R}^{\mu\nu}\mathbf{R}_{\mu\nu}$, and $\mathbb{R}$ is the Riemann curvature tensor of the connection $\mathbb{V}$ of $\mathbb{E}$.

F - COMPUTATION OF FIBER-TRACES.

[E.1] *Lemma.* We have the following traces on the fiber $\mathbb{E}_x$ *of* $x\in\mathbf{M}$:[35]

$$(42a)\quad \begin{cases} tr_x\mathbb{1}=144 \\ tr_x\mathbb{1}=36 \\ tr_x\mathbb{1}=180 \end{cases},$$

$$(42b)\quad tr_x\Phi_q=tr_x\,\Phi_l=tr_x\,\Phi=0$$

$$(42c)\quad \begin{cases} tr_x(\Phi_q{}^2)=8A_q|\Phi|^2 & \text{with} & A_q=3tr_N(\mu_u+\mu_d) \\ tr_x(\Phi_l{}^2)=8A_l|\Phi|^2 & \text{with} & A_l=tr_N\mu_e \\ tr_x(\Phi^2)=8A|\Phi|^2 & \text{with} & A=tr_N[3(\mu_u+\mu_d)+\mu_e] \end{cases},$$

$(\mu_u=M_u{}^*M_u,\ \mu_d=M_d{}^*M_d,\ \mu_e=M_e{}^*M_e)$

$$(42d)\quad \begin{cases} tr_x(\Phi_q{}^4)=8B_q|\Phi|^4 & \text{with} & B_q=3tr_N(\mu_u{}^2+\mu_d{}^2) \\ tr_x(\Phi_l{}^4)=8B_l|\Phi|^4= & \text{with} & B_l=tr_N\mu_e{}^2 \\ tr_x(\Phi^4)=8B|\Phi|^4 & \text{with} & B=tr_N[3(\mu_u{}^2+\mu_d{}^2)+\mu_e{}^2] \end{cases},$$

$$(42e)\quad tr_x(c^\mu[\mathbb{V}_{q\mu},\Phi_q])=tr_x(c^\mu[\mathbb{V}_{l\mu},\Phi_l])=tr_x(c^\mu[\mathbb{V}_\mu,\Phi])=0$$

$$(42f)\quad \begin{cases} tr_x\{(ic^\mu[\mathbb{V}_{q\mu},\Phi_q])^2\}=8A_q|\mathbf{D}\Phi|^2 \\ tr_x\{(ic^\mu[\mathbb{V}_{l\mu},\Phi_l])^2\}=8A_l|\mathbf{D}\Phi|^2 \\ tr_x\{(ic^\mu[\mathbb{V}_\mu,\Phi])^2\}=8A|\mathbf{D}\Phi|^2 \end{cases},$$

$$(42g)\quad tr_x[c(R^E)^2]=-8tr_x^E(R^E{}_{\mu\nu}R^{E\mu\nu}),$$

$$(42h)\quad -tr_x^E(R^E{}_{\mu\nu}R^{E\mu\nu})=2(\frac{20}{3}N\mathbf{f}_{\mu\nu}\mathbf{f}^{\mu\nu}+N\mathbf{h}^s{}_{\mu\nu}\mathbf{h}_s{}^{\mu\nu}+N\mathbf{g}^a{}_{\mu\nu}\mathbf{g}_a{}^{\mu\nu}),$$

after modular adjustment (here:

[35] We use the shorthand tr_x for the trace on the fiber $\mathbb{E}_x$, and denote the trace on the fiber E_x by tr_x^E

- $\mathbf{f}$ and $\mathbf{g}_0$ are classical U(1)-curvatures: $\tilde{\mathbf{f}}=\mathbf{f},\in\Omega(\mathbf{M},\mathbb{C})^2$,

- $\mathbf{h}^{\cdot}.$ is a classical U(2)--curvatures: $\mathbf{h}^{\cdot}.=\begin{pmatrix} \mathbf{h}^1{}_1=\tilde{\mathbf{h}}^1{}_1 & \mathbf{h}^1{}_2=\tilde{\mathbf{h}}^2{}_1 \\ \mathbf{h}^2{}_1 & \mathbf{h}^2{}_2=-\mathbf{h}^1{}_1 \end{pmatrix}\in\Omega(\mathbf{M},i\mathbb{H}_{traceless})^2$,

- $\mathbf{g}^{\cdot}=(\mathbf{g}^a)\frac{\lambda_a}{2}$ is a SU(3)--curvatures (the λ_a, a=1,...,8, are the Gell-Mann matrices)).

Proof: We refer to [33] for the routine matrix computations. [36] Check of (42g):

(43)
$$\begin{aligned} r_X[c(R^E)^2]&=tr_X[((\gamma^\mu\gamma^\nu\otimes R^E(e_\mu,e_\nu))^2]=tr_X[(\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\otimes R^E(e_\mu,e_\nu)R^E(e_\alpha,e_\beta)] \\ &=4(g^{\mu\nu}g^{\alpha\beta}+g^{\mu\beta}g^{\nu\alpha}-g^{\mu\alpha}g^{\nu\beta})tr_E(R^E{}_{\mu\nu}R^E{}_{\alpha\beta}) \\ &=4(g^{\mu\beta}g^{\nu\alpha}-g^{\mu\alpha}g^{\nu\beta})tr_E(R^E{}_{\mu\nu}R^E{}_{\alpha\beta})=4tr_E(R^{E\beta}{}_\nu R^{E\nu}{}_\beta-R^{E\alpha}{}_\nu R^E{}_\alpha{}^\nu) \\ &=4tr_E(R^E{}_{\beta\nu}R^{E\nu\beta}-R^E{}_{\alpha\nu}R^{E\alpha\nu})=-8tr_E(R^E{}_{\mu\nu}R^{E\mu\nu}) \end{aligned}$$

Check of (42h):

(44)
$$\begin{aligned} tr_X[\mathbb{R}_{\mu\nu}\mathbb{R}^{\mu\nu}]&=tr_X[(\tilde{R}_{\mu\nu}\otimes id_E+id_{\mathbb{S}_M}\otimes R^E{}_{\mu\nu})(\tilde{R}^{\mu\nu}\otimes id_E+id_{\mathbb{S}_M}\otimes R^{E\mu\nu}).] \\ &=tr_X(\tilde{R}_{\mu\nu}\tilde{R}^{\mu\nu}\otimes id_E)+tr_X(id_{\mathbb{S}_M}\otimes R^E{}_{\mu\nu}R^{E\mu\nu})+2tr_X(\tilde{R}_{\mu\nu}\otimes R^{E\mu\nu}) \\ &=(rank\mathbf{E})tr_{\mathbb{S}_M}(\tilde{R}_{\mu\nu}\tilde{R}^{\mu\nu})+4tr_E(R^E{}_{\mu\nu}R^{E\mu\nu}) \\ &=-\frac{1}{2}(rank\mathbf{E})\mathbf{R}^2+4tr_E(R^E{}_{\mu\nu}R^{E\mu\nu}), \end{aligned}$$

G - GATHERING THE PIECES.

Computation of $a_0(x,\mathcal{D}_A{}^2)$: we have $(4\pi)^2a_0(x,\mathcal{D}_A{}^2)=tr_X(\mathbb{1})=4rank\mathbf{E}$.

Computation of $a_2(x,\mathcal{D}_A{}^2)$: we have to compute $(4\pi)^2a_2(x,\mathcal{D}_A{}^2)=tr_X(\frac{1}{6}s-E)$, recalling that we found $E=\frac{1}{4}s\mathbb{1}+\frac{i}{2}c(R^E)+ic^\mu[\nabla_\mu,\Phi]+\Phi^2$ (cf.(36)). With $\equiv$ denoting equality under tr_X, we have $\frac{1}{6}s\mathbb{1}-E\equiv\frac{1}{6}s-\frac{1}{4}s\mathbb{1}-\Phi^2$, whence $(4\pi)^2a_2(x,\mathcal{D}_A{}^2)=-\frac{rank\mathbf{E}}{3}s\mathbb{1}-8A|\Phi|^2\mathbb{1}$

Computation of $a_4(x,\mathcal{D}_A{}^2)$: We have, with the shorthands $\mathbf{r}^2=\mathbf{r}_{\mu\nu}\mathbf{r}^{\mu\nu}$, $\mathbf{R}^2=\mathbf{R}_{\mu\nu\alpha\beta}\mathbf{R}^{\mu\nu\alpha\beta}$:

(45)
$$\begin{aligned} &360(4\pi)^2a_4(x,\mathcal{D}_A{}^2) \\ &=tr_X\{5s^2\mathbb{1}-2\mathbf{r}^2\mathbb{1}+2\mathbf{R}^2\mathbb{1}-60sE+180E^2+30\mathbb{R}_{\mu\nu}\mathbb{R}^{\mu\nu}\} \\ &=tr_X\{(\frac{5}{4}s^2-2\mathbf{r}^2+2\mathbf{R}^2)\mathbb{1}+30s\Phi^2+180\Phi^4+180(ic^\mu[\nabla_\mu,\Phi])^2+45c(R^E)^2+30\mathbb{R}_{\mu\nu}\mathbb{R}^{\mu\nu}\} \\ &=(rank\mathbf{E})(5s^2-8\mathbf{r}^2-7\mathbf{R}^2)+240As|\Phi|^2+1440B|\Phi|^4+1440A|\mathbf{D}\Phi|^2+120tr_V(R^E{}_{\mu\nu}R^{E\mu\nu}) \\ &=-18(rank\mathbf{E})\mathbf{C}^2+240As|\Phi|^2+1440B|\Phi|^4+1440A|\mathbf{D}\Phi|^2-240tr_E(R^E{}_{\mu\nu}R^{E\mu\nu}). \end{aligned}$$

We first took account of the fact that we have:

(46)
$$\begin{aligned} -60sE+180E^2&\equiv-60[\frac{1}{4}s^2\mathbb{1}+s\Phi^2]+45[\frac{1}{4}s^2\mathbb{1}+c(R^E)^2+4(ic^\mu[\nabla_\mu,\Phi])^2+4\Phi^4+2s\Phi^2] \\ &\equiv-\frac{15}{4}s^2\mathbb{1}+30s\Phi^2+180\Phi^4+180(ic^\mu[\nabla_\mu,\Phi])^2+45c(R^E)^2, \end{aligned}$$

neglecting the cross-terms in E^2 involving $c(R^E)$ or/and $c^\mu[\nabla_\mu,\Phi]$ (these vanish under tr_X,

[36] In fact formula (30) of [33] is erroneous: the lower right 2×2 sub matrix has non-vanishing off-diagonal entries which however do not affect the computation of traces.

owing to $\mathrm{tr}\gamma^{\mu}=0$ and $\mathrm{tr}\gamma^{\mu}\gamma^{\nu}=\mathrm{tr}\gamma^{\nu}\gamma^{\mu}$). We then plugged into (45) the values (43) and (44), and finally effected the replacement:

(47) $$5s^2-8\mathbf{r}^2-7\mathbf{R}^2=88\pi^2\chi_4-18C^2,$$

where $C^2=\mathbf{R}^2-2\mathbf{r}^2+\frac{1}{3}s^2$ is the square of the Weyl tensor, and $\chi_4 dv=2(4\pi)^{-2}(\mathbf{R}^2-4\mathbf{r}^2+s^2)dv$ is the Euler characteristics. We found, neglecting the latter:

(48) $$\begin{cases}(4\pi)^2a_0(x,\mathcal{D}'_{\mathbb{A}}{}^2)=180\\ (4\pi)^2a_2(x,\mathcal{D}'_{\mathbb{A}}{}^2)=-15s\mathbb{1}-8A|\Phi|^2\\ (4\pi)^2a_4(x,\mathcal{D}'_{\mathbb{A}}{}^2)=\frac{1}{8}(5s^2-8\mathbf{r}^2-7\mathbf{R}^2)+\frac{2}{3}As|\Phi|^2+4B|\Phi|^4+4A|\mathbf{D}\Phi|^2\\ \qquad\qquad +\frac{4}{3}[\frac{20}{3}N\mathbf{f}_{\mu\nu}\mathbf{f}^{\mu\nu}+N\mathbf{h}^s{}_{\mu\nu}\mathbf{h}_s{}^{\mu\nu}+N\mathbf{g}^a{}_{\mu\nu}\mathbf{g}_a{}^{\mu\nu}]\end{cases}$$

Sticking this into:

(49) $$\mathrm{Tr}F(\frac{1}{\Lambda^2}\ F\mathcal{D}_{\mathbb{A}}{}^2)=\Lambda^4f_0a_0(\mathcal{D}_{\mathbb{A}}{}^2)+\Lambda^2f_2a_2(\mathcal{D}_{\mathbb{A}}{}^2)+f_4a_4(\mathcal{D}'_{\mathbb{A}}{}^2)\quad,\begin{cases}f_0=\int F(u)udu\\ f_2=\int F(u)du\\ f_4=F(0)\end{cases},$$

yields the two first lines of the spectral action (12). The surface terms in the third line stem from χ_4 and the surface-term $\frac{1}{360}\mathrm{tr}_X\{12s_{;\alpha}{}^{\alpha}\mathbb{1}\ -60E_{;\alpha}{}^{\alpha}\}$ in a_4 [35] which we ignored .

H - A VARIANT OF THE ACTION. I had the perverse curiosity to look at what happens if one suppresses the γ^5 in the definition of the Dirac operator (tensoring the space-time and the inner K-cycles as odd K-cycles) [37]. In fact it is just as simple to treat the convex combination:

(50) $$\mathcal{D}'=\tilde{D}\otimes id_E+(\alpha\gamma^5+\beta\mathbb{1})\otimes D,\qquad\qquad,\ \alpha,\beta\in\mathbb{R},\ \alpha+\beta=1,$$

all other items being unchanged. The interpretation of $\mathcal{D}'_{\mathbb{A}}$ as a generalized Clifford operator is analoguous to the previous case, yielding $\mathcal{D}'=D^{\nabla}+\alpha\Phi+\beta\Psi$. But the canonical decomposition $\mathcal{D}'_{\mathbb{A}}{}^2=\mathbb{A}^{\nabla'}+E$ of its square is now more subtle, using the full force of the theory, Cf. **App 1**: one finds the horizontal connection $\nabla'_{\mu}=\nabla_{\mu}-i\beta\Psi c_{\mu}$ (with an accordingly modified curvature $\mathbb{R}'_{\mu\nu}$) and the endomorphism:

(51) $$E'=\frac{1}{4}s\mathbb{1}-\frac{1}{2}c(R^E)+(\alpha^2-3\beta^2)\Psi^2+2\alpha\beta(\gamma^5+id_E)\Psi^2+i\alpha c^{\mu}[\nabla_{\mu}\Phi],$$

now leading to the heat equation expansion coefficients:

(52) $$\begin{cases}(4\pi)^2a_0(x,\mathcal{D}'_{\mathbb{A}}{}^2)=180\\ (4\pi)^2a_2(x,\mathcal{D}'_{\mathbb{A}}{}^2)=-15s\mathbb{1}-8(\alpha^2-3\beta^2)A|\Phi|^2\\ (4\pi)^2a_4(x,\mathcal{D}'_{\mathbb{A}}{}^2)=\frac{1}{8}(5s^2-8\mathbf{r}^2-7\mathbf{R}^2)+\frac{2}{3}(\alpha^2-3\beta^2)As|\Phi|^2+\frac{4}{3}\beta^2A\mathbf{R}^2|\Phi|^2\\ \qquad +4(\alpha^4-2\alpha^2\beta^2+5\beta^4)B|\Phi|^4+4(\alpha-\beta)A|\mathbf{D}\Phi|^2\\ \qquad\qquad +\frac{4}{3}[\frac{20}{3}N\mathbf{f}_{\mu\nu}\mathbf{f}^{\mu\nu}+N\mathbf{h}^s{}_{\mu\nu}\mathbf{h}_s{}^{\mu\nu}+N\mathbf{g}^a{}_{\mu\nu}\mathbf{g}_a{}^{\mu\nu}].\end{cases},$$

reducing, as it should, to (48) in the "cum γ^5 case" $\alpha=1$, $\beta=0$, and yielding in the "sine γ^5 case" $\alpha=0$, $\beta=1$ to the same type of terms plus a $\mathbf{R}^2|\Phi|^2$-term, but with the sign of the Higgs

[37] The non-perverse reader is invited to skip this paragraph!

kinetic and potential term reversed (thus unstable) - these terms curiously stemming now, from the $\mathbb{R}'_{\mu\nu}\mathbb{R}'^{\mu\nu}$-term rather than the E'^2-term of a_4.

OUTLOOK.

The spectral action rests on the combination of a general conceptual apparatus (non-commutative spin manifolds, alias spectral geometry), and a heuristically constructed object: the real spectral triple (described in [II]C above) patterned after phenomenology. Now, as signaled by Alain Connes in [20], the quantum group $Sl_q(2)$ for $q=e^{2\pi i/3}$ (a covering of Sl(2) [36]) has its quotient by its nilradical isomophic to the algebra $M_1(\mathbb{C})\oplus M_2(\mathbb{C})\oplus M_3(\mathbb{C})$, thus almost coïnciding with the inner algebra $M_1(\mathbb{C})\oplus\mathbb{H}\oplus M_3(\mathbb{C})$ of the standard model [37][38][39][40]! This quantum group or some analogue may thus play a fundamental role in the basic structure of the fermions - hopefully ultimately providing a first-principle construction of the so far heuristic real spectral triple: this would make the theory entirely conceptual. The perspective of such a development would raise the hope for two major advances. On the one hand a first-principle Dirac operator would procure a theoretical Cabibbo-Kobayashi-Maskawa matrix, therefore theoretical (primal) fermion masses hopefully yielding realistic masses via renormalization group - at last a prospective scenario for the computation of fermion masses! Second, it would be tempting to try and base the theory on the full root-of-unity-quantum group rather than on its semi-simple quotient, hoping to obtain a sensible (generalized, possibly ternary?) supersymmetric model. [38] Other interestig questions and perspectives:

- one wishes to understand the deeper relationship between the present spectral action and the former quantum Yang-Mills (also for gravitation, cf. [41]).
- would a theory of "non-commutative spin structures" and accompanying Dirac operators obtain the standard model as one of the first non-classical examples lying around?
- are the physical notions of anomalies and BRS in fact general items of spectral geometries?
- can one work out a minkowskian version of the theory (as suggested in [42])?
- more ambitiously: could one reach a further level of the theory by basing it on a more deeply noncommutative algebra? - seeking inspiration in the fascinating "fuzzy sphere" modification of gauge field theories [43][44] - see also the attempt [45]. I heard Alain Connes say that one gets a deep hint from algebraic K-theory (impenetrable to me!) that the ultimate non-commutative

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[38] Quite generally, one should probably try to evade from the (protective, since hilbertian and semi-simple) ghetto to investigate interesting and mysterious non-semi-simple environments (which we call "medusae", we presume that they act on Hilbert spaces with indefinite metric, possessing at least semipositivity synonimous with semi-simplicity - this holds for electrodynamics in Lorentz gauge, with the longitudinal photons the null-space - and also for $Sl_q(2)$ at cubic (and at fourth) root of unity where the radical is the null-space - the trace of the adjoint representation having the nilradical as its kernel, inducing the matrix-trace on the semi-simple quotient [38][38a].

algebra might be of the nature of the discrete C*-algebras considered by logicians - in our case coding the procedures at the accelerator [39] - a landscape clearly more fundamental than the kantian space we learn to know in the cradle!

- progress is expected in the important (largely open) direction of "differential calculi" on modules, bimodules, quantum groups... etc, related to quantum principal bundles ...etc.- a subject worthy of more than this furtive reference and arbitrary quotations amongst a wealth of contributions [46][47] [48][49][50][51][52][53] (see the monographs [54][55]).

One of the main future concerns is of course the passage to the (field-theoretic) quantum level - functional integration... and all that. Here, owing to formal beauty and apparent fundamentality of the spectral action, one is tempted to try and quantize directly from there rather than from the asymptotic approximation through the heat expansion. Such a program would require phrasing physics in terms of the eigenvalues of the Dirac operator - a program which already started [56].

Carried by enthusiasm, I once mythologized the standard model as a shakspearian king disguised as a beggar (no mediocre role!). Alain retorted that he preferred more soberly to consider him as a beggar carrying diamonds in his pockets. So far he has tricked the beggar to show us some of the diamonds [40] - but there should be more of them!

I end up with an anecdote. Long before the standard model project [8], Alain Connes was trying to dress electrodynamics in noncommutative guise. Having realized the importance of the Dirac [41], he showed me an essay featuring a sum of two Hilbert spaces, the first with the Dirac acting as $d=[D,\cdot]$, the second with a kind of a screwed action in order to fight the annoying feature $d^2\neq 0$. I studied this conscientiously and was happy to provide a lemma removing an obscurity. The kind of electrodynamics thus concocted had two exciting features: sorts of non-local kernels (improving divergencies?), and curious objects floating around which vagely resembled Higgs particles! When Alain discovered the embryonal Higgs we understood what was actually happening: the Higgs aspect came just from doubling the space! - and the non local kernels were a sort of a lame (?) "junk"!... The Gods must smile when watching the halfgods stumble (- stumbling of ordinary mortals is not worth their attention).

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[39] congruuous with Einstein-Podolsky-Rosen?

[40] See [57] for a physicist's comment.

[41] Dirac reached a form of immortality where he became an operator!

APPENDIX 1. Generalized Laplacians and their canonical decomposition.

[App1.1] *Definitions*. Let $\mathcal{E}$ be a smooth vector-bundle over **M**, with $C^\infty(\mathbf{M})$-module of smooth sections **E**.

(i): A **generalized laplacian of E** is a second-order differential operator H of **E** with principal symbol: [42]

(App1,1) $$\sigma_2(H)(\mathbf{d}f)=-\frac{1}{2}[f,[f,H]]=-\frac{1}{2}[[H,f],f]=|\mathbf{d}f|^2 \qquad , f\in C^\infty(\mathbf{M}),$$

(implying by polarization via Jacobi identity: [43]

(App1,1a) $$[[H,f],g]=[[H,g],f]=-2(\mathbf{d}f,\mathbf{d}g) \qquad , f,g\in C^\infty(\mathbf{M})).$$

Local formulation: $\Delta\in\mathrm{Diff}^2\mathbf{E}$ is a generalized laplacian iff one has for the coordinate patch (x_μ) of **M** and the trivializing frame (e_i) of **E**

(App1,1b) $$\Delta=-g^{\mu\nu}\partial_\mu\partial_\nu+\Delta_1 \qquad \text{with} \qquad \Delta_1\in\mathrm{Diff}^1\mathbf{E}.$$

The set of generalized laplacians of **E** is denoted Lapl**E**.

(ii): With ∇ a connection of **E**, the following composition of $\mathbb{R}$-linear maps:

(App1,2) $$\mathbf{E}\xrightarrow{\nabla}\Omega(\mathbf{M})^1\otimes\mathbf{E}\xrightarrow{\mathrm{id}\otimes\nabla+\nabla^M\otimes\mathrm{id}}\Omega(\mathbf{M})^1\otimes\Omega(\mathbf{M})^1\otimes\mathbf{E}\xrightarrow{-g(.,.)\otimes\mathrm{id}}\mathbf{E}$$

defines the **connection-laplacian** Δ^∇ **of** ∇, locally given as follows in the coordinate patch $\{x_\mu\}$ of **M** and the trivializing frame $\{e_i\}$ of **E**: with $\nabla_\alpha=\nabla_{\partial_\alpha}$, and $\Gamma^\alpha_{\mu\nu}$ are the Christoffel symbols:

(App1,2a) $$\Delta^\nabla=-g^{\mu\nu}(\nabla_\mu\nabla_\nu-\Gamma^\alpha_{\mu\nu}\nabla_\alpha).$$

(ii): One has:

(App1,3) $$-i\sigma_1(\mathbf{d}v)=[v,\Delta^\nabla]=2\nabla_{\mathrm{grad}v}-\Delta v \qquad , v\in C^\infty(\mathbf{M}),$$

i.e. Δ^∇ determines in turn ∇ as follows:

(App1,3a) $$\nabla_{u\,\mathrm{grad}v}=\frac{1}{2}u\{[v,\Delta^\nabla]+\Delta v\} \qquad , u,v\in C^\infty(\mathbf{M}),$$

thus the connections and the connection-laplacians are one-to-one: we have a bijection: Conn**E**$\ni\nabla\leftrightarrow\Delta^\nabla\in$Connlapl**E**.

[App1.2] *Proposition*. Let $\mathcal{E}$ be a smooth vector-bundle over **M**, with $C^\infty(\mathbf{M})$-module of smooth sections **E**. And *let* H$\in$Lapl**E**. Then, identifying elements *of* $C^\infty(\mathbf{M})$ with their multiplicative action on **E** :

(i): H determines both:

- a connection ∇^H of **E** called *the horizontal connection of* H, *specified by:*

(App1,4) $$\nabla^H_{u\,\mathrm{grad}v}=\frac{1}{2}u\{[v,H]+\Delta v\}(=\frac{1}{2}u\{-i\sigma_2(H)(\mathbf{d}v)+\Delta v\} \qquad , u,v\in C^\infty(\mathbf{M}),$$

where Δ denotes the scalar laplacian. [2]

- a n element $\Phi^H\in\mathrm{End}\mathbb{A}$ called *the endomorphism of* H, given by the difference:

(App1,5) $$\Phi^H=H-\Delta^{\nabla^H}.$$

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[42] We recall the definitions $\sigma_1(H)(df)=-i[f,H]$, $\sigma_2(H)(df)=-\frac{1}{2}[f,[f,H]]$.

[43] In what follows we consistently identify elements of $C^\infty(M)$ with their multiplicative action on E.

(ii): In fact the splitting $H=\Delta^{\nabla^H}+\Phi^H$ is unique: one has the implication:

(App1,6) $\quad H=\Delta+\Phi$ with $\Delta\in\mathrm{Connlap}\mathbf{E}$, $\Phi\in\mathrm{End}\mathbb{A}$ $\quad\Rightarrow\quad \Delta=\Delta^{\nabla^H}$, $\Phi=\Phi^H$.

(iii): Consequently $\mathrm{Lap}\mathbf{E}$ is a fiber-space with basis $\mathrm{Conn}\mathbf{E}\approx\mathrm{Connlap}\mathbf{E}\approx\sigma_2(\mathrm{Lap}\mathbf{E})$, and fiber $\mathrm{End}\,C^\infty(\mathbf{M})$ acting on $\mathrm{Lap}\mathbf{E}$ by translations: $\mathrm{Lap}\mathbf{E}\ni\Delta\to\Delta+\Phi$.

APPENDIX 2. Lichnérowicz formula.

We first have the *Bochner formula* valid for any Dirac operator on a Clifford bundle associated with a Clifford connection $\mathbb{\nabla}$ with curvature $\mathbb{R}$:

(App2,1) $$\mathbb{D}^2=\Delta^{\mathbb{\nabla}}+\frac{1}{2}c(\mathbb{R}) \qquad \text{where} \qquad c(\mathbb{R})=c^\mu c^\nu\mathbb{R}_{\mu\nu},$$

Proof:

(App2,2) $$\begin{aligned}\mathbb{D}^2&=(c^\mu\mathbb{\nabla}_\mu)(c^\nu\mathbb{\nabla}_\nu)=c^\mu c^\nu\mathbb{\nabla}_\mu\mathbb{\nabla}_\nu+c^\mu[\mathbb{\nabla}_\mu,c^\nu]\mathbb{\nabla}_\nu=c^\mu c^\nu\mathbb{\nabla}_\mu\mathbb{\nabla}_\nu-\Gamma^{\ \nu}_{\mu\alpha}c^\mu c^\alpha\mathbb{\nabla}_\mu\\&=\frac{1}{2}(c^\mu c^\nu+c^\mu c^\nu)\mathbb{\nabla}_\mu\mathbb{\nabla}_\nu+\frac{1}{2}(c^\mu c^\nu-c^\mu c^\nu)\mathbb{\nabla}_\mu\mathbb{\nabla}_\nu-\ -\frac{1}{2}(c^\mu c^\nu+c^\mu c^\nu)\Gamma^{\ \nu}_{\mu\alpha}\mathbb{\nabla}\\&=-g^{\mu\nu}(\mathbb{\nabla}_\mu\mathbb{\nabla}_\nu-\Gamma^{\ \alpha}_{\mu\nu}\mathbb{\nabla}_\alpha)+\frac{1}{2}c^\mu c^\nu(\mathbb{\nabla}_\mu\mathbb{\nabla}_\nu-\mathbb{\nabla}_\nu\mathbb{\nabla}_\mu)\\&=-g^{\mu\nu}(\mathbb{\nabla}_\mu\mathbb{\nabla}_\nu-\Gamma^{\ \alpha}_{\mu\nu}\mathbb{\nabla}_\alpha)+\frac{1}{2}c^\mu c^\nu[\mathbb{\nabla}_{[\partial_\mu,\partial_\nu]}+\frac{1}{2}c^\mu c^\nu\mathbb{R}(\partial_\mu,\partial_\nu)],\end{aligned}$$

where we took account of the fact that:

(App2,3) $$[\mathbb{\nabla}_\mu,c^\nu]=c(\delta^M_\mu\gamma(dx^\nu))=(c\circ\gamma)(\nabla^M_\mu dx^\nu)=-(c\circ\gamma)(\Gamma^{\ \nu}_{\mu\alpha}dx^\alpha)=-\Gamma^{\ \nu}_{\mu\alpha}c^\alpha.$$

For the Dirac operator $\mathbb{D}$, acting on the twisted bundle $\mathbb{S}(\mathbf{M})\otimes\mathbf{E}$, associated to the compound connection $\mathbb{\nabla}=\tilde{\nabla}\otimes id_E+id_{\mathbb{S}(M)}\otimes\nabla^E$, ∇^E a connection of $\mathbf{E}$ with curvature $\mathbf{R}$ we have the *Lichnérowicz formula*:

(App2,4) $$\mathbb{D}^2=\Delta^{\mathbb{\nabla}}+\frac{1}{4}s\mathbb{1}-\frac{1}{2}c(R^E) \qquad , \qquad \gamma^i\gamma^j\otimes R^E(e_i,e_j)$$

where s is the scalar curvature, and $\{e_i,\varepsilon^i\}_{i=1,\dots,d}$ a local orthonormal frame.

Proof:. Plugging $\mathbb{R}=R^E+\tilde{R}$ in (App2,1), we get:

(App2,5) $$\mathbb{D}^2=\mathbb{\Delta}+\frac{1}{2}\,c(\mathbb{R})=\mathbb{\Delta}+\frac{1}{2}\,c(F^E)+\frac{1}{2}\,c(R^E)=\mathbb{\Delta}+\frac{1}{2}\,c(F^E)-\frac{1}{8}R_{ijmn}c^mc^nc^ic^j$$

taking account of the fact that:

(App2,6) $$R_{mn}=\frac{1}{2}R_{ijmn}\varepsilon^i\wedge\varepsilon^j=\frac{1}{2}R_{ijmn}(\varepsilon^i\otimes\varepsilon^j-\varepsilon^j\otimes\varepsilon^i)=R_{ijmn}\varepsilon^i\otimes\varepsilon^j$$

whence $c(R_{mn})=R_{ijmn}c^ic^j$, whence $c(R^E)=\frac{1}{4}R_{ijmn}c^mc^nc^ic^j$. Now, owing to the orthonormality of the ε^i, and to the Clifford relations we have that: [44]

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[44] One verifies that this holds (i): for m,n, i all different (r.h.s. reducing to its first term); (ii): for m≠n=i (r.h.s. reducing to its second term); (iii): for n≠m=i (r.h.s. reducing to its third term); (iii): for i≠m=n (r.h.s. reducing to its fourth term).

(App2,7) $$c^m c^n c^i = \frac{1}{6}\Sigma_{\sigma\in S_3}\chi(\sigma)c^{\sigma m}c^{\sigma n}c^{\sigma i} - \delta^{ni}c^m + \delta^{mi}c^n - \delta^{mn}c^i,$$

implying

(App2,8) $$R_{ijmn}\, c^m c^n c^i c^j = R_{mnij}\, c^m c^n c^i c^j = R_{mnij}[-\delta^{ni}c^m + \delta^{mi}c^n]c^j = 2R_{mnij}\delta^{mi}c^n c^j$$
$$= R_{mnij}\delta^{mi}(c^n c^j + c^j c^n) = -2R_{mnij}g^{mi}g^{nj} = -2R^{ij}{}_{ij} = -2s$$

which turns (App2,5) into (App2,4) (we took account of the relations $R_{mnij}+R_{nimj}+R_{imnj}=0$ and $R_{mnij}+R_{nmij}=0$ making the first, resp. the last term of (App2,8) ineffective; and also of the fact that R_{inij} is symmetric in n and j).

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Gravity on Fuzzy Space-Time

J. Madore

Simply stated, 'fuzzy space-time' is a space-time in which the 'coordinates' do not commute. One typically replaces the four Minkowski coordinates x^μ by four generators q^μ of a noncommutative algebra which satisfy commutation relations of the form

$$[q^\mu, q^\nu] = i\bar{k}q^{\mu\nu}.$$

The parameter $\bar{k}$ is a fundamental area scale which we shall suppose to be of the order of the Planck area: $\bar{k} \simeq \mu_P^{-2} = G\hbar$.

Perhaps not the simplest but certainly the most familiar example of a 'fuzzy space' is the quantized version of a 2-dimensional phase space, described by the 'coordinates' p and q. This example has the advantage of illustrating what is for us the essential interest of the non-vanishing commutator as expressed in the Heisenberg uncertainty relations. Since one cannot measure simultaneously p and q to arbitrary precision quantum phase space has no longer a notion of a point. It can however be thought of as divided into cells of volume $2\pi\hbar$. If the classical phase space is of finite total volume there will be a finite number of cells and the quantum system will have a finite number of possible states. A 'function' then on quantum phase space will be defined by a finite number of values and can be represented by a matrix.

By analogy with quantum mechanics we shall suppose that the generators q^μ can be represented as hermitian operators on some (complex) Hilbert space. The q^μ have real eigenvalues but they cannot be simultaneously diagonalized; points are ill-defined and space-time consists of elementary cells of volume $(2\pi\bar{k})^2$. Now when a physicist calculates a Feynman diagram he is forced to place a cut-off Λ on the momentum variables in the integrands. This means that he renounces any interest in regions of space-time of volume less than Λ^{-4}. As Λ becomes larger and larger the forbidden region becomes smaller and smaller but it can never be made to vanish. There is a fundamental length scale, much larger than the Planck length, below which the notion of a point is of no practical importance. The simplest and most elegant, if certainly not the only, way of introducing such a scale in a Lorentz-invariant way is through the introduction of the 'coordinates' q^μ. The analogs of the Heisenberg uncertainty relations imply then that

$$\Lambda^2\bar{k} \lesssim 1.$$

The existence of a forbidden region around each point in space-time means that the standard description of Minkowski space as a 4-dimensional continuum is redundant. There are too many points. Heisenberg already in the early days of quantum field

theory proposed to replace the continuum by a lattice structure. A lattice however breaks Poincaré invariance and can hardly be considered as fundamental. It was Snyder [5] who first had the idea of using non-commuting coordinates to mimic a discrete structure in a covariant way.

We argued above that the noncommutative structure gives rise to an ultraviolet cut-off. This idea has been developed by several authors since the original work of Snyder. It is the right-hand arrow of the diagram

$$\begin{array}{ccc} \mathcal{A}_{\hbar} & \Longleftarrow & \Omega^*(\mathcal{A}_{\hbar}) \\ \Downarrow & & \Uparrow \\ \text{Cut-off} & & \text{Gravity} \end{array} \tag{1.4}$$

The top arrow is a mathematical triviality; the $\Omega^*(\mathcal{A}_{\hbar})$ is what gives a differential structure to the algebra [1, 2, 3]. We have argued [4], not completely successfully, that each gravitational field is the unique 'shadow' in the limit $\hbar \to 0$ of some differential structure over some noncommutative algebra. This would define the left-hand arrow of the diagram. The composition of the three arrows is an expression of an old idea due to Pauli that perturbative ultraviolet divergences will one day be regularized by the gravitational field.

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The Modular Theory as Tool in Quantum Field Theory

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These investigations will use the quantum field theory in the frame of Araki, Haag and Kastler (see the textbook of R. Haag [Ha] for details). Due to the Reeh–Schlieder property of quantum field theory every algebra $\mathcal{M}(D)$ associated with a domain D, which has a spacelike complement with interiour points, has cyclic and separating vectors. The vacuum vector is an example. Therefore, we obtain by the Tomita–Takesaki theory [Tak] for every of these algebras a symmetry group

$$\sigma^t(\mathcal{M}(D)) = \Delta_D^{it}\mathcal{M}(D)\Delta_D^{-it} = \mathcal{M}(D).$$

Since all these local symmetry groups are present one should use them for the structure analysis of quantum field theory. In some special cases the modular group has been computed. For the general field theory our knowledge about the informations one can extract from the modular groups is very limited. The situation will be better if also half–sided translations exist besides the modular group.

Definition 1.:

Let $\mathcal{M}$ be a von Neumann algebra acting on $\mathcal{H}$ with cyclic and separating vector $\Omega \in \mathcal{H}$. The modular operator and the modular conjugation of this pair will be denoted by Δ and J.

1. *$\mathcal{H}str(\mathcal{M})^+$ denotes the set of one–parametric continuous unitary groups $U(t)$, $t \in \mathbb{R}$ with the properties:*

   α. $U(t)$ has a positive generator, i.e. we can write

$$U(t) = \exp\{iHt\}, \quad \text{with} \quad H \geq 0.$$

   β. $U(t)\Omega = \Omega \quad \forall \quad t \in \mathbb{R}$.
   γ. $\mathrm{Ad}\, U(t)\mathcal{M} \subset \mathcal{M}$ for all $t \geq 0$.

We call the groups belonging to $\mathcal{H}str(\mathcal{M})^+$ +half–sided translations associated with $\mathcal{M}$.

One also can define $\mathcal{H}str(\mathcal{M})^-$ if one replaces $\mathbb{R}^+$ by $\mathbb{R}^-$.

The importance of the concept of half–sided translations are given by the following result:

Theorem 1:

Let $U(s)$ be a continuous unitary group fulfilling $U(s)\mathcal{M}U(-s) \subset \mathcal{M}$ for $s \geq 0$. Then any two of the three conditions imply the third

a. $U(s) = e^{iHs}$ with $H \geq 0$.
b. $U(s)\Omega = \Omega$ for all $s \in \mathbb{R}$.

c. $\mathrm{Ad}\,\Delta^{\mathrm{i}t}(U(s)) = U(\mathrm{e}^{-2\pi t}s)$,
$JU(s)J = U(-s)$.

