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Maximal solutions of nonlinear parabolic equations with absorption

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Abstract We study the existence and the uniqueness of the solution of the problem (P): $\partial_t u - \Delta u + f(u) = 0$ in $Q := \Omega \times (0, \infty)$, $u = \infty$ on the parabolic boundary $\partial_p Q$ when Ω is a domain in \mathbb{R}^N with a compact boundary and f a continuous increasing function satisfying super linear growth condition. We prove that in most cases, the existence and uniqueness is reduced to the same property for the associated stationary equation in Ω .

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Key words. Parabolic equations, singular solutions, self-similarity, removable singularities

1 Introduction

Let Ω be a bounded domain in \mathbb{R}^N with boundary $\partial\Omega := \Gamma$, $Q_T^\Omega := \Omega \times (0, T)$ ($0 < T \leq \infty$) and $\partial_p Q = \bar{\Omega} \times 0 \cup \partial\Omega \times (0, T]$. We denote by $\rho_{\partial\Omega}(x)$ the distance from x to $\partial\Omega$ and by $d_P(x, t) = \min\{\rho_{\partial\Omega}(x), t\}$ the product distance from $(x, t) \in Q_\infty^\Omega$ to $\partial_p Q_\infty^\Omega$. If $f \in C(\mathbb{R})$, we say that a function $u \in C^{2,1}(Q_\infty^\Omega)$ solution of

$$u_t - \Delta u + f(u) = 0, \quad (1.1)$$

in Q_∞^Ω is a large solution of (1.1) in Q_∞^Ω if it satisfies

$$\lim_{d_P(x,t) \rightarrow 0} u(x, t) = \infty. \quad (1.2)$$

The existence of such a u is associated to the existence of large solutions to the stationary equation

$$-\Delta w + f(w) = 0, \quad (1.3)$$

in Ω , i.e. solutions which satisfy

$$\lim_{\rho_{\partial\Omega}(x) \rightarrow 0} w(x) = \infty, \quad (1.4)$$

and solutions of the ODE

$$\phi' + f(\phi) = 0 \quad \text{in } (0, \infty). \quad (1.5)$$

subject to the initial blow-up condition

$$\lim_{t \rightarrow 0} \phi(t) = \infty. \quad (1.6)$$

A natural assumption on f is to assume that it is nondecreasing with $f(0) \geq 0$. If $f(a) > 0$, a necessary and sufficient condition for the existence of a maximal solution \bar{w}_Ω to (1.3) is the Keller-Osserman condition,

$$\int_a^\infty \frac{ds}{\sqrt{F(s)}} < \infty, \quad (1.7)$$

where $F(s) = \int_0^s f(\tau) d\tau$. A necessary and sufficient condition for the existence of a solution ϕ of (1.6) with initial blow-up is

$$\int_a^\infty \frac{ds}{f(s)} < \infty. \quad (1.8)$$

Furthermore the unique maximal solution $\bar{\phi}$ is obtained by inversion from the formula

$$\int_{\bar{\phi}(t)}^\infty \frac{ds}{f(s)} = t \quad \forall t > 0. \quad (1.9)$$

It is known that, if f is convex, (1.7) implies (1.8). If (1.7) holds and there exists a maximal solution to (1.3), it is not always true that this maximal solution is a large solution. In the case of a general nonlinearity, only sufficient conditions are known, independent of the regularity of $\partial\Omega$. We recall some of them.

If $N \geq 3$ and f satisfies the weak singularity assumption

$$\int_a^\infty s^{-2(N-1)/(N-2)} f(s) ds < \infty \quad \forall a > 0. \quad (1.10)$$

If $N = 2$ and the exponential order of growth of f defined by

$$a_f^+ = \inf \left\{ a \geq 0 : \int_0^\infty f(s) e^{-as} ds < \infty \right\} \quad (1.11)$$

is finite.

When $f(u) = u^q$ with $q > 1$, (1.10) means that $q < N/(N-2)$. When $q \geq N/(N-2)$ the regularity of $\partial\Omega$ plays a crucial role in the existence of large solutions. A necessary and sufficient condition involving a Wiener type test which uses the $C_{2,q}^{\mathbb{R}^N}$ -Bessel capacity has been obtained by probabilistic methods by Dhersin and Le Gall [4] in the case $q = 2$ and extended to the general case by Labutin [6].

Uniqueness of the large solution of (1.3) has been obtained under three types of assumptions (see [7], [10] and [11]):

If $\partial\Omega = \partial\bar{\Omega}^c$ and $f(u) = u^q$ with $1 < q < N/(N-2)$ or if $N = 2$ and $f(u) = e^{au}$.

If $\partial\Omega$ is locally a continuous graph and $f(u) = u^q$ with $q > 1$ or $f(u) = e^{au}$.

If $f(u) = u^q$ with $q \geq N/(N-2)$ and $C_{2,q}^{\mathbb{R}^N}(\partial\Omega \setminus \tilde{\Omega}^c) = 0$, where \tilde{E} denotes the closure of a set in the fine topology associated to the Bessel capacity $C_{2,q}^{\mathbb{R}^N}$.

In this article we extend most of the above mentioned results to the parabolic equation (1.1). We first prove that, if f is super-additive, i. e.

$$f(x+y) \geq f(x) + f(y) \quad \forall (x,y) \in \mathbb{R} \times \mathbb{R}, \quad (1.12)$$

and satisfies (1.7) and (1.8), there exists a maximal solution \bar{u}_{Q^Ω} to (1.1) in Q^Ω , and it satisfies

$$\bar{u}_{Q^\Omega}(x,t) \leq \bar{w}_\Omega(x) + \bar{\phi}(t) \quad \forall (x,t) \in Q^\Omega. \quad (1.13)$$

If we assume also that $\partial\Omega = \partial\bar{\Omega}^c$, there holds

$$\max\{\bar{w}_\Omega(x), \bar{\phi}(t)\} \leq \bar{u}_{Q^\Omega}(x,t) \quad \forall (x,t) \in Q^\Omega. \quad (1.14)$$

Under the assumption $\partial\Omega = \partial\bar{\Omega}^c$, it is possible to consider a decreasing sequence of smooth bounded domains Ω^n such that $\bar{\Omega}^n \subset \Omega^{n-1}$, $\bar{\Omega} = \bigcap \Omega_n$, and prove that the increasing sequence of large solutions $\bar{u}_{Q^{\Omega^n}}$ of (1.1) in $Q^{\Omega^n} := \Omega^n \times (0, \infty)$, converges to the exterior maximal solution \underline{u}_{Q^Ω} of (1.1) in Q^Ω . If we proceed similarly with the large solutions \bar{w}_{Ω^n} of (1.3) in Ω^n and denote by \underline{w}_Ω their limit, then we prove that

$$\max\{\underline{w}_\Omega(x), \bar{\phi}(t)\} \leq \underline{u}_{Q^\Omega}(x,t) \quad \forall (x,t) \in Q^\Omega. \quad (1.15)$$

The main result of this article is the following

Theorem 1. *Assume Ω is a bounded domain such that $\partial\Omega = \partial\bar{\Omega}^c$, $f \in C(\mathbb{R})$ is nondecreasing and satisfies (1.7), (1.8) and (1.12). Then, if $\underline{w}_\Omega = \bar{w}_\Omega$, there holds $\underline{u}_{Q^\Omega} = \bar{u}_{Q^\Omega}$.*

Consequently, if (1.3) admits a unique large solution in Ω , the same holds for (1.1) in Q_∞^Ω .

2 The maximal solution

In this section Ω is a bounded domain in \mathbb{R}^N and $f \in C(\mathbb{R})$ is nondecreasing and satisfies (1.7) and (1.8). We set $k_0 = \inf\{\ell \geq 0 : f(\ell) > 0\}$ and assume also that, for any $m \in \mathbb{R}$ there exists $L = L(m) \in \mathbb{R}_+$ such that

$$\forall (x,y) \in \mathbb{R}^2, x \geq m, y \geq m \implies f(x+y) \geq f(x) + f(y) - L. \quad (2.1)$$

Theorem 2.1 *Under the previous assumptions there exists a maximal solution \bar{u}_{Q^Ω} in Q_∞^Ω .*