That a and b imply c has been shown in [Bch92]. The implication $b + c \longrightarrow a$ is a result of H.-W. Wiesbrock [Wie92]. The implication $a + c \longrightarrow b$ has been shown by Borchers [Bch97]. The last implication can be strengthened.

Lemma:

Assume $U(s)$ fulfils the conditions a. and c. of Thm. 1.

a. *If E_0 denotes the projection onto the eigenspace to the value 0 of H then E_0 commutes with $\Delta^{\mathrm{i}t}$.*
b. *If F_1 denotes the projection onto the eigenspace to the value 1 of Δ then $F_1 \leq E_0$.*

As a consequence of Thm.1 and the commutation of E_0 with F_1 one finds:

Proposition:

Let $U(s) \in \mathcal{H}str(\mathcal{M})^+$ be a half-sided translation with $U(s) = \mathrm{e}^{\mathrm{i}Ht}$, $H \geq 0$. Let E_0 be the projection onto the $U(s)$ invariant vectors. Then $\Delta^{\mathrm{i}t}$ commutes with E_0. $\Delta^{\mathrm{i}t}$ and $H^{\mathrm{i}s}$ fulfil the Weyl relation

$$\Delta^{\mathrm{i}t}H^{\mathrm{i}s} = \mathrm{e}^{-2\pi \mathrm{i}ts}H^{\mathrm{i}s}\Delta^{\mathrm{i}t}$$

on $(\mathbb{1} - E_0)\mathcal{H}$.

¿From this result one can conclude:

Theorem 2:

Assume $\mathcal{M}$ is a factor. If there exists a non-trivial group $U(s) \in \mathcal{H}str(\mathcal{M})^+$ or $U(s) \in \mathcal{H}str(\mathcal{M})^-$ then $\mathcal{M}$ is of type III_1.

Closely related to the half-sided translation is the concept of half-sided modular inclusions introduced by Wiesbrock [Wie93].

Definition 2:

Let $\mathcal{M}$ be a von Neumann algebra acting on $\mathcal{H}$ with cyclic and separating vector $\Omega \in \mathcal{H}$. The modular operator and the modular conjugation of this pair will be denoted by Δ and J.

By $\mathcal{H}smi(\mathcal{M})^-$ we denote the set of von Neumann subalgebras $\mathcal{N}$ of $\mathcal{M}$ with the properties:

α. *Ω is cyclic for $\mathcal{N}$. It is also separating for $\mathcal{N}$ since $\mathcal{N} \subset \mathcal{M}$.*
β. $\Delta^{\mathrm{i}t}\mathcal{N}\Delta^{-\mathrm{i}t} =: \mathrm{Ad}\,\Delta^{\mathrm{i}t}\mathcal{N} \subset \mathcal{N}$ *for* $t \leq 0$.

The elements of $\mathcal{H}smi(\mathcal{M})^-$ will be called the von Neumann algebras fulfilling the condition of $-$half-sided modular inclusion.

The set $\mathcal{H}smi(\mathcal{M})^+$ can be defined correspondingly. The relation between the half-sided translations and the half-sided modular inclusions is given by the following result:

Theorem 3:

Let $\mathcal{N} \in \mathcal{H}smi(\mathcal{M})^-$. Then there exists a group $U(t) \in \mathcal{H}str(\mathcal{M})^+$ such that the equation

$$\mathcal{N} = \mathrm{Ad}\,U(1)\mathcal{M}$$

holds.

The set $\mathcal{H}smi(\mathcal{M})^-$ consists of von Neumann subalgebras of $\mathcal{M}$. Hence it is semi-ordered (by inclusion). This order can be transported to $\mathcal{H}str(\mathcal{M})^+$ by the following result:

Theorem 4:

Let $\mathcal{N}_1, \mathcal{N}_2 \in \mathcal{H}smi(\mathcal{M})^-$ and $U_1(s), U_2(s)$ be their associated half-sided translations. Then $\mathcal{N}_1 \subset \mathcal{N}_2$ iff $\mathrm{e}^{-H_1 t} \leq \mathrm{e}^{-H_2 t}$ for all $t \geq 0$. Hence the bijection between $\mathcal{H}smi(\mathcal{M})^-$ and $\mathcal{H}str(\mathcal{M})^+$ preserves the order.

Applications:

A wedge in Minkowki space can be characterized by two different lightlike vectors ℓ_1, ℓ_2 belonging to the boundary of the forward lightcone V^+.

$$W(\ell_1, \ell_2) = \{\lambda \ell_1 - \mu \ell_2 + \ell^\perp; \lambda, \mu > 0, (\ell^\perp, \ell_i) = 0, i = 1, 2\}.$$

If $\mathcal{M}(W(\ell_1, \ell_2))$ is the algebra of this wedge then $U(\lambda \ell_1)$ belongs to $\mathcal{H}str(\mathcal{M}(W(\ell_1, \ell_2)))^+$. This example led to the concept of half-sided translations.

If we assume that the modular groups of the wedges act geometric (they do it always in the plane spanned by the two lightlike vectors) then the algebra associated with the intersection $W(\ell_1, \ell_2) \cap W(\ell_1, \ell_3)$ fulfils the condition of half-sided modular inclusion with respect to the two wedge algebras. This has been used in order to construct the whole Poincaré group starting from the modular group of the wedges [Bch96].

Recently H.-W. Wiesbrock has used these intersection of wedges with a common lightray in order to give an algebraic characterization of quantum field theories on the three-dimensional Minkowski space [Wie97].

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SCHROEDINGER EQUATION WITH MOVING POINT SOURCES

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Dedicato ad Alex Grossmann, con affetto e stima

I shall describe recent results, obtained in collaboration with R.Figari and A.Teta, on the theory of point interaction, one of the fields on which Alex has left his inprint.
Point interactions are useful in case of short range potentials when the wave length of the quantum particle is very long compared with the range of the potential. One expects then that the detailed shape of the potential be irrilevant and only the effetive parameters (typically the scattering lenght) come to play. A point interaction, properly defined, can then be used instead of the potential, with the advantage that the resolvent of the Schroedinger operator (and therefore the propagator and the scattering matrix) has an expicit form which lends itself to detailed calculations.
Examples of this procedure are the analysis made by E.Fermi of the scattering of slow neutrons and the Kronig-Penney model of a cristal. The results I describe are preliminary steps towards a more general theory in which the effect of the quantum particle on the motion of the point source is taken into account.
A mathematical study of point interactions, with applications, were made by A.Grossmann and other Authors [1].
For dimension $d = 1$ a point interaction is a small form perturbation of the Laplacian. For $d \geq 4$ there is no point interaction. For $d = 2, 3$ point interactions are best described using the theory of selfadjoint extensions of symmetric operators.
The Schroedinger operator with a point interaction of strenght $\alpha \in R$ placed in the point $y \in R^3$ is a selfadjoint extension of the Laplacian Δ defined on smooth functions of compact support and vanishing in some neihborhood of y.
Its domain is

$$D(H_{\alpha,y}) = \{u \in L^2(R^3) \,|u = \phi_\lambda + qG_\lambda(\cdot - y),\ \phi_\lambda \in H^2(R^3),$$

$$lim_{|x-y|\to 0}(u(x) - qG_0(x-y)) = \alpha q\}$$

where $G_\lambda(x - x')$ is the free Green's function with parameter λ. On $D(H_{\alpha,y})$ one has

$$(H_{\alpha,y} + \lambda)u = (-\Delta + \lambda)\phi_\lambda$$

We refer to [1] for a complete study of this operator, and for the case on N points.
We consider the Schroedinger equation with a point interaction of constant strenght moving on a preassigned path $y(t)$ of class C^2. Along the same lines one can treat the case of varying strenght and of N points moving on nonintersecting paths.
Consider therefore the initial value problem

$$i\frac{\partial u(t)}{\partial t} = H_{\alpha,y(t)}u(t), \qquad u(0) = u_0 \tag{1}$$

The problem is far from trivial because the form domain of the generator depends strongly on time and therefore the standard theory of non-autonomous evolution equations cannot

be applied. In fact we show that if $y(t)$ is not constant (1) cannot have a strong solution even for a smooth intial datum, and we must therefore look for weak solutions.
Notice that the situation is entirely different for the heat equation with point interactions [2]; for that case, exploiting the smoothing properties of the heat kernel, we proved existence and uniqueness of a strong solution, i.e. a function $w(t)$ which for all t belongs to $D(H_{\alpha,y(t)})$ and satifies in the L^2 sense the corresponding parabolic equation.
Let $\Phi_{\alpha,y}$ be the bilinear form associated to the selfadjoint operator $H_{\alpha,y(t)}$; one has

$$D(\Phi_{\alpha,y}) = \{u \in L^2(R^3) \mid u = \phi_\lambda + qG_\lambda(\cdot - y), \quad \phi_\lambda \in H^2(R^3)\}$$

$$\Phi_{\alpha,y}(u) = \int_{R^3} dx|\nabla\phi_\lambda(x)|^2 + \lambda\int_{R^3} dx|\phi_\lambda(x)|^2 - \int_{R^3} dx|u(x)|^2 + (\alpha + \frac{\sqrt{\lambda}}{4\pi}|q|^2 \qquad 2$$

It is easily seen that (2) is indipendent of λ.
To specify in which sense the solution we find is weak, let $V_{y(t)}$ be the Hilbert space $D(\Phi_{\alpha,y(t)})$ equipped with the scalar product $\Phi_{\alpha,y(t)}(u,v) + \lambda(u,v)$ with $\lambda > \alpha^2$ and let $V^*_{y(t)}$ be its dual with respect to L^2, with pairing $<,>$.
We prove

Theorem 1

Let $y(t)$ be a curve of class C^2 in R^3 and $u_0 \in C^\infty(R^3 - \{y(0)\})$. Then there exists a unique $u(t) \in V_{y(t)}$, $t \in R$ such that the Bochner derivative $\frac{du(t)}{dt}$ is in $V^*_{y(t)}$ and for all $v(t) \in V_{y(t)}$ the following identity is satisfied

$$i < v(t), \frac{du(t)}{dt} >= \Phi_{\alpha,y(t)}(v(t), u(t)) \qquad \forall t \qquad 3$$

Moreover $u(t)$ has the following representation

$$u(t) = U_t u_0 + i\int_0^t ds U_{t-s}(\cdot - y(s))q(s) \qquad 4$$

where U_t is the free Schroendiger propagator and the charge $q(t)$ satisfies a Volterra equation

$$q(t)\alpha\int_0^t ds\frac{q(s)}{\sqrt{t-s}} + \int_0^t dsq(s)C(t,s) = \int_0^t ds\frac{(U_s u_0)(y(s))}{\sqrt{t-s}}$$

where the kernel $C(t,s)$ is given explicitely in terms of $y(s)$ and $\dot{y}(s)$. ♡

Using the representation of the solution given in Theorem 1 one can moreover prove

Theorem 2

For any t the map $u_0 \to u(t)$ extends uniquely to a unitary map in $L^2(R^3)$. ♡

The proof of Theorem 1 is given in several steps. First one proves that if $y(t)$ is of class C^2 and

$$q(t) \in C^2_{loc}, \quad q(0) = 0, \ \dot{q}(0) = 0 \qquad 5$$

then $u(t)$ given by (4) belongs to $V_{y(t)}$.

Then one verifies that such $u(t)$ solves the problem (3) if and only if $q(t)$ satisfies and Abel-type integral equation of the form

$$\alpha q(t) + \int_0^t ds\dot{q}(s)\frac{B(t,s)}{\sqrt{t-s}} - i\int_0^t dsq(s)\frac{A(t,s)}{\sqrt{t-s}} = (U_t u_0)(y(t)) \qquad 6$$

with suitable kernels A, B.
Using the smoothing properties of the Abel kernel it is possible to prove that (6) admits a unique solution $q(t)$ which satisfies (5).

Remark 1

In Fourier trasform the solution we obtain has the following form

$$u(k,t) = \xi(k,t) + \frac{q(t)e^{-ik\cdot y(t)}}{(k+\dot{y}(t))^2+\lambda} \qquad \hat{\xi}(x,t) \in H^2(R^3) \qquad 7$$

It should be remarked that in the case $y(t) = y(0) + wt$, $w \in R^3$ the solution (7) can be obtained from the case $y(t) = y_0$ $\forall t$ through a Galilei transformation.
Pictorially, one may interpret (7) as the fact that the structure of the solution is best described in the inertial reference frame in which the point source is at rest. $\diamondsuit$

Remark 2

One can think of applying the techniques described here to the case of a wave equation with a point interaction. This has been already considered in [3] in the dipole approximation, clarifying in this case the role of the operator $H_{\alpha,0}$.
A straighforward application does not seem to be possible in the fully covariant case: indeed the integration by parts which is used to isolate the most singular terms, fail to do so in the relativisic case. A possible way out could be to use Remark 1 but this has not been attempted so far. $\diamondsuit$

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DE LA FIXATION DE JAUGE CONSIDEREE COMME UN DES BEAUX ARTS ET DE LA SYMETRIE DE SLAVNOV QUI S'ENSUIT

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Résumé

La fixation de jauge est définie comme l'opération permettant d'exprimer une intégrale sur un espace d'orbite comme intégrale sur le fibré principal correspondant. Quand la fibre est non compacte cette opération met en jeu une classe de cohomologie à support compact -ou à décroissance rapide- de celle-ci. La symétrie de Slavnov est l'expression algébrique de l'ambiguité de cette construction.

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1 Intégration de certaines classes de cohomologie équivariantes*

La fixation de jauge est l'un des détails techniques inévitables qui embarrasse le paysage des théories de jauge. Sa nécessité est dûe à la non compacité du groupe de jauge. La situation est la suivante : soit $\mathcal{A}$ un fibré principal de fibre $\mathcal{G}$, un groupe de Lie connexe non compact. On supposera $\mathcal{A}$ non trivial, en général. Par exemple, $\mathcal{A}$ sera l'espace des connexions principales $\underline{a}$ sur un fibré principal $P(M,G)$ où M est une variété compacte, G un groupe de Lie compact, et où $\mathcal{G}$ est le groupe de jauge de ce fibré défini de sorte que $\mathcal{A}$ soit un fibré principal. Les difficultés liées à la dimension infinie sont rappelées dans l'appendice B.

Soit Θ un représentatif d'une classe de cohomologie équivariante de $\mathcal{A}$ de dimension $|\mathcal{A}/\mathcal{G}|$ et $\tilde{\Theta}$ la forme basique obtenue en choisissant une connexion $\tilde{\omega}$ sur $\mathcal{A}$, de courbure $\tilde{\Omega}$. De façon générale, soit $\tilde{\Theta}$ une forme basique sur $\mathcal{A}$, de dimension $|\mathcal{A}/\mathcal{G}|$. On cherche une représentation intégrale sur $\mathcal{A}$ de l'intégrale $\int_{\mathcal{A}/\mathcal{G}}\tilde{\Theta}$, où, par abus de notation, on a confondu $\tilde{\Theta}$ avec la forme qu'elle définit sur $\mathcal{A}/\mathcal{G}$. Dans le cas des théories de jauge, on part d'une forme Θ $\mathcal{G}$ invariante sur $\mathcal{A}$, de degré maximum et on construit la forme de Ruelle Sullivan $\tilde{\Theta}_{RS}$ [?] associée à une forme volume invariante sur Lie $\mathcal{G}$, définie à un scalaire multiplicatif positif près (cf. paragraphe 2). Pour exprimer l'intégrale sur $\mathcal{A}/\mathcal{G}$, on peut choisir, en supposant $\mathcal{A}/\mathcal{G}$ paracompacte un recouvrement $\{U_i, i \in I\}$ localement fini et une partition de l'unité $\{\theta_i(\dot{a}_i), i \in I\}$ où $\{\dot{a}_i\}$ désigne un choix de coordonnées dans l'ouvert U_i, ainsi que des sections locales σ_i au dessus des U_i, représentées par des équations locales $g_i(a_i) = 0$ où les a_i sont des coordonnées locales de $\mathcal{A}$ au dessus de U_i. On peut alors écrire

$$\int_{\mathcal{A}/\mathcal{G}}\tilde{\Theta} = \sum_{i\in I}\int_{\mathcal{A}/\mathcal{G}}\theta_i(\dot{a})\tilde{\Theta})\int_{fibre}\delta(g_i)(\wedge\delta g_i) \tag{1.1}$$

où on a inséré l'intégrale sur la fibre d'un représentatif du dual de Poincaré de l'image $\sum_i$ de σ_i, au dessus de U_i. δ dénote la différentielle sur $\mathcal{A}$. Les g_i sont des fonctions à valeur dans un espace vectoriel de dimension $|\mathcal{G}|$ égale à la dimension de $\mathcal{G}$, et $\delta(g_i)(\wedge\delta g_i)$ est par construction indépendant du choix d'une base dans cet espace.

Comme $\tilde{\Theta}$ est de degré maximum, on peut restreindre $\wedge\delta g_i$ à la fibre en utilisant une connexion $\tilde{\omega}$ arbitraire sur $\mathcal{A}$ en écrivant

$$\delta g_i = \frac{\delta g_i}{\delta a_i}\delta a_i = \frac{\delta g_i}{\delta a_i}(\tilde{\psi}_i + \ell(\tilde{\omega})a_i) \tag{1.2}$$

où $\tilde{\psi}_i$ est la partie horizontale de δa_i pour la connexion $\tilde{\omega}$, $\ell(\tilde{\omega})$ désigne la dérivée de Lie le long du champ de vecteur fondamental correspondant à l'élément $\tilde{\omega}$ de Lie $\mathcal{G}$. La restriction à la fibre consiste à oublier $\tilde{\psi}_i$, autrement dit à travailler modulo l'idéal différentiel $\mathcal{I}_h^+$ engendré par les formes horizontales. Du même coup le choix de $\tilde{\omega}$ n'importe pas, la différence de deux connexions étant horizontale. Introduisant une "fonction δ fermionique" au moyen d'une intégration de Berezin, on peut récrire

$$\int_{\mathcal{A}/\mathcal{G}}\tilde{\Theta} = \int_a\int \mathcal{D}\omega\tilde{\Theta}(\wedge\tilde{\omega})\gamma(a,\omega) \tag{1.3}$$

*Les notions relatives à l'équivariance sont rappelées dans l'appendice A

avec

$$\gamma(a,\omega) = \sum_i \theta_i(\dot{a})\delta(g_i) \wedge m_i\omega \tag{1.4}$$

où

$$m_i\omega = \frac{\delta g_i}{\delta a_i}\ell(\omega)a_i \tag{1.5}$$

$\gamma(a,\omega)$ représente la projection sur la fibre de la forme

$$\gamma(a,\widetilde{\omega}) = \sum_i \theta_i(\dot{a})\delta(g_i) \wedge m_i\widetilde{\omega} \tag{1.6}$$

indépendamment du choix de $\widetilde{\omega}$.

Les volumes utilisés dans $\mathcal{D}\omega$ et dans $\wedge\widetilde{\omega}$ sont duaux l'un de l'autre.

Par construction, $\gamma(a,\widetilde{\omega})$ est de degré $|\mathcal{G}|$ puisque, les sections Σ_i étant transverses aux fibres, les opérateurs m_i sont inversibles. De plus, indépendamment de la fibre

$$\int_{fibre} \gamma(a,\widetilde{\omega}) = 1 \tag{1.7}$$

On peut prendre γ à support compact ou à décroissance rapide le long des fibres.

Il est clair que, dans le calcul précédent, on aurait pu remplacer γ par une forme de (fixation de) jauge avec les propriétés suivantes: γ représente la projection sur la fibre d'une forme de degré $|\mathcal{G}|$ d'intégrale sur la fibre égale à 1 quelque soit la fibre. Il s'ensuit que si s dénote la différentielle induite par projection sur la fibre $(\Omega^*(\mathcal{A}) \to \Omega^*(\mathcal{A})/\mathcal{I}_h^+)$, où $\mathcal{I}_h^+$ est l'idéal engendré par les formes horizontales de degré strictement positif)

$$\begin{aligned} s\omega &= -\frac{1}{2}[\omega,\omega] \\ sa &= \ell(\omega)a \end{aligned} \tag{1.8}$$

on reconnaît ici la partie géométrique de la symétrie de Slavnov[?] qui, ainsi qu'on le voit est géométriquement naturelle.

La différentielle s est bien la projection sur la fibre de l'opération s^{top} dérivée de la différentielle de l'algèbre de Weyl et de celle liée à l'action de $\mathcal{G}$ sur $\mathcal{A}$ (cf. Appendice A)

$$\begin{array}{ll} s^{top}\widetilde{\omega} = \widetilde{\Omega} - \frac{1}{2}[\widetilde{\omega},\widetilde{\omega}] & s^{top}a = \widetilde{\psi} + \ell(\widetilde{\omega})a \\ s^{top}\widetilde{\Omega} = -[\widetilde{\omega},\widetilde{\Omega}] & s^{top}\widetilde{\psi} = -\ell(\widetilde{\Omega})a - \ell(\widetilde{\omega})\widetilde{\psi} \end{array} \tag{1.9}$$

puisque $\widetilde{\psi}, \widetilde{\Omega}, [\widetilde{\omega},\widetilde{\Omega}], \ell(\widetilde{\omega})\widetilde{\psi}$ appartiennent à $\mathcal{I}_h^+$.

Comme γ est de degré maximum on a trivialement

$$s\gamma(a,\omega) = 0 \tag{1.10}$$

De plus, comme γ représente une classe de cohomologie à support compact (ou à décroissance rapide) de $\mathcal{G}$ (qu'on a supposé connexe), γ est défini à un cobord près :

$$\gamma \to \gamma + s\chi \tag{1.11}$$

où χ est la projection sur la fibre d'une forme à support compact ou à décroissance rapide.

Si $\mathcal{G}$ était compact, on pourrait choisir $\gamma = 1$ (la jauge unitaire des physiciens), mais dans le cas non compact ce choix est impossible.

Remarque

La construction ci-dessus, qui préconise l'intégration sur la fibre comme elle est esquissée dans le livre de J. Zinn Justin[?], fournit une alternative à la méthode initiale de Faddeev et Popov[?] basée sur la factorisation du volume du groupe de jauge qui n'est pas adaptée au cas d'un groupe $\mathcal{G}$ non compact, même en dimension finie.

L'espace des formes de jauge est non vide, par construction, convexe. Reste à construire des formes de jauge "commodes" et respectant la géométrie. A ce point, faute de mieux, on conjecture l'existence de formes de jauge du type

$$\int \mathcal{D}\bar{\omega}\mathcal{D}b \; e^{s(\bar{\omega}g(a,\dot{a})+i\bar{\omega}\varphi(b,\dot{a}))} \tag{1.12}$$

où l'opération s est étendue aux variables d'intégration selon

$$\begin{aligned} s\bar{\omega} &= ib \\ sb &= 0, \end{aligned} \tag{1.13}$$

où $g(a,\dot{a})$ est une fonction $\mathcal{A}$ jauge telle que $m(a,\dot{a})$ est partout inversible, et où $b\varphi(b,\dot{a})$ est positive croissante à l'infini. Si cette classe de formes de jauge est non vide, elle conduit vraisemblablement à des choix de jauge non renormalisables et ou non locaux, à moins qu'on ne réussisse à simuler les effets non locaux au moyen de l'introduction de champs locaux auxiliaires.

2 Le cas des théories de jauge

On se donne sur $\mathcal{A}$ une forme Θ $\mathcal{G}$ invariante de degré maximum. Si on se donne une forme volume μ invariante sur $\mathcal{G}$, il lui correspond une forme volume duale sur Lie $\mathcal{G}$ $\tilde{\mu}$. On définit

$$\Theta_{RS} = i(\tilde{\mu})\Theta \tag{2.1}$$

où le symbole de contraction $i(\tilde{\mu})$ est défini en représentant Lie $\mathcal{G}$ par les champs de vecteur fondamentaux appropriés. L'invariance de Θ assure que Θ_{RS} est fermée, et, Θ_{RS} est horizontale, par construction. On se ramène ainsi au problème décrit dans le paragraphe 1 avec

$$\tilde{\Theta}_{RS} \wedge \tilde{\omega} = \Theta. \tag{2.2}$$

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Appendice A

On rappelle la terminologie classique relative à la cohomologie équivariante.

Soit M une variété sur laquelle opère un groupe de Lie connexe $\mathcal{G}$ d'algèbre de Lie Lie $\mathcal{G}$ représentée par des champs de vecteurs sur M :

$$\lambda \in Lie\, \mathcal{G} \rightarrow \underline{\lambda} \in Vect\, M \tag{A.1}$$

tels que le crochet de Lie $\mathcal{G}$ soit représenté par le crochet de Lie de Vect M.

Soit $\Omega^*(M)$ l'algèbre des formes extérieures sur M munie de la différentielle d_M.

On définit sur $\Omega^*(M)$

$$i_M(\lambda) = i_M(\underline{\lambda}) \tag{A.2}$$

le produit intérieur par $\underline{\lambda} \in\ Vect\, M$

$$\ell_M(\lambda) = \ell_M(\underline{\lambda}) = [i_M(\underline{\lambda}), d_M] \tag{A.3}$$

la dérivée de Lie le long de λ.

Les formes horizontales ω_h sont celles pour lesquelles

$$i_M(\lambda)\omega_h = 0 \quad \forall \lambda \in Lie\, \mathcal{G} \tag{A.4}$$

Les formes invariantes ω_{inv} sont celles pour lesquelles

$$\ell_M(\lambda)\omega_{inv} = 0 \quad \forall \lambda \in Lie\, \mathcal{G} \tag{A.5}$$

Les formes qui sont à la fois horizontales et invariantes sont appelées basiques. La cohomologie basique de M est la cohomologie des formes basiques pour la différentielle d_M. Ces notions se généralisent à toute algèbre différentielle graduée commutative (E, d_E) avec une action de Lie $\mathcal{G}$: $i_E(\lambda)$, dérivation graduée de degré -1, $\ell_E(\lambda) = [i_E(\lambda), d_E]_+$ telles que

$$\begin{aligned} [i_E(\lambda), i_E(\lambda')] &= 0 \\ [\ell_E(\lambda), i_E(\lambda')] &= i_E([\lambda, \lambda']) \end{aligned} \tag{A.6}$$

Par exemple, l'algèbre de Weil de Lie $\mathcal{G}$ $\mathcal{W}(\mathcal{G})$ est définit au moyen des générateurs ω, Ω à valeur dans Lie $\mathcal{G}$, de degrés respectifs 1 et 2 avec les équations de structure

$$\begin{aligned} d_W\omega &= \Omega - \frac{1}{2}[\omega,\omega] \\ d_W\Omega &= -[\omega,\Omega] \\ i_W(\lambda)\omega &= \lambda \quad i_W(\lambda)\Omega = 0 \end{aligned} \tag{A.7}$$

La cohomologie équivariante de M est la cohomologie basique de $\Omega^*(M) \otimes \mathcal{W}(\mathcal{G})$ munie de la différentielle $d_M + d_W$ et de l'action $i_M(\lambda) + i_W(\lambda)$.

Appendice B

Nous rappellerons brièvement quelques-unes des difficultés bien connues que l'on rencontre en dimension infinie par exemple dans le cas des théories de jauge à la Yang Mills. Le point de départ, c'est-à-dire la forme invariante sur $\mathcal{A}$

$$\Theta = e^{-S_{inv}(a)}\mathcal{D}a \tag{B.1}$$

n'existe pas pour la même raison que la mesure de Haar sur le groupe de jauge n'existe pas. La forme de Ruelle Sullivan a des chances d'exister, modulo les problèmes ultraviolets que, précisément, on ne sait pas régler sur l'espace des orbites, mais on n'en a pas de forme explicite. Le point de départ est donc remplacé par une classe d'équivalence dont l'unicité n'est pas garantie. Il est concevable qu'il en résulte l'unicité pour les observables locales mais pas pour des "observables à l'infini". Il ne reste plus qu'à donner un sens à des formes différentielles de dimension infinies et à leurs intégrales qui font apparaître les comportements ultraviolets usuels qu'on ne sait maîtriser que grâce à la localité dans l'espace des champs. Quant à leur existence, basée sur l'existence de partitions de l'unité sur $\mathcal{A}/\mathcal{G}$, il faut se rappeler que ce dernier espace n'est métrique que pour la topologie L^2 [?]. Le problème de réconcilier des fixations de jauge géométriquement licites avec la localité de la théorie des champs reste donc ouvert.

Regular spectral triples and good algebras

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1. Regular spectral triples

a) Let us recall that, according to the terminology initiated by A. Connes [3], a smooth unital spectral triple is a triplet $\mathcal{K} = (\mathcal{A}, (\mathcal{H}, \pi), \mathcal{D})$ in which :

(1) : $\mathcal{A}$ is a unital involutive algebra, with unit denoted by $\mathbb{1}$ and involution denoted by * ;

(2) : π is a faithful *-morphism from $\mathcal{A}$ into the *-algebra $\mathcal{L}(\mathcal{H})$ of bounded linear operators on some separable Hilbert space $\mathcal{H}$;

(3) : $\mathcal{D}$ is an unbounded self-adjoint operator on $\mathcal{H}$ with compact resolvent, and such that for any element a in $\mathcal{D}$ the operator $[\mathcal{D}, \pi(a)] = \mathcal{D}.\pi(a) - \pi(a).\mathcal{D}$ lies in $\mathcal{L}(\mathcal{H})$;

(4) : the unital *-subalgebra $\pi(\mathcal{A})$ of $\mathcal{L}(\mathcal{H})$ is contained in the domain $\mathrm{Dom}(\delta^n)$ of the n-th power of the derivation $\delta = [|\mathcal{D}|, \cdot]$ on $\mathcal{L}(\mathcal{H})$ for every n in $\mathbb{N}$, where $|\mathcal{D}| = \sqrt{\mathcal{D}.\mathcal{D}^*}$.

Conditions (1) and (2) are the properties required in order that the triple $\mathcal{K}$ is a K-cycle over $\mathcal{A}$; the smoothness condition (3) ensures that for any a in $\mathcal{A}$, and for any $n \geq 0$ the n-th derivative of $\pi(a)$ with respect to the derivation $\partial = [\mathcal{D}, \cdot]$ is well defined.

b) Given a smooth unital spectral triple $\mathcal{K} = (\mathcal{A}, (\mathcal{H}, \pi), \mathcal{D})$ we shall denote by $\| \ \|$ the C*-norm operator on $\mathcal{L}(\mathcal{H})$, and for each n in $\mathbb{N}$, by $T^{(n)}$ the family : $T^{(n)} = (T_k)_{0 \leq k \leq n}$, where T_k denotes the seminorm defined on $\mathcal{A}$ by $T_0(a) = \|\pi(a)\|$ and $T_k(a) = \frac{1}{k!}\|\partial^k(\pi(a))\|$, $1 \leq k \leq n$.

Endly, we shall we let : $\| \ \|_{(n)} = \sum_{0 \leq k \leq n} T_k$. One easily proves (Cf : [1] and [4]) that :

Lemma. *(1) For any natural integer the followig properties hold :*

(1) : For any pair (a, b) of elements in in $\mathcal{A}$ one has : $\|a.b\|_{(n)} \leq \|a\|_{(n)}.\|b\|_{(n)}$.

(ii) : For any element a in $\mathcal{A}$ one has : $\|a^*\|_{(n)} = \|a\|_{(n)}$.

(iii) : For any element a in $\mathcal{A}$ one has : $\|a\|_{(n+1)} = \|a\|_{(n)} + \frac{1}{n+1!}\|\partial^{n+1}(\pi(a))\|$.

(2) : For any natural integer n, let $\mathcal{A}_n$ be the completion of $\mathcal{A}$ w. r. t. $\| \ \|_{(n)}$. Then :

*(i) : $\mathcal{A}_n$ is a unital Banach *-algebra, $n \in \mathbb{N}$ and $\mathcal{A}_0$ is a unital C*-algebra ;*

(ii) : $\mathcal{A}(\mathcal{K}) = \varprojlim \mathcal{A}_n = \bigcap_{n \geq 0} \mathcal{A}_n$ *is a unital Fréchet *-algebra .*

The topology of $\mathcal{A}(\mathcal{K})$, that we shall called the *$\mathcal{K}$-topology*, is of course given by the increasing countable family of *-norms $(\| \ \|_{(n)})_{n \in \mathbb{N}}$.

c) **Definition** . A smooth unital spectral triple $\mathcal{K}=(\mathcal{A}, (\mathcal{H}, \pi), \mathcal{D})$ such that $\mathcal{A} = \mathcal{A}(\mathcal{K})$ will be called a *regular spectral triple*.

It is easy to checks that, given any smooth unital spectral triple $\mathcal{K}=(\mathcal{A}, (\mathcal{H}, \pi), \mathcal{D})$, the triplet $\mathcal{K}^{\#} = (\mathcal{A}(\mathcal{K}), (\mathcal{H}, \pi), \mathcal{D})$ is a regular unital spectral triple. Replacing $\mathcal{A}$ by $\mathcal{A}(\mathcal{K})$ if necessary, we can then deal only with regular spectral triples. A first result is ([4]) :

Proposition : *Let $\mathcal{K}=(\mathcal{A}, (\mathcal{H}, \pi), \mathcal{D})$ be a regular spectral triple. Provided with the $\mathcal{K}$-topology, $\mathcal{A}$ is a unital Fréchet *-algebra. Morover, the group Inv$(\mathcal{A})$ of invertible elements in $\mathcal{A}$ is a generalized Lie group (in the Omori's sense given in [Omo]) with Lie algebra $\mathcal{A}$.*

2. Regularity of Inv($\mathcal{A}$) and goodness of $\mathcal{A}$

A new and important result is the following ([4], Theorem 1) :

Theorem. *Let $\mathcal{K}=(\mathcal{A}, (\mathcal{H}, \pi), \mathcal{D})$ be a regular spectral triple.The topological group.Inv$(\mathcal{A})$ is open in $\mathcal{A}$, and for any integer $k \geq 0$ one has : Inv$(\mathcal{A}_k) = \mathcal{A}_k \cap$ Inv$(\mathcal{A}_0)$.*

Let us recall now the notion of *good algebra* developped by J.-B. Bost [2] : a topological unital algebra $\mathcal{A}$ is a *good algebra* if the group Inv($\mathcal{A}$) is open in $\mathcal{A}$ and if the mapping $u \mapsto u^{-1}$ is continuous on Inv($\mathcal{A}$). One deduces important consequences of our theorem, in particular :

Corollary. *Let $\mathcal{K}=(\mathcal{A}, (\mathcal{H}, \pi), \mathcal{D})$ be a regular spectral triple. Then :*

*(i) : For any integer $n > 0$ the matrix algebra $M_n(\mathcal{A})$ of n by n matrices over $\mathcal{A}$ is a good unital Fréchet *-algebra, and Inv$(M_n(\mathcal{A}))$ is a generalized Lie group with Lie algebra $M_n(\mathcal{A})$.*

*(ii) : The path algebra $C(I, \mathcal{A})$ provided with the pointwise defined operations and the uniform convergence topology is a good unital Fréchet *-algebra. Moreover, at the level of the topological K-theory, one has an isomorphism from $K_j(C(I, \mathcal{A}))$ onto $K_j(\mathcal{A})$, $j = 1, 2$.*

(iii) : One has an isomorphism from $K_j(\mathcal{A}))$ onto $K_j(\mathcal{A}_0)$, $j = 1, 2$.

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SOME RECENT RESULTS ON RANDOM OPERATORS
Dedicated to Alex Grossmann

P. D. Hislop

ABSTRACT. We outline some recent developments in the theory of random operators concentrating on the spectral properties of additive and multiplicative random perturbations. These models describe the propagation of electrons and of classical waves in randomly perturbed media, respectively. The results presented concern the localization of states and the behavior of the integrated density of states at energies in intervals near the band edges of the spectrum of the unperturbed Hamiltonian.

1. Introduction

This is an overview of some recent results in the spectral theory of random operators obtained in collaborations with Jean-Marie Barbaroux, Jean Michel Combes, Eric Mourre, and Adriaan Tip. One of the goals of the study of randomly perturbed operators is to describe the propagation of quantum and classical waves in media which is randomly perturbed. One is particularly interested in the conditions under which random perturbations cause localization. Often, the background or unperturbed system is described by an operator H_0 which has only absolutely continuous spectrum. The spectrum is usually semibounded with a lower bound $E_0 > -\infty$. In the half-line $[E_0, \infty)$, the spectrum of H_0, consists of a union of closed intervals $\sigma(H_0) = \cup_{i=1}^{N}[a_i, b_i]$, where $N \leq \infty$ and $a_1 = E_0$. The energies satisfy $a_i < b_i$ and $b_i \leq a_{i+1}$. We call the interval $B_i = [a_i, b_i]$, the ith band. In the case that $b_i < a_{i+1}$, we say that the ith gap is open. A bounded perturbation of H_0 will broaden the bands B_i, adding energy levels in a neighborhood of the band edges a_i and b_i. If the perturbation is random, the almost sure spectral type near the band edges may be quite different from that of H_0. In particular, randomness tends to localize the states. Much recent work has been directed toward proving the assertion that randomness causes the states with energies near the band edges to localize almost surely, and that the corresponding eigenfunctions decay exponentially. We will discuss many of these results below. We are also interested in the integrated density of states for these models at energies near the band edges.

Key words and phrases. random operators, Wegner estimates, localization.

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1.1. Additive Perturbations. Additively perturbed families of random operators on $L^2(R^d)$, for $d \geq 1$, have the form

$$H_\omega = H_0 + V_\omega, \tag{1.1}$$

where the random potential $V_\omega(x)$ is a stochastic process on R^d. These families of operators describe the propagation of single electrons through a randomly perturbed medium.

1.2. Multiplicative Perturbations. The wave equation for acoustic and electromagnetic waves lead us to families of operators having the form

$$H_\omega = A_\omega^{-1/2} H_0 A_\omega^{-1/2}, \tag{1.2}$$

where $A_\omega(x)$ belongs to a permissible class of stochastic processes.

In the discussion to follow, we will assume that the background operator H_0 has an open spectral gap $G = (B_-, B_+)$, with $B_- = b_i$ and $B_+ = a_{i+1}$, with $b_i < a_{i+1}$ for some i, in its spectrum.

2. Single Electron Transport

There are several models of physical relevance which describe the behavior of a single electron in randomly perturbed media. We have the choice of the background operator and of the random potential. The most well known situation is when the background operator $H_0 = -\Delta + V_{per}$, where V_{per} is a periodic function on R^d. This unperturbed operator describes a single electron moving in an infinite, periodic lattice. The basic random perturbation is the Anderson-type potential described by a family of functions u_i, with $i \in \mathbb{Z}^d$, called the *single-site potentials*, and coupling constants $\{\lambda_i(\omega) \mid i \in \mathbb{Z}^d\}$. The potential is defined by

$$V_\omega(x) \equiv \sum_{i \in \mathbb{Z}^d} \lambda_i(\omega) u_i(x). \tag{2.1}$$

The family of coupling constants $\{\lambda_i(\omega) \mid i \in \mathbb{Z}^d\}$ forms a stochastic process on $\mathbb{Z}^d$. In the simplest model, the coupling constants form a collection of independent, identically distributed (*iid*) random variables. In the *iid* case, the random operators are ergodic if $u_i(x) = u(x - i)$, for all $i \in \mathbb{Z}^d$. More complicated models treat the case of correlations between the random variables. We can also introduce another family of vector-valued random variables $\{\xi_i(\omega) | \, i \in \mathbb{Z}^d\}$ with $\xi_i(\omega) \in B_R(0)$, $0 < R < \frac{1}{2}$. We assume that the random variable ξ_i has an absolutely continuous distribution, for example, a uniform distribution. These random variables will model thermal fluctuations of the scatterers with random strengths about the lattice points $\mathbb{Z}^d$. The random potential has the form

$$V_\omega(x) \equiv \sum_{i \in \mathbb{Z}^d} \lambda_i(\omega) u_i(x - \xi_i). \tag{2.2}$$

The Anderson model on the lattice has been extensively studied. We refer to the books by Carmona and Lacroix [**6**] and by Pastur and Figotin [**38**] for the references. The basic papers, of relevance to the approaches described here for the continuous cases, include [**26, 27, 41, 42, 48**]. There are now many results on localization at the bottom of the spectrum and at the band edges for the continuous Anderson model. These include [**3, 7, 33, 30, 31, 32**]. The case of correlated random variables for lattice models is described in [**49**] and for the continuous models in [**10, 11**].

In general, the background operator for electrons may have the form

$$H_0 = (p - A)^2 + V_0, \tag{2.3}$$

where $p \equiv -i\nabla$, the vector-valued function A is a reasonable magnetic potential and V_0 is a background potential, cf. [**3**]. The Landau Hamiltonian is in two dimensions is a special case with a constant magnetic field B generated by the potential $A = (B/2)(-x_2, x_1)$, and $V_0 = 0$. Localization and the integrated density of states for this model have been extensively studied, see [**4, 8, 14, 15, 50, 51**], because of the role it plays in the theory of the integer quantum Hall effect (cf. [**37**]).

There are other models of randomness of physical interest. We give a brief description of them here. In sections 3–5, we will concentrate on results for the Anderson-type potential, although they also hold for some of the other models.

2.1. Breather Model. Let $u > 0$ be a single-site potential of compact support. We require a repulsive condition on u: $-x \cdot \nabla u(x) \geq 0$, and a relative boundedness condition on the Hessian of u. We assume that the random variables λ_i are *iid* with a common density supported some positive interval. The breather potential has the form

$$V_\omega(x) \equiv \sum_{i \in \mathbb{Z}^d} u(\lambda_i(\omega)(x - i)). \tag{2.4}$$

One can show that the random family $H_\omega = -\Delta + V_\omega$ has an interval of localized states $[0, E_1]$, almost surely, for some $E_1 > 0$ [**9**].

2.2. Random Displacement Model. The random potential for this model has the form

$$V_\omega(x) \equiv \sum_{i \in \mathbb{Z}^d} u(x - i - \xi_i(\omega)), \tag{2.5}$$

where the vector-valued random variables $\{\xi_i(\omega)\}$ are distributed in a ball of radius $R < 1/2$, as above. This model was studied in the case of $u \leq 0$ so that the single-site potential represents a potential well. Klopp [**34**] showed that in this case the model exhibits localization at negative energies provided the semiclassical parameter h, appearing in the Hamiltonian as $H(h) = -h^2\Delta + V_\omega$, is sufficiently small. Quantum tunneling plays a major role in the localization of states for this model. Localization in the general case is an open problem.

2.3. Poisson Model. One of the most realistic models of impurities randomly distributed in a perfect crystal is given by a Poisson potential. Let $X_i(\omega)$, $i \in \mathbb{Z}$, represent the points of a Poisson process in R^d. Suppose that $u \geq 0$ is a single-site potential of compact support. The Poisson potential is given by

$$V_\omega(x) \equiv \sum_{i \in \mathbb{Z}} u(x - X_i(\omega)). \tag{2.6}$$

Surprisingly, very little is known about these potentials except in one dimension. Stolz [**45**] recently proved that all states are localized for the one-dimensional model. In arbitrary dimensions, Tip [**46**] has proved that for a class of repulsive potentials u, the integrated density of states is absolutely continuous at high energies. As above, localization in the general case is an open problem.

2.4. Gaussian Models. This random family of Schrödinger operators has the form (1.1), with $H_0 = -\Delta$ and the random potential $V_\omega(x)$ a Gaussian process indexed by $\mathbb{R}^d$. This model has been recently studied [**24, 25**]. The authors prove a Wegner estimate and show the absence of absolutely continuous spectrum.

3. Classical Wave Propagation

The localization of acoustic waves and of light is of theoretical and practical importance. Because of the absence of the electron-electron interaction, which might tend to obscure localization effects, it may be easier to detect the localization of light experimentally. For a review of these questions, we refer to [**5, 28**]. The techniques used to prove band-edge localization for electrons can also be used for various models of classical waves propagating in inhomogeneous media. Some of the early work on localization for lattice models describing waves was done by Faris [**16, 17**] and Figotin and Klein [**19**]. We describe here the models studied in [**12**].
(1) Acoustic waves. The wave equation for acoustic waves propagating in a medium with sound speed C and density ρ is

$$\partial_t^2 \psi + \hat{H}\psi = 0, \tag{3.1}$$

where the propagation operator $\hat{H}$ is given by

$$\hat{H} \equiv -C^2 \rho \nabla \rho^{-1} \nabla. \tag{3.2}$$

By a standard unitary transformation, it suffices to consider the operator H, unitarily equivalent to $\hat{H}$, given by

$$\begin{aligned} H &\equiv -C\Delta C - \frac{1}{2}\left\{\frac{C^2\Delta\rho}{\rho} - \frac{3}{2}\frac{C^2|\nabla\rho|^2}{\rho^2}\right\} \\ &= -(C^2\rho)^{\frac{1}{2}}\nabla\cdot\rho^{-1}\nabla(C^2\rho)^{\frac{1}{2}}, \end{aligned} \tag{3.3}$$

acting on the Hilbert space $\mathcal{H} = L^2(R^d)$, $d \geq 1$. We consider perturbed sound speeds of the form

$$C_\omega(g) \equiv (1 + g\tilde{C}_\omega)^{-1/2} C_0, \tag{3.4}$$

for $g \geq 0$. To relate this to (1.2), we factor out the unperturbed sound speed C_0 and define the unperturbed acoustic wave propagation operator H_0 by

$$H_0 \equiv -C_0 \rho^{\frac{1}{2}} \nabla \cdot \rho^{-1} \nabla \rho^{\frac{1}{2}} C_0. \tag{3.5}$$

The coefficient A_ω appearing in (1.2) is given by

$$A_\omega \equiv (1 + g\tilde{C}_\omega). \tag{3.6}$$

(2) Electromagnetic waves. The wave equation for electromagnetic waves can be written in the form of equation (1.2) for vector-valued functions ψ. In this case, the operator H describing the propagation of electromagnetic waves in a medium characterized by a dielectric function ϵ and a magnetic permeability $\mu = 1$ is given by

$$H \equiv -\epsilon^{-1/2}\Delta\Pi\epsilon^{-1/2}, \tag{3.7}$$

acting on the Hilbert space $\mathcal{H} = L^2(R^3, \mathbb{C}^3)$. The matrix-valued operator Π is the orthogonal projection onto the subspace of transverse modes. We consider random

perturbations of a background medium described by a dielectric function ϵ_0 and given by

$$\epsilon_\omega(g) \equiv 1 + \epsilon_0 + g\tilde{\epsilon_\omega}, \tag{3.8}$$

where $\tilde{\epsilon_\omega}$ is a stochastic process. The unperturbed operator describing the background medium is defined by

$$H_0 \equiv -(1+\epsilon_0)^{-1/2}\Delta\Pi(1+\epsilon_0)^{-1/2}, \tag{3.9}$$

and the coefficient A_ω in (1.2) is given by

$$A_\omega \equiv (1 + g(1+\epsilon_0)^{-1}\tilde{\epsilon_\omega}). \tag{3.10}$$

We note that A_ω is the velocity of light for the realization ω.

There have been several recent papers proving band-edge localization for these and related models. Figotin and Klein [**20, 21**] have results on band edge localization quite similar to ours. Stollmann [**43**] recently studied random perturbations of metrics. The random processes $\tilde{C_\omega}$ and $\tilde{\epsilon_\omega}$ can be of any of the type described in section 2.

4. The Main Results on Band-Edge Localization

We now list our hypotheses necessary to prove localization at energies near the unperturbed band-edges $B_\pm$. We will begin with conditions of the deterministic operator H_0.

HYPOTHESIS H1. The self-adjoint operator H_0 is essentially self-adjoint on $C_0^\infty(\mathbb{R}^d)$. It is semi-bounded and has an open spectral gap. That is, there exist constants $-\infty < -C_0 \le B_- < B_+ \le \infty$ so that

$$\sigma(H_0) \subset (-C_0, B_-] \cup [B_+, \infty).$$

HYPOTHESIS H2. The operator H_0 is strongly locally compact in the sense that for any $f \in L^\infty(\mathbb{R}^d)$ with compact support, the operator $f(H_0)(H_0+C_0+1)^{-1} \in \mathcal{I}_q$, for some even integer q, with $1 \le q < \infty$. Here, $\mathcal{I}_q$ denotes the qth-Schatten class, cf. [**40**].

HYPOTHESIS H3. Let $\rho(x) \equiv (1+\|x\|^2)^{1/2}$. The operator

$$H_0(\alpha) \equiv e^{i\alpha\rho} H_0 e^{-i\alpha\rho} ,$$

defined for $\alpha \in \mathbb{R}$, admits an analytic continuation as a type-A analytic family to a strip

$$S(\alpha_0) \equiv \{x + iy \in \mathbb{C} \mid |y| < \alpha_0\} ,$$

for some $\alpha_0 > 0$.

Schrödinger operators with periodic potentials provide examples of operators H_0 with open spectral gaps, cf. [**39**]. As for classical waves, certain models of photonic crystals are known to have open gaps, cf. [**22, 23, 47**]. Hypothesis (H3) is satisfied for a large class of operators H_0, see [**3, 12**].

We now give hypotheses on the random potential given in (2.1), which are conditions of the single-site potentials u_j and the coupling constants λ_j. We will use the following notation. We denote by $B_r(x)$ the ball of radius r centered at $x \in \mathbb{R}^d$. A cube of side length ℓ centered at $x \in \mathbb{R}^d$ is denoted by $\Lambda_\ell(x)$. The characteristic function of a subset $A \subset \mathbb{R}^d$ is denoted by χ_A.

HYPOTHESIS H4. The single-site potentials $u_i \geq C_i \chi_{B_{r_i}(0)}$ for some constant $C_i > 0$ and some radius $r_i > 0$. Furthermore, we assume that

$$\sum_{j \in \mathbb{Z}^d} \left\{ \int_{\Lambda_1(0)} |u_j(x)|^p \right\}^{1/p} < \infty, \tag{4.1}$$

for $p \geq d$ when $d \geq 2$ and $p = 2$ when $d = 1$.

We will assume that the random variables $\lambda_j(\omega)$, appearing in the Anderson-type potential (2.1), form a stationary stochastic process indexed by $\mathbb{Z}^d$. The probability space for this process is $\Omega = [-m, M]^{\mathbb{Z}^d}$, for some constants $m \neq M$ and $0 \leq m, M \leq \infty$. In the case that the range of the random variables is unbounded, we will need to control some of the moments of λ_j.

HYPOTHESIS H5. The random variables λ_j have p finite moments:

$$\mathbb{E}\{\lambda_j^k\} < \infty, \qquad k = 1, 2, \ldots, p, \tag{4.2}$$

where p is the dimension-dependent constant given in (H4).

We refer to the review article of Kirsch [**29**] for a proof of the fact that hypotheses (H4)–(H5) imply the essential self-adjointness of H_ω on $C_0^\infty(\mathbb{R}^d)$.

HYPOTHESIS H6. The conditional probability distribution of λ_0, conditioned on ${\lambda_0}^\perp \equiv \{\lambda_i \mid i \neq 0\}$, is absolutely continuous with respect to Lebesgue measure. It has a density h_0 satisfying $\|h_0\|_\infty < \infty$, where the sup norm is defined with respect to the probability measure $\mathbb{P}$.

Hypothesis (H5) implies that the correlation function $C(i,j) \equiv \mathbb{E}\{\lambda_i \lambda_j\} - \mathbb{E}\{\lambda_i\}\mathbb{E}\{\lambda_j\}$ exists and is finite. An example of a process satisfying (H5) is a Gaussian process on $\mathbb{Z}^d$ with each local covariance function $C(i,j), i, j \in \Lambda$ being a bounded, invertible matrix. In the case that the random coupling constants are *iid* random variables, hypothesis (H6) reduces to the usual assumption that a density exists.

HYPOTHESIS H7. The density h decays sufficiently rapidly near $-m$ and near M in the following sense:

$$0 < \mathbb{P}\{|\lambda + m| < \varepsilon\} \leq \varepsilon^{3d/2+\beta},$$

$$0 < \mathbb{P}\{|\lambda - M| < \varepsilon\} \leq \varepsilon^{3d/2+\beta},$$

for some $\beta > 0$.

Recent work of Klopp [**35**] on the existence of internal Lifshitz tails may allow us to remove hypothesis (H7).

We need to assume the existence of deterministic spectrum Σ for families of randomly perturbed operators as in (1.1)–(1.2) with H_0 satisfying (H1)-(H7). If H_0 is periodic with respect to the translation group $\mathbb{Z}^d$, and for Anderson-type perturbations described above, the random families of operators H_ω are measurable, self-adjoint, and $\mathbb{Z}^d$-ergodic. In this case, it is known (cf. [**6, 38**]) that the spectrum of the family is deterministic.

HYPOTHESIS H8. There exists constants $B'_\pm$ satisfying $B_- < B'_- < B'_+ < B_+$ such that

$$\Sigma \cap \{(B_-, B'_-) \cup (B'_+, B_+)\} \neq \emptyset\,.$$

In light of hypothesis (H8), we define the band edges of the almost sure spectrum Σ near the gap G, as follows:

$$\tilde{B_-} \equiv \sup\{E \leq B'_- \mid E \in \Sigma\},$$

and

$$\tilde{B_+} \equiv \inf\{E \geq B'_+ \mid E \in \Sigma\},$$

Examples of operators satisfying these conditions may be found in [**3, 12, 20, 21**]. The main results are the following two theorems.

THEOREM 4.1. *Assume (H1) - (H8). There exist constants $E_\pm$ satisfying $B_- \leq E_- < B_-$ and $B_+ < E_+ \leq B_+$ such that $\Sigma \cap (E_-, E_+)$ is pure point with exponentially decaying eigenfunctions.*

THEOREM 4.2. *Assume (H1) - (H8). The integrated density of states is Lipschitz continuous on the interval (B_-, B_+).*

5. Basic Ideas: Local Hamiltonians and Wegner Estimates

A key aspect of the proof of localization is the analysis of local Hamiltonians which describe a local, compactly supported perturbation of the background operator H_0. To unify notation, we let V_ω denote the additive random potential or the random variables $\tilde{C_\omega}$ or $\tilde{\epsilon_\omega}$ in the multiplicative cases. The local potential is defined by

$$V_\Lambda(x) = V_\omega(x) \mid \Lambda. \tag{5.1}$$

The local Hamiltonian associated with Λ is defined by

$$H_\Lambda \equiv H_0 + V_\Lambda, \tag{5.2}$$

in the additive case, and by

$$H_\Lambda \equiv (1 + gV_\Lambda)^{-1/2} H_0 (1 + gV_\Lambda)^{-1/2}, \tag{5.3}$$

in the multiplicative case, where H_0 is the deterministic background operator appearing in (1.1) and (1.2). We remark that if the single-site potential u has non-compact support or if the coupling constants $\lambda_j(\omega)$ occurring in (1.1)-(1.2) are correlated, the local Hamiltonians associated with two disjoint regions are not independent. In these cases, one has to modify the usual arguments (cf. [**32**] and [**10, 11**]). Let us assume here that the local Hamiltonians for two disjoint regions are independent.

We want to emphasize that the choice of a local Hamiltonian of the form (5.2)–(5.3) has many advantages over the choice of a self-adjoint extension of $H_\omega \mid \Lambda$, which depends on the boundary conditions. First among them is the fact that the essential spectrum of the unperturbed operator H_0 is preserved under the relatively-compact perturbation V_Λ. This allows one to work at energies in the spectral gap G of H_0. All known models for which localization has been proved are of this type (cf. [**3, 8, 12, 20, 21, 31, 33, 34**]). Secondly, the resolvent $R_0(E) \equiv (H_0 - E)^{-1}$, for $E \in G$, decays exponentially in the distance (cf. [**3, 13**]) when localized between two functions with separated supports. This fact contributes significantly to the proof of the Wegner estimate which has a $|\Lambda|$ volume dependence.

There are two main estimates on the local Hamiltonians which are necessary for the multiscale analysis: a Wegner estimate, and an exponential decay estimate (cf. [**7, 26, 41, 42, 48**]). A Wegner estimate is a measure of the probability of

resonance between the spectrum of a local random Hamiltonian and a fixed energy E. It is an upper bound on the probability that the distance ϵ from the spectrum of a local Hamiltonian H_Λ, associated with a region Λ, to a fixed energy E, is less than a specified value depending on the volume $|\Lambda|$ and the distance ϵ.

The main theorem on the eigenvalue distribution of the finite volume Hamiltonians is the following. The Lipschitz continuity of the integrated density of states follows from this result. For other versions of Wegner's estimate, see [**20, 33, 30, 31, 44**]

THEOREM 5.1. *Assume* (H1)–(H6). *Let Λ be a bounded, open region in $\mathbb{R}^d$. For any $E \in G = (B_-, B_+) \subset \rho(H_0)$, and for any $\epsilon < \frac{1}{2}\text{dist}\ (E, \sigma(H_0))$, there exists a finite constant C_E, depending on $d, \|h_0\|_\infty$, and* dist $(E, \sigma(H_0))$, *such that*

$$\mathbb{P}\{\text{dist}(\sigma(H_\Lambda), E) < \epsilon\} \leq C_E \epsilon |\Lambda|. \tag{5.4}$$

The proof of this theorem follows the lines of that given in [**3**] and [**4**]. The integration-by-parts used in the proof of the spectral averaging theorem is possible due to hypothesis (H5) on the absolute continuity of the conditional probability distribution.

The second main result on the local Hamiltonians is partly probabilistic. We show that there exists $\delta > 0$ such that the probability that $H_{\Lambda_{\ell_0}}$, has no eigenvalue in $[\tilde{B}_- - \delta, \tilde{B}_-] \cup [\tilde{B}_+, \tilde{B}_+ + \delta]$ is greater than $1 - \ell_0^{-\xi}$, for some ℓ_0 sufficiently large and some $\xi > 2d$. An application of the Combes-Thomas argument [**3, 13**] shows that the Green's function for $H_{\Lambda_{\ell_0}}$, at energies in $[\tilde{B}_- - \delta/2, \tilde{B}_-] \cup [\tilde{B}_+, \tilde{B}_+ + \delta/2]$ decays exponentially in $|x-y|$ with the same probability. These results, the Wegner estimate, and the multiscale analysis allow us to prove almost sure exponential decay of the resolvent of the Hamiltonian H_ω for all energies in an interval $[\tilde{B}_- - \delta/2, \tilde{B}_-] \cup [\tilde{B}_+, \tilde{B}_+ + \delta/2]$, near the edges of the spectral gap of Σ. This estimate has the form

$$\mathbb{P}\{\sup_{\epsilon>0} \|\chi_x R_\omega(E + i\epsilon)\chi_y\| \leq e^{-m_E|x-y|}\} = 1, \tag{5.5}$$

for a constant $m_E > 0$, and for all $x, y \in \mathbb{R}^d$ such that $|x - y|$ is sufficiently large. The functions χ_x and χ_y are compactly supported in a small neighborhood of x and y, respectively.

The proof of localization is competed using a result of the perturbation of singular spectra, see [**7, 9, 27, 36, 41**].

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Deterministic spin models with glassy behaviour and quantized toral symplectomorphisms

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Roughly speaking, the spin glasses are spin systems not ferromagnetic (or anti) on any scale, non translation invariant. The ground state(s) oriented so that $\frac{1}{N}\sum_{x=1}^{N}\sigma_x = 0$; at $T = T_c : M = \frac{1}{N}\sum_{x=1}^{N} m_x = 0$ but $q = \frac{1}{N}\sum_{x=1}^{N} m_x^2 > 0$ (EA order parameter). $\sigma_i = \pm 1$ and $m_i = \langle\sigma_i\rangle$ are spin and magnetization at site i, respectively. The systems freezes and settes into the ground state(s) through a complex low temperature phase (there are many pure magnetization states in competition).

To reproduce glassy behaviour, one assignes a "random potential" $J_{x,y}$ to the spins. The simplest "solvable" case is (Sherrington-Kirkpatrick 1975): $d = 1$, $J_{x,y}$ i.i.d. (Gaussian), mean 0, variance $\overline{J}/N$ (infinite-range).
"Solution" 1 (Thouless-Anderson-Palmer, 1977): The Gibbs free energy $\Phi(m_1, \ldots, m_N; \beta)$ is computed in closed form. Its stationarity conditions (TAP equations)

$$m_x = \tanh\left(\beta\sum_{y=1}^{N} J_{x,y}m_y - \beta m_i(1 - q_{EA})\right) \tag{1}$$

yield the pure magnetizetion states. Linearization and spectral theory of J yield $\beta_c = \overline{J}$ (variance). Howver, in the temperature phase, the number of solutions (=pure states) increase exponentially. What is the order parameter?
"Solution" 2 (Parisi, 1980-1983). How to decompose the Gibbs state on the pure ones? Set:

$$q_{\alpha,\beta} = \frac{1}{N}\sum_{x=1}^{N} m_x^\alpha m_x^\beta \qquad \text{overlap}$$

$$P(q) = \overline{P_J(q)} = \sum_{\alpha,\beta}\overline{w_\alpha^J w_\beta^J \delta(q_{\alpha,\beta} - q)}$$

(overlap probability distribution)

$$w_\alpha \approx e^{-\beta F_\alpha}, \qquad F_\alpha = \text{free energy of } m_\alpha$$

$$x(q) = \int_0^q P(u)\,du, \quad q = q(x)$$

$q(x) \in C[0,1], q(1) = q_{EA}$ is the Parisi order parameter describing the glassy phase. Computation of $q(x)$: replica method+broken replica symmetry.

Problem: is there a *deterministic* interaction J reproducing (at least partially) the glassy behaviour of SK? Prerequisites: J should oscillate, and be of "size" $1/\sqrt{N}$. Marinari-Parisi-Ritort 1994:

$$J_{xy} = \frac{1}{\sqrt{2N+1}} \sin \frac{2\pi}{N} xy \tag{2}$$

Ground-state: for $2N+1 = 4p+3$,

$$\sigma_x = \left(\frac{x}{N}\right) = \begin{cases} +1, x = p^2 \bmod N \\ -1, x \neq p^2 \bmod N \end{cases} \tag{3}$$

Glassy behaviour detected by numerical implementation of the replica method; the average is performed over the Haar measure of the orthogonal group.
Now, remark:
$J_{xy} \equiv ImV_A$, V_A- unitary propagator quantizing the unit symplectic matrix on T^2:

$$V_A(x,y) = \frac{1}{\sqrt{N}} e^{\frac{2\pi i}{N} xy}$$

Recall the quantization procedure for any symplectomorphism $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $a, b, c, a \in Z$, $ad - bc = 1$ of T^2 through representations of the discrete Weyl-Heisenberg group (Degli Esposti 1991). For $n \in Z^2, \phi \in T^2$ set $t(n) = e^{i\langle n, \phi\rangle}$ and look for all unitary representations in $L^2(S^1)$ of the discrete Weyl-Heisenberg algebra

$$Op^W(t(n))Op^W(t(m)) = e^{\frac{2\pi i}{\hbar}\omega(m,n)}$$
$$\omega(m,n) = m_1 n_2 - m_2 n_1$$

For $\hbar = 1/N$, ∞^2 N-dimensional representations indexed by $\theta \in [0, 2\pi[^2$. Hence $Op^W(t(n)) = T_\theta(n)$. Now $T_\theta(An) = T_{A(\theta)}(n) \Longrightarrow \exists\ V_{A,\theta}$ such that $V_{A,\theta} T(n) V_{A,\theta}^{-1} = T_\theta(An)$. This defines $V_A(\theta)$. If $A = \begin{pmatrix} 2g & 1 \\ 4g^2 - 1 & 2g \end{pmatrix}$, we can take $\theta = 0$.
$A \mapsto V_A$ is metaplectic representation; moreover

$$V(A)_{xy} = \frac{1}{\sqrt{N}} \exp \frac{2\pi i}{N} (gx^2 - xy + gy^2) \tag{4}$$

For $g = 0$, we get the discrete Fourier transform: $(V(A)\psi)_x = \frac{1}{\sqrt{N}} \sum_{y=0}^{N-1} \exp -\frac{2\pi i xy}{N} \psi_y$.

APPLICATIONS OF THE FORMAL IDENTITY.
1. What happens to the ground state for general N?
Reply (I.Borsari,F.Unguendoli, S.G., J.Phys.A, 1996): it disappears for N even, while it becomes asymptotically degenerate ($\sim N^\alpha$ states with energy differing by less than $N^{-\alpha}$ from the minimum) for N odd.

Does it imply the existence of two thermodynamic limits? This is an open question (related to the thermodynamic chaos).
Proof: Explicit construction of an eigenvector set through the Gaussian sums containing primitive multiplicative characters of the ring Z_N.
2. TAP (or mean field) equations:

$$m_x = \tanh\left(\beta \sum_{y=1}^{N} J_{x,y} m_y - 2\beta m_x G'(\beta(1-q))\right) \tag{5}$$

$$G(\beta) = -\frac{1}{4}[\ln\sqrt{1+4x^2} + 1 + \sqrt{1+4x^2} - 1] \tag{6}$$

(Parisi-Potters (1996) through Legendre transform of the resummation of the high temperature expansion for the Helmholtz free energy: TAP $G = x^2/4$). # of solutions exponentially increasing ; the linearization however does not determine the critical temperature numerically detected.
Consider the models whose coupling is defined by (Re or Im of) quantized hyperbolic (hence chaotic) maps: example, (4) with $g \neq 0$:

$$J_{x,y} = \frac{1}{\sqrt{N}} \cos\frac{2\pi}{N}(gx^2 - xy + gy^2) \tag{7}$$

Then (I.Borsari, M.Degli Esposti, F.Unguendoli, S.G, J.Phys.A 1997) (6) becomes

$$G(\beta) = \beta^2/(8 + 4\beta^2) \tag{8}$$

TAP-PP linearized yield a "glassy" (staggered total magnetization 0, $q_{EA} > 0$) critical temperature at $T = 0.8$.
Proof: resummation of the high temperature expansion for the Helmholtz free energy $\beta F(\beta)$ through the Gaussian sums estimates.
3. Simplified models (spherical, XY, etc.).
In the random case they show glassy behaviour. In the deterministic case, numerical evidence (Parisi and coworkers): $\sigma_x = \pm 1$ is necessary to have "glassy" behaviour.
This fact is actually true (I.Borsari, F.Camia, F.Unguendoli, S.G., J.Phys.A in press): none of the above described models has critical points when considered in the spherical or XY case, even though their ground state is "glassy" and degenerate.
Proof: computation of the Gibbs free energy in closed form.

The square root problem

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Abstract: The square root problem was raised by T.Kato in the 60's, motivated by perturbation theory for some hyperbolic partial differential equations. the question is whether the domain of the square root of a divergence form operator is the natural Sobolev space or not. While it is trivial in the self-adjoint case, it turned out to be a profound problem in the general case, connected to abstract operator theory, modern harmonic analysis and the study of elliptic partial differential equations. The conference will describe a new approach to this problem (still unsolved in general), due to P. Auscher and the speaker, which takes advantage of recent developments in functional calculus and in harmonic analysis. This approach allows to unify previous results and to obtain new ones. In particular, it sheds light on relations between the study of the square root of an opeartor L and the properties of weak solutions to the inhomogeneous equation $Lu = 3\ Df$ with nice f.

A New Class of Stable Time Discretization Schemes for the Solution of Nonlinear PDEs

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1. Sparse representations of several classes of operators in wavelet bases (up to finite but arbitrary accuracy) lead to numerical calculus of operators [2]. What makes such calculus possible is an algorithm for operator multiplication of order N^d, where d is the dimension (instead of the usual multiplication of order N^{3d}). The classes of operators for which fast multiplication algorithm is applicable are sufficiently broad and not restricted to operators arising from problems with constant coefficients. In this presentation we desribe several consequences of these facts for the development of PDE solvers.

2. We restrict our attention to a class of advection-diffusion equations of the form

$$u_t = \mathcal{L}u + \mathcal{N}(u), \qquad x \in \Omega \subset \mathbf{R}^d, \tag{1}$$

where $u = u(x,t)$, $x \in \mathbf{R}^d$, $d = 1,2,3$ and $t \in [0,T]$ with the initial conditions,

$$u(x,0) = u_0(x), \qquad x \in \Omega, \tag{2}$$

and the linear boundary conditions

$$\mathcal{B}u(x,t) = 0, \qquad x \in \partial\Omega, \quad t \in [0,T]. \tag{3}$$

In (1) $\mathcal{L}$ represents the linear and $\mathcal{N}(\cdot)$ the nonlinear terms of the equation, respectively. We note that the incompressible Navier-Stokes equations can be written in this form (see below).

3. We use the semigroup approach to rewrite the partial differential equation (1) as a nonlinear integral equation in time,

$$u(x,t) = e^{(t-t_0)\mathcal{L}}u_0(x) + \int_{t_0}^{t} e^{(t-\tau)\mathcal{L}}\mathcal{N}(u(x,\tau))\, d\tau, \tag{4}$$

and describe a new class of time-evolution schemes based on its discretization. A distinctive feature of these new schemes is exact evaluation of the contribution of the linear part. Namely, if the non-linear part is zero, then the scheme reduces to the evaluation of the exponential function of the operator (or matrix) $\mathcal{L}$ representing the linear part. We show that such schemes have very good stability properties and describe explicit schemes with stability regions similar to those of typical implicit schemes used in e.g. fluid dynamics applications.

4. The incompressible Navier-Stokes equations in three spatial dimensions can be written in the form (1). We start with the usual form of the Navier-Stokes equations for $x \in \Omega \subset \mathbf{R}^3$,

$$\mathbf{u}_t = \nu \Delta \mathbf{u} - (u_1\partial_1 + u_2\partial_2 + u_3\partial_3)\,\mathbf{u} - \nabla p, \tag{5}$$

$$\partial_1 u_1 + \partial_2 u_2 + \partial_3 u_3 = 0, \tag{6}$$

$$\mathbf{u}(x,0) = \mathbf{u_0}, \tag{7}$$

where p denotes the pressure, and the boundary condition

$$\mathbf{u}(x,t) = 0 \quad x \in \partial\Omega, \quad t \in [0,T]. \tag{8}$$

Here we use the following notation $\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and $\partial_k = \frac{\partial}{\partial x_k}$. Let us introduce the Riesz transforms which are defined in the Fourier domain as

$$(\widehat{R_j f})(\xi) = \frac{\xi_j}{|\xi|}\widehat{f}(\xi), \quad j = 1,2,3, \tag{9}$$

where $\widehat{f}$ denotes the Fourier transform of the function f. It is not difficult to show that the projection operator on the divergence free functions (the Leray projection) may be written with the help of the Riesz transforms,

$$\mathbf{P} = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} - \begin{pmatrix} R_1^2 & R_1R_2 & R_1R_3 \\ R_2R_1 & R_2^2 & R_2R_3 \\ R_3R_1 & R_3R_2 & R_3^2 \end{pmatrix}. \tag{10}$$

Using projection $\mathbf{P}$ it is not difficult to obtain

$$\mathbf{u}_t = \nu\Delta\mathbf{u} - \mathbf{P}(\sum_{m=1}^{3} u_m \partial_m \mathbf{u}), \tag{11}$$

instead of (5) and (6). Equations (11) are now in the form (1), where $\mathcal{L} = \nu\Delta$ and $\mathcal{N}(\mathbf{u}) = -\mathbf{P}(\sum_{m=1}^{3} u_m \partial_m \mathbf{u})$.

The transformation from (5) and (6) to (11) is well known and appears in a variety of forms in the literature. Here we followed a derivation presented by Yves Meyer at Summer School at Luminy.

The apparent problem with (11) for use in numerical computations is that the Riesz transforms are integral operators (which makes (11) into an integro-differential equation). Let us point out that the presence of the Riesz transforms does not create serious difficulties if we represent operators $R_j, j = 1, 2, 3$ in a wavelet basis with a sufficient number of vanishing moments (for a given accuracy). The reason is that these operators are nearly local on wavelets, and thus, have a sparse representation. This approximate locality follows directly from the vanishing moments property. Vanishing moments imply that the Fourier transfom of the wavelet and its several first derivatives vanish at zero, and therefore, the discontinuity of the symbol of the Riesz transform at zero has almost no effect. The precise statements about such operators can be found in [2] and [1].

Finally, in rewriting (11) as

$$\mathbf{u}_t = \mathcal{L}\mathbf{u} + \mathcal{N}(\mathbf{u}), \tag{12}$$

we incorporate the boundary conditions into the operator $\mathcal{L}$. For example, $\mathbf{u} = \mathcal{L}^{-1}\mathbf{v}$ means that u solves $\mathcal{L}\mathbf{u} = \mathbf{v}$ with the boundary conditions $\mathcal{B}u = 0$. Similarly, $u(x,t) = \mathrm{e}^{\mathcal{L}t}u_0(x)$ means that u solves $u_t = \mathcal{L}u$, $u(x,0) = u_0(x)$ and $\mathcal{B}u(x,t) = 0$.

5. Computing and applying the exponential or other functions of operators in the usual manner typically requires evaluating dense matrices and is highly inefficient. If $\mathcal{L}$ is a circulant matrix which is diagonalized by the Fourier Transform (FT), then computing functions of operators can be accomplished by a fast algorithm. It is clear, however, that in this case the need of FT for diagonalization prevents one from extending this approach to the case of variable coefficients.

Computing exponentials of strictly elliptic operators with variable coefficients in the wavelet system of coordinates results in sparse matrices and using exponentials of operators for numerical purposes is an efficient option [3]. Further development of the approach of [3] can be found in [4], where issues of stability of time-discretization schemes with exact treatment of the linear part (ELP) schemes are considered. The ELP schemes are shown

to have distinctly different stability properties as compared with known implicit-explicit schemes. The stability properties of traditional time-discretization schemes for advection-diffusion equations are controlled by the linear term, and typically, these equations require implicit treatment to avoid choosing an unreasonably small time step. As it is shown in [4], using an explicit ELP scheme, it is possible to achieve stability usually associated with implicit predictor-corrector schemes.

Even if an implicit ELP scheme is used, as it is done in [3], an approximation is used only for the nonlinear term. This changes the behavior of the corrector step of implicit schemes. The corrector step iterations of usual implicit schemes for advection-diffusion equations involve either both linear and nonlinear terms or only the linear term (see [5]). Due to the high condition number of the matrix representing the linear (diffusion) term, convergence of the fixed point iteration requires very small time step making the fixed point iteration impractical. Implicit ELP schemes do not involve the linear term and the fixed point iteration is sufficient as in [3].

6. There are several possible approaches to solving (4), and for the purposes of this presentation, we choose a procedure that leads to multistep schemes [4]. We would like to note, that (4) in effect reduces the problem to an ODE–type setup, and for that reason, a variety of methods can be applied for the solution of (4). We will obtain operator valued coefficients and our main point is that these coefficients can be represented by sparse matrices and applied in an efficient manner.

Let us consider the function $u(x,t)$ at the discrete moments of time $t_n = t_0 + n\Delta t$, where Δt is the time step so that $u_n \equiv u(x,t_n)$ and $N_n \equiv \mathcal{N}(u(x,t_n))$. Discretizing (4) yields

$$u_{n+1} = e^{l\mathcal{L}\Delta t} u_{n+1-l} + \Delta t \left(\gamma N_{n+1} + \sum_{m=0}^{M-1} \beta_m N_{n-m} \right), \tag{13}$$

where $M+1$ is the number of time levels involved in the discretization, and $l \leq M$. The expression in parenthesis in (13) may be viewed as the numerical quadrature for the integral in (4). The coefficients γ and β_m are functions of $\mathcal{L}\Delta t$. In what follows we restrict our considerations to the case $l = 1$.

We observe that the algorithm is explicit if $\gamma = 0$ and it is implicit otherwise. Typically, for a given M, the order of accuracy is M for an explicit scheme and $M+1$ for an implicit scheme due to one more degree of freedom, γ. We refer to this family of schemes as exact

linear part (ELP) schemes.

M	γ	β_0	β_1	β_2	order
1	Q_2	$Q_1 - Q_2$	0	0	2
2	$Q_2/2 + Q_3$	$Q_1 - 2Q_3$	$Q_3 - Q_2/2$	0	3
3	$Q_2/3 + Q_3 + Q_4$	$Q_1 + Q_2/2 - 2Q_3 - 3Q_4$	$-Q_2 + Q_3 + 3Q_4$	$Q_2/6 - Q_4$	4

Table 1: Coefficients of implicit ELP schemes for $l = 1$, where $Q_k = Q_k(\mathcal{L}\Delta t))$.

M	β_0	β_1	β_2	order
1	Q_1	0	0	1
2	$Q_1 + Q_2$	$-Q_2$	0	2
3	$Q_1 + 3Q_2/2 + Q_3$	$-2(Q_2 + Q_3)$	$Q_2/2 + Q_3$	3

Table 2: Coefficients of explicit ELP schemes for $l = 1$, where $Q_k = Q_k(\mathcal{L}\Delta t)$.