Proof. Step 1- Approximation and estimates. Let Ω_n be an increasing sequence of smooth domains such that $\bar{\Omega}_n \subset \Omega_{n+1}$ and $\bigcup \Omega_n = \Omega$. For each of these domains and $(n,k) \in \mathbb{N}_*^2$ we denote by $w = w_{n,k}$ the solutions of

$$\begin{cases} -\Delta w + f(w) = 0 & \text{in } \Omega_n \\ w = k & \text{in } \partial\Omega_n. \end{cases} \quad (2.2)$$

where $\partial_p Q_\infty^{\Omega_n} := \partial\Omega_n \times (0, \infty) \cup \overline{\Omega}_n \times \{0\}$. By [5] there exists a decreasing function g from \mathbb{R}_+ to \mathbb{R} , with limit ∞ at zero, such that

$$w_{n,k}(x) \leq g(\rho_{\partial\Omega_n}(x)) \quad \forall x \in \Omega_n. \quad (2.3)$$

The mapping $k \rightarrow w_{n,k}$ is increasing, while $n \rightarrow w_{n,k}$ is decreasing. If we set

$$\overline{w}_\Omega = \lim_{n \rightarrow \infty} \lim_{k \rightarrow \infty} w_{n,k}, \quad (2.4)$$

it is classical that \overline{w}_Ω is the maximal solution of (1.3) in Ω , and it satisfies

$$w(x) \leq g(\rho_{\partial\Omega}(x)) \quad \forall x \in \Omega. \quad (2.5)$$

We denote also by $u = u_{n,k}$ the solution of

$$\begin{cases} u_t - \Delta u + f(u) = 0 & \text{in } Q_\infty^{\Omega_n} \\ u = k & \text{in } \partial_p Q_\infty^{\Omega_n}. \end{cases} \quad (2.6)$$

By the maximum principle $k \rightarrow u_{n,k}$ is increasing and $n \rightarrow u_{n,k}$ decreasing. If we denote by $\bar{\phi}$ the maximal solution of the ODE (1.5), then $\bar{\phi}(t)$ is expressed by inversion by (1.9). If $t_k = \bar{\phi}^{-1}(k)$, there holds, since $\bar{\phi}$ is decreasing,

$$\bar{\phi}(t + t_k) \leq u_{n,k}(x, t) \quad \text{in } Q_\infty^{\Omega_n}. \quad (2.7)$$

Furthermore, if $f(k) \geq 0$ (which holds if $k \geq k_0$), $w_{n,k} \leq k$. Therefore

$$w_{n,k}(x) \leq u_{n,k}(x, t) \quad \text{in } Q_\infty^{\Omega_n}. \quad (2.8)$$

Combining (2.7) and (2.8), we derive

$$\max\{w_{n,k}(x), \bar{\phi}(t + t_k)\} \leq u_{n,k}(x, t) \quad \forall (x, t) \in Q_\infty^{\Omega_n}. \quad (2.9)$$

Next we obtain an upper estimate. Let $T > 0$ and $m \in \mathbb{R}$ such that

$$\min\{\overline{w}_\Omega(x) : x \in \Omega\} > m \geq \bar{\phi}(T).$$

For $n \geq n_1$ and $k \geq k_1$ there holds $\min\{w_{n,k}(x) : x \in \Omega\} \geq m$. Let $L = L(m) \geq 0$ be the corresponding damping term from (2.1). If $v_{n,k} = w_{n,k}(x) + \bar{\phi}(t + t_k)$, then it satisfies

$$v_t - \Delta v + f(v) = f(v) - f(\bar{\phi}(\cdot + t_k)) - f(w_{n,k}) \geq -L \quad \text{if } (x, t) \in \Omega_n \times [0, T - t_k]. \quad (2.10)$$

Since $L \geq 0$, the function $\tilde{v}_{n,k} := v_{n,k} + Lt$ is a supersolution for (1.1) in $Q_{T-t_k}^{\Omega_n} := \Omega_n \times (0, T - t_k)$ which dominates $u_{n,k}$ on $\partial_p Q_{T-t_k}^{\Omega_n}$, thus in $Q_{T-t_k}^{\Omega_n}$ by the maximum principle. Therefore

$$u_{n,k}(x, t) \leq w_{n,k}(x) + \bar{\phi}(t + t_k) + Lt \quad \forall (x, t) \in Q_{T-t_k}^{\Omega_n}. \quad (2.11)$$