For implicit schemes $\gamma \neq 0$ and

$$\begin{aligned} \gamma + \sum_{m=0}^{M-1} \beta_m &= Q_1 \\ \frac{1}{k!}\left(\gamma + (-1)^k \sum_{m=1}^{M-1} m^k \beta_m\right) &= Q_{k+1}, \qquad k = 1, ..., M, \end{aligned} \tag{14}$$

whereas for explicit schemes $\gamma = 0$ and

$$\begin{aligned} \sum_{m=0}^{M-1} \beta_m &= Q_1, \\ \frac{(-1)^k}{k!} \sum_{m=1}^{M-1} m^k \beta_m &= Q_{k+1}, \qquad k = 1, ..., M-1. \end{aligned} \tag{15}$$

where

$$Q_j(\mathcal{L}) = \frac{e^{\mathcal{L}} - E_j(\mathcal{L})}{\mathcal{L}^j}, \tag{16}$$

and

$$E_j(\mathcal{L}) = \sum_{k=0}^{j-1} \frac{\mathcal{L}^j}{j!} \tag{17}$$

For $l = 1$ we provide Tables 1 and 2 for $M = 1, 2, 3$ with expressions for the coefficients of the implicit ($\gamma \neq 0$) and the explicit ($\gamma = 0$) schemes in terms of $Q_k = Q_k(\mathcal{L}\Delta t)$.

In Tables 1 and 2 we have presented the so-called "bare" coefficients. Modified coefficients [3] differ in high order terms: these terms do not affect the order of accuracy but do affect

the stability properties. Modified coefficients depend on a particular form of the non-linear term.

7. Let us describe algorithms for the evaluation of the operator-valued quadrature coefficients as sparse matrices. The coefficients of ELP schemes are written in terms of operators $\mathrm{Q}_k = \mathrm{Q}_k(\mathcal{L}\Delta t)$, where

$$\mathrm{Q}_j(\mathcal{L}\Delta t) = \frac{\mathrm{e}^{\mathcal{L}\Delta t} - \mathrm{E}_j(\mathcal{L}\Delta t)}{(\mathcal{L}\Delta t)^j}, \tag{18}$$

and

$$\mathrm{E}_j(\mathcal{L}\Delta t) = \sum_{k=0}^{j-1} \frac{(\mathcal{L}\Delta t)^j}{j!}, \tag{19}$$

for $j = 0, 1, \ldots$. We have

$$\mathrm{Q}_0(\mathcal{L}\Delta t) = \mathrm{e}^{\mathcal{L}\Delta t}, \tag{20}$$

$$\mathrm{Q}_1(\mathcal{L}\Delta t) = (\mathrm{e}^{\mathcal{L}\Delta t} - \mathcal{I})(\mathcal{L}\Delta t)^{-1}, \tag{21}$$

$$\mathrm{Q}_2(\mathcal{L}\Delta t) = (\mathrm{e}^{\mathcal{L}\Delta t} - \mathcal{I} - \mathcal{L}\Delta t)(\mathcal{L}\Delta t)^{-2}, \tag{22}$$

etc.

We will now describe a method that permits us to compute operators Q_0, Q_1, Q_2, etc. without computing $(\mathcal{L}\Delta t)^{-1}$. The problem with using the Taylor expansion directly is that it will results in a loss of accuracy due to possibly large singular values of $\mathcal{L}\Delta t$. To avoid this problem in computing the exponential, Q_0, we use the scaling and squaring method. This method results in a fast algorithm if the evaluation is performed in a wavelet basis with a sufficient number of vanishing moments (for a chosen accuracy) [3].

The scaling and squaring method for the exponential is based on the identity

$$\mathrm{Q}_0(2x) = (\mathrm{Q}_0(x))^2 . \tag{23}$$

First we compute $\mathrm{Q}_0(\mathcal{L}\Delta t 2^{-l})$ for some l chosen so that the largest singular value of $\mathcal{L}\Delta t 2^{-l}$ is less than one. This computation is performed using the Taylor expansion. Using (23), the resulting matrix is then squared l times to obtain the final answer. In all of these computations it is necessary (and possible) to remove small matrix elements to maintain sparsity, and at the same time, maintain a predetermined accuracy.

A similar algorithm may be used for computing $\mathrm{Q}_j(\mathcal{L}\Delta t)$, $j = 1, 2, \ldots$ for any finite j. Let us illustrate this approach by considering $j = 1, 2$. It is easy to verify that

$$\mathrm{Q}_1(2x) = \tfrac{1}{2}\left(\mathrm{Q}_0(x)\mathrm{Q}_1(x) + \mathrm{Q}_1(x)\right), \tag{24}$$

$$Q_2(2x) = \tfrac{1}{4}\left(Q_1(x)Q_1(x) + 2Q_2(x)\right), \tag{25}$$

etc.

Thus, a modified scaling and squaring method for computing operator-valued quadrature coefficients for ELP schemes starts by the computation of $Q_0(\mathcal{L}\Delta t 2^{-l})$, $Q_1(\mathcal{L}\Delta t 2^{-l})$ and $Q_2(\mathcal{L}\Delta t 2^{-l})$ for some l selected so that the largest singular value of all three operators is less than one. For this evaluations we use the Taylor series. We then proceed by using identities in (23), (24) and (25) l times to compute the operators for the required value of the argument.

The speed of evaluation and application of operator valued coefficients in spatial dimensions two and three is one of the important factors in practicality of ELP schemes. Although algorithms described above scale properly with size in all dimensions, establishing ways of reducing constants in operation counts remains an important task in dimensions two and three. This is an area of the ongoing research.

8. Let us compare (linear) stability regions of ELP schemes with that of the stiffly stable method of [5]. Here we provide just one example and refer to [4] for more details. The method of [5] is of the third order and is implicit with respect to the linear and explicit with respect to the nonlinear terms. The resulting stability regions are depicted in Figure 1 and should be compared with those of the third order explicit ELP scheme in Figure 2. By direct examination of Figures 1 and 2 we observe that the explicit third order ELP scheme requires the time step only about one half of that of the implicit scheme.

9. As an example consider Burgers' equation

$$u_t + uu_x = \nu u_{xx}, \qquad 0 \le x \le 1, \quad t \ge 0, \tag{26}$$

for $\nu > 0$, together with an initial condition,

$$u(x,0) = u_0(x), \qquad 0 \le x \le 1, \tag{27}$$

and periodic boundary conditions $u(0,t) = u(1,t)$. Burgers' equation is the simplest example of a nonlinear partial differential equation incorporating both linear diffusion and nonlinear advection. In [3] a spatially adaptive approach is used to compute solutions of Burgers' equation via

$$u_{n+1} = e^{\mathcal{L}\Delta t}u_n - \frac{\Delta t}{2}Q_1(\mathcal{L}\Delta t)\left[u_n\partial_x u_{n+1} + u_{n+1}\partial_x u_n\right], \tag{28}$$

where

$$Q_1(x) = \frac{e^x - 1}{x}.$$

In Figure 3 the number of available scales $n = 15$, the time step $\Delta t = 0.001$, $\nu = 0.001$ and the cutoff $\epsilon = 10^{-6}$.

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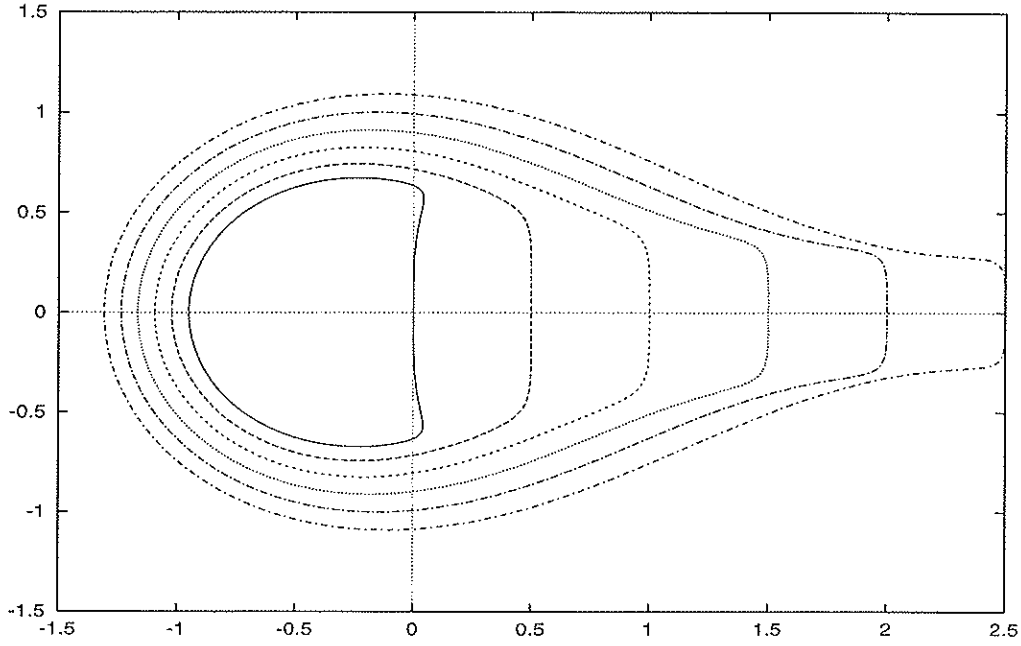

Figure 1: Stability diagram of the third order implicit stiffly stable scheme of [KIO].

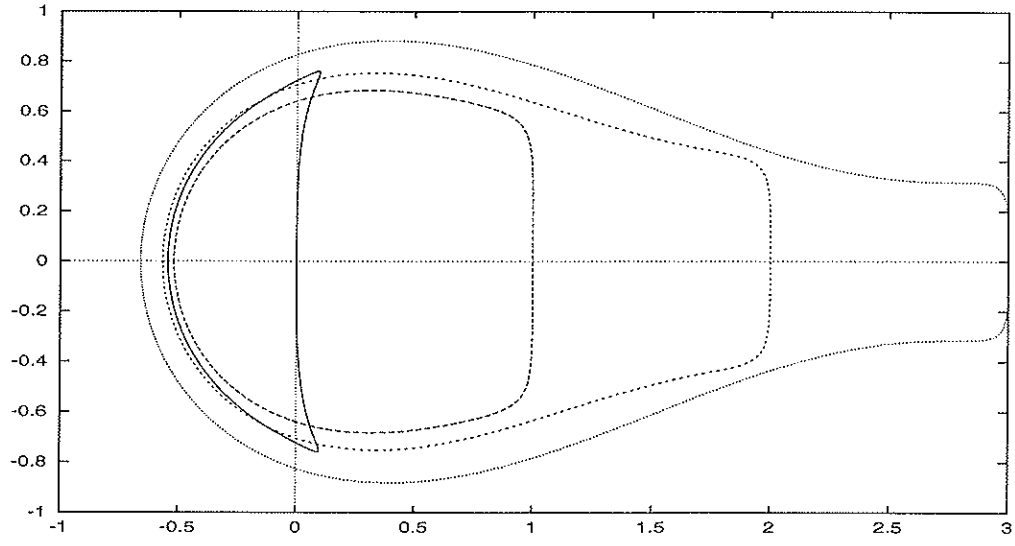

Figure 2: Stability diagram of the third order explicit scheme with $\kappa = 1$, $M = 3$ and coefficients from Table 2 (stability regions are comparable with those of the implicit stiffly stable KIO scheme)

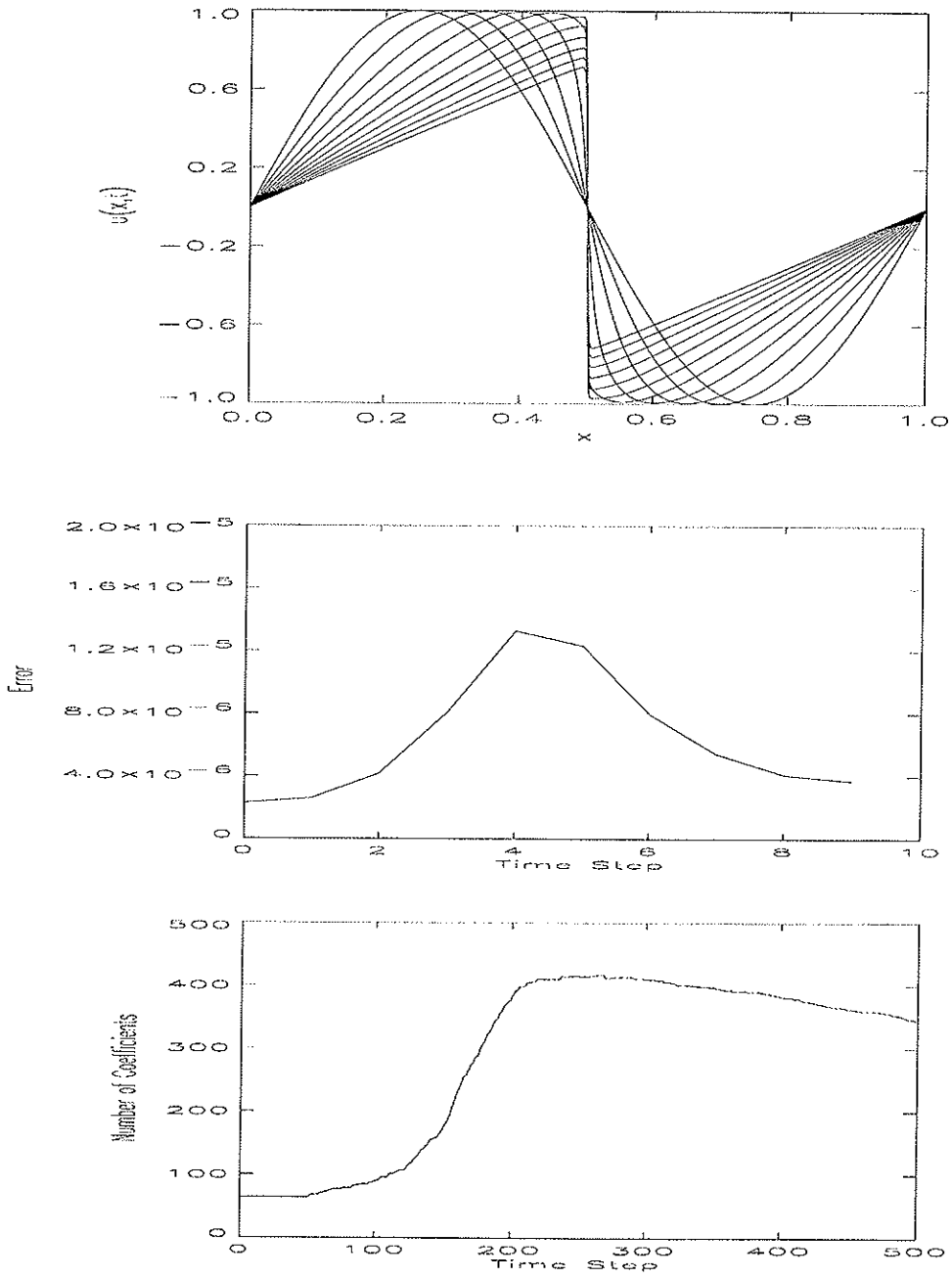

Figure 3: Solution of Burgers' equation, accuracy and the number of significant coefficients as a function of the time step

In terms of the polyphase matrices

$$P(z) = \begin{pmatrix} h_e(z) & g_e(z) \\ h_o(z) & g_o(z) \end{pmatrix}, \quad \tilde{P}(z) = \begin{pmatrix} \tilde{h}_e(z) & \tilde{g}_e(z) \\ \tilde{h}_o(z) & \tilde{g}_o(z) \end{pmatrix} \tag{0.2}$$

the condition for reconstruction is

$$P(z)\tilde{P}(z)^* = 1 \tag{0.3}$$

where $h_e(z), h_o(z), g_e(z), g_e(z)$ denote the even and odd parts of the Laurent polynomials $h(z) = \sum h_n z^{-n}$ and $g(z) = \sum g_n z^{-n}$ respectively. Hence each polyphase matrix is a loop in the group of invertible two by two matrices $GL(2, \mathcal{F})$.

We consider only the case of finite impulse filters (FIR). In this case, the entries in both matrices $P(z)$ and $\tilde{P}(z)$ are Laurent polynomials. This implies that $\det P(z)$ is also a Laurent polynomial and, by the reconstruction condition (0.3), it is in fact a monomial. The problem of finding biorthogonal filters with finite impulse response is thus equivalent to finding polynomial loops in $GL(2, \mathcal{F})$.

For the purpose of classification of filters, it suffices to consider the following restricted class of loops:

Definition 0.1. *Let X be the group of Laurent polynomial loops $P : \mathbb{T} \to GL(2, \mathcal{F})$ such that the determinant is a circle map $\det P : \mathbb{T} \to \mathbb{T}$. In the following we refer to elements in X as polynomial loops.*

It turns out that every polynomial loop is a product of elementary loops of three different types which we call upper triangular, lower triangular and diagonal loops.

Definition 0.2. *Let $k, m, n \in \mathbb{Z}$ and $a \in \mathbb{R}$ Then,*

$$U_+(k, a) := \begin{pmatrix} 1 & z^{-k}a \\ 0 & 1 \end{pmatrix}$$

and

$$U_-(k, a) := \begin{pmatrix} 1 & 0 \\ z^k a & 1 \end{pmatrix}$$

are special loops in $SL(2, \mathcal{F})$ referred to as upper and lower triangular loops, respectively.

$$D(m, n) = \begin{pmatrix} z^m & 0 \\ 0 & z^n \end{pmatrix}$$

is an elementary diagonal loop.

The following factorization of polynomial loops holds

Theorem 0.1. *Consider a polynomial loop $P \in X$. There exist finitely many upper and lower triangular loops and an elementary diagonal loop such that: $P = U_{\sigma(1)} \cdots U_{\sigma(N)} D$. $\sigma(1) \ldots \sigma(N)$ denotes an appropriate choice of signs.*

A consequence of Theorem 0.1 is that every loop in X can be written as a product of shifts, and of constant loops. More precisely we have:

Corollary 0.1. *Consider the two subgroups of X:*

$$\begin{aligned} &1. \quad \mathcal{C} = \{C(z) = C_0 \in SL(2, \mathcal{F})\} \\ &2. \quad \mathcal{S} = \{S^m \mid m \in \mathbb{Z}\} \end{aligned} \tag{0.4}$$

Then X is generated algebraically by $\mathcal{C}$ and $\mathcal{S}$, i.e. each $P \in X$ is of the form:

$$P = \lambda S^m C_1 S^{n_1} C_2 S^{n_2} \ldots C_l S^{n_l} C_{l+1} \tag{0.5}$$

where $n_i \in \{-1, 0, +1\}$, $l \in \mathbb{N}$, $m \in \mathbb{Z}$, and $\lambda \in \mathbb{T}$; all factors S are to the left and the factors S^{-1} to the right.

The computational complexity of the factorized wavelet transform is essentially determined by the number of triangular matrices in the factorization of the loop $P(z)$. This is because the upper and lower shifts affect only the indexing.

Each term in the factorization adds to the total complexity $C(P)$ according to the following rule:

$$C(triangular\, matrix) = 2 \qquad \text{(one multiplication and two sums)} \tag{0.6}$$

$$C(B) = 6 \qquad \text{(four multiplications and one sum)} \tag{0.7}$$

$$C(shift) = C(D) = 0 \qquad \text{(reindexing)} \tag{0.8}$$

A rough estimate leads to the two bounds

$$C(P) \leq 2(n_{e,f} - n_{o,f}) + 2(n_{o,f} - n_{o,s}) + 6 \tag{0.9}$$

$$C(P) \leq l(h) + l(g) + 2 \tag{0.10}$$

where $l(h)$ and $l(g)$ denote the two smallest even numbers larger or equal to the lengths of the two filters.

We conclude with some examples of factorisation, computed with the help of a Mathematica program, implementing the algorithm based on the proof, given elsewhere [1].

The first example is a biorthogonal basis having the cubic B-spline as primal scaling function. The polyphase matrix is defined by:

$$\begin{pmatrix} \frac{1}{8}(z^{-1} + 6 + z) & \frac{1}{32}(-3z^{-2} - 5z^{-1} - 5 - 3z) \\ \frac{1}{2}(1 + z) & \frac{1}{8}(-3z^{-1} + 10 - 3z) \end{pmatrix}$$

The factorization of the cubic B-spline reads:

$$\begin{pmatrix} 1 & \frac{1}{4} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{4}z^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{3}{16}z^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{3}{8} \\ 0 & 2 \end{pmatrix} \tag{0.11}$$

This is an instance of a filter pair with coefficients in the ring of dyadic rationals that can be decomposed into factors with coefficients that remain in this ring.

The second example is from an article by Vetterli and Le Gall [7]. As the two authors note, it cannot be factorized into simple pseudo-unitary terms. [1] it is, however, factorizable in the sense of theorem .1. The defining polyphase matrix is:

$$\begin{pmatrix} z^{-2}+3z^{-1}+2+z & -\frac{1}{14}z^{-1}--\frac{5}{14}+\frac{2}{7}z+\frac{1}{14}z^2 \\ z^{-2}+2z^{-1}+3+z & -\frac{1}{14}z^{-1}--\frac{2}{7}+\frac{5}{14}z+\frac{1}{14}z^2 \end{pmatrix}$$

The factorization of this filter is:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -z & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ z^{-1} & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix}\begin{pmatrix} 1 & \frac{1}{7}z \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \frac{1}{7} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\frac{7}{2}z^{-1} & 1 \end{pmatrix}\begin{pmatrix} -2 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} =$$
$$= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -z+z^{-1}-4 & 1 \end{pmatrix}\begin{pmatrix} 1 & \frac{1}{7}(1+z) \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -\frac{7}{2}z^{-1} & 1 \end{pmatrix}\begin{pmatrix} -2 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

The last examples is a filter pair with coefficients in $\mathbb{Z}$ that can be decomposed into factors with coefficients again in $\mathbb{Z}$ with our algorithm:

$$\begin{pmatrix} 1+2z & 4 \\ -z^2 & 1-2z \end{pmatrix}$$
$$= \begin{pmatrix} z^{-1} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & z^{-1} \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & -2z^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -z^2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2z^{-1} \\ 0 & 1 \end{pmatrix}$$

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MUSIC: A TIME-SCALE GAME?

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My contribution to the conference in the honor of Alex Grossmann dealt with digital music: digital sound, synthesized or processed, and also digital skills - interactive piano music. The term digital is ambiguous - but musical keyboards were the first keyboards: how would you use computers today without a keyboard? Music was involved in a number of innovations: it was probably the first time-scale game.

My presentation was illustrated with sound examples, and a musical demonstration was given in the evening. The examples will be lacking in this written version, although some are described in the text.

1 Music, numbers, time-scale

Music has a long history of relations with science. Pythagoras insisted that numbers rule the world - the intervals in music as well as the movement of planets. According to Jean-Marie Souriau, this was the point of departure for modern science, and for virtual reality as well. Numbers permit to control sound - privileged musical intervals correspond to simple ratios - and space - one can construct a right angle by assembling three segments with lengths proportional to 3, 4, 5.

It seems reasonable to posit that the notion of a time-frequency space or a time-scale space first appeared in music. A musical interval corresponds to a given frequency ratio. Early lithophones appear to be the first instance of a log frequency scale. Scales involve specific relations between the frequencies of the successive steps - this is the "out of time" part of music, as Xenakis calls it. Certain Indonesian scales are close to a division of the octave in five

equal steps. Music involves time. According to Saint Augustinus, music is "ars bene movandi". In the middle ages, musical notation developed in the western world as needed by polyphony: it uses a time-scale representation. Musical scores and musical automata have exemplified cartesian coordinates long before Oresmus and Descartes: in fact the British historian Geoffroy Hindley believes that musical notation brought about the idea of representation in terms of cartesian coordinates, which played such an important role in the explosion of western science since Newton.

2 Representations of musical sounds

The following point of view was held for a long time: musical sounds are periodic and characterized by attributes - pitch, intensity, timbre - which can be assigned to measurable parameters of the periodic wave - frequency, energy, Fourier spectrum. This point of view is insufficient, but today it is easy to understand how it developed. In the second half of the nineteenth century, sound waves could be recorded and submitted to Fourier analysis. It was not convenient to look at a long stretch of periods for tones with frequencies of hundred of Hertz, and it was tiresome to extract the Fourier spectra for many periods. Sounds with slowly evolving spectra were considered as pseudo-periodic, with a fixed spectrum modulated in amplitude by an envelope shaping the attack, the changes in amplitude and the decay. Other processes of analysis, such as the flame spectrum analyzer using Helmholtz resonators, were slow and averaged variations. For each different timbre, the analysis yielded a different spectrum, and this spectrum was considered to be responsible for the timbre differences. There was no possibility to check by synthesis the aural validity of this grossly simplified model; hence this view was not questioned for many years, especially since it was advocated in the authoritative treatise by Helmholtz, Sensations of tone.

With modern ways to transform or to synthesize sounds, however, the simplistic view described above was recognized as inaccurate. In the fifties, one could transform recorded sounds recorded on magnetic tape - for instance play them backwards, as though the arrow of time was reversed. Now a piano tone reversed in time is no longer recognizable, even though it consists of the same spectra - in a different order. Also there is no clear-cut distinction between musical sounds and noise. One can say that, to be musical, a sound has to deviate from strict periodicity (which means ever-lasting) or pseudo-

periodicity, corresponding to a steady spectrum modulated in amplitude. There are changes throughout interesting sounds: the aural importance of spectral changes is easy to demonstrate via digital synthesis. For instance my analysis-by-synthesis of brass tones showed that such tones were impossible to imitate using a fixes spectrum, since their characteristic feature is the dependency of the spectrum upon the instantaneous amplitude - the louder, the brighter. Thus representation by a single fixed Fourier spectrum is generally inadequate. The Fourier integral supposedly captures the whole information contained in evolving sounds. However it does not provide a convenient display of the spectral evolution in the course of the sounds. In effect, Fourier analysis attempts to represent finite duration functions in terms of sine waves, that is, functions of infinite duration, which seems far-fetched and leads to difficulties.

Ad hoc modifications of the Fourier representation have been proposed for some time. The sound spectrograph, initially developed during the last world war under the name visible speech, permitted to obtain so-called sonagrams. The sonagram is probably the most useful portrayal for sound, the closest to what might be termed a sound photograph. It is a time-frequency representation: the sound goes through a set of band-pass filters; the time-dependent output of each filter is rectified to control the darkness of a line, which thus gives an indication of the amplitude. Representing the waveform itself does not bring much useful insight on how the sound is going to sound. Certain waves can look almost identical yet sound distinctly different. Certain waves can look extremely different, yet sound the same: this is the case for periodic tones with the same amplitude Fourier spectrum but with different values for the relative phases of the harmonics. In contradistinction, looking at sonagrams, one gets a feeling for sounds, for their behavior and their evolution, for their figure, their gait, which is very significant to the ear. The structure of the Corti organ, part of the inner ear, entails a frequency analysis of the incoming sound, and the nervous fibers conveying the information to the higher structures of the brain are organized tonotopically: different frequencies trigger nerve impulses in different fibers. Hence it is not surprising that a proper representation of musical sound should allow to follow the variation of spectrum in time. Emile Leipp resorted a great deal to sonagrams in his insightful acoustic studies of musical practice. In his book New images of musical sound, Robert Cogan has compiled sonagram-like representations for various types of music, classical and contemporary, instrumental and electronic: on these images, one can follow many musical

features.

However the sonagram cannot provide a unique optimal representation. If the filters are narrow, the resolution in frequency will be high, but the ringing of the filters will smear fine temporal details. Wide filters yield a good temporal resolution but do not allow to separate components which are close in frequency. One cannot escape this trade-off, akin to Heisenberg uncertainty relations1. Hence there is no "best way" to represent sound along these lines: it depends upon what the representation must portray best. Wide-band sound spectrograms are useful for speech: adjacent harmonics are not resolved, but shown as groups corresponding to the "formants", the groups of harmonics reinforced by the resonances of the vocal tract - the position of the first formants characterizes a given vowel. Narrow-band spectrograms are generally more adapted to music, since the representation can then separate the notes of a scale.

One should remember that the sonagram provides a visual representation which is useful, but not always faithful: it can even be misleading, since the auditory result does not always correspond to what one sees on the sonagram representation. Different frequency components, represented as separate on the sonagram, can either be discriminated by the ear or fuse into a single entity: this depends crucially on their frequency ratio (octave and simple rational ratios favor fusion). This is not easily seen on a sonagram: two simultaneous sine waves one octave apart look similar on a sonagram to two simultaneous sine wave one octave and a semi-tone apart, but they sound quite different!

3 Reconstruction: analysis-synthesis and wavelets

The sonagram basically represents a running short-term Fourier spectrum. It does not permit to reconstruct the sound: some of the information is lost. To be complete, the representation should include the phase spectrum. Indeed, a Fourier transform comprising an amplitude spectrum and a phase spectrum can be inverted to reconstitute the original sound. However such a representation is cryptic: it does not represent clearly the way the sound evolves in time. The amplitude spectrum gives some kind of average distribution of energy along the frequency axis, and the time evolution is not explicit in the phase spectrum. Our hearing, on the contrary, tracks the evolution of sounds in time. Running Fourier analysis use time windows to

evaluate instantaneous spectra, thus taking the evolution in account2. But it runs into certain difficulties, which seems normal considering that Fourier analysis tries to describe signals in terms of a combination of sine waves, with an infinite exte! nt in time: if the signal is time- limited, artifacts are required to have this combination limit itself in time.

Clearly, the granular approach adopted by Dennis Gabor, with the Gabor representation, and by Morlet and Grossmann, with the wavelet transform, are in principle much more satisfactory. Gabor's idea to represent signals on the basis of functions limited in both frequency and time was published shortly after the second world war. It was presented in the proceedings of a meeting presided by Louis de Broglie in 1950 - among the participants were Robert Fortet, Andr Blanc-Lapierre, Pierre Aigrain, Jean Ville, and Gabor himself. These basic functions - Gabor grains, i.e. sine waves modulated in amplitude by Gaussian "bells" - were used by Curtis Roads and Barry Truax in their granular synthesis, but not as a way to reconstruct analyzed sounds: rather, grains were assembled with diverse random localizations in time and frequency to produce novel sounds. Szasz had demonstrated that functions of time could be expanded in terms of complex exponentials (Cf. Paley & Wiener, 1934). T! here were a few attempts to repres ent sounds in terms of specific functions, notably sine wave multiplied by exponentially decaying exponentials by Dolansky and Huggins: The possibility to analyze and reconstitute a given signal in terms of Gabor grains was demonstrated by Baastians (1979)3. I discussed these possibilities in a joint article with David Wessel published in 1982, and which comprised a paragraph entitled signal representation and analysis-synthesis processes. Shortly after, I heard about the work of Alex Grossmann about wavelets: it seemed clearly of high interest for the representation of sound. Although I had known Alex for ten years, I was unaware of what Alex was working on: I thought it dealt mainly with out of my reach4 advanced mathematics of quantum mechanics. In preparation for the RCP ondelettes, I wrote in early 1985 a brief memo for Alex to describe our team and our interest in exploring the wavelet transform for the representation and the modification of sounds.

At that time, Richard Kronland-Martinet was initiating his thesis work in our team. Our Equipe d'Informatique Musicale had been created in 1972 at Luminy thanks to Mohammed Mebkhout, and moved to Laboratoire de Mécanique et d'Acoustique of CNRS (LMA) with the help of Bernard Nayroles. Daniel Arfib had been working with me on digital synthesis of musical sound, and we were just beginning to tackle digital processing. One has

to inject richness, suppleness and musicality into synthetic sounds: acoustic sounds - for instance those of musical instruments - are already endowed with those qualities; however it is difficult to transform then with the very ductility one benefits from with synthesis. Thus we were looking for processes that would permit to extend ductility to processing. One can try to represent acoustic sounds in term of a synthesis model to take advantage of the ductility of synthesis.

Richard was a very motivated young scientist with a lot of musical training and interest in music. After his master in Physics at Luminy, he had followed the DEA of Acoustics, hoping to do research on digital synthesis of music - and he was assigned to work on digital synthesis for active noise absorption! He was longing to get to more musical material, and the project for his thesis was to develop sonic analysis so as to extract models from sounds: these models could then be the point of departure for synthesis, with the possibility to produce variants - to perform intimate variations on the initial sound. A research grant on analysis-synthesis from the French Navy, obtained with the help of Françoise Briolle, made it possible for us to work on the audio digital processor SYTER, built by the company Digilog, which permitted faster processing (computers were not as fast then as they are today).

Richard collaborated with Alex Grossmann and Jean Morlet to implement the wavelet transform so as to process digital sounds. With analysis-synthesis processes, intimate changes can be performed on the original signal by altering the analysis parameters before reconstruction. Indeed, the wavelet worked very well to reconstruct sound: the copy was indistinguishable form the original, whether one looked at the waveforms or listened to the sounds. Defective synthesis immediately showed the possibility of intimate transformations. Richard demonstrated this on flute sounds, separating low and high scales in convincing ways. Later Frdric Boyer used wavelets separated by one semi-tone: by adding only those components corresponding to a given chord of the well-tempered scale, analyzed speech could be reconstructed as perfectly intelligible, yet sounding as though this chord was imprinted in the vocal chords of the speakers. Richard also demonstrated time-stretching: but in this cas! e a low-frequency quality was adde d. Time-stretching was easier using Gabor grains, as Daniel Arfib demonstrated. Some wavelet and Gabor features were incorporated into the LMA version of the MUSIC V synthesis program and in a SOUND MUTATIONS, a program to modify sounds. The SYTER processor speeded up the computations of the transforms. Wavelet transform programs developed at LMA were incorporated in a work station,

Onyx, developed by the company Digilog for signal processing. A number of possibilities were explored and described in fundamental articles. Among them, the use of converging patterns in the phase diagram to locate discontinuities: this was served to localize scratches in vinyl disk recordings. Richard also showed that by using a wavelet made up of two sine waves at an interval of an octave, one would cause the octave intervals to stand out clearly on the module diagram of the transform of a musical sequence - this figure was selected in 1989 for the cover of Computer Music Journal 12 n 4. Oli! vier Rioul spent a training period in Marseille at both CPT (Centre de Physique Thorique) and LMA. In 1987, a 40 mn pedagogic video on wavelets was realized at LMA by Alex and Richard, using SYTER and computer animations to explain the wavelet transform and to show wavelet translation and dilation as well as wavelet transform diagrams for modulus and phase: this video was shown at the first congress on wavelets, organized by Elf Aquitaine and the RCP Ondelettes in Pau. The same year, Richard produced the color figures presented in the first article published for a large audience, "L'analyse par ondelettes" by Yves Meyer, Stphane Jaffard and Rioul, which appeared in Pour la Science, and he also contributed to edit this article.

Later the cooperation with Alex continued, with the occasional participation of the late Bernard Escudi: the notion of ridge or skeleton of a time-scale or a time-frequency transform helped to extract modulation laws from a signal. Philippe Guillemain developed a clever process to model a signal in term of additive synthesis - that is, as a sum of sine waves, each of which is controlled in both amplitude and frequency by separate curves. The curves are extracted automatically by the analysis program, although the user can intervene to improve things in difficult cases. Then the sound be reconstructed. The process works remarkably well if the sounds are made up of a limited set of discrete "spectral components": this is the case of quasi-periodic tones, but also of inharmonic sounds such as those of bells or gongs. The fact that the reconstruction is practically indistinguishable from the original demonstrates that the analysis does capture the aspects which are relevant f! or the ear. But the interest is al so to produce variants of the original. If, instead of merely reconstructing it, one alters the amplitudes or frequencies, one can achieve very interesting transformations: the sound becomes highly ductile. For instance, the rate of the amplitude and frequency envelopes can be altered: the sound can be stretched or compressed in time without changing its frequencies. Richard Kronland and Philippe Guillemain made substantial progress towards non-linear resynthesis of instrumental tones. The goal here

is to estimate the parameters of a non-linear synthesis process - frequency-modulation or waveshaping - best approximating a given instrumental sound. This is a considerable challenge - the problem may have no solution or no unique one. To get around the difficulty, frequency components were grouped in subclasses with simpler behavior.

There are a number of other applications of wavelets to represent, reconstruct and modify signals in novel ways, and many more to come.

4 A musical demonstration

In the honor of Alex Grossmann, whose culture is broad and deep in art as well as in science, a musical demonstration was also given, illustrating certain points discussed in the conference.

A milestone of "kinetic" music, Turenas by John Chowning demonstrated swift movements of virtual sound sources in illusory space. The sounds were actually emitted by four loudspeakers which were fixed in position: but Chowning carefully contrived the synthesized sounds so as to emulate the cues which suggest to our ear that the sounds it receives comes from moving sources (Chowning, 1971). Among these cues, Chowning simulated the Doppler effect5. The space here is generated by the music, specifically by the interaction of the sonic signals with the auditory perceptual organization. This may remind one of the status of space in models based upon the algebra of non-commutative operators, presented at the opening of the conference by Daniel Kastler: space is no longer a priori, it results from the interactions.

Also in four tracks, Electron-Positron, a short "metaphoric event" realized by Richard Beaudemont and myself for the inauguration of the LEP in Geneva in 1989 - the visual part was shown in video, in synchrony with the music. In the first part of this audio-visual piece, the accelerating particles are symbolized by rotating trajectories. While the sound circles around at increasing speed, one can hear rhythmic beats at a tempo which also goes faster and faster - in fact these beats accelerate for ever, a cyclical auditory illusion analogous to Shepards tones or to my endlessly descending glissandi. Tension builds up to the "collision". Then the end of the event is supposed to evoke the work of the scientists. The film shows trajectories of high-energy particles. The music includes the voice of Alex Grossmann, transformed using both Gabor and wavelet techniques. The processes presented in the audio illustrate the work of the team of Acoustique Musicale on processing (spatial!

ization, time-frequency alteration s) and on perception (auditory illusions).

Then followed the demonstration of a piano "duet" with a single pianist, resorting to a novel process: the pianist has a "partner" - but an invisible, virtual one. A computer program "listens" to what the pianist plays, and instantly adds its own musical part on the same piano: this part is not a mere recording, it depends upon what the pianist plays and how he plays. Hence we have a genuine duet: the pianist's partner, although unreal and computerized, is sensitive and responsive. Different processes of interaction between what the pianist plays and what his or her shadow partner responds were shown, from simple, "geometrical" relations such as translation (musical transposition) or symmetry (inversion of intervals) to proliferating processes such as canons, automatic arpeggiation of fractalization, and to acoustic resonances through the sound board of the acoustic piano.

The presentation was concluded by Invisible: In this work, inspired by Italo Calvino's Invisible cities, a soprano speaks and sings texts by Tchouang-tseu, Wang Wei, Lao Tseu, Dante, Basho, Heine, Goethe, Longfellow et Leopardi, in dialogue with sounds from an "invisible" partner, sounds produced by synthesis or processing resorting in particular to wavelets. Wavelets which are - hopefully - good to hear!

Many warm thanks to Alex, and best wishes.

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6 Recordings

Chowning, J.M. Turenas. On CD Wergo 2012-50.

Risset, J.C. Invisible. On CD GRM EI-06 (Irène Jarsky, soprano)

Risset, J.C. Duet for one pianist. On CD Neuma 450-87.

1 It is worth insisting here on the well-known fact that the same mathematical formalism can apply to different phenomena. Electric analogies are very helpful in mechanics and acoustics. Heisenberg's uncertainty principles relate to the trade-off between frequency and temporal resolution in linear filters. An analogy across several fields which does not seem widely known is Callen and Welton's theorem: for a causal system, the fluctuations - the

behavior in a free state - relate to the susceptibility- the response under constraint - according to integral formulas, which can be derived by expressing causality. These formula yield the Nyquist formula giving the noise in resistors as well as the Mandelstam formulas, or dispersion formulas, in nuclear physics. Note 3, below, gives another instance.

2 The Wiener-Kinchine theorem, linking the power spectrum and the autocorrelation function, has been generalized to the case of short-time spectra by Fano, Schroeder and Atal.

3 As I found later, work published in the context of quantum mechanics by Bacry, Grossmann, and Zak five years before the article of Bastiaans in the field of signal processing, was very close to demonstrating the validity of the representation of sounds by grains - Gabor grains are akin to the wave packets used to represent the wave assciated with a particle.

4 I had initially worked in high-energy physics at the Orsay linear accelerator with George Bishop and the late Bernard Grossette (1961), but I had soon switched to the domain of audible waves.

5 Christian Doppler first experimented his effect with musicians of the Conservatory of Vienna - performers on a railway cart and listeners by the railroad.

Are Statistical and Deterministic Approaches Compatible to study Fully-developed Turbulence?

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Fluid mechanics is governed by the Navier-Stokes equations, which are deterministic. *Fully-developed turbulence*, which corresponds to very large Reynolds number flows (for which the micro-scale Reynolds number Re should be larger than 10^4), is the limit where the non-linear advective term of Navier-Stokes equations strongly dominates the linear dissipative term. In this limit, the solutions are highly chaotic, namely very sensitive to initial conditions, and we are unable to integrate the Navier-Stokes equations, even with the largest computers presently available or proposed for the future. Therefore, in order to compute fully-developed turbulent flows we should combine a *deterministic numerical integration* and a *statistical model.*

The present state of the art, so called *Large Eddy Simulation* (LES), consists of integrating the dynamics of the large-scale motions (corresponding to the scales resolved by the computational grid) and of statistically modelling the effect of the small-scale motions (corresponding to the subgrid-scales) on the large scales. But this programme is not adequate for fully-developed turbulence because it requires a *scale separability*, namely a decoupling between large-scale and small-scale dynamics. Such a decoupling exists between the inertial range scales and the dissipative scales, which therefore allows the numerical simulation of weakly nonlinear flows (corresponding to low Reynolds numbers). However, if we want to compute very large Reynolds number flows, i.e. fully-developed turbulent flows, at a reasonable computing cost we should find another way to separate the modes to be deterministically computed from those to be statistically modelled.

We will first focus on two-dimensional turbulence, and then extend our results to three-dimensional turbulence. From a physical point of view the two-dimensional approximation is relevant to study large scale geophysical flows, due to the combined effect of stable stratification at scales larger than 10 Km and rotation of the Earth. From a mathematical point of view there are existence, unicity and regularity theorems for two-dimensional Navier-Stokes equations, which do not yet exist in dimension three. These theorems are necessary to validate the numerical procedure we use to solve Navier-Stokes equations. Therefore, from a numerical analysis point of view, the integration of two-dimensional turbulent flows is much 'safer' than for three-dimensional turbulent flows. Moreover, the *existence of an inertial manifold* has been proven for two-dimensional turbulent flows, and upper bounds for the dimension of the attractor have been given, but this is still an open problem for three-dimensional turbulent flows. Last argument to justify our interest for two-dimensional turbulent flows: according to the present estimation, the minimal number N of degrees of freedom necessary to compute fully-developed turbulent flows without turbulence model,

namely by *Direct Numerical Simulation* (DNS), scales as Re in dimension two and as $Re^{9/4}$ in dimension three. Therefore we are able to compute by DNS much larger Reynolds flows in two dimensions than in three dimensions. For two-dimensional turbulence we have already reached the fully-developed turbulent regime without using ad hoc turbulence models, but this is not yet the case for fully-developed three-dimensional turbulence.

Twelve years ago we proposed to use the wavelet representation to analyze, model and compute fully-developed turbulent flows. We have shown that the strong wavelet coefficients correspond to the coherent vortices, while the weak wavelet coefficients correspond to the incoherent background flow (see for 2D turbulence /FR88/, /FS89/, /FHC90/ and for 3D /FMGH90/). Both components are multiscale and therefore cannot be separated by Fourier filtering (/F91/, /F92/). We have developed a method, inspired by Donoho's denoising technique (wavelet shrinkage) used for image compression, to separate coherent vortices and background flow (/FSK97/). The figures illustrating this paper show an example of this separation applied to a vorticity field which has been computed with resolution 256^2. Only 0.7% of the wavelet coefficients correspond to the coherent vortices, which exhibit the same non-Gaussian Probability Distribution Function (PDF) as the total field, while the 99.3% remaining weaker wavelet coefficients correspond to the incoherent background flow, which presents a Gaussian PDF. Higher the resolution stronger will be the compression ratio we obtain with the wavelet representation. We have also shown that the incoherent background flow is slaved to the coherent vortices, due to their straining effect which inhibits any nonlinear instability to develop from the background flow (/KF97/). We have then developed a wavelet-based method to force two-dimensional turbulent flows, more physically sound than the Fourier forcing presently used (/SF97/). Our method excites vortices locally in physical space and as smoothly as possible without affecting the background flow. This procedure does not interfere with the emergence of vortices and does not impose them a scale, contrarily to the Fourier forcing. We have reached a statistically steady regime, for which energy, enstrophy, spectra and PDF are steady.

We are presently developing a new method to compute fully-developed two-dimensional turbulent flows based on wavelet phase-space segmentation (/FGMPW92/, /FKPG96/, /SKF97/, FKPS97/). This method proposes to *compute the dynamics of the coherent vortices* with a limited number of wavelet modes, keeping only the most excited ones which correspond to coherent vortices, and remapping the wavelet basis at each time step. The discarded coefficients, which are characterized by a Gaussian one-point Probability Distribution Function (PDF) and correspond to the incoherent background flow, should be *statistically model* in order to take into account their effect on the coherent vortices. We can either model them by a stochastic forcing having the same statistical behaviour, or compute the linear equation characterizing their motions, or design a one-point turbulence model (such as Boussinesq, Smagorinsky or k-ϵ). The justification for this procedure is that the coherent vortices are not numerous enough and their encounters are too rare events to have reached a statistical equilibrium state, and therefore we have to compute their dynamics with a deterministic method. On the contrary, for the well-mixed background flow we can

assume stationarity, homogeneity and ergodicity, in order to define a statistical equilibrium state from which we can design an appropriate statistical model.

Legend of the figures

Wavelet compression of vorticity.
(a) The vorticity. (b) The modulus of velocity. (c) The stream function. (d) The coherence scatter plot. (e) Cut of vorticity. (f) PDFs of velocity and vorticity. (f) Energy spectrum. The solid lines correspond to the total vorticity ω, the dashed lines to the coherent part $\omega_>$, and the dotted lines to the incoherent part $\omega_<$.
We observe that only 0.7% of the total number of wavelet coefficients are sufficient to represent all coherent structures, while the remaining 99.3%correspond to the incoherent background flow, which is much weaker and homogeneous. The coherent vorticity $\omega_>$ contains 94.3% of the total enstrophy. Moreover, the velocity associated with the coherent structures is quasi-identical to the total velocity and contains 99.2% of the total energy. As for the coherent stream function, $\psi_>$ is perfectly identical to the total stream function ψ. The fact that the scatter plot of the background, $F_<$ such that $\omega_< = F_<(\psi_<)$, is isotropic proves that our method has extracted all coherent structures. The PDFs of velocity and vorticity show that only 0.7% of the wavelet coefficients are sufficient to capture the non-Gaussian one-point statistical distributions of vorticity and velocity, while the remaining 99.3% correspond to Gaussian distributions. The energy spectrum, on the contrary, is dominated at small scales by the incoherent background flow and therefore is insensitive to coherent structures because they are too rare to affect the energy spectrum (which is the Fourier transform of the two-point correlation function).

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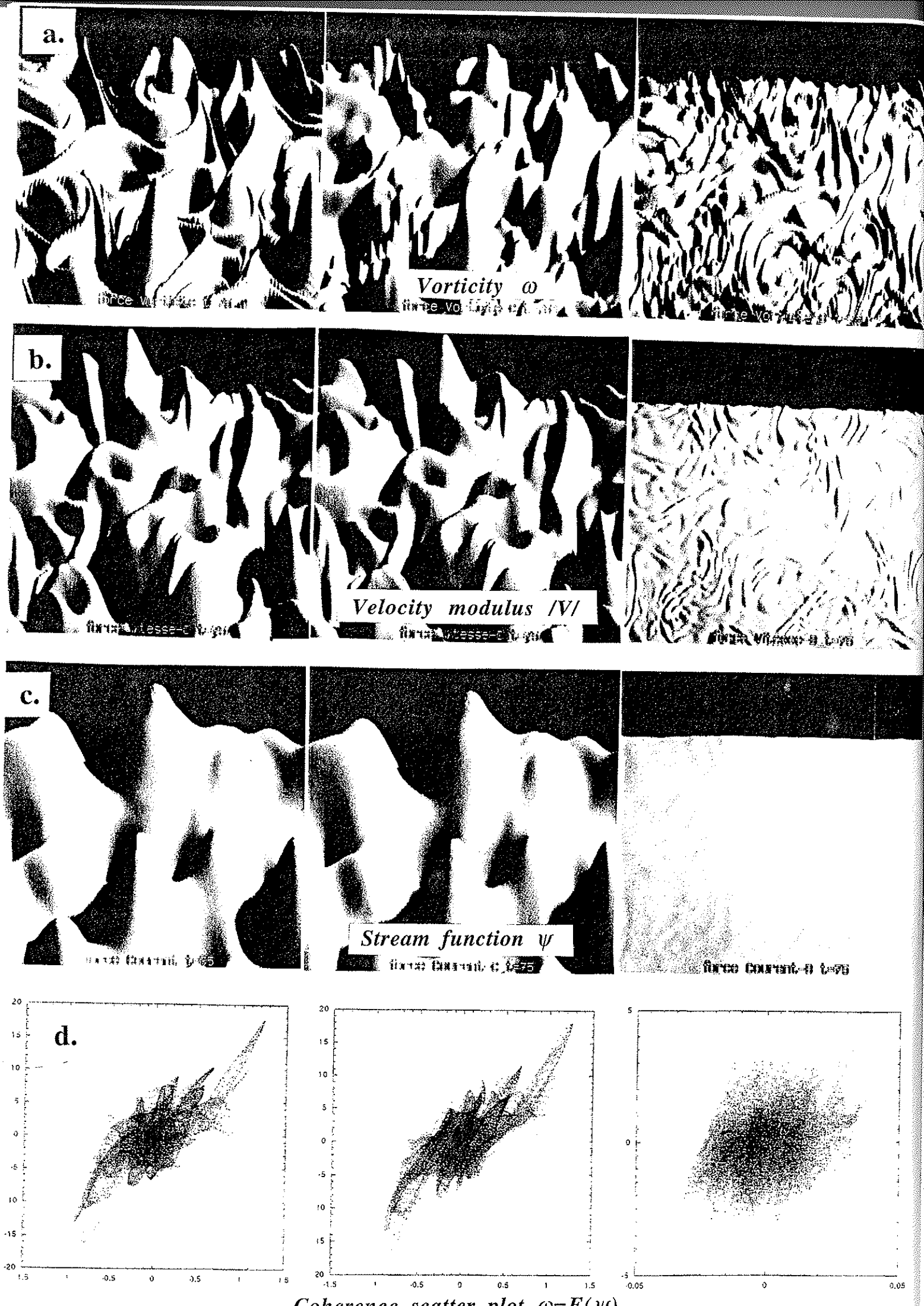
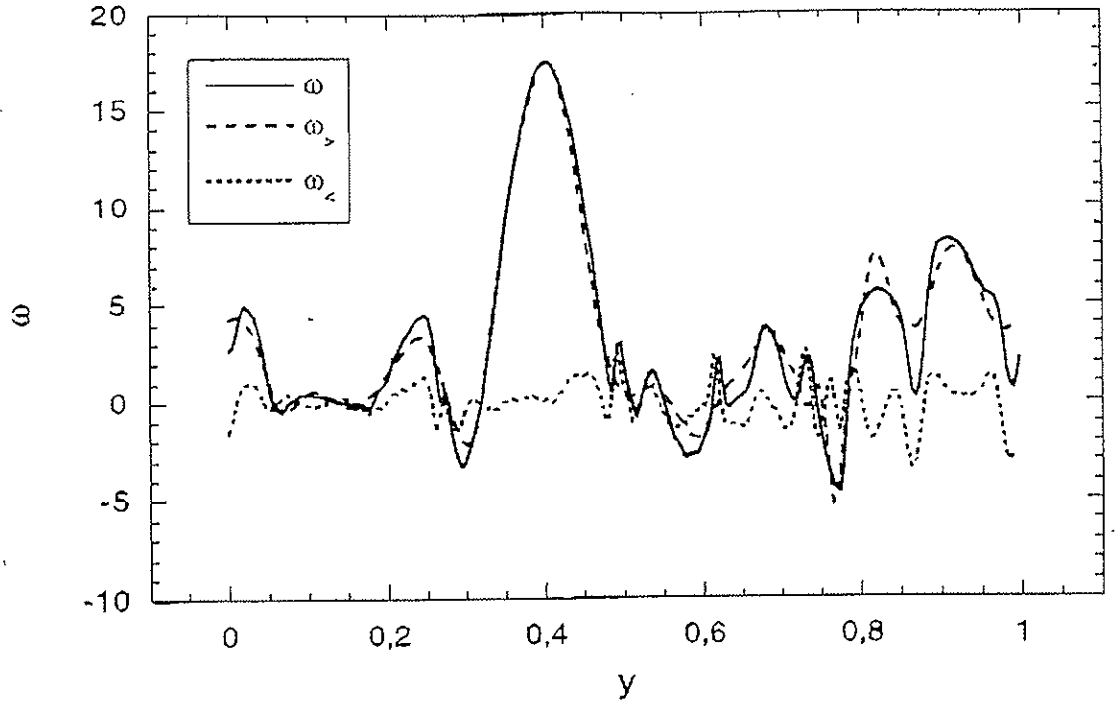
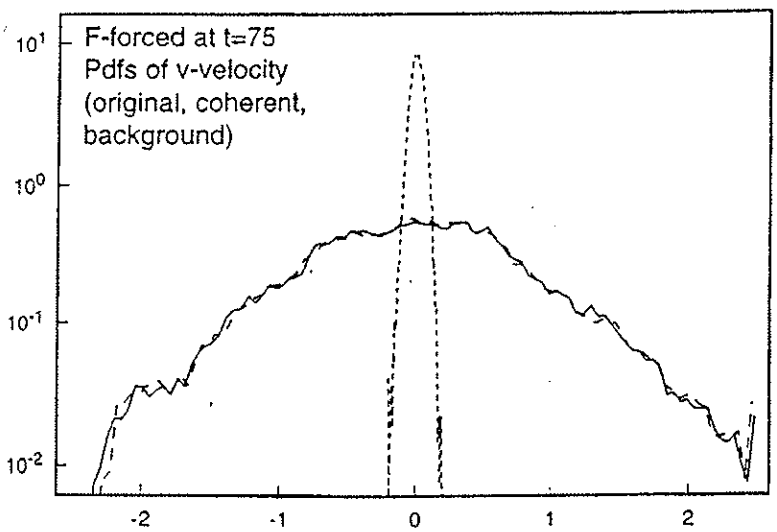


Coherence scatter plot ω=F(ψ)

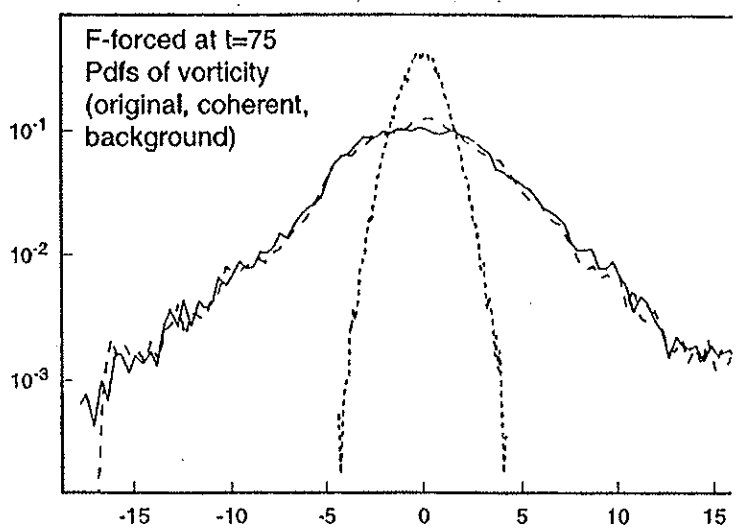
e. *Cut of vorticity*



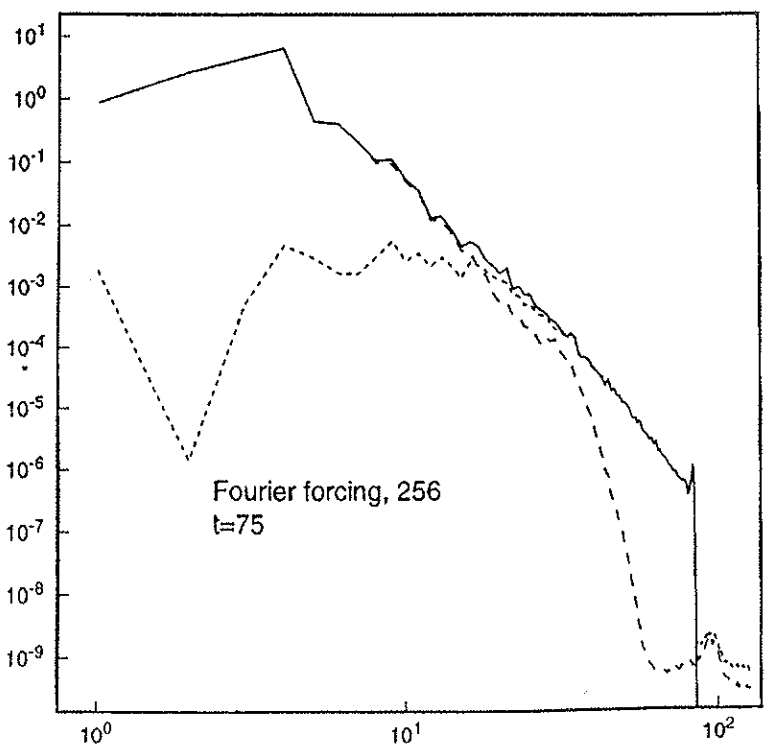
f. *Velocity probability distribution*



Vorticity probability distribution



g. *Energy spectrum*



Moment-Wavelet Quantization of Schrödinger Operators

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Abstract

The space of polynomials maps onto itself under affine transformations, $x \to \frac{x-b}{a}$. This suggests that a moment reformulation of Continuous Wavelet Transform (CWT) theory should lead to significant simplifications in the analysis of Schrödinger quantum operators. We review the implications of this for one and two space dimension problems.

Overview

The space of polynomials of degree N, $\mathcal{P}_N(x)$, maps onto itself under the affine transformation, $x \to \frac{x-b}{a}$. This simple observation underlies the theoretical simplicity that a moment representation formulation can offer in applying continuous wavelet transform (CWT) analysis [1,2] to one dimensional Sturm-Liouville problems in quantum mechanics. By working with properly scaled and translated moments, $\{\tilde{\mu}_{a,b}(p)\}$, one is able to transform the Schrödinger eigenvalue problem (for an arbitrary rational fraction potential) into an exact representation involving a finite number of coupled first order differential equations in the inverse scale variable, $\gamma \equiv \frac{1}{2a^2}$[3-5]. Contrary to other applications of wavelet analysis to quantum systems, the present approach does not involve a truncated variational wavelet basis expansion [6-8] or other type of Galerkin analysis [9].

An essential part of our approach is obtaining the infinite scale moments, $\mu_{\gamma,b}(p) \equiv \tilde{\mu}_{a,b}(p)$, for $\gamma = 0$ or $a = \infty$. Utilizing *moment quantization* methods, the infinite scale problem ($a = \infty$) can be solved, enabling the numerical integration of the coupled differential moment equations, to arbitrary small scale ($a \to 0$). This approach readily yields high precision estimates for the bound state energies and corresponding wavefunction [3-5]. This formalism, to be referred to as *moment-wavelet quantization*, has yielded excellent results for various one dimensional problems. Despite the manifestly non-wavelet structure of the formalism, it is inherently (equivalent to) a wavelet analysis [10]. We review the relevant formalism and briefly outline the essential issues necessary for its multidimensional extension.

Some General Results

Consider the (normalized) one dimensional Schrödinger problem

$$-\partial_x^2 \Psi(x) + V(x)\Psi(x) = E\Psi(x), \tag{1}$$

where $V(x)$ is any rational polynomial. The following presentation applies to mother wavelet functions $\mathcal{W}(x)$ of the form $\mathcal{W}(x) = -\mathcal{N}\partial_x^i e^{-Q(x)}$, where $i \geq 1$, $Q(x) = \sum_{n=0}^{2N} q_n\, x^n$, and $q_{2N} > 0$ (i.e. $\mathrm{Lim}_{x\to\infty}\mathcal{W}(x) = 0$); however, for simplicity, we limit the discussion to the *Mexican hat* wavelet case, $\mathcal{W}(x) = -\mathcal{N}\partial_x^2 e^{-\frac{1}{2}x^2}$. $\mathcal{N}$ is a normalization factor.

The Mexican hat wavelet transform for the unknown wavefunction is given by

$$W\Psi(a,b) = \frac{\mathcal{N}}{\sqrt{a}} \int_{-\infty}^{\infty} dx[-\partial_{\frac{x}{a}}^2 e^{-\frac{1}{2}(\frac{(x-b)}{a})^2}]\Psi(x). \tag{2}$$

Performing the indicated differentiation one obtains:

$$W\Psi(a,b) = \mathcal{N}(2\gamma)^{\frac{1}{4}}\Big[\mu_{\gamma,b}(0) - 2\gamma\mu_{\gamma,b}(2)\Big], \tag{3}$$

where

$$\mu_{\gamma,b}(p) \equiv \int_{-\infty}^{+\infty} dx\; x^p\; e^{-\gamma x^2}\Psi(x+b),\; p \;\geq\; 0 \tag{4}$$

are the moments of the measure $\Phi_{\gamma,b}(x) \equiv e^{-\gamma x^2}\Psi(x+b)$. Clearly, the wavelet transform is a linear superposition of the appropriately scaled and translated moments, $\mu_{\gamma,b}(p)$.

Another important relationship is

$$\partial_\gamma \mu_{\gamma,b}(p) = -\mu_{\gamma,b}(p+2). \tag{5}$$

For the class of one space dimension problems being considered, the moments are linearly dependent on the first $1+m_s$ moments (the *missing moments*) :

$$\mu_{\gamma,b}(p) = \sum_{l=0}^{m_s} M_{E,\gamma,b}(p,l)\mu_{\gamma,b}(l), \tag{6}$$

where m_s is problem dependent. The energy, E, dependent coefficients $M_{E,\gamma,b}(p,l)$ are numerically or algebraically determinable, and must satisfy $M_{E,\gamma,b}(i,j) = \delta_{i,j}$, for $0 \leq i,j \leq m_s$. Inserting these relations into the right hand side of Eq.(5) results in a coupled set of first order differential equations for the missing moments:

$$\partial_\gamma \mu_{\gamma,b}(p) = -\sum_{l=0}^{m_s} M_{E,\gamma,b}(p+2,l)\mu_{\gamma,b}(l), \text{ for } 0 \leq l \leq m_s. \tag{7}$$

The Quartic Anharmonic Oscillator

In order to fix the above formalism in the reader's mind, consider the example of the quartic anharmonic oscillator defined by:

$$-\frac{d^2\Psi}{dx^2} + (mx^2 + gx^4)\Psi(x) = E\Psi(x). \tag{8}$$

We need to define a moment equation for $\Phi_{\gamma,b}(x) \equiv e^{-\gamma x^2}\Psi(x+b)$. One first obtains the differential equation [4]:

$$\Big[-\big[\frac{d^2}{dx^2} + 4\gamma x\frac{d}{dx} + 2\gamma + 4\gamma^2 x^2\big] + \big[m(x+b)^2 + g(x+b)^4\big]\Big]\Phi_{b,\gamma}(x) = E\Phi_{b,\gamma}(x). \tag{9}$$

The moments satisfy the equation (for $p \geq 0$):

$$\mu_{\gamma,b}(p+4) =$$

$$\sum_{i=1}^{3} C_i[\gamma]\mu_{\gamma,b}(p+i) + \{-4\frac{\gamma p}{g} + C_0[\gamma]\}\mu_{\gamma,b}(p) + \frac{p(p-1)}{g}\mu_{\gamma,b}(p-2), \tag{10}$$

where the coefficients are $C_3 = -4b$, $C_2[\gamma] = g^{-1}[4\gamma^2 - m - 6gb^2]$, $C_1 = -g^{-1}[2bm + 4gb^3]$, and $C_0[\gamma] = -g^{-1}[2\gamma + mb^2 + gb^4 - E]$.

The moment equation corresponds to a linear, homogeneous, fourth order finite difference equation (for nonnegative p values). All of the moments are linearly dependent on the initialization or missing moments $\{\mu_{\gamma,b}(l)|0 \leq l \leq 3\}$. This is expressed through the relation

$$\mu_{\gamma,b}(p) = \sum_{l=0}^{3} M_{E,\gamma,b}(p,l)\mu_{\gamma,b}(l), \tag{11}$$

where

$$M_{E,\gamma,b}(i,j) = \delta_{i,j}, \text{ for } 0 \leq i,j \leq 3, \tag{12}$$

and the remaining $M_{E,\gamma,b}(p > 3, l)$ coefficients satisfy the same moment equation as in Eq.(10).

The relation (from Eq.(5))

$$\frac{\partial}{\partial\gamma}\begin{pmatrix}\mu_{\gamma,b}(0)\\ \mu_{\gamma,b}(1)\\ \mu_{\gamma,b}(2)\\ \mu_{\gamma,b}(3)\end{pmatrix} = -\begin{pmatrix}\mu_{\gamma,b}(2)\\ \mu_{\gamma,b}(3)\\ \mu_{\gamma,b}(4)\\ \mu_{\gamma,b}(5)\end{pmatrix}, \tag{13}$$

now becomes (upon substituting Eq.(11))

$$\frac{\partial}{\partial\gamma}\begin{pmatrix}\mu_{\gamma,b}(0)\\ \mu_{\gamma,b}(1)\\ \mu_{\gamma,b}(2)\\ \mu_{\gamma,b}(3)\end{pmatrix} = \begin{pmatrix}0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\\ \mathcal{M}_{2,0}[\gamma] & \mathcal{M}_{2,1}[\gamma] & \mathcal{M}_{2,2}[\gamma] & \mathcal{M}_{2,3}[\gamma]\\ \mathcal{M}_{3,0}[\gamma] & \mathcal{M}_{3,1}[\gamma] & \mathcal{M}_{3,2}[\gamma] & \mathcal{M}_{3,3}[\gamma]\end{pmatrix}\begin{pmatrix}\mu_{\gamma,b}(0)\\ \mu_{\gamma,b}(1)\\ \mu_{\gamma,b}(2)\\ \mu_{\gamma,b}(3)\end{pmatrix}$$

(14)

(note, $\mathcal{M}_{2,0\leq j\leq 3} = -M_{E,\gamma,b}(4,j)$ and $\mathcal{M}_{3,0\leq j\leq 3} = -M_{E,\gamma,b}(5,j)$) where $\mathcal{M}_{2,0}[\gamma] = -C_0[\gamma]$, $\mathcal{M}_{2,1}[\gamma] = -C_1$, $\mathcal{M}_{2,2}[\gamma] = -C_2[\gamma]$, $\mathcal{M}_{2,3}[\gamma] = -C_3$, $\mathcal{M}_{3,0}[\gamma] = -C_0[\gamma]C_3$, $\mathcal{M}_{3,1}[\gamma] = -[C_1C_3 + C_0[\gamma] - 4\gamma/g]$, $\mathcal{M}_{3,2}[\gamma] = -[C_1 + C_3C_2[\gamma]]$, and $\mathcal{M}_{3,3}[\gamma] = -[C_2[\gamma] + C_3^2]$.

Generating the Wavefunction

For arbitrary 'b', given the physical energy and starting missing moment values $\{\mu_{0,b}(l)|\ 0 \leq l \leq m_s\}$ (through methods to be described shortly) one can numerically integrate Eq.(14) and proceed to determine the wavelet transform (Eq.(3)). Facilitating this is the relation:

$$\mu_{0,b}(p) = \int_{-\infty}^{+\infty} dx\ x^p \Psi(x+b) = \int_{-\infty}^{+\infty} dx\ (x-b)^p \Psi(x), \tag{15}$$

or (expanding)

$$\mu_{0,b}(p) = \sum_{q=0}^{p} \binom{p}{q} (-b)^{p-q} \mu_{0,0}(q). \tag{16}$$

Thus, knowledge of the $\mu_{0,0}(l)$ moments generates the initial configuration, $\mu_{0,b}(l)$, necessary for integrating Eq.(14).

Instead of generating the wavelet transform, one can directly recover the wavefunction upon numerically generating the $\mu_{\gamma,b}(l)$ for $\gamma \to \infty$. This follows from the simple observation (obtained after performing the change of variables $y = \sqrt{\gamma}x$)

$$\mathrm{Lim}_{\gamma\to+\infty}\ \mu_{\gamma,b}(p) = (\frac{1}{\sqrt{\gamma}})^{(p+1)}\ \theta(\frac{p}{2})\ \Psi(b),\ p = \text{even}, \tag{17}$$

where $\theta(\rho) = \int_{-\infty}^{+\infty} dy\ y^{2\rho} exp(-y^2)$. For $p = 0, 2$, we have $\theta(0) = \sqrt{\pi}$ and $\theta(1) = \sqrt{\pi}/2$. Usually, for the quantum systems studied so far, moderately small γ values ($\gamma < O(10)$) are sufficient to generate the low lying discrete state wavefunctions (despite the fact that $\gamma \to \infty$ corresponds to the local, configuration space, domain for the wavefunction).

It is immediate to show that the asymptotic relations in Eq.(17) lead to the usual signal-wavelet transform inversion formula [10]

$$\Psi(b) = \frac{1}{\nu} \int_0^{\infty} da \frac{\sqrt{a}}{a^3} \int_{-\infty}^{\infty} d\xi\ \mathcal{D}\Big(\frac{\xi - b}{a}\Big)\ W\Psi(a,\xi), \tag{18}$$

where $\nu = \int_{-\infty}^{\infty} dx\ e^{-\frac{1}{2}x^2}$, and the dual expression, $\mathcal{D}$, satisfies (in terms of the Fourier transforms)

$$\hat{\mathcal{F}}(k) = \sqrt{2\pi}\ \hat{\mathcal{W}}(k)\hat{\mathcal{D}}(k), \tag{19}$$

where $\mathcal{F}(x) = -\partial_x^2 e^{-\frac{1}{2}x^2}$ and $\mathcal{W}(x) = -\mathcal{N}\partial_x^2 e^{-\frac{1}{2}x^2}$.

Moment Quantization for the Infinite Scale Problem

As noted above, an important component of the underlying analysis is the generation of the starting moment values and eigenenergy. If the integration of Eq.(14) is to begin at $\gamma = \gamma_s \neq 0$, then one cannot use the relation in Eq.(16). Instead, the missing moments must be generated at fixed γ_s and each translation variable value, 'b', within the desired range: $-B \leq b \leq B$. If $\gamma_s = 0$, then from Eq.(16) one only requires that the missing moments be generated for the $b = 0$ case.

For the infinite scale problem, $\gamma_s = 0$, there are various moment quantization methods available. One of the earliest is that of Blankenbecler et al [11] which involves a complicated analysis making use of the asymptotic ($p \to \infty$) behavior of the moments. Subsequent reformulations by Killingbeck et al [12] realized that the asymptotic requirements could be weakened without affecting the accuracy of the eigenenergy estimates. Additional works by Killingbeck et al [12], Fernandez and Ogilvie [13], and Witwit [14] used a moment quantization representation in order to generate, wavefunction independent, regular perturbation expansions for the energy.

Paralleling these developments, Handy, Bessis and co-workers [15 - 17] realized that for bosonic systems, the positivity of the ground state wavefunction permitted the exploitation of well known theorems arising from the classic Moment Problem in pure mathematics [18]. Their investigations led to a novel method (that combines moment representation analysis, Moment Problem theorems, and linear programming [19]) producing highly accurate eigenenergy values (and the corresponding missing moments) through the generation of rapidly converging lower and upper bounds : $E_i^{(L)} < E_{physical} < E_i^{(U)}$, with $\mathrm{Lim}_{i\to\infty}(E_i^{(U)} - E_i^{(L)}) = 0$. This method is generally referred to as the Eigenvalue Moment Method (EMM).

The application of EMM, coupled with moment-wavelet analysis, as described above, has yielded impressive results for both the (low lying) discrete state energies and the associated wavefunctions [4,5,10].

Application of any of the above moment quantization methods to the case $\gamma_s \neq 0$ becomes cumbersome. Instead, a very powerful moment quantization method has been recently developed by Tymczak et al [20]. An overview of this is presented following the discussion on the two dimensional extension of our moment-wavelet formalism.

Extension of Moment-Wavelet Quantization to Multidimensions

The extension of the preceding formalism to two dimensional space problems presents one important complication. Instead of working with a finite number of coupled first order differential equations, one is forced to work within a finite hierarchy of coupled partial differential equations, first order in γ, and of Q-th order in 'b', depending on the nature of the potential. For the case $V(x,y) = m(x^2+y^2) + g(xy)^2$, whose moment equation structure is reviewed in ref.[21] (particularly with respects to implementation of our moment-wavelet analysis), the maximum ∂_b^Q order is $Q \leq 2$. A simple analogue of the necessary multidimensional modifications to the one space dimension moment-wavelet analysis formalism is afforded by the quartic anharmonic oscillator problem.

Reconsider the expression $\mu_{\gamma,b}(p) \equiv \int dx x^p e^{-\gamma x^2} \Psi(x+b)$. The first order partial derivative with respect to 'b' gives

$$\partial_b \mu_{\gamma,b}(p) = \int dx x^p e^{-\gamma x^2} \partial_x \Psi(x+b), \tag{20};$$

or (integrating by parts)

$$\partial_b \mu_{\gamma,b}(p) = -p\mu_{\gamma,b}(p-1) + 2\gamma\mu_{\gamma,b}(p+1). \tag{21}$$

We can rewrite this as

$$\mu_{\gamma,b}(p+1) = \frac{[p\mu_{\gamma,b}(p-1) + \partial_b \mu_{\gamma,b}(p)]}{2\gamma}. \tag{22}$$

Utilizing these relations, we can transform the quartic anaharmonic first order equations (refer to Eq.(13)) into either of three versions:

$$\frac{\partial}{\partial\gamma}\begin{pmatrix}\mu_{\gamma,b}(0)\\ \mu_{\gamma,b}(1)\\ \mu_{\gamma,b}(2)\\ \mu_{\gamma,b}(3)\end{pmatrix} = -\begin{pmatrix}\mu_{\gamma,b}(2)\\ \mu_{\gamma,b}(3)\\ (2\gamma)^{-1}[3\mu_{\gamma,b}(2) + \partial_b\mu_{\gamma,b}(3)]\\ \frac{1}{2\gamma}[4\mu_{\gamma,b}(3) + \frac{3}{2\gamma}\partial_b\mu_{\gamma,b}(2) + \frac{1}{2\gamma}\partial_b^2\mu_{\gamma,b}(3)]\end{pmatrix} \tag{23a};$$

$$\frac{\partial}{\partial\gamma}\begin{pmatrix}\mu_{\gamma,b}(0)\\ \mu_{\gamma,b}(1)\\ \mu_{\gamma,b}(2)\\ \mu_{\gamma,b}(3)\end{pmatrix} = -\begin{pmatrix}\mu_{\gamma,b}(2)\\ \mu_{\gamma,b}(3)\\ (2\gamma)^{-1}[3\mu_{\gamma,b}(2) + \partial_b\mu_{\gamma,b}(3)]\\ \sum_{l=0}^{3} M_{E,\gamma,b}(5,l)\mu_{\gamma,b}(l)\end{pmatrix} \tag{23b};$$

$$\frac{\partial}{\partial\gamma}\begin{pmatrix}\mu_{\gamma,b}(0)\\ \mu_{\gamma,b}(1)\\ \mu_{\gamma,b}(2)\\ \mu_{\gamma,b}(3)\end{pmatrix} = -\begin{pmatrix}\mu_{\gamma,b}(2)\\ \mu_{\gamma,b}(3)\\ \sum_{l=0}^{3} M_{E,\gamma,b}(4,l)\mu_{\gamma,b}(l)\\ \frac{1}{2\gamma}[4\mu_{\gamma,b}(3) + \frac{3}{2\gamma}\partial_b\mu_{\gamma,b}(2) + \frac{1}{2\gamma}\partial_b^2\mu_{\gamma,b}(3)]\end{pmatrix}. \tag{23c}$$

For either of these three formulations, one numerically integrates in the γ direction starting at $\gamma = \gamma_s \neq 0$, since the inverse γ dependence complicates any numerical integration starting from zero. This implicitly requires that we know the starting missing moment

values $\{\mu_{\gamma_s,b}(l)|0 \leq l \leq 3\}$ for $-B \leq b \leq B$, as well as the corresponding physical energy value. Clearly, these quantities correspond to large scale information about the physical configuration. As γ increases, the detailed small scale structure of the underlying wavefunction is systematically recovered through the integration of any of the three coupled partial differential equations.

The first partial differential equation formulation (Eq.(23a)) involves no manifest information about the physical system (the quartic anharmonic oscillator). The numerical integration required for obtaining the physical configuration, as $\gamma \to \infty$, critically depends on a very accurate description of the initial moment configuration $\{\mu_{\gamma_s,b}(l)|0 \leq l \leq 3\}$.