Step 2- Final estimates and maximality. Using the different monotonicity properties of the mapping $(k, n) \mapsto w_{n,k}$ and the estimates (2.9) and (2.11), it follows that the function defined by

$$\overline{u}_{Q^\Omega} := \lim_{n \rightarrow \infty} \lim_{k \rightarrow \infty} u_{n,k} \quad (2.12)$$

is a solution of (1.1) in Q_∞^Ω . Furthermore

$$\max\{\bar{w}_\Omega(x), \bar{\phi}(t)\} \leq \bar{u}_{Q^\Omega}(x, t) \quad \forall (x, t) \in Q_\infty^\Omega, \quad (2.13)$$

and

$$\bar{u}_{Q^\Omega}(x, t) \leq \bar{w}_\Omega(x) + \bar{\phi}(t) + tL(\phi(T)) \quad \forall (x, t) \in Q_T^\Omega. \quad (2.14)$$

since $\phi(T) \leq \min\{\bar{w}_\Omega(x) : x \in \Omega\}$. Next, we consider $u \in C^{2,1}(Q_\infty^\Omega)$, solution of (1.1) in Q_∞^Ω . Then, for $\epsilon > 0$ and $n \in \mathbb{N}$, there exists $k^* > 0$ such that for $k \geq k^*$,

$$u_{n,k}(x, t - \epsilon) \geq u(x, t) \quad \forall (x, t) \in \Omega_n \times (\epsilon, \infty).$$

Letting successively $k \rightarrow \infty$, $n \rightarrow \infty$ and $\epsilon \rightarrow 0$, yields to $\bar{u}_{Q^\Omega} \geq u$ in Q_∞^Ω . \square

Since \bar{w}_Ω be a large solution in Ω implies the same boundary blow-up for \bar{u}_{Q^Ω} on $\partial\Omega \times (0, \infty)$, we give below some conditions which implies that \bar{u}_{Q^Ω} is a large solution.

Corollary 2.2 *Assume the assumptions of Theorem 2.1 are fulfilled. Then \bar{u}_{Q^Ω} is a large solution if one of the following additional conditions is satisfied:*

- (i) $N \geq 3$ and f satisfies the weak singularity condition (1.10).
- (ii) $N = 2$ and the exponential order of growth of f defined by (1.11) is positive.
- (iii) $N \geq 3$ and $\partial\Omega$ satisfies the Wiener regularity criterion.

Proof. Under condition (i) or (ii), for any $x_0 \in \partial\Omega$, there exists a solution w_{c,x_0} of

$$\begin{cases} -\Delta w + f(w) = c\delta_{x_0} & \text{in } B_R(x_0) \\ w = 0 & \text{in } \partial B_R(x_0), \end{cases} \quad (2.15)$$

where $R > 0$ is chosen such that $\bar{\Omega} \subset B_R(x_0)$ and $c > 0$ is arbitrary under condition (i) and smaller than $2/a_f^+$ in case (ii). The function w_{c,x_0} is radial with respect to x_0 and

$$\lim_{x \rightarrow x_0} w_{c,x_0}(x) = \infty.$$

If $x \in \Omega$, we denote by x_0 a projection of x on $\partial\Omega$. Since

$$w_n(x) \geq w_{c,x_0}(x) \implies \bar{w}_\Omega(x) \geq w_{c,x_0}(x),$$

we derive from (2.13),

$$\lim_{\rho_{\partial\Omega}(x) \rightarrow 0} \bar{u}_{Q^\Omega}(x, t) = \infty,$$

uniformly with respect to $t > 0$. In case (iii) we see that, for any $k > 0$

$$\bar{w}_\Omega(x) \geq w_{k,\infty}(x) \quad \forall x \in \Omega, \quad (2.16)$$

where $w_{k,\infty}$ is the solution of (2.2), with Ω_n replaced by Ω . This again implies (2.13). \square

Using estimate (2.13) leads to the asymptotic behavior of $\bar{u}_{Q^\Omega}(x, t)$ when $t \rightarrow \infty$.