The other partial differential equation formulations (Eqs.(23b,c)) involve more manifest information about the underlying physical problem, as evidenced through the explicit incorporation of the $M_{E,\gamma,b}(p,l)$ coefficients. One expects that the corresponding numerical integration is less sensitive (more stable) to the accuracy of the initial moment configuration. This is confirmed by our preliminary numerical investigations. For the ground state, taking $\gamma_s = .1$, we have succesfully integrated the second and third equations up to $\gamma = O(1)$. This inverse scale value ($\gamma = \frac{1}{2a^2}$) is sufficient to approximate the basic structure of the ($\gamma = \infty$) ground state wavefunction for $|b| < O(1.5)$.

For two dimensional problems, such as the $\mathcal{H}_{xy} \equiv -\nabla^2 + m(x^2+y^2) + g(xy)^2$ Hamiltonian, the relevant coupled partial differential moment equations are of the type represented in Eq.(23c).

Moment Quantization for the Large Scale Problem

An effective procedure for determining not only the physical energies but also the required initial moments, for sufficiently small γ values and arbitrary 'b', has been found by Tymczak, Japaridze, Handy, and Wang [20]. This method, to be referred to as TJHW moment quantization, has been shown to generate excellent results for one and two dimensional problems. The simplicity of the approach belies its robustness in solving a large assortment of multidimensional systems, including the famous quadratic Zeeman effect for hydrogenic atoms.

We outline the momentum space version of TJHW moment quantization though its one space dimension formulation. The extension to multidimensional configurations follows in a similar manner.

Consider the Fourier transform of the $\Phi_{\gamma,b}(x) = e^{-\gamma x^2}\Psi(x+b)$ configuration, $\hat{\Phi}_{\gamma,b}(k) = \frac{1}{\sqrt{2\pi}}\int dx e^{-ikx}\Phi_{\gamma,b}(x)$. Its momentum space power series expansion becomes:

$$\hat{\Phi}_{\gamma,b}(k) = \frac{1}{\sqrt{2\pi}}\sum_{p=0}^{\infty}(-ik)^p\frac{\mu_{\gamma,b}(p)}{p!}. \tag{24}$$

Now consider transforming the Fourier transform into the representation $\hat{\Phi}_{\gamma,b}(k) = \frac{1}{\sqrt{2\pi}}\left[\sum_{n=0}(-ik)^n a_n\right]R(k)$, where the *reference function* $R(k)$ can be taken to be $R(k) = e^{-\beta k^2}$ (β is an arbitrary constant). One readily establishes that the a_n coefficients become ($e^{-\beta(-ik)^2}\sum_{p=0}^{\infty}(-ik)^p\frac{\mu_{\gamma,b}(p)}{p!} = \sum_{n=0}(-ik)^n a_n$)

$$a_n = \sum_{2q+p=n} \frac{(-\beta)^q \mu_{\gamma,b}(p)}{q!p!}. \tag{25}$$

However, the moments are linearly dependent on the missing moments through $\mu_{\gamma,b}(p) = \sum_{l=0}^{m_s} M_{E,\gamma,b}(p,l)\mu_{\gamma,b}(l)$. Accordingly,

$$a_n = \sum_{2q+p=n} \frac{(-\beta)^q}{q!p!} \sum_{l=0}^{m_s} M_{E,\gamma,b}(p,l)\mu_{\gamma,b}(l), \tag{26}$$

or

$$a_n = \sum_{l=0}^{m_s} \Big[\sum_{2q+p=n} \frac{(-\beta)^q M_{E,\gamma,b}(p,l)}{q!p!} \Big] \mu_{\gamma,b}(l). \tag{27}$$

That is,

$$a_n[E; \mu_{\gamma,b}(0), \ldots, \mu_{\gamma,b}(m_s)] = \sum_{l=0}^{m_s} D_{n,l}[E,\gamma,b]\mu_{\gamma,b}(l). \tag{28}$$

Quantization results from taking $a_{N+n'}[E; \mu_{\gamma,b}(0), \ldots, \mu_{\gamma,b}(m_s)] = 0$, for arbitrarily large N and $0 \leq n' \leq m_s$. Thus, TJHW quantization becomes

$$Det(D_{N+n',l}[E,\gamma,b]) = 0, \text{for } 0 \leq n', l \leq m_s, \ \textit{and } N \to \infty. \tag{29}$$

This prescription yields excellent, high accuracy, results for the low lying energy states of many multidimensional quantum systems. Upon obtaining the physical energy, E, one can then determine very accurate values for the moment ratios $\mathcal{R}_{\gamma,b}(l) \equiv \frac{\mu_{\gamma,b}(l)}{\mu_{\gamma,b}(0)}$, $1 \leq l \leq m_s$.

To obtain $\mu_{\gamma,b}(0)$ (and thereby the $\mu_{\gamma,b}(l)$'s as well) we can use Eq.(21) divided by $\mu_{\gamma,b}(0)$, for $p = 0$:

$$\frac{1}{\mu_{\gamma,b}(0)} \partial_b \mu_{\gamma,b}(0) = 2\gamma \mathcal{R}_{\gamma,b}(1). \tag{30}$$

Since the right hand side is known numerically, one can integrate this equation in the b-direction, for fixed γ_s (the moments are appropriately normalized at $\gamma = \gamma_s$ and $b = 0$). Thus this entire procedure generates the necessary initial missing moment configurations corresponding to $\gamma = \gamma_s$.

The TJHW quantization method works provided γ is sufficiently small. This is not too surprising since quantization is a global, multiscale process, and moments are extensive (non-local) structures. The smaller the chosen value for γ_s, the larger B becomes (for a specified accuracy for the energy and missing moments). In the case of the quartic anharmonic oscillator we find that for $N = O(40)$, $\beta = .5$, and $\gamma_s = .1$, we can predict the ground state energy to at least ten decimal places for $|b| < O(5)$.

Conclusion

The preceding analysis (i.e. Eq.(23b,c) and TJHW quantization) works reasonably well for the quartic anharmonic oscillator ground state. More numerically efficient implimentations are under investigation, together with the extension of the method to the two dimenisonal problem $\mathcal{H}_{xy}$. A more detailed discussion will appear elsewhere.

The moment-wavelet quantization formalism presented shows that for a large class of problems, specifically those corresponding to (multidimensional) rational polynomial potentials, and mother wavelets of the type defined at the outset, one can transform the Schrödinger equation into an exact set of coupled (partial) differential equations in the inverse scale and translation variables. Utilization of moment quantization methods then permit the numerical integration of these coupled equations yielding either the wavelet transform itself, or pointwise recovery of the (multidimensional) wavefunction and energy. This process recovers the wavefunction through a multiscale analysis starting from large (infinite) scale information ($\gamma_s << O(1)$), systematically generating more detailed 'local information' as the inverse scale approaches infinity, $\gamma \to \infty$. This analysis has been firmly established by Handy and Murenzi within the context of one space dimension quantum mechanical systems. The extension to two dimensions follows in a similar manner through the analysis of multidimensional versions of Eqs.(23b,c). The numerical analysis of this is currently under investigation. The details will be presented elsewhere.

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Uncovering hidden symmetries with directional wavelets

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It is well-known that the two-dimensional continuous wavelet transform (CWT) is a powerful tool for detecting various features in a picture or a pattern, as opposed to the discrete WT, which is the prime choice for data compression. If the relevant features have a preferred direction, the tool analysis necessary for detecting them must fulfill two conditions: (i) One must use the full 2-D CWT, including the rotation parameter, in addition to the usual translations and dilations; (ii) One must choose a wavelet with some directional selectivity, such as a Morlet or a Cauchy wavelet. This is standard, for instance, in directional filtering or edge detection, two classical problems in image processing. In this talk, we describe a novel application of such directional wavelets, namely the determination of the symmetries of a pattern, even approximate or local ones.

In order to fix the notations, let us recall first the 2-D CWT, in position and in spatial frequency space, respectively:

$$S(\vec{b}, a, \theta) = a^{-1} \int d^2\vec{x}\, \bar{\psi}(a^{-1} r_{-\theta}(\vec{x} - \vec{b})) s(\vec{x}) = a \int d^2\vec{k}\, e^{i\vec{b}\cdot\vec{k}}\, \overline{\widehat{\psi}}(a r_{-\theta}(\vec{k}))\, \widehat{s}(\vec{k}), \tag{1}$$

where $s, \psi \in L^2(\mathbb{R}^2, d^2\vec{x})$ and ψ is admissible, which essentially reduces to the familiar zero mean condition

$$\widehat{\psi}(\vec{0}) = 0 \iff \int d^2\vec{x}\, \psi(\vec{x}) = 0. \tag{2}$$

In the sequel we will use a directional wavelet, that is, a wavelet ψ whose Fourier transform $\widehat{\psi}(\vec{k})$ has (essential) support in a convex cone in spatial frequency space, with apex at the origin. In particular, we will use a Cauchy wavelet, since this class is particularly well-suited for the problem at hand, as shown by various calibration tests [1]. The wavelet we will use is defined as follows:

$$\widehat{\psi}^{(10)}_{44}(\vec{k}) = \begin{cases} (\vec{k}\cdot\vec{e}_{-10})^4\, (\vec{k}\cdot\vec{e}_{10})^4\, e^{-k_x}, & -10^\circ \le \arg(\vec{k}) \le 10^\circ, \\ 0, & \text{otherwise}, \end{cases} \tag{3}$$

where $\vec{e}_{\pm 10}$ denote unit vectors with argument $\pm 10^\circ$, respectively. This particular Cauchy wavelet, whose supporting cone has an opening angle of $\Phi = 20^\circ$, has the same angular selectivity as the familiar 2-D Morlet wavelet with $|\vec{k}_o| \simeq 5.5$ [1].

We turn now to the symmetry problem. Given a 2-D signal (an object, a pattern,...), let $S(\vec{b}, a, \theta)$ be its wavelet transform with respect to a directional wavelet. Our main tool for analyzing the symmetries of the signal is its *scale-angle measure*, defined as the positive-valued, bounded function:

$$\mu_S(a, \theta) = \int d^2\vec{b}\, |S(\vec{b}, a, \theta)|^2,$$

which may also be viewed as a partial energy density in the scale and angle variables, that is, in spatial frequency space.

We begin with a simplified version and eliminate the scale dependence by integrating over a, thus ending with a function α_S of the rotation angle only, called the *angular measure*

of the object. In general, $\alpha_s(\theta)$ is 2π-periodic. But when the analyzed object has rotational symmetry n, that is, it is invariant under a rotation of angle $\frac{2\pi}{n}$, then α_s is in fact $\frac{2\pi}{n}$-periodic. This is illustrated on simple geometrical figures, such as a square, a rectangle and a regular hexagon.

If the object has not only a rotational symmetry, but a combined rotation-dilation symmetry, one has to use the full scale-angle measure μ_s, which is then a doubly-periodic function of a and θ. This is exemplified on several types of pictures: a 'twisted snowflake', a Penrose tiling with local 10-fold symmetry, an octagonal dot pattern. The latter is particularly interesting (see the figure). It is invariant under a rotation by $\pi/4$, and under dilation by a factor $1+\sqrt{2}$, as follows from its construction rule, but it has, in addition, two distinct combined rotation-dilation symmetries, one exact, the other one approximate. The remarkable fact is that these two additional symmetries were discovered on the graph of the scale-angle measure, *not* on the tiling itself! In each case, one gets a semigroup, with seemingly infinitely many orbits, and some defects in the second case. This example suggests a systematic wavelet analysis of 2-D lattices, which often show rich geometric and arithmetic properties. The technique could also be used for uncovering hidden symmetries of physical objects, such as quasicrystals or nanotubes, through their X-ray diffraction patterns.

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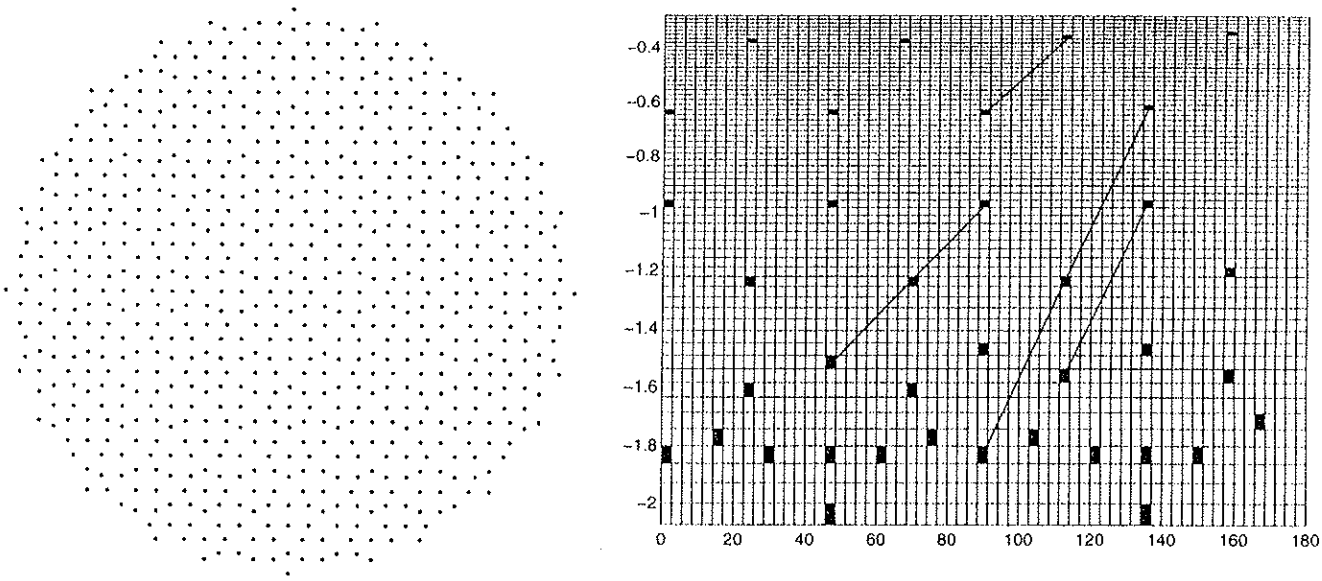

Analysis of an octagonal tiling: (left) the tiling; (right) the set of local maxima of its angular measure $\mu_s(a,\theta)$. The latter shows two symmetries consisting of a rotation by $\pi/8$ combined with a dilation by $\sqrt{2}\cos(\pi/8)$, resp. $2\cos(\pi/8)$. The variables are θ (in degrees) on the x-axis and $-\ln a$ on the y-axis.

Darwinian evolution of proteins

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According to the most recent evaluations based on fossil records, it is estimated that life appeared on earth about 3.5 billions years ago ... only 1 billion years after the birth of earth itself. Thus, following the Darwinian view on the evolution of life, all the present day living creatures are descendants of the primitive bacteria-like, unicellular organisms that populated the earth at that time. Consequently, it is a nonsense to state that any presently living organism is more "primitive" than any other one or, stated differently, that any organism is "more developed" than any other one: all the contemporary living organisms have the very same age and any bacterium, like any human, is the result of 3.5 billion years of evolution. The bacteria are perfectly well adapted to their own environments and, in many respects, are much more "robust" than the mammalians! However, it is quite true that a human being is more complicated than a bacterium, which is a different thing. Although the Darwinian "theory" of evolution has now gained wide acceptance, there still often remains some sort of unconscious belief that evolution proceeds towards more complexity and that the driving force of evolution is the search for perfection (indeed, the driving force of evolution is natural selection). According to this view, the "tree of life" is generally drawn like a tree with a root (the primitive bacteria) and a top (the Man) and many branches holding the leaves of the different species. In fact, and this is now well documented, the tree of life is a bush with no particular prominent top and with a great number of dead branches. Let us give some examples. Horses appeared some 50 My ago in the Americas, and many fossil records enabled to trace their evolution, in particular the progressive disappearance of all but one of their fingers. Fortunately, some American horses managed to migrate to Asia and Europe and proliferate there, because at one time they totally disappeared from the Americas! A dead branch in the bush... Horses reappeared in the Americas recently when Cortez invaded Mexico. About 500 My ago, an extraordinary "explosion" of multicellular organisms took place. The well-known "Burgess fossils" kept the record of animals looking like crustaceans or arthropodes, but showing some morphological traits totally unshared by modern animals. One of them, for example, had five eyes! Everything happened as

if evolution became over- imaginative and performed a great number of trial-and-error experiments. Most of these species have disappeared from the earth. Some 250 My ago, a massive extinction occured and it is currently estimated that more than 80 extinct. Another massive -but less important- extinction occured about 60 My ago, that led among other things to the disparition of the famous dinosaurs. Think that before being swept from the surface of the earth, dinosaurs had been the kings of the animals for more than 100 million years, and that Homo sapiens appeared about 200,000 years ago..... Interested readers will find many such beautiful stories in the books by Stephen Jay Gould. So far, our knowledge of the evolution of living species has been based solely on the study of fossils. But we can go somewhat deeper into the biochemical origin of the evolution of organisms. As everybody knows, all the information that is necessary to make an organism develop, live and duplicate -or make babies- is entirely hold into the chromosomes of the cells, the so-called genome. The chromosomes are huge nucleic acid macromolecules (that can be more than one millimeter long) built up by only four elementary bricks called the bases. The human genome contains approximately 3 billions of such bases, while that of a simple bacterium amounts to ca. 4-6 million bases. A genome can be considered as a text, the significance of which results from the precise linear order of the bases along the macromolecule, called the sequence. A mutation is the result of any modification in this sequence -substitution of a base by another one, insertion or deletion of one or more bases, etc... It is essentially through mutational events, whatever their precise nature, that evolution proceeds. The cells also contain important macromolecules called proteins (e.g. muscles, hemoglobin). The proteins are built up by amino acids. There are 20 different amino acids in biological proteins, and a typical protein contains about 300 amino acids. The sequence of a protein -the order of the amino acids- is dictated univocally by the sequence of the corresponding gene in the genome. Hence a mutation in a gene will generally result in a mutation in the corresponding protein. The point is that nowadays, it has become feasible to determine the sequence of nucleic acids. One can therefore compare the sequences of genes or proteins having the same function but coming from different organisms. For example, one can compare the sequences of all the hemoglobins of all the animals. The basic idea is that all these proteins (in fact all their genes) have a common ancestor (they are said to be homologous) and that they have independently accumulated mutations as time passed. Mammalians and birds have diverged quite a long time ago (their common ancestor lived hundreds of million years ago) and it is anticipated that the sequences of their respective hemoglobins will show a great number of differences. On the contrary, humans and chimps diverged only 5 million years ago and their hemoglobins will have very similar sequences. Hence, by counting the number of differences between homologous sequences, it is possible to have an idea of the relationships between different species and to build a "genealogical tree" or "phylogenic tree" so as to trace back the path of evolution. Such studies based on sequence comparisons are

the core of an exciting field of research called molecular evolution. Now comes the time when Alex Grossmann turned to biology and when, together with his colleagues Bruno Torresani and Mathias Holschneider from Marseilles, he could solve a long-standing problem. Since we are plain simple-minded biologists, we shall give only the outline of his work. The observed differences between two homologous sequences are the result of two different phenomena: i) the intrisic mutability of amino acids, and ii) the time course. Depending on their respective chemical properties, two amino acids can or can not be easily exchanged for one another. Hence, if the comparison of two sequences shows that two amino acids (say A and C) are seldom exchanged for one another, this can mean that i) either A and C can hardly be exchanged or ii) they can be exchanged but the sequences diverged recently. Here is the problem: so far, it has not be possible in sequence comparison studies to disentangle the contribution of the intrinsic mutability of amino acids from that of the time. By comparing two homologous sequences A and B and counting the numbers of substitutions, one can build a 20x20 transition matrix M(A/B). Its element Mij(A/B) is the probability that, if the amino acid i is found at one site in sequence A, the amino acid j will occupy the same position in sequence B. Thus, every comparison (alignment) of two sequences can be represented by a transition matrix and, therefore, every alignment determines a point in a 400-dimensional space. Let us now consider the action of a rate matrix Q. An alignment generated a time t will give rise to a transition matrix M = etQ. M can be determined experimentally and the problem is to disentangle the respective influences of t and Q. Suppose that we have in our hands the sequences of n homologous proteins from n organisms. We can perform N = n(n-1)/2 pairwise comparisons and build N transition matrices. For the kth pair of sequences (1 ¡ k ¡ N) coming from the kth pair of species and assuming that the rate matrix Q is the same for all the pairs, we write Mk = etkQ or log(Mk) = tkQ where tk is the time elapsed since the divergence of the two species. Hence in the 400- dimensional space, all the points corresponding to Mk will lie along the line of multiples of Q and the relative distances of the points along this line will give the relative divergence times of the different species. The method is presently being tested on different sequences from different families, and the first results are definitely encouraging. They show, beyond any doubt, that a theoretical physicist can be scientifically original and productive even in fields such as biology.....

What can we learn about DNA sequences from wavelet analysis?

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The wavelet transform has recently received much attention in the "fractal" community [1-4]. This mathematical microscope is actually well adapted to the large hierarchy of scales involved in fractal patterns. Increasing the magnification factor provides a natural way to explore the intricate internal structure of fractals and to resolve local scaling invariance. Our purpose in this talk, is to report on some interesting and rather successful application of the wavelet transform to the analysis of the complexity of DNA sequences.

We first provide some background definitions concerning the continuous wavelet transform [5,6] and its ability to capture the hierarchical distribution of singularities of a fractal (1D) signal [1-4]. We elaborate on a unified multifractal description of singular distributions, including measures and functions, based on the wavelet decomposition [7-9]. This new approach relies on the definition of partition functions from the wavelet transform modulus maxima. We demonstrate that very much like thermodynamic functions, the generalized fractal dimensions and the singularity spectrum can be readily determined from the scaling behavior of these partition functions [9]. We show that this method provides a natural generalization of the classical box-counting and structure function techniques to fractal functions. We illustrate our theoretical considerations on pedagogical examples including devil's staircases and fractional Brownian motions [7-9]. We briefly review some experimental applications to fully developed turbulence data [7] and financial time series [10].

Then we use the wavelet transform to investigate the fractal scaling properties of DNA sequences [11,12]. Mapping nucleotide sequences onto a "DNA walk" [13] produces fractal landscapes that can be studied quantitatively by applying the wavelet transform modulus maxima method. By considering analyzing wavelets that make the wavelet transform microscope blind to "patches" of different nucleotide compositions which are ubiquitous to genomic sequences, we demonstrate and quantify the existence of long-range correlations in both the (protein) coding and the non coding regions of the human genome [14]. Moreover, the fluctuations in the patchy landscapes of DNA walks are found to be homogeneous with Gaussian statistics. When looking at the introns, one notices some significant tendancy to the long-range correlation exponent H to increase with the GC content of the sequence [14]. In particular, a few introns with a low percentage of GC do not display long-range correlations ($H = 1/2$) and therefore cannot be distinguished from actual exons. We show that similar GC content dependent long-range correlations also exist in exons when undersampling these coding sequences by retaining the third base of each codon only [14]. This observation seems to corroborate the attractive biological conjecture that the correlations in coding DNA sequences could be attained through the degeneracy of the genetic code (most of the synonyms are due to change in the third base in the codon). Finally, we comment about the possible understanding of the origin of the observed long-range correlations in terms of the nonequilibrium dynamical processes that produce the "isochose structure" of the human genome [14].

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Balian-Low Theorem for Landau Levels*

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In this paper we connect the Bailian[1]-Low[2] theorem with the dynamics of an electron in a magnetic field B. When the motion is in the xy-plane with the magnetic field $\vec{B}$ in the z-direction, Schrödinger's equation assumes the form [the Landau gauge $\vec{A} = (0, Bx)$ is chosen]

$$\left[\frac{p_x^2}{2m} + \frac{(p_y + \frac{\hbar}{\lambda^2}x)^2}{2m}\right]\psi(x,y) = E\psi(x,y),\ \lambda^2 = \frac{\hbar c}{eB} \tag{1}$$

Eq. (1) is written in the xy-representation. For the magnetic field problem it is convenient to work in the WX-representation according to the transformation

$$\begin{aligned} W = \frac{m\lambda^2}{\hbar}v_y = \frac{\lambda^2}{\hbar}\left(p_y + \frac{\hbar}{\lambda^2}x\right), \qquad & P_W = mv_x = p_x \\ X = \frac{\lambda^2}{\hbar}\left(p_x + \frac{\hbar}{\lambda^2}y\right), \qquad & P_X = p_y = \frac{\hbar}{\lambda^2}Y \end{aligned} \tag{2}$$

Here WP_W denotes the velocity degree of freedom and XP_X – the orbit center degree of freedom. λ is the cyclotron radius (as was already used above).

* This paper is a follow-up of the talk presented at the Conference in honor of Alex Grossmann. It was a pleasure to come to this conference, to present the talk and to write this paper as well.

¿From Eqs. (1) and (2) it follows that the Hamiltonian depends on the degree of freedom WP_W only (it does not depend on the orbit center) The wave function $\phi(W, X)$ can therefore be chosen as a product function

$$\phi_\ell(W, X) = \gamma_\ell(W)\chi(X) \tag{3}$$

where ℓ labels the Landau levels. For connecting with the Balian-Low Theorem we use the magnetic translations which depend on the orbit center operators X and P_X only.[3,4] In particular, the commuting finite magnetic translations in x and y-directions can be written in the following way[5]

$$\begin{aligned} \tau_x(d) &= \exp\left[\frac{i}{\hbar}(p_x + \frac{\hbar}{\lambda^2}y)Nd\right] \equiv \exp(iX\frac{2\pi}{d}) \\ \tau_y(Nd) &= \exp\left(\frac{i}{\hbar}p_y d\right) \equiv \exp\left(\frac{i}{\hbar}P_X d\right) \end{aligned} \tag{4}$$

where N is an integer which is defined by the following rationality condition[3]

$$N = \frac{hc/e}{Bd^2} = \frac{2\pi\lambda^2}{d^2} \tag{5}$$

This relation has a simple meaning of the ratio of the elementary fluxon hc/e to the flux of the magnetic field B through a unit cell of area d^2 [d is an arbitrary constant[6]]. As is seen from Eq. (4), the magnetic translations depend only on the XP_X degree of freedom of the orbit center. We can therefore use them directly for constructing a von Neumann set for a Landau level, by replacing $\chi(X)$ in Eq. (3) by $\chi_{mn}(\chi)$ in Eq. (4). We get

$$\chi_{mn}(X) = (-1)^{mn}\exp(-\frac{i}{\hbar}P_X dm)\exp(iX\frac{2\pi}{d}n)\chi(X) \tag{6}$$

¿From Eq. (3) we then get the von Neumann set for a Landau level

$$\phi_{\ell mn}(W,X) = \gamma_\ell(W)\chi_{mn}(X) \tag{7}$$

The properties of the von Neumann set in Eq. (6) are determined by the initial function $\chi(X)$. The Balian-Low theorem states that if the functions in Eq. (6) are orthogonal for $mn \neq m'm'$, then at least one of the uncertainties ΔX or ΔP_X in the $\chi(X)$ -state diverges. An example of a normalized χ-function, that leads to an orthogonal von Neumann set in Eq. (6), is

$$\chi(X) = \begin{cases} \frac{1}{\sqrt{d}} & |X| < d/2 \\ 0 & |X| > d/2 \end{cases} \tag{8}$$

For the χ in Eq. (8), $\Delta X = d/\sqrt{12}$ and $\Delta P_X = \infty$, in agreement with the Balian-Low theorem.

The condition on χ for an orthogonal von Neumann set [Eq. (6)] was first formulated in the kq-representation[4,5] [See also Ref. 1]. This condition is that the absolute value of the kq-function, $C(k,q)$, is constant[7]

$$|C(k,q)| = \text{ const}, \tag{9}$$

where (a is an arbitrary constant). $C^{(a)}(k,q)$ and $\chi(X)$ are related in the following way[4]

$$C^{(a)}(k,q) = \left(\frac{q}{2\pi}\right)^{\frac{1}{2}} \sum_n e^{ikan}\chi(q-na) \tag{10}$$

It is easy to check that the function $\chi(X)$ in Eq. (8) leads to $\left|C^{(d)}(kq)\right| =$ const. A normalized kq-function that satisfies Eq. (9) is necessarily a pure phase factor

$$C(k,q) = \frac{1}{\sqrt{2\pi}} \exp[i\varphi(k,q)] \tag{11}$$

where $\varphi(k,q)$ is real. In the X-representation the condition given by Eq. (9) [or Eq. (11)] assumes the following form [Eq. (10) is used]

$$a\sum_n \chi^*(X-na)\chi(X-na-\ell a)=\delta_{\ell 0} \tag{12}$$

Again it is easy to check that the function in Eq. (8) with $d=a$ satisfies Eq. (12).

Now, when $\chi(X)$ satisfies the condition given by Eq. (9) or (12), the set of functions in Eq. (16) for each Landau level ℓ is orthogonal. By using the canonical transformation for going from WX to xy-coordinates (see Ref. 8) we get the orthonormal von Neumann set $\psi_{\ell mn}(x,y)$ in the xy-representation

$$\psi_{\ell mn}(x,y)=(-1)^{mn}\exp\left(\frac{i}{\lambda^2}yNdn\right)\psi_\ell(x+Ndn,y+dm) \tag{13}$$

where the function $\psi_\ell(x,y)$ in Eq. (13) can be given the two alternative forms

$$\begin{aligned}\psi_\ell(x,y) &= \frac{1}{\sqrt{2\pi}}\int \exp(iyk)\gamma_\ell(k\lambda^2+x)F_\chi(k)dk \\ &= \frac{\exp\left(-\frac{ixy}{\lambda^2}\right)}{\frac{1}{\sqrt{2\pi}}\lambda^2}\int \exp(\frac{i}{\lambda^2}xz)F_\gamma\left(\frac{z-y}{\lambda^2}\right)\chi(z)dz\ ,\end{aligned} \tag{14}$$

and where F_χ and F_γ are the Fourier transforms of χ and γ respectively (γ and χ are the functions in Eq. (3)). It should be pointed out that the condition given by Eq. (9) [or Eq. (12)] is necessary and sufficient for the von Neumann set [Eqs. (7) or (14)] for each Laudau level ℓ to be orthogonal.

By using the results for the single degree of freedom XP_X ($P_X=\frac{\hbar}{\lambda^2}Y$), the Balian-Low theorem implies that the coordinates X and Y of the orbit center in the state ψ_ℓ [Eq. (14)] for a magnetic field cannot both be well localized, or in a more precise language, at least one of the two uncertainties

ΔX or ΔY in the ψ_ℓ-state [Eq. (14)] diverges. It is interesting to point that for obtaining the result there is no need to go to the configuration plane xy. The coordinates x and y of the electron in a magnetic field are related to the operators WP_W and XP_X ($P_X = \frac{\hbar}{\lambda^2}Y$) in Eq. (2), one has

$$x = W - Y \ , \quad y = X - \frac{\lambda^2}{\hbar}P_W \ . \tag{15}$$

Bearing in mind that the uncertainties ΔW and ΔP_W are finite in the state given by Eq. (3) [or alternatively, by Eq. (14)], we come to the conclusion [by using Eq. (15)] that at least one of the uncertainties Δx or Δy diverges in the state $\psi_\ell(x, y)$ of Eq. (14). This also means that if orthongonality is required on von Neumann type functions [see Eq. (7)] then the electron in a magnetic field cannot be well localized in both the x and y- directions.

It is of interest to compare the consequences of the Balian-Low theorem with the known results for the Wannier function in a magnetic field.[9,10] In Ref. 9 it was shown that Wannier functions for a Bloch electron in the xy-plane, with the magnetic field B perpendicular to the plane, cannot fall off at infinity faster than r^{-2} ($r^2 = x^2 + y^2$). In Ref. 10 Wannier functions with a r^{-2} fall off were actually constructed for such a two-dimensional problem where only a magnetic field is present. One can see that the Balian-Low theorem for Landau levels is in good agreement with the results of Refs. 9 and 10. In order to see it, we notice that as a consequence of the Balian-Low theorem the quantity $\langle x^2 + y^2 \rangle = \langle r^2 \rangle$ diverges, (the triangular brackets denote the expectation value) because either $\langle x^2 \rangle$ or $\langle y^2 \rangle$ has to diverge. This divergence of $\langle r^2 \rangle$ also follows when the function falls off as r^{-2} at infinity, like in Ref. 10.

It should, however, be pointed out that the Balian-Low theorem is completely general in nature. For example, when $\chi(X)$ is given by Eq. (8), then $\psi_0(x,y)$ in Eq. (14) assumes the form (for the lowest Landau level, $\ell = 0$)

$$\psi_0(x,y) = \left(\frac{1}{4\pi^3 d^2 \lambda^6}\right)^{\frac{1}{4}} \exp\left(-\frac{ixy}{\lambda^2}\right) \int_{-d/2}^{d/2} \exp\left[\frac{i}{\lambda^2} xz - \frac{(y-z)^2}{2\lambda^2}\right] dz \quad (16)$$

In deriving Eq. (16), use was made of the second line in Eq. (14). For large y, Eq. (16) becomes

$$\psi_0(x,y) \sim \left(\frac{d^2}{4\pi^3 \lambda^6}\right)^{\frac{1}{4}} \exp\left(-\frac{ixy}{\lambda^2} - \frac{y^2}{2\lambda^2}\right) \sin\left(\frac{xd}{2\lambda^2}\right) / \left(\frac{xd}{2\lambda^2}\right) \quad (17)$$

This function has a $\frac{1}{x}$ fall off in the x-direction and is Gaussian in the y-direction.

Finally, it can be directly checked that the von Neumann set in Eq. (13) is orthonormal, when $\psi_\ell(x,y)$ is given by Eq. (14). In the particular case of the localized function in Eq. (16), the explicit orthonormal von Neumann set is [See Eq. (17)]:

$$\psi_{0mn}(x,y) = \frac{(-1)^{mn}}{(4\pi^3 d^2 \lambda^6)^{\frac{1}{4}}} \exp\left(-\tfrac{i}{\lambda^2} xy - \tfrac{i}{\lambda^2} xmd\right) \int_{-d/2}^{d/2} \exp\left[\tfrac{i}{\lambda^2}(x + nNd)z - \right.$$
$$\left. - \tfrac{(y+md-z)^2}{2\lambda^2}\right] dz \quad (18)$$

In conclusion, the Balian-Low theorem was applied to the motion of an electron in the xy-plane, with a magnetic field perpendicular to this plane. By using this theorem it was shown that orthonormality on a von Neumann lattice and localizability are incompatible features. In particular, it is shown that even for the best localized eigenfunctions for the Landau levels the uncertainties Δx and Δy can never both be made finite. On a qualitative level,

incompatibility of orthonormality and localizability is not an unexpected feature for wave functions in quantum mechanics[11]. In this paper we derive for the first time a precise quantitative result for Landau level wave functions in a magnetic field. Although the proof was carried out in the Landau gauge, the result that Δx and Δy can never both be finite in a Landau level state $\psi_\ell(x, y)$ [See Eq. (14)], for any ℓ, is gauge independent. This is seen from the expression of the wave function in the WX-representation [Eq. (7)]. The part of the wave function $\chi_{mn}(X)$ in Eq. (7) that leads to the von Neumann set does not depend on the gauge because the X-coordinate does not appear in the Hamiltonian of the problem [Eq. (1)].

As is well known, in the symmetric gauge the wave function for any Landau level can be chosen well localized in both x and y-directions[12]. However, for such a well localized wave function, the von-Neumann set in Eq. (13) will *not be orthogonal.*[13]

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The Quantum Hall Effect

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January 25, 1998

To Alex–A teacher and friend

The talk gave a historical and mathematical review of the Hall effect. It started in the previous century with an observation of Maxwell which lead Hall, a graduate student at Johns Hopkins, to his classical experiment which proved Maxwell wrong. The next spectacular development took place in 1980: K. von Klitzing found that, under appropriate conditions, the Hall conductance of rather poorly define materials, is quantized to a great precision. Such measurements now serve as an accurate determination of the fundamental constant e^2/h. I outlined two complementary theoretical frameworks, both of geometric character, that address the precise quantization that occurs in the integral quantum Hall effect. One associates the Hall conductance with a Chern number and another associates it with an index of a Fredholm operator. The Chern number point of view was started by D. Thouless and collaborators [4]. The Index approach was started by J. Bellissard [2]. My review was based on a series of works on the subject I have carried out in collaboration mainly with R. Seiler [3]. A review of the subject from such a point of view is given in [1] which can be downloaded from `http://physics.technion.ac.il/~avron/B3.ps.gz`

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Formules de traces locales[‡]

T. Paul[§]

On se propose dans cette courte note de montrer comment le concept d'état cohérent peut être utilisé comme filtre spectral pour des opérateurs de Schrödinger, faisant ainsi le lien entre deux des sujets les plus importants traités jusqu'ici par Alex Grossmann.

Un état cohérent est un vecteur de l'espace de Hilbert qui "ressemble" beaucoup à un point de l'espace de phases: il permet en effet, par action d'un hamiltonien quantique H d'évaluer l'énergie en un point,

$$H\psi_{p,q} \sim h(p,q)\psi_{p,q} \tag{1}$$

et il "se souvient", lors de son évolution (quantique), de lévolution classique ainsi que de l'action lagrangienne associée,

$$e^{i\frac{tH}{\hbar}}\psi_{p,q} \sim e^{i\frac{\mathcal{L}(t)}{\hbar}}\psi_{p(t),q(t)} \tag{2}$$

Ici $(p(t), q(t))$ est le flot classique et $\mathcal{L}(t) := \int_0^t (p\dot{q} - h(p,q)ds$ est l'action lagrangienne.

En fait les relations d'incertitude de Heisenberg le rendent sensible non seulement au flot mais à toutes ses dérivées (par rapport aux conditions initiales). C'est ainsi que dés l'ordre dominant en $\hbar$ la "forme" de l'état cohérent (son "vide" en théorie des champs) se transporte lors de l'évolution quantique par un opérateur unitaire "guidé" par le flot linéarisé autour de la trajectoire.

Grâce à leur localisation (autour d'un point) en espace de phases, les états cohérents permettent de déterminer la partie du spectre d'un opérateur de Schrödinger correspondant à une région de l'espace de phases. Soyons plus précis.

A tout fonction test a, par exemple (mais pas nécessairement) une gaussienne, et tout fonction "cut-off" ρ (égale à 1 autour de 0 et à support compact) on associe la famille de vecteurs indicée par $\mathbb{R}^{2n}$,

$$\psi^a_{(p,q)}(x) \;=\; \rho(x-q)\,(2\pi\hbar)^{-\frac{3n}{4}}\,2^{-n/4}\,e^{-ipq/2\hbar}\,e^{ipx/\hbar}\,a\left(\frac{x-q}{\sqrt{\hbar}}\right). \tag{3}$$

Soit $H = -\hbar^2\Delta + V(x)$, V lisse, un opérateur de Schrödinger sur $\mathbb{R}^n$ (ou sur une variété sans bord), de spectre (discret) $\{E_j\}$ et fonctions propres φ_j. La formule de traces de Gutzwiller exprime la densité spectrale de H convolée avec une fonction test φ à transformée de Fourier

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à support compact. Si l'on veut localiser en espace de phases on est amené à considérer la quantité suivante:

$$\sum \varphi\left(\frac{E_j - E}{\hbar}\right) \mid (\psi^a_{(p,q)}, \varphi_j) \mid^2 \tag{4}$$

La localisation de φ assure que l'on "regarde" le spectre autour de l'énergie E et la localisation de $\psi^a_{(p,q)}$ ne retient dans cette somme que les valeurs propres dont la fonction propre correspondante est localisée en (p, q).

Le premier résultat est un développement asymptotiuqe lorsque $\hbar \to 0$ de la forme:

$$\sum \varphi\left(\frac{E_j - E}{\hbar}\right) \mid (\psi^a_{(p,q)}, \varphi_j) \mid^2 \sim \sum_{k=0}^{\infty} \nu_k(\varphi)\hbar^{-n+1/2+k} \tag{5}$$

On peut de plus exprimer les premiers coéfficients en fonction des caractéristiques de stabilité ou d'instabilité linéaire de l'éventuelle trajectoire périodique γ passant par (q, p). Dans tous les cas on exhibe ainsi une partie "linéaire" de "spectre", *rélle* si γ est stable et *complexe* si γ est instable. Plus précisement on a (dans la cas $n = 2$), si γ est linéairement stable:

$$(6\sum \varphi\left(\frac{E_j - E}{\hbar}\right) \mid (\psi^a_{(p,q)}, \varphi_j) \mid^2 \sim \sum_{j\in\mathbb{Z},k\in\mathbb{N}} c_l\varphi\left(\frac{1}{T_\gamma}(2\pi j + (k + \frac{1}{2})\theta + \frac{S_\gamma}{\hbar} + \sigma_\gamma)\right)\hbar^{-n+1/2}$$

Ici T_γ, S_γ et σ_γ sont les période, action et indice de Maslov de γ. De plus θ est l'angle de l'application de Poincaré. Si γ est instable on a une formule presque identique mais "complexe":

$$\sum \varphi\left(\frac{E_j - E}{\hbar}\right) \mid (\psi^a_{(p,q)}, \varphi_j) \mid^2 \sim \sum_{j\in\mathbb{Z},k\in\mathbb{Z}} c_l\varphi^{signe(k)}\left(\frac{1}{T_\gamma}(2\pi j + \sqrt{-1}(k + \frac{1}{2})\mu + \frac{S_\gamma}{\hbar} + \sigma_\gamma)\right)\hbar^{-n+1/2}$$

(7)

où μ est l'exposant de Liapounov de γ et $\varphi^{+/-}$ sont les parties "Hardy" et "anti-Hardy" de φ.

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Unitary Extensions of Convolutive Time Evolutions

Contribution to "Perspectives in Mathematical Physics"

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Abstract

Starting from a convolutive time evolution we sketch a construction of a unitary one.

1 Introduction

In this note we consider linear, non-unitary time evolutions in a separable Hilbert space that allow a unitary extension in a larger space. This situation has been studied extensively for dissipative time evolutions where unitary "dilations" have been constructed [1]. Thus let A be a dissipative operator in the Hilbert space $\mathcal{H}$, generating the time evolution

$$V(t) = \exp[At]. \tag{1}$$

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Then there is a larger Hilbert space $\mathcal{K}$, a projector P, projecting $\mathcal{K}$ upon $\mathcal{H}$ and a unitary time evolution $U(t)$ on $\mathcal{K}$, such that

$$V(t) = PU(t)P. \tag{2}$$

Here we address a similar problem that arises within the context of Maxwell's equations for lossy, linear dielectric systems. There the displacement $\mathbf{D}$ is related to the electric field $\mathbf{E}$ by the relation

$$\mathbf{D}(\mathbf{x},t) = \mathbf{E}(\mathbf{x},t) + \int_0^t ds\, \chi(\mathbf{x}, t-s)\mathbf{E}(\mathbf{x}, s), \tag{3}$$

which contains a convolutive term. Since dielectric systems have recently become important in connection with photonic crystals (spatially periodic dielectrics) [2], it is convenient to extend, if possible, the convolutive time evolution by a unitary one in a larger space. Once a unitary time evolution is available, a canonical setup and quantization of the system offers no further problems [3]. This is important for a description of atomic decay and the production of X-ray radiation by fast electrons travelling through a dielectric, see [4].

In order to understand how convolutive time evolutions appear in physical processes, consider the situation where part of the system is "integrated out". Thus

$$\partial_t \psi(t) = -iH\psi(t) \tag{4}$$

in some linear space $\mathcal{H}$, is rewritten as (P and $Q = 1 - P$ are complementary projectors, $Q\psi(0) = 0$)

$$\begin{aligned} \partial_t P\psi(t) &= -iPHP\psi(t) - iPHQ\psi(t), \\ \partial_t Q\psi(t) &= -iQHP\psi(t) - iQHQ\psi(t). \end{aligned} \tag{5}$$

Solving the second and substituting into the first results in the convolutive equation

$$\partial_t P\psi(t) = -iPHP\psi(t) - \int_0^t dsPHQ\exp[-iQHQ(t-s)]QHP\psi(s). \tag{6}$$

This expression is the starting point for the construction of generalized master equations in statistical mechanics, where ψ is a density operator and H the Liouville operator, i.e., the commutator with the Hamiltonian. In quantum theory ψ is the state vector and H the Hamiltonian. This procedure suggests following the inverse route, i.e., reconstructing a unitary time evolution from (6). Below we show how this is done.

2 Reconstruction of a unitary time evolution

Let $\mathcal{H} = L^2(\mathbb{R}^N, d\mathbf{x})$ and H a self-adjoint operator acting in $\mathcal{H}$. Consider the equation of motion for $f(t) \in \mathcal{H}$,

$$\partial_t f(t) = -iHf(t) - \int_0^t ds W(\mathbf{x}, t-s) f(s). \tag{7}$$

Note that only $t \geqslant 0$ appears, so we can consider $W(|t|)$ without penalty. Suppose it has a Fourier transform

$$W(\mathbf{x}, |t|) = \int M(\mathbf{x}, d\omega) \exp[-i\omega t], \tag{8}$$

with $M(\mathbf{x}, d\omega) \geqq 0$ an $\mathbf{x}$-dependent measure. In physical applications W is real, so $M(\mathbf{x}, -d\omega) = M(\mathbf{x}, d\omega)$. Usually also $M(\mathbf{x}, d\omega) = g(\mathbf{x}) m(d\omega)$ in applications. Now let $f_1(t) = f(t)$ and $f_2(t, \omega)$ be a second function with the property $f_2(0, \omega) = 0$. Consider the set

$$\partial_t f_1(t) = -iHf(t) - \int M(\mathbf{x}, d\omega) f_2(t, \omega), \tag{9}$$

$$\partial_t f_2(t, \omega) = -i\omega f_2(t, \omega) + f_1(t). \tag{10}$$

Solving the second and substituting into the first then gives back (7) and we have removed the convolution at the price of introducing f_2. With $F = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ we have

$$\partial_t F = -iKF, \tag{11}$$

where

$$K = \begin{pmatrix} H & -i\int M(\mathbf{x}, d\omega)\ldots \\ i & \omega \end{pmatrix} = K_0 + K_1,$$

$$K_0 = \begin{pmatrix} H & 0 \\ 0 & \omega \end{pmatrix}, K_1 = \begin{pmatrix} 0 & -i\int M(\mathbf{x}, d\omega)\ldots \\ i & 0 \end{pmatrix}. \tag{12}$$

Let us now check whether these formal manipulations can be made precise. Thus let $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}'$, $\mathcal{H}' = L^2(\mathbb{R}^{N+1}, d\mathbf{x} M(\mathbf{x}, d\omega))$, $f_1 \in \mathcal{H}$, $f_2 \in \mathcal{H}'$, so $F \in \mathcal{K}$.

Proposition: Let $M(\mathbf{x}, \mathbb{R}) \leqslant c < \infty$. Then K is self-adjoint in $\mathcal{K}$.

Proof: Obviously K_1 is symmetric on a suitable dense domain. Moreover it is bounded. We have

$$||K_2F||^2 = \int d\mathbf{x}\{|\int M(\mathbf{x}, d\omega)f_2(\mathbf{x}, \omega)|^2 + \int M(\mathbf{x}, d\omega)|f_1(\mathbf{x})|^2\},$$

so, since

$$|\int M(\mathbf{x}, d\omega)f_2(\mathbf{x}, \omega)| \leqslant [\int M(\mathbf{x}, d\omega).1]^{1/2}[\int M(\mathbf{x}, d\omega')|f_2(\mathbf{x}, \omega')|^2]^{1/2},$$

we obtain

$$|\int M(\mathbf{x}, d\omega)f_2(\mathbf{x}, \omega)|^2 \leq c\int M(\mathbf{x}, d\omega')|f_2(\mathbf{x}, \omega')|^2,$$

so

$$||K_2F||^2 \leq c||F||^2.$$

Thus K_1 is bounded, self-adjoint and K is self-adjoint with domain $\mathcal{D}(K_0)$.■

Corollary: Let $U(t) = \exp[-iKt]$ and P the projector upon $\mathcal{H}$, i.e., $P\mathcal{K} = \mathcal{H}$. Then $f(t) = PU(t)Pf(0)$.

We see that, as in the dilation case (2), we retrieve the original motion by projecting back upon $\mathcal{H}$.

3 Discussion

Above we sketched a special case. Obviously we can replace H by a dissipative operator, thus making K dissipative as well. Actually dissipative structures sometimes appear in physical applications by making the approximation

$$\int_0^t dsW(\mathbf{x}, t-s)f(s) = \int_0^t dsW(\mathbf{x}, s)f(t-s)$$
$$\approx \int_0^t dsW(\mathbf{x}, s)f(t) \approx \int_0^\infty dsW(\mathbf{x}, s)f(t) = Vf(t),$$

which makes sense if $W(\mathbf{x}, s)$ is a rapidly decaying function of s. From this we can conclude that in dissipative time evolutions details of the original structure are lost that are still present in a convolutive situation. A notorious case is the quantization of the damped harmonic oscillator [5].

Depending upon the precise structure of M, the procedure can be extended to more general Hilbert spaces $\mathcal{H}$. As discussed in [4], further properties of M are required in the Maxwell case . In that reference some applications can also be found.

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REGGE POLES 38 YEARS LATER

André MARTIN

CERN and Annecy

Sorry: I do not think that Alex ever worked on Regge poles, but I believe he worked a lot on the Schrödinger equation and Regge poles are part of it. In 1959, Tullio Regge had the idea to make ℓ, the orbital angular momentum, non-integer and even complex in the radial reduced Schrödinger equation [1]

$$\left[-\frac{d^2}{dv^2}+\frac{\ell(\ell+1)}{r^2}+V(r)-E(n,\ell)\right]U_{n,\ell}=0\ .$$

Then resonances and bound states become poles in the complex ℓ plane. The purpose of Regge was to study the asymptotic behaviour of the scattering amplitude for unphysical $\cos\theta$ (θ scattering angle) going to infinity. This was controlled by the Regge pole most to the right for potentials of the type of Yukawa. Explicit calculations [2] showed that for that kind of potential Regge poles move in a compact region when energy runs from - infinity to + infinity.

Shortly afterwards, it was proposed, in particular by Chew and Frautschi [3], to exchange the role of energy and angle and to assume that high energy scattering amplitudes, for finite square of the momentum transfer t were dominated by the exchange of Regge poles. They postulated that the Regge trajectories (the position of the pole),

$$J=\alpha(t)$$

were linear in t. There did not seem to be any good reason for this except simplicity. However, with the years, there was growing evidence that Regge trajectories were linear, like the ρ trajectory linear for negative t [4] (scattering) and positive t (resonances) [5].

Around 1967 the idea of duality (equivalence of sum of resonances and sum of exchanged Regge poles) appeared [6] and after various episodes [7], Veneziano [8] came out with a formula which had explicit duality and strictly linear trajectories, but at the fundamental level, there was no understanding of this.

The miracle is that if mesons are quark-antiquark pairs, and baryons 3-quark colourless systems, interacting by some sort of relativistic potential behaving like a string at large distances, we understand now that Regge trajectories are linear if we use relativistic kinematics

[8]. Furthermore one can prove that, at large angular momentum, the preferred configuration of a 3-quark system is a quark-diquark. The diquark, which must be in the representation $\bar{3}$ of the colour group, is exactly equivalent to an antiquark, and therefore we have an understanding of the fact that baryons and mesons have parallel trajectories [9],[10], so that in the end, Chew and Frautschi were right, ever if nobody understands why they had this intuition!

Even if we know that the fundamental theory is QCD, we shall have to live with Regge poles which dominate fixed momentum transfer high energy scattering. They are a tool, essential in many places, like for instance to control the phase of the $K_{0L} - K_{0S}$, regeneration amplitude [11] in matter, which is needed in at least one of the CP and CPT major experiments.

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Arguments de symétrie
en faveur d'une réinterprétation d'une gravure de Dürer (1471-1528),
celle connue sous le nom de *Melencolia I*

Henri Bacry

Deux gravures sur cuivre marquent l'année 1514: *Saint Jérôme dans sa cellule* et *MELENCOLIA I*. Tous les critiques s'accordent pour les considérer comme des chefs-d'œuvre et, si on les associe, c'est parce que Dürer lui-même les appariaient; en effet, il ne les offrait jamais l'une sans l'autre.

La plupart des spécialistes de Dürer donne la mélancolie comme thème de la gravure, alors que Dürer n'a pas donné le mot *mélancolie* comme titre. Il faudrait expliquer pourquoi Dürer a placé le mot *Melencolia* dans le ciel. Le numéro I qui l'accompagne reste mystérieux. D'autre part, les éléments usuellement associés à la mélancolie n'apparaissent pas dans la gravure.

Dans le *Saint Jérôme* apparaît le caractère exemplaire de la perspective. A l'opposé, la perspective ne semble jouer qu'un rôle tout à fait secondaire dans *la Mélancolie*. Dans cette gravure, un point de fuite se déduit sans difficulté en prolongeant le fléau de la balance et les lignes des corniches du bâtiment qui lui sont parallèles. Le point de convergence se situe, comme il se doit, sur la ligne d'horizon, au-dessous du *c* du mot *MELENCOLIA*.. On constate que l'échelle, elle est *étrangement adossée en biais*. Elle semble tout droit sortie d'un dessin d'Escher.

Une construction simple permet de déterminer la forme exacte de cette échelle en la projetant horizontalement sur le mur sur lequel elle s'appuie, à l'aide du point de fuite mentionné; on est obligé de lui attribuer une forme inadmissible, les échelons étant bien plus écartés dans la partie inférieure que dans la partie supérieure (voir la figure correspondante). Il est donc *impossible* de la considérer comme une échelle ordinaire, car Dürer, connaissant parfaitement les règles de la perspective, a certainement voulu attirer notre attention sur son caractère étrange.

On se trouve devant l'obligation d'identifier l'échelle avec la seule échelle extraordinaire qui apparaisse dans le texte sacré que traduit saint Jérôme, à savoir *l'échelle de Jacob*. Rappelons le texte de ce rêve de Jacob (*Genèse*, XXVIII, 12-13):*Il eut un rêve que voici: une échelle était dressée sur la terre. Son sommet atteignait le ciel. Et des messagers d'Élohim montaient et descendaient le long de cette échelle. Et voici que YHVH se tenait au-dessus d'elle et disait: Je YHVH, l'Élohim d'Abraham, ton père, et l'Élohim d'Isaac*. Nous avons pris soin, dans cette traduction, de distinguer entre les deux noms de la divinité qui apparaissent dans la Bible: *Élohim* et *YHVH*.

Le symbolisme de l'échelle est le suivant. L'esprit de l'homme peut gravir et descendre les degrés de l'échelle et démontrer ainsi qu'elle relie la terre au ciel, où siège *Élohim*. La terre est d'ailleurs nommée en premier. Cependant *YHVH* se tient AU-DESSUS de l'échelle. Ne veut-on pas souligner ici que, contrairement à *Élohim*, *YHVH* est inaccessible? Un autre élément va nous conforter dans notre interprétation. Selon Maïmonide[1], le nombre de barreaux est de quatre, certains, dit-il, en mentionnent sept. Fallait-il dessiner quatre ou sept barreaux pour l'échelle? Curieusement, Dürer semble avoir suggéré quatre barreaux et indiqué qu'il pourrait y en avoir sept, puisque trois d'entre eux n'apparaissent qu'à moitié.

S'arrêter au ciel, c'est atteindre la banderole *MELENCOLIA I*, le premier stade de la divinité (*Élohim*). Il faut voir au-delà de ce premier stade, dépasser l'état de mélancolie. Ce n'est rien d'autre que ce que Maïmonide écrit: *En effet, tous les hommes alors, à l'exception de quelques-uns, ignoraient l'existence de YHVH, et leur plus haute méditation N'ALLAIT PAS AU-DELÀ DE LA SPHÈRE CÉLESTE, DE SES FORCES ET DE SES EFFETS.*[2] Ajoutons que *Le Guide* est un best seller de la Renaissance et que Dürer est un humaniste.

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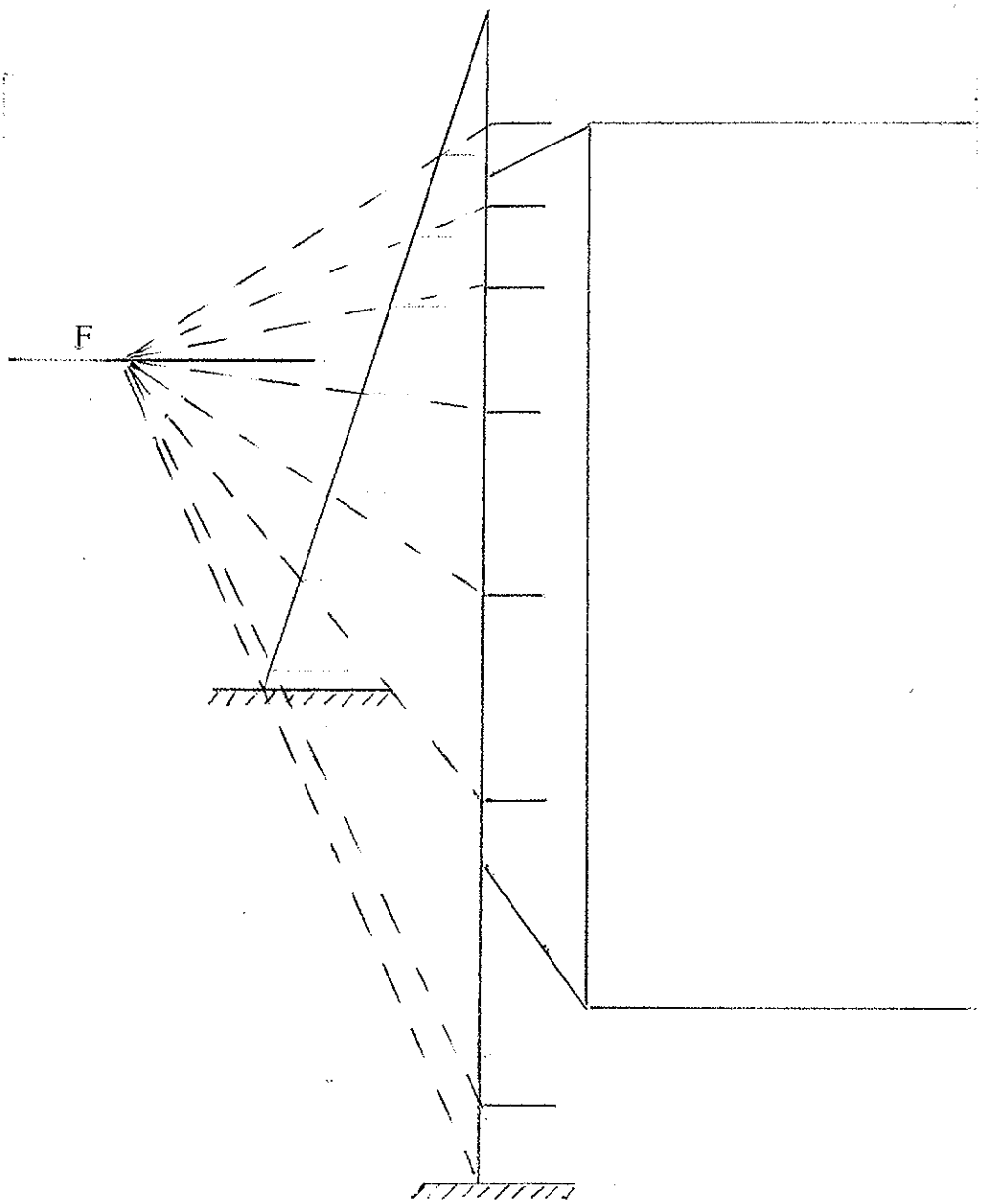
[1] Moïse Maïmonide, *Le guide des égarés*, II, 11, Verdier, 1979.

[2] *Op. cit.*, I, 63. Notons que le traducteur a remplacé *YHVH* par Dieu, ce qui rend le texte inintelligible.

Fig. 190 : Dürer, Melencolia I, 1514 (premier état). Gravure sur cuivre, 239×168 mm.



Projection horizontale de l'échelle sur le mur d'appui
(on suppose l'échelle de longueur minimale)

Deux anomalies:
1) L'espacement irrégulier des barreaux
2) L'échelle repose sur un plan situé au-dessous de la marche

Si l'on allonge l'échelle, la première anomalie s'atténue,
mais la seconde s'accentue considérablement

Alex Grossmann features large in my life, as he does in so many other lives, and I am grateful for the many insights I gained from him, in science as well as in non scientific matters. If I was sure that my students got from me even half the amount of wisdom that I gained from Alex, I would be a happy advisor. Thank you, Alex, for everything.

REGULARITY OF IRREGULAR SUBDIVISION IN 1D

INGRID DAUBECHIES[†]

May 1997

ABSTRACT. This talk presents joint work with Igor Guskov (Princeton University) and Wim Sweldens (Bell Laboratories). We study the smoothness of the limit function for one dimensional unequally spaced interpolating subdivision schemes. The new grid points introduced at every level can lie in irregularly spaced locations between old, adjacent grid points and not only midway as is usually the case. For the natural generalization of the four point scheme introduced by Dubuc and Dyn, Levin, and Gregory, we show that, under some geometric restrictions, the limit function is always C^1; under slightly stronger restrictions we show that the limit function is almost C^2, the same regularity as in the regularly spaced case.

Subdivision is a powerful mechanism for the construction of smooth curves and surfaces. The main idea behind subdivision is to iterate upsampling and local averaging to build complex geometrical shapes. The following is a typical subdivision scheme, used to construct interpolating curves. In this four-point scheme [1, 6], one constructs a real function f on $\mathbf{R}$, starting from the pre-assigned (but arbitrary) values $f(n)$ to be taken by f at the integers, via a simple iterative procedure: first, the value $f(k+\frac{1}{2})$ of f at each odd multiple of 1/2 is computed by cubic interpolation from its two left neighbors $f(k-1)$, $f(k)$ and its two right neighbors $f(k+1), f(k+2)$; next, the same procedure is repeated to compute the values of f at the odd multiples of 1/4, and so on, "filling in" the graph of f. The limiting function for this subdivision scheme is "almost C^2"; more precisely it is C^1, and $|f'(x+t)-f'(x)| \leq C[\sup_{n\in \mathbf{N}} |f(n)|]|t|\ |log|t||$. If arbitrary points (x_n, y_n) in the plane are given, this scheme can be used to construct an interpolating curve by taking $f(n) = x_n$, $g(n) = y_n$, and finding the corresponding limit functions f and g; the map $t \mapsto (f(t), g(t))$ then parameterizes the interpolating curve. Originally subdivision schemes were studied in the context of corner cutting [7, 10] as well as for building piecewise polynomial (or *spline*)

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curves [11, 8, 9]. Later subdivision was studied independently of spline functions [1, 6, 2, 3, 4, 5]. Around the same time it was noted that subdivision fits into the framework of wavelets and multiresolution analysis [12, 13].

Smoothness of spline functions follows from simple algebraic conditions on the polynomial segments at the knots. However, when the limit function of a subdivision scheme is not a spline, convergence and smoothness are usually harder to prove. Various approaches have been explored to find the Hölder exponent of the limit function for a general subdivision scheme [6, 1, 2, 15, 16, 17, 18, 19, 3, 20] or to determine its Sobolev class [21, 22]; all these results are concerned with a regular (or sometimes called uniform) grid, i.e, at each stage new grid points are introduced in the middle of two old grid points. The most common tools used are the commutation formula (by which the order of the subdivision can be reduced), the Fourier transform, and spectral analysis.

In the spline context, knot insertion algorithms allowed for splines with non-equally spaced knots from very early on. This extra flexibility is crucial in developing algorithms for computer aided geometric design, see e.g. [23]. Later a global subdivision scheme for non-uniform splines was introduced in [24]. Again smoothness results are relatively easy given that the analytic form of the limit function is known.

Only recently have people started working on subdivision for non-equally spaced knots. Here we distinguish two settings. The *semi-regular* case, where the original samples are non-equally spaced, but the subdivision scheme still introduces new grid points midway between old ones, and the *irregular* case, were new grid points need not be in the middle between old ones even ad infinitum. For example, the family of Lagrange interpolating schemes [25] of which the four point described above scheme is the cubic case, can easily be generalized to both the semi-regular case [26, 27] and the irregular case [28].

Almost all work on smoothness for non-equally spaced grids concerns the semi-regular case; the subdivision scheme becomes spatially variant, so that the Fourier transform can no longer be used in a simple way. On the other hand, at very fine scales, the grids consist of long concatenated finite stretches of equispaced points, with a sudden transition in spacing distance at the original data points. One can then combine results from the uniformly spaced case with the spectral analysis of a matrix that characterizes the behavior through scale of the scheme near the transition points, exploiting the stationarity through scale of the scheme near those points. In [26] Warren shows that the four point interpolating scheme in the semi-regular case still yields an C^1 "almost" C^2 function. Several results have been obtained in the higher dimensional semi-regular setting. The problem is then harder because the topology can be irregular as well. We refer to [29, 27, 30] for more details.

In the irregular case the subdivision scheme becomes both spatially variant and non-stationary, meaning that near every point the weights used for

interpolation can change from scale to scale. Smoothness results are then not straightforward; because the subdivision is spatially variant the Fourier transform can no longer be used and because it is non-stationary no simple matrix spectral analysis can help. In [14] we study the regularity of limit functions of subdivision in the irregular (ad infinitum) one dimensional case. We show that the commutation formula still holds in the irregular case and use it as the main tool in our analysis. In particular, if an interpolating subdivision scheme preserves polynomials up to a certain order (that is, if the data happen to be samples, at the irregularly spaced coarse grid points, of a polynomial, then the subdivision scheme will produce the correct polynomial values at every finer scale), then we can define "derived" schemes that preserve lower order polynomials only, but that are simpler to analyze. For the cubic four-point scheme, we can use up to four such reduction steps; after the fourth reduction, the resulting subdivision scheme is no longer convergent, but we can control the rate of its exponential growth through scale, and this helps us prove smoothness for the original, more complicated scheme. For example, with a very mild condition on the non-uniformity of the grid, the four point scheme converges to a C^1 function in the irregular case; we also show how the irregularity of the grid affects the fractional smoothness exponent. In fact, with a more restrictive condition on the irregular grid, the four point scheme actually converges to a $C^{2-\epsilon}$ function, so that we obtain the same regularity as in the regular case. This holds under remarkably general conditions. For instance, let us start from arbitrary (but uniformly bounded) function values at irregularly spaced points such that the successive spacings are bounded uniformly, and bounded below away from zero. Now choose the "new" points at every level under the following fairly weak restriction. Every time you introduce a new point between two old points, it splits the interval between the two old points into two subintervals; suppose that the ratio of the lengths of the left and right subintervals always lies between .5 and 2. (This still allows for fairly crazy subdivisions; if the ratio is always .5, for instance, then the ratio of the subintervals on both sides of a coarse grid point grows exponentially with the number of scales.) For all such cases, the cubic four-point scheme still leads to a limit function that is "almost" C^2. The proof of this result is fairly technical, and we refer to [14] for all the details. It is interesting that even for the regular case, our approach leads to a proof that is different from earlier proofs, and that is conceptually very simple.

We conclude this summary with a short discussion of why one should even care about the irregular setting. Is not the semi-regular setting sufficient? The user provides the coarse level grid points and after that the subdivision might as well use the midpoints to synthesize a curve. In this setup, indeed, the semi-regular setting is sufficient to generate smooth functions, although, to have more control over the geometric shape of a curve a designer may want to insert new points at arbitrary locations independent of the underlying parameterization. There is another framework, however, in which the

irregular setting comes up naturally, namely in the case of compression of or multiresolution analysis for irregular samples. Here the user provides data, sampled on a closely spaced but irregular grid, which we can think of as the "finest" level grid. Resampling onto a regular grid is typically costly and may generate unwanted artifacts. In [28] it is shown how to then build a multiresolution analysis and an associated wavelet transform on the original grid. The main idea is to downsample the original grid and introduce spatially variant filter banks using the lifting scheme. Once the multiresolution is defined, wavelet based algorithms such as compression and denoising, familiar from the regular case, can be carried out in the same way in the irregular setting. The wavelet basis functions from the coarsest level are now generated with a subdivision scheme where the new points are no longer midpoints but are dictated by the finest level grid on which the data was sampled. They are no longer translates and dilates of one fixed function, but form an instance of so-called "second generation wavelets" [31]. One could now use the semi-regular setting to argue that using midpoints *beyond* the finest level leads to a smooth limit function. However, in a practical setting one often cannot afford or one does not care to synthesize functions on levels finer than the original finest level. Instead all processing is done on the original grid or the coarser grids. Given that the finest and coarsest level can be arbitrarily far apart, the irregular setting then becomes the correct model.

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TIME-FREQUENCY LOCALIZATION, SYMMETRIES AND GENERALIZED MEANS *

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to Alex Grossmann, for his 65th birthday

"Chirps" are generally introduced so as to model monocomponent signals modulated in both amplitude and frequency. Intuitively, chirps should therefore localize on some curve of the time-frequency plane (instantaneous frequency or group delay), and a key question is, given a chirp, to find a time-frequency description with such a localization property.

In the case of (unimodular) *linear* chirps, it is well-known that the Wigner-Ville distribution is a solution and, moreover, that it is the only solution in the class of quadratic energy distributions [7, 2]. This can be readily checked by a direct calculation, but this also admits a geometrical interpretation which allows for generalizations beyond the restrictive case of linear frequency modulations. In fact, A. Grossmann first showed in 1976 [4] (then followed independently by A. Royer [8]) that the Wigner-Ville distribution can be expressed as the expectation value of a *symmetry* operator in the plane. This has the consequence that energy contributions located at any two points in the plane create an additional contribution at the midpoint of the line joining the two considered points. This interpretation is indeed a key for understanding the localization property of the Wigner-Ville distribution on lines in the plane, since straight lines are the only curves which exactly coincide with all of their midpoints.

The situation of linear chirps can be extended to the case of *hyperbolic* chirps for which J. and P. Bertrand have established that suitable affine generalizations of the Wigner-Ville distribution may guarantee localization [1]. As shown in [3],

*The results reported here are mostly based on joint work with P. Gonçalvès (formerly with ENS Lyon and now with INRIA Rocquencourt).

localization properties of Bertrand distributions can be given a geometrical interpretation almost identical to that pertaining to the Wigner-Ville distribution, provided that the usual *arithmetic* mean (which underlies the ordinary concept of midpoint) is replaced by some suitable *generalized* mean which turns out to be the generalization of the logarithmic mean introduced by K.B. Stolarsky in [9].

In order to preserve the duality relation which held — in the case of the Wigner-Ville distribution — between arithmetic mean and symmetry, one has to look for the conditions under which a Stolarsky's mean is indeed *quasi-arithmetic* in the sense of G.H. Hardy *et al.* [6]. It can be shown [3] that the class of solutions is fairly restricted and that, apart from the arithmetic mean, only two other cases are admissible solutions to this problem, corresponding respectively to the *square-root* mean and the *geometric* mean. Given this interpretation, the geometric mean can then serve as a basis for proposing the definition of a time-frequency distribution [5] which happens to coincide with the one referred to as an "Unterberger distribution" in Bertrands's terminology [1].

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Designing a Custom Wavelet Packet Image Compression Scheme, with Applications to Fingerprints and Seismic Data*

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1 Introduction

No single image compression algorithm can be expected to work well for all images, and designing a transform coding image compression algorithm for a given application is itself a meta-algorithm. Sampling rates, frequency content, and pixel quantization all influence the compressibility of the original data. Subsequent machine or human analyses of the compressed data, or its presentation at various magnifications, all influence the nature and visibility of distortion and artifacts. Thus, algorithms like JPEG [1], established for a "natural" images intended to be viewed by humans, do not satisfy the requirements for compressing fingerprint images intended to be scanned by machines. In that particular example, it was necessary to develop a new algorithm *WSQ* [2].

One procedure focuses on the transform portion of the compression algorithm: the *best basis method* automatically finds a transform which provides the best average compression of a representative set of images, selected from a set of "fast" transforms. A version of this method was used to design the WSQ fingerprint image compression algorithm, while another was used to design compression algorithms for various types of seismic exploration data.

2 Transform coding image compression

The generic transform coding compression scheme is depicted in Figure 1. It consists of three pieces:

- *Transform:* Apply a function, invertible in exact arithmetic, to decorrelate nearby pixels in the image. Do this by decomposing the image into a superposition of independent patterns, producing a sequence of floating-point amplitudes of the new components.
- *Quantize:* Replace the transform amplitudes with (small) integer approximations. This is *lossy*, or non-invertible; all distortion is introduced here.
- *Code:* Rewrite the integer stream into a more efficient alphabet, so as to approach the information-theoretic minimum bit rate. This operation is invertible.

These three steps are depicted at the left of Figure 1; to recover an image from the coded, stored data, they are inverted.

Compression and decompression are judged together by their *rate-distortion curve*. The *Unquantize* block does not in general produce the same amplitudes that were given to the *Quantize* block during compression, but the errors thus introduced can be reduced in exchange for a lower *compression ratio*, which is computed by dividing the size of the input image file by the size of the stored file. This must take into account all of the side information that is needed for reconstruction.

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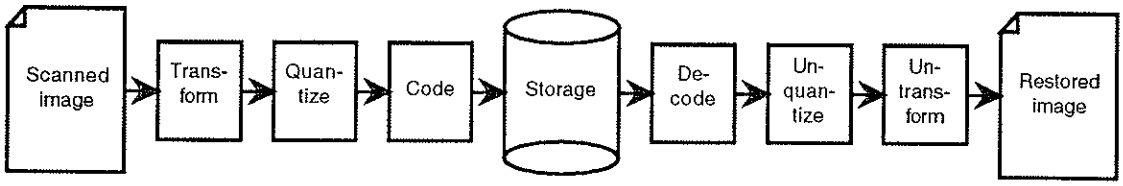

Figure 1: Generic transform coding image compression device and its inverse.

If the coding step is perfectly efficient, the compression ratio is maximized for a given distortion when the transform and quantize steps produce a sequence with minimal entropy. However, since minimal entropy is hard to characterize and harder to achieve, it is better to aim at a broader target: a sequence with almost all of the values being zero. Any such sequence will have low entropy, since its value distribution with be highly peaked at zero, and a best one can be chosen from any set by maximizing the number of zeroes. To produce these sequences, there are large families of wavelet, wavelet packet, and local trigonometric transforms, all of which have low-complexity implementations. All of them are orthogonal or nearly orthogonal, so that their condition number is close to 1. The best is the one which produces the largest fraction of negligibly small amplitudes.

3 Custom transforms

Two fast ways to implement transforms are: splitting into small blocks of pixels and then applying some fast transform to the blocks, or splitting the whole image into frequency subbands by convolving with short filters. Both methods cost $O(P \log P)$ operations for an P-pixel image. Detailed formulas and a proof of the complexity statement is omitted here; they can be found in Reference [6].

In the pixel splitting scheme, the image is cut into blocks small enough so that the intensities of all pixels contained within a block are correlated. This cutting is depicted in the left half of Figure 2. Decorrelation is performed by applying the two-dimensional discrete cosine transform (DCT) to the blocks, as in JPEG [4], or by Malvar transform (LCT) as in the Amoco seismic data compression algorithm [3]. The resulting amplitudes represent spatial frequency components in the blocks. Digitized images are limited in their spectral content, so most of the amplitudes in each block will be negligible. To maximize the proportion of negligible amplitudes, the blocks are chosen as large as possible subject to the constraints that (1) only a few spatial frequencies are present in each block, and (2) describing the block boundaries does not create too much side information.

In the subband splitting scheme, a low-pass and a high-pass filter are used along rows and columns to split the image into four subimages characterized by restricted frequency content. This process is repeated on the subimages, down to some maximum depth of decomposition, resulting in a segmentation of frequency space into subbands. The segmentation used for WSQ [2] is depicted in the right half of Figure 2. The resulting amplitudes again represent spatial frequency components. Again, for images of limited spectral content, most of these amplitudes will be negligible.

4 The joint best basis

Both splitting schemes can be organized as quadtrees to a specified depth, with the selected transform determined by the leaves of a subtree like the one depicted in Figure 3. To choose the subtree and thus the transform, each member of a representative training set of images is decomposed into the complete quadtree of amplitudes. Then the squares of these amplitudes are summed into a sum-of-squares quadtree. Using an information cost function such as "fraction of non-negligible amplitudes", the sum-of-squares quadtree is searched for its *best basis*, which is the one that minimizes this cost ([6], p. 282). Figure 4 depicts this algorithm. The best basis for the Σ quadtree is the *joint best basis* for the training set of images $1, 2, \ldots, N$. That is the transform which produces, on average, the largest number of negligible output coefficients.

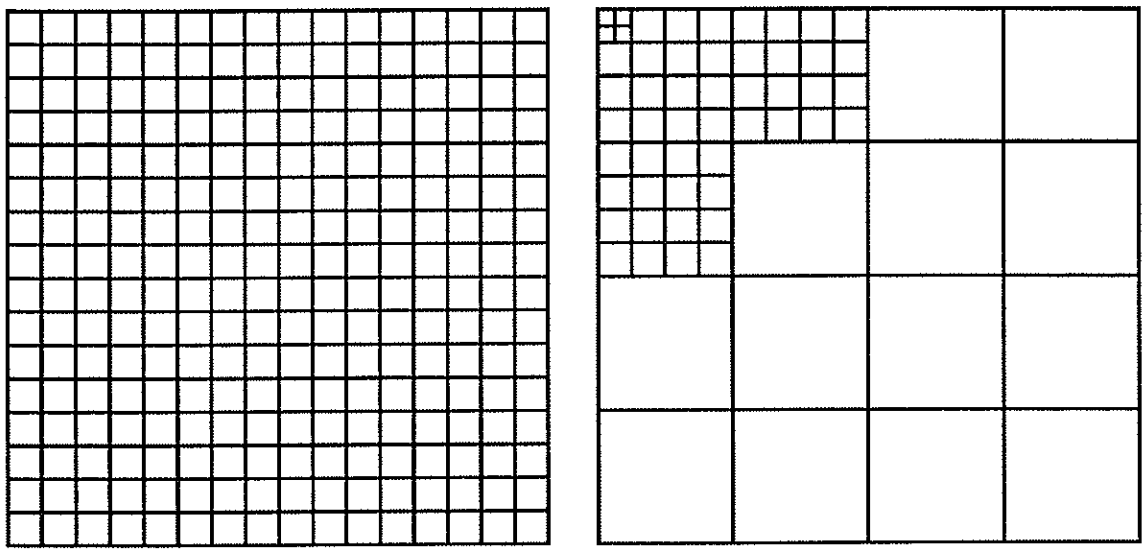

Figure 2: Left: division of a 128×128 pixel image into 8×8 blocks, as in JPEG. Right: Division of an image into orthogonal subbands, as in WSQ.

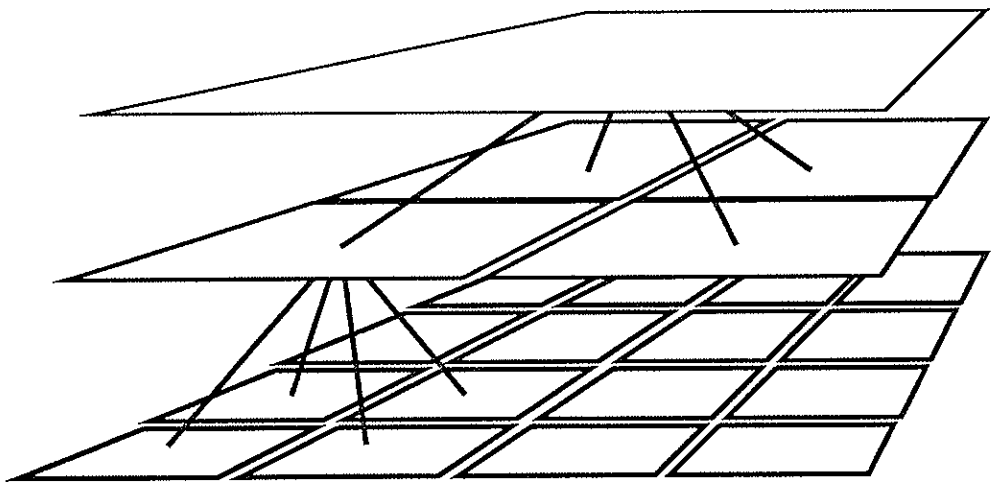

Figure 3: Splitting schemes produces quadtrees; custom bases are determined by the leaves of a subtree such as the one shown here, shaded for emphasis.

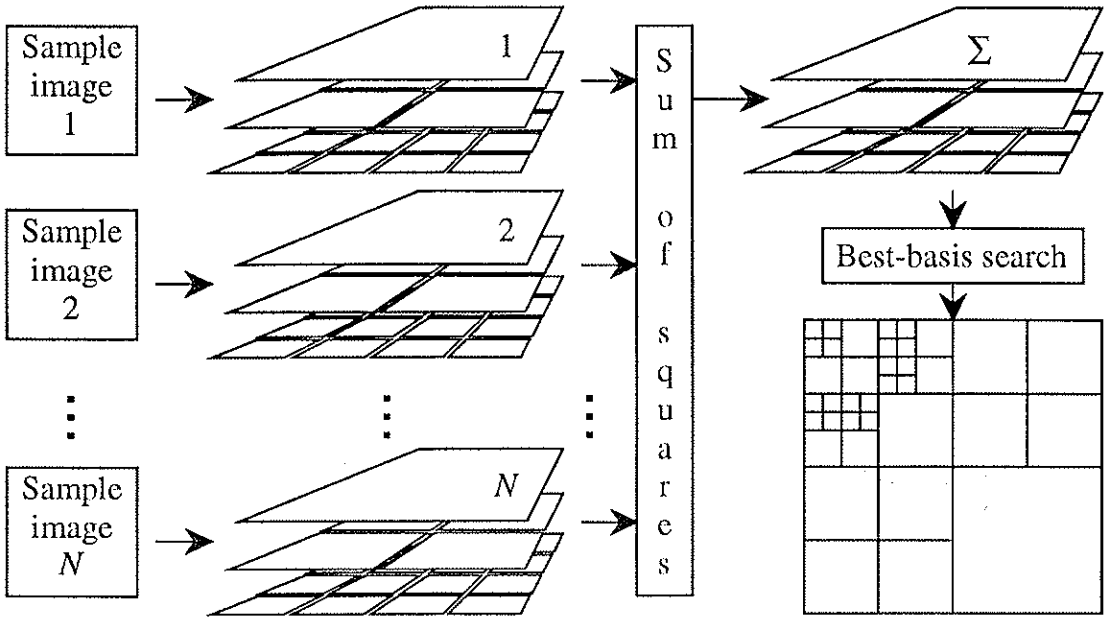

Figure 4: A joint best basis from a class of splitting algorithms is determined by a sample set of N images.

To find the best basis requires examining each coefficient in the quadtree and examining each subband or pixel block at most twice, which means that the complexity is $O(P \log P)$ for P-pixel images. To find the joint best basis requires building the sum-of-squares tree first, which dominates the total complexity with its $O(NP \log P)$ cost for a training set of N P-pixel images.

Of course, the joint best basis transform is only optimal within its own class, and the class is determined by the technical details and mathematical properties of the splitting algorithm. If these constraints were removed and the search performed over all orthonormal transforms, then the joint best basis will be the *Karhunen Loève* (KL) or *principal orthogonal* basis [5], which is known to be the minimizer of the fraction of non-negligible amplitudes. With the constraints, whose purpose is to speed things up, the chosen transform is just an approximation to KL.

5 Choosing the best transform from multiple classes

There is another meta-algorithm for relaxing the constraints a bit while preserving the speed. Namely, a custom transform can be chosen by checking many classes of splitting algorithms in order to further increase the expected number of negligible coefficients. This scheme was first proposed by Yves Meyer, and is depicted in Figure 5. At the end of each path is a cost figure, the expected fraction of non-negligible coefficients for the training set of images. The path that leads to the lowest cost determines which algorithm should be used to find the custom transform for compressing the images represented by the training set.

Examples of different classes are the different subband splitting schemes associated to different conjugate quadrature filters ([6], Chapter 5 and Appendix C), or the adapted local trigonometric bases determined by different windows ([6], Chapters 3 and 4).

6 Conclusion

Given a training set of images, a transform coding image compression algorithm may be rationally chosen from a class of fast splitting algorithms. The choice criterion is a cost function that, when low, yields high compression ratios for transform coding image compression. The method works for wavelet packet and local trigonometric transforms and thus produces well-conditioned compression and decompression methods of

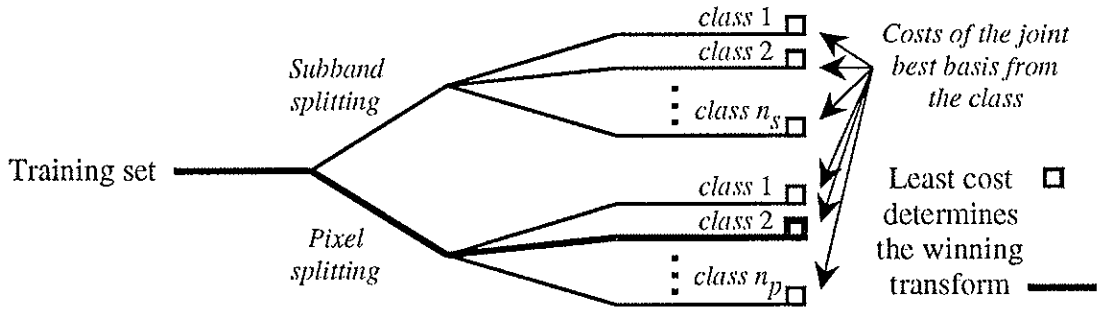

Figure 5: A meta-algorithm for deciding which splitting algorithm to use with a particular class of images.

complexity $O(P \log P)$ for P-pixel images. Searching for the best choice itself costs $O(NP \log P)$, where N is the number of training images.

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Noiselets and fast rotations

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Abstract

A general method to generate adaptively orthogonal transforms with a multiscale FFT structure will be described. In particular a nonlinear version of the Haar transform, as well as noiselet transforms will be constructed. These are connected to automatic sequences and paper folding mechanisms.

1 Introduction

As the reader undoubtedly knows, various effective algorithms exist for using wavelets and wavelet packets to process data, for example for compression, or noise removal. In these algorithms, analysis of data is achieved because one is able to find rapid decay in the distribution of values of the data, when it is transformed into wavelet or wavelet packet bases.

In practice one finds that the few large values in the transformed data describe the "interesting" part of the data, and the vast majority of values, which are small, represent a noise term. See, for example, [3].

The performance of these algorithms is impressive, and might lull one into the belief that analysis of any "interesting" structure can be carried out via wavelet packet analysis.

However, there are counter-examples to the preceding point of view. *Noiselets* (see [1]) are functions which give worst case behavior for the aforementioned type of orthogonal wavelet packet analysis. In particular, there are explicit examples of (complex-valued) noiselets for which all Haar-Walsh wavelet packet coefficients have exactly the same absolute value. So, in some sense, noiselets are "noise-like".

Although noiselets are noise like in the sense of being spread in time and frequency, there are patterns lurking in them. Certain families of noiselets arise as bases for the spaces of the Haar multi-resolution analysis. These bases are computationally good, in the same way that wavelet packets are; they come with fast algorithms for forward and inverse transforms, and there are trees of bases with the structure needed to support the best-basis algorithm. These good properties of noiselets are no coincidence. Noiselets are constructed via a multiscale iteration in exactly the same way as wavelet packets, but with a twist. So in some sense noiselets have the structure of wavelet packets.

Another source of pattern within noiselets is that one finds within their construction certain classical fractal generating mechanisms. In fact, a whole class of noiselets are nothing but the distributional derivatives of the classical paper folding curves (see [2] for an introduction to paper folding). Hence noiselets provide a counter-example to the philosophical view of analysis with which this note began. Indeed, one sees that certain interesting multiscale mechanisms can produce well organized data which are none the less invisible to our standard analysis tools.

Because of all of the above properties, rather than simply representing counter-examples, the possibility exists that noiselets will be valuable tools for certain applications. The reader should note that the authors are pursuing certain applications of noiselets and their generalizations, and some such applications may be patented or patent pending. The interested reader should contact the authors for more information.

2 Some Noiselets

2.1 An Example Of A Noiselet

Here is a recursive scheme for generating a discrete noiselet. Start with the length one sequence $s_1 = \langle\ 1\ \rangle$. Now at each stage, replace the sequence s_i with

$$s_{i+1} = s_i \wedge (i * s_i).$$

The symbol "$\wedge$" denotes sequence juxtaposition, and "$*$" point-wise multiplication. One has $s_2 = \langle\ 1\ i\ \rangle$, $s_3 = \langle\ 1\ i\ i\ -1\ \rangle$, $s_4 = \langle\ 1\ i\ i\ -1\ i\ -1\ -1\ -i\ \rangle$, $s_5 = \langle\ 1\ i\ i\ -1\ i\ -1\ -1\ -i\ i\ -1\ -1\ -i\ -1\ -i\ -i\ 1\ \rangle$, etc.

The sequence s_i has length 2^{i-1}, and it's inner product with any discrete Haar wavelet packet of length 2^{i-1} or less, has absolute value 1. Indeed, such a wavelet packet is simply a Haar-Walsh sequence: that is, a dyadicly shifted rescaled Walsh function, sampled on a dyadic grid. For example, the Haar-Walsh sequences of length 4 are: $\langle\ 1\ 0\ 0\ 0\ \rangle$, $\langle\ 0\ 0\ 1\ 0\ \rangle$, $\langle\ 0\ 0\ 1\ 0\ \rangle$, $\langle\ 0\ 0\ 0\ 1\ \rangle$, $\left\langle\ \frac{1}{\sqrt{2}}\ \frac{1}{\sqrt{2}}\ 0\ 0\ \right\rangle$, $\left\langle\ 0\ 0\ \frac{1}{\sqrt{2}}\ \frac{1}{\sqrt{2}}\ \right\rangle$, $\left\langle\ \frac{1}{\sqrt{2}}\ -\frac{1}{\sqrt{2}}\ 0\ 0\ \right\rangle$, $\left\langle\ 0\ 0\ \frac{1}{\sqrt{2}}\ -\frac{1}{\sqrt{2}}\ \right\rangle$, $\left\langle\ \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\ \right\rangle$, $\left\langle\ \frac{1}{2}\ \frac{1}{2}\ -\frac{1}{2}\ -\frac{1}{2}\ \right\rangle$, $\left\langle\ \frac{1}{2}\ -\frac{1}{2}\ \frac{1}{2}\ -\frac{1}{2}\ \right\rangle$, and $\left\langle\ \frac{1}{2}\ -\frac{1}{2}\ -\frac{1}{2}\ \frac{1}{2}\ \right\rangle$.

Now one can see that the inner product of s_k with any Haar wavelet packet of length 2^{k-1} has absolute vale 1, by induction as follows. It is clearly true for s_1. If it is true for s_{i-1}, consider the inner product of s_i with any Haar-Walsh sequence of length 2^{i-1}. Such a Haar-Walsh sequence either has no zeros, or the non-zero part is contained in its left or right half, and this half is a Haar-Walsh sequence of length 2^{i-2}. In the latter case, the induction hypothesis implies the right conclusion since both the left and right halves of s_i are unit multiples of s_{i-1}. In the former case, the Haar-Walsh sequence has the property that it's left half is either the same as, or -1 times it's right half, and each of the halves is $\frac{1}{\sqrt{2}}$ times a Haar-Walsh sequence of length 2^{i-2}. Also, the left half of s_i is s_{i-1}, and the right half of s_i is i times s_{i-1}. So in this case, the inner product is $\frac{1\pm i}{\sqrt{2}}$ times a complex number of absolute value 1, and this is a complex number of absolute value 1.

2.2 A Noiselet Basis

One can generalize the above construction, to produce a full basis of noiselets for sequences of length 2^j, for any j, by allowing i or $-i$ in the "string

composition rules". This amounts to making "wavelet packets" with the two filters: $\langle\ 1\ i\ \rangle$, and $\langle\ 1\ -i\ \rangle$ (in the same sense in which the Haar-Walsh sequences are built from the filters $\left\langle\ \frac{1}{\sqrt{2}}\ \frac{1}{\sqrt{2}}\ \right\rangle$, and $\left\langle\ \frac{1}{\sqrt{2}}\ -\frac{1}{\sqrt{2}}\ \right\rangle$).

This basis has the property that all basis functions are "totally flat" in the Haar-Walsh system, and all Haar-Walsh wavelet packets are totally flat in this basis.

2.3 Noiselet functions and distributions

In the same way that the discrete Walsh sequences correspond to the Walsh functions defined by:

$$\begin{aligned} W_0(x) &= \chi_{[0,1)}(x) \\ W_{2n}(x) &= W_n(2x) + W_n(2x-1) \\ W_{2n+1}(x) &= W_n(2x) - W_n(2x-1), \end{aligned}$$

one also has noiselet function defined by:

$$\begin{aligned} N_1(x) &= \chi_{[0,1)}(x) \\ N_{2n}(x) &= (1-i)N_n(2x) + (1+i)N_n(2x-1) \\ N_{2n+1}(x) &= (1+i)N_n(2x) + (1-i)N_n(2x-1). \end{aligned}$$

Note that we have multiplied the two filters by $1+i$ and $1-i$ respectively, since we observe that in doing so, we can view N_{2n} and N_{2n+1} as refinements of N_n. In this way, if we change notation and write $N_{\langle\epsilon_0,\ldots,\epsilon_{j-1}\rangle}$ for N_k, when $k = 2^j + \sum_{m=0}^{j-1} \epsilon_m 2^m$, then for an infinite binary sequence $\langle\epsilon\rangle$, the limit has Fourier transform:

$$\widehat{N_{\langle\epsilon\rangle}}(\xi) = \prod_{j=2}^{\infty} \left(\cos\frac{\xi}{2^j} + \epsilon_{j-2} \sin\frac{\xi}{2^j} \right),$$

and exists as a $|\xi|^{1/2}$-tempered distribution. See [1] for details.

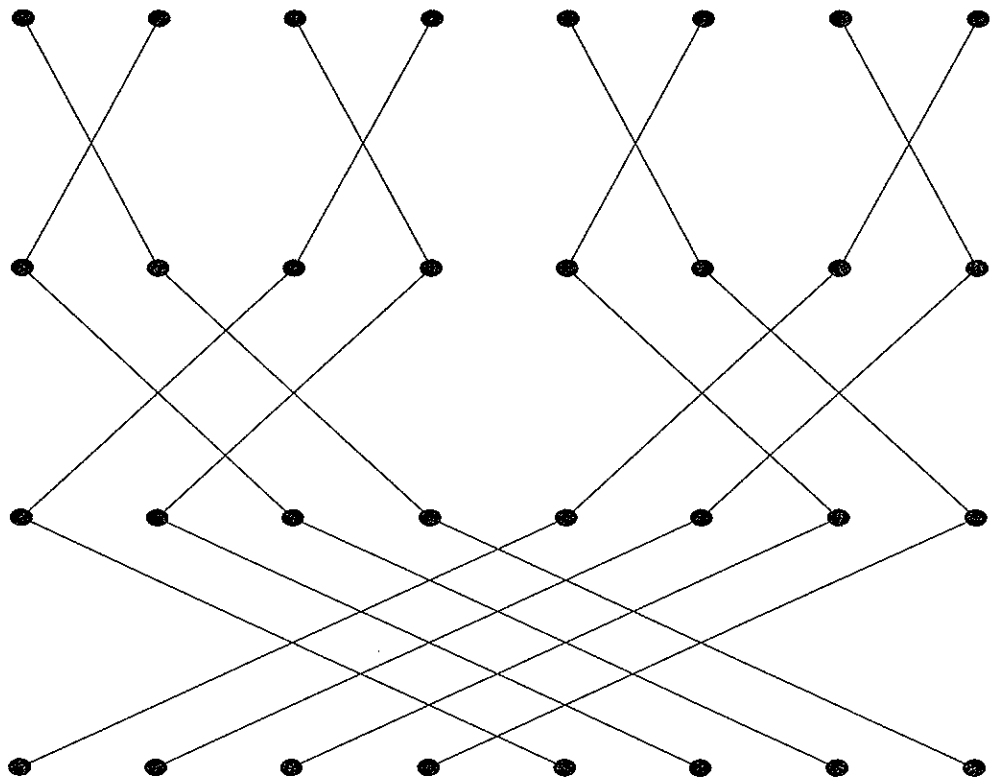
3 Looking For Structure

Notice that the discrete Haar wavelet packet transforms can all be computed as follows. Start with some discrete sequence $x_0^0, x_1^0, \ldots, x_{N-1}^0$, where $N = 2^n$. Now form a new sequence $x_0^1, \ldots, x_{2^n-1}^1$, as follows. form (x_0^1, x_1^1) by rotating

the vector (x_0^0, x_1^0) by 45 degrees. Form (x_2^1, x_3^1), from (x_2^0, x_3^0), by 45 degree rotation, and similarly for (x_4^1, x_5^1), ..., $(x_{2^n-2}^1, x_{2^n-1}^1)$. That is, we think of adjacent pairs of numbers as being coordinates of a vector in two dimensions, and we get new coordinates by 45 degree rotation.

Now repeat at the next scale, rotating (x_0^1, x_2^1) by 45 degrees to get (x_0^2, x_1^2), (x_1^1, x_3^1) by 45 degrees to get (x_2^2, x_3^2), etc. At the jth iteration, we get $(x_0^j, x_{2^j-1}^j)$ from $(x_0^{j-1}, x_{2^j-1}^{j-1})$ by 45 degree rotation, etc. When one does this $n-1$ times, one has produced all of the $N \log N$ Haar wavelet packet coefficients of the original sequence.

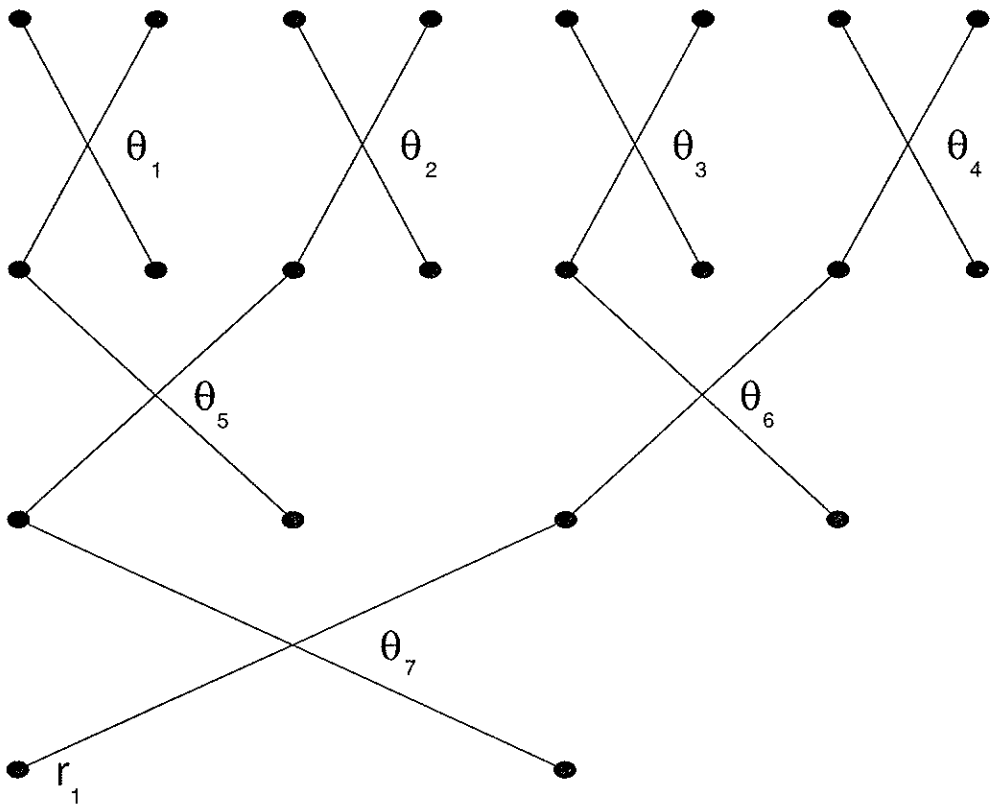
The following figure illustrates this scheme for the case $n = 3$.



When one replaces the 45 degree rotations above with any rotations, one gets what we call a *fast rotation*. It is fast because we can compute everything in $O(N \log N)$.

Given any vector of length $N = 2^n$, one can find a fast rotation which sends all of it's energy to the first coordinate. We call this rotation it's *non-*

linear wavelet transform, and it is found as follows. First note that for any two dimensional vector, (x, y), there is a unique rotation which sends it to $(\sqrt{x^2+y^2}, 0)$. Now one simply fills in the angles in the tree structure such as the one illustrated for length 8 in the following figure:



so as to always put all of the energy towards the left. We also sometimes call the data $\theta_1, \ldots, \theta_{n-1}, r$ the non-linear wavelet transform of the original data.

Note that given an orthogonal $N \times N$ matrix, one can find in this way a fast rotation with the structure of figure 1, so that the matrix is transformed into a lower triangular matrix. We can think of this as a kind of *non-linear Walsh transform* of the original matrix columns.

In particular, the non-linear wavelet transform of any Walsh function is the Haar wavelet transform, and the non-linear Walsh transform of the Walsh basis is the Walsh transform.

If one replaces the rotations by 2×2 unitary matrices with appropriate constraints, then one can discover the noiselet transform from the noiselets.

So the idea of fast rotation allows one to find a method of analysis, which generalized the traditional ones, and which "sees" the noiselets.

4 Some More Noiselets

4.1 Dragon curve noiselets

There are however other ways to create noiselets. Here is another construction of noiselets which can not be "seen" using wavelet packets, nor can they be seen via the fast rotations of the previous section.

The construction is analogous to the Rudin-Shaprio construction, but with different initial conditions. This time we start with two sequences $P_0 = \langle\ 1\ i\ \rangle$, and $P_1 = \langle\ 1\ -i\ \rangle$. We let $P_{n+1} = P_n \wedge Q_n$, and $Q_{n+1} = P_n \wedge -Q_n$.

The noiselet functions in this case are given by:

$$\begin{aligned} D_1(x) &= \chi_{[0,1)}(x) \\ D_{2n}(x) &= (1-i)D_n(2x) + (1+i)D_n(2-2x) \\ D_{2n+1}(x) &= (1+i)D_n(2x) + (1-i)D_n(2-2x), \end{aligned}$$

and the limiting objects, with Fourier transforms given by

$$\begin{pmatrix} \widehat{D_{\langle\epsilon\rangle}}(\xi) \\ -e^{-2\pi i\xi}\widehat{D_{\langle\epsilon\rangle}}(-\xi) \end{pmatrix} = \prod_{j=0}^{\infty} \begin{pmatrix} \frac{1-\epsilon_j i}{2} & \frac{1+\epsilon_j i}{2} e^{-\pi i\xi/2^j} \\ \frac{1+\epsilon_j i}{2} & \frac{1-\epsilon_j i}{2} e^{-\pi i\xi/2^j} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

exist as $|\xi|^0$-tempered distributions.

These noiselets are called *dragon noiselets*, since if one graphs the indefinite integrals of the functions D_N one gets the classical paer-folding dragon curves.

The Rudin-Shaprio argument, or an examination of the $\mathbf{L}^\infty$ bound on the above Fourier transform matrix product implied by the fact that each matrix in the product is unitary, shows that the dragon noiselets are not only flat in the Haar wavelet packet bases, but are also *semi-flat* in the trigonometric bases, and their windowed cousins.

4.2 A challenge

The dragon noiselets arise from a fast unitary map, which simply has its "X's" tied together by a scheme other than the usual wavelet packet scheme

of figure 1. Let us call such a transform a *generalized fast rotation.* Since there are many permutations, it seems that there are many ways in which one can generate a fast basis, the vectors of which will not be particularly visible to those who do not know the scheme.

As a challenge problem, the reader is asked to consider whether there might be some fast algorithm which is able to search through the possible generalized fast rotations, in order to have an adaptive analysis which sees all of these structures.

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Spatio-temporal wavelets: application to tracking of moving targets in noisy environment

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Abstract: In this work we apply the spatio-temporal continuous wavelet transform to tracking moving targets in noisy environment. We focus our attention on handling more general classes of motion, such as acceleration. To accomplish this task the spatio-temporal wavelet transform is adapted to the motion parameters on a frame-by-frame basis. Three different energy densities, associated with velocity, location and size are used to determine motion parameters. Tracking results on synthetically generated images sequences demonstrate the capabilities of the proposed methods.

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Wavelet analysis of signals with gaps

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In many domains of science, there are inevitable gaps in the time (or space) domains at which data can be recorded. When one applies spectral methods to such data, one is faced with the problem of separating the spectral properties of the signal from the spectral properties of the set of gaps.

This situation is typical, e.g. for astronomy, where the gaps are caused by seasonal windows of observation, cloudy skies, telescope maintenance etc. In order to overcome these problems, different kinds of interpolation are currently used. However, extended gaps in time series are difficult to fill by interpolation (in some cases the total size of gaps is comparable with the duration of observations) and, secondly, any interpolation also introduces additional artifacts, leading at least to smoothing of the higher frequencies of the signal. These problems appear in Fourier analysis as well as in wavelet analysis. An extension of Fourier transform to an uneven data set is known in astronomy as the Lomb-Scargle periodogram [1, 2]. The idea of this technique is to correct the basic functions $cos(\omega t)$ and $sin(\omega t)$ by a phase shift and a mean value subtraction to preserve their normalization conditions on a given set of observations.

Foster [3] recently introduced an analogous algorithm for wavelet transform on an irregular data set. He considers the Morlet wavelets and proposes to reorthogonalyze the three basic functions (the real and the imaginary parts of Morlet wavelet and a constant) by rotating the matrix of their scalar productions.

Another technique for the wavelet transform of signals with gaps has been inbtroduced in the context of astronomical problems by Frick *et al.* [4]. The main idea of this method, called the gapped wavelet transform, was to considere the wavelet transform not as a convolution of signal with gaps with a given analysing wavelet, but as the convolution of a signal and of the wavelet with gaps. Transfering the gap problem from the unknown signal function to the known wavelet function, one tries to correct the broken wavelet in a way to provide at least the admissibility condition for any dilation and translation of analyzing wavelet.

Gapped wavelets. Let $\Omega \subset \mathbb{R}^n$ be an open set, for the moment arbitrary, and let $f \in L^2(\mathbb{R}^n)$ a function defined on Ω. Assume that we know only the restriction of f to Ω, i.e. the function $f\chi_\Omega$, where χ_Ω is the characterisitic function of Ω.

One would like to develop a Littlwood-Paley like analysis of $f\chi_\Omega$, which characterizes the time-scale (or space-scale in case of many dimensions) localization of function f and not of $f\chi_\Omega$.

To do it one introduces for the given wavelet ψ a scaling (envelope) function ϕ such that $|\psi| \leq \phi$. If $a > 0$ and $b \in \mathbb{R}^n$ one defines

$$\theta_{a.b} = \psi_{a,b}G - C(a,b)\phi_{a,b}G \tag{1}$$

where for simplicity we write $G = \chi_\Omega$ and $C(a,b)$ is a parameter, which should insure that

$$\int_{\mathbb{R}^n} \theta_{a,b} = 0. \tag{2}$$

We use for wavelets the notation $\psi_{a,b} = \frac{1}{a^n}\psi(\frac{\cdot - b}{a})$ as well as for $\phi_{a,b}$.

Let us note that as $|\psi| \leq \phi$, one has

$$|C(a,b)| \leq 1 \tag{3}$$

for all a, b.

As it was mentioned above, the replacement of a given wavelet $\psi_{a,b}$ by its gapped version $\theta_{a,b}$ is motivated by the intention to reconstruct at least its admissibility condition. Theorems proved in [5] showed that no dramatical changes (which lead to the growth of the spectral energy) are expected in the spectra of the signal calculated using the functions $\theta_{a,b}$ and that the density of spectral energy which reflects the artefacts comming from the borders and gaps should be suppressed in both high and low frequencies.

The proposed technique could be generalized by requiring the functions $\theta_{a,b}$ to have vanishing values of moments of higher orders [5]. To do that one replaces the formula for $\theta_{a,b}$ by

$$\theta_{a,b} = \Big[\psi_{a,b} - \Big(C_0(a,b) + C_1(\frac{t-b}{a}) + C_2(\frac{t-b}{a})^2 + ...\Big)\phi_{a,b}\Big]G \tag{4}$$

The formulae for C_j are readily obtained from the corresponding conditions for all the moments under the consideration

$$\int_{\mathbb{R}^n} \theta_{a,b}(\frac{t-b}{a})^k = 0. \tag{5}$$

In the simplest case one asks only for zeroth and first moments and (4) reduses up to

$$\theta_{a,b} = \Big[\psi_{a,b} - \Big(C_0(a,b) + C_1(\frac{t-b}{a})\Big)\phi_{a,b}\Big]G \tag{6}$$

with explecit formula for the both papameters C_0 and C_1 :

$$C_0(a,b) = \frac{\int_\Omega \psi \int_\Omega (\frac{t-b}{a})^2\phi - \int_\Omega (\frac{t-b}{a})\psi \int_\Omega (\frac{t-b}{a})\phi}{\int_\Omega \phi \int_\Omega (\frac{t-b}{a})^2\phi - \Big(\int_\Omega (\frac{t-b}{a})\phi\Big)^2}, \tag{7}$$

$$C_1(a,b) = \frac{\int_\Omega \phi \int_\Omega (\frac{t-b}{a})\psi - \int_\Omega \psi \int_\Omega (\frac{t-b}{a})\phi}{\int_\Omega \phi \int_\Omega (\frac{t-b}{a})^2\phi - \left(\int_\Omega (\frac{t-b}{a})\phi\right)^2} \tag{8}$$

.

Let us note that C_0 and C_1 are not more inevitable less as 1. The values of these coefficients were checked in numerical calculations. They really become more as 1, but only in very large a, where the caracteristic lenght of the wavelet essentialy exceeds the whole interval of data, and anyway the results of wavelet transform become doubtfull.

No strick results are obtained concerning the wavelets (6), but in numerical simulations we used them as well as the functions (1). We call as G0 the wavelet (1), which enables only the admissibility of the wavelet (the zeroth moment vanishs), and as $G1$ the wavelet (6) which adopts to satisfy the admissibility condition for the first order moment as well as for the zeroth. We compare the both with the standart wavelet transform with Morlet wavelet (noted in the figure as M). Let us emphasize, that in the later case (taken as the reference) we also do not use any interpolation of signals in the gap, and we do not try to eliminate the mean value, the trend etc. Surely, there are a lot of possibilities to ameliorate the results of wavelet transform of signals with gaps, espessialy in a concrete situation when the expected caracter of the signal is known *a priori*. Our aim is to show that the proposed algorithm eliminates automaticaly the influence of constants and linear trends and not only in the cas when these factors are global, but also when they are local.

One example which illustrates the properties of gapped wavelets is presented in figure, where an signal with two frequencies on a linear background and its wavelet spectra are shown. In this case the standart technic produces an additional peak in intermediat scales. The gapped algorithm destroys it. One can see that the wavelets G1 provides further improvement of spectral resolution of the wavelet transform. The position of maxima in spectra given by G1 better coinside with the frequency in the signal. The large scale noise is better supressed by the wavelets G1 as by G0.

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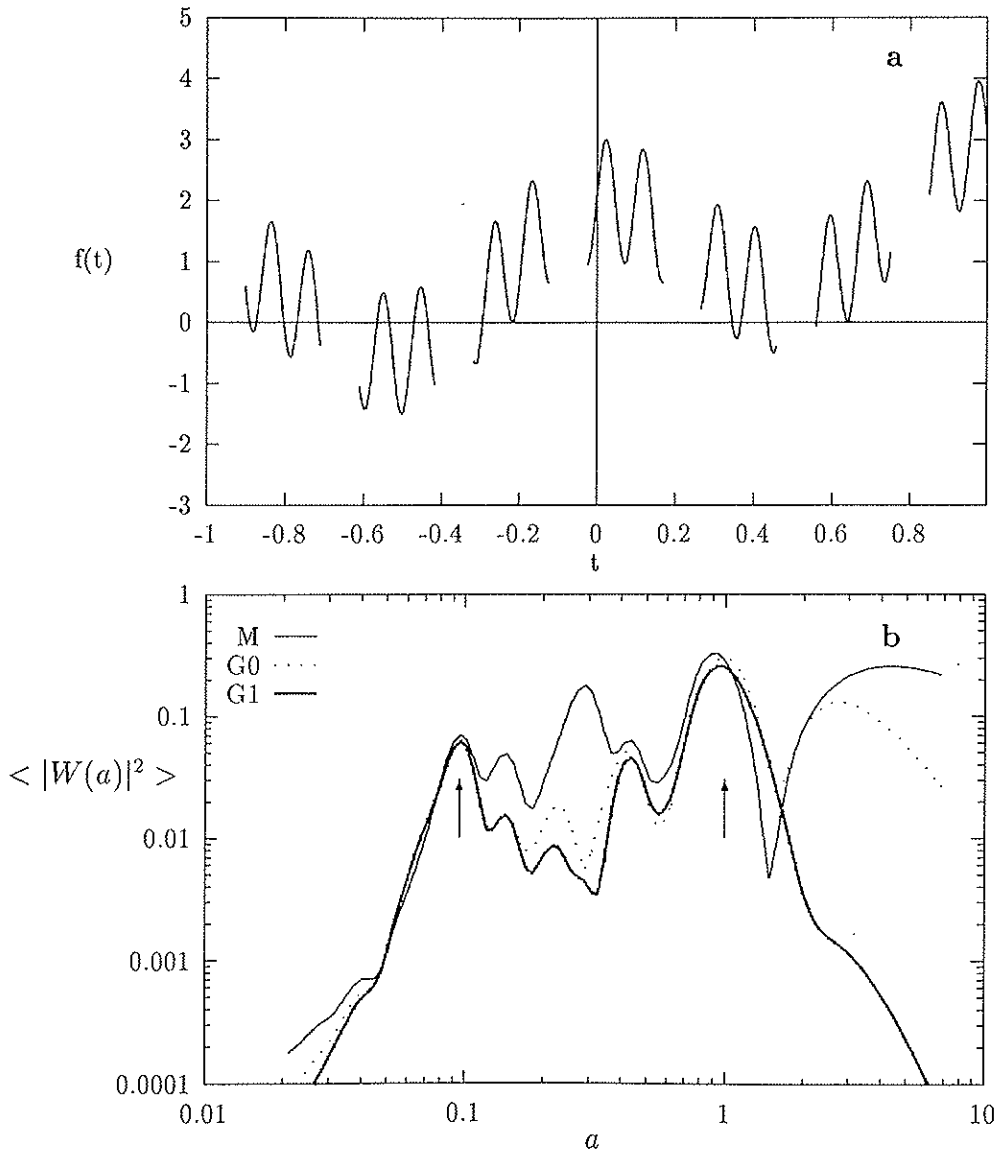

Figure 1: The analyzed signal $f(t) = t + c + cos(2\pi t) + sin(21\pi t)$ with gaps and its spectra. The both frequencies are marked in the scales by arrows.

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L'analyse multi-échelle

"C'est de loin le plus proche du sujet.

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Alexander Grossmann, 1985.