Corollary 2.3 *Assume the assumptions of Theorem 2.1 are fulfilled. Then $\bar{u}_{Q^\Omega}(x, t) \rightarrow \bar{w}_\Omega(x)$ locally uniformly on Ω when $t \rightarrow \infty$.*

Proof. For any $k > k_0$ and $n \in \mathbb{N}_*$ and any $s > 0$, there holds by the maximum principle,

$$u_{n,k}(x, s) \leq k = u_{n,k}(x, 0) \quad \forall x \in \Omega_n.$$

Using the monotonicity of f , we derive $u_{n,k}(x, t+s) \leq u_{n,k}(x, t)$ for any $(x, t) \in Q_\infty^\Omega$. Letting $k \rightarrow \infty$ and then $n \rightarrow \infty$ yields to

$$\bar{u}_{Q^\Omega}(x, t+s) \leq \bar{u}_{Q^\Omega}(x, t) \quad \forall (x, t) \in Q_\infty^\Omega. \quad (2.17)$$

It follows that $\bar{u}_{Q^\Omega}(x, t)$ converges to some $W(x)$ as $t \rightarrow \infty$ and $\bar{w}_\Omega \leq W$ from (2.13). Using the parabolic equation regularity theory, we derive that the trajectory $\mathcal{T} := \bigcup_{t \geq 0} \{\bar{u}_{Q^\Omega}(\cdot, t)\}$ is compact in the $C_{loc}^1(\Omega)$ -topology. Therefore W is a solution of (1.3) in Ω . It coincides with \bar{w}_Ω because of the maximality. \square

3 Large solutions

In this section we construct a minimal-maximal solution of (1.1) which is the minimal large solution whenever it exists. If $\partial\Omega$ is regular enough, the construction of the minimal large solution is easy.

Theorem 3.1 *Let Ω be a bounded domain in \mathbb{R}^N the boundary of which satisfies the Wiener regularity condition. If $f \in C(\mathbb{R})$ is nondecreasing and satisfies (1.7), (1.8) and (2.1), then there exists a minimal large solution \underline{u}_{Q^Ω} to (1.1) in Q_∞^Ω . Furthermore*

$$\max\{\underline{u}_\Omega(x), \bar{\phi}(t)\} \leq \underline{u}_{Q^\Omega}(x, t) \quad \forall (x, t) \in Q_\infty^\Omega, \quad (3.1)$$

and, for any $T > 0$,

$$\underline{u}_{Q^\Omega}(x, t) \leq \underline{u}_\Omega(x) + \bar{\phi}(t) + tL(\bar{\phi}(T)) \quad \forall (x, t) \in Q_T^\Omega, \quad (3.2)$$

where $L(\bar{\phi}(T))$ is as in (2.16), and \underline{u}_Ω denotes the minimal large solution of (1.3) in Ω .

Proof. For $k \geq k_0$ (see Section 2), we denote by u_k the solution of

$$\begin{cases} u_t - \Delta u + f(u) = 0 & \text{in } Q_\infty^\Omega \\ u = k & \text{in } \partial_p Q_\infty^\Omega. \end{cases} \quad (3.3)$$

When k increases, u_k increases and converges to some large solution \underline{u}_{Q^Ω} of (1.1) in Q_∞^Ω . If u is any large solution of (1.1) in Q_∞^Ω , then the maximum principle and (1.2) implies $u \geq u_k$. Therefore $u \geq \underline{u}_{Q^\Omega}$. The same assumption allows to construct the solution w_k of

$$\begin{cases} -\Delta w + f(w) = 0 & \text{in } \Omega \\ w = k & \text{in } \partial\Omega, \end{cases} \quad (3.4)$$

and, by letting $k \rightarrow \infty$, to obtain the minimal large solution \underline{w}_Ω of (1.3) in Ω . Next we first observe, that, as in the proof of Theorem 2.1, (2.10) applies under the form

$$\bar{\phi}(t + t_k) \leq u_k(x, t) \quad \text{in } Q_\infty^\Omega, \quad (3.5)$$

where, we recall it, $t_k = \bar{\phi}^{-1}(k)$. In the same way, for $k \geq k_0$ (with $f(k) \geq 0$), (2.11) holds under the form

$$w_k(x) \leq u_k(x, t) \quad \text{in } Q_\infty^\Omega. \quad (3.6)$$

Letting $k \rightarrow \infty$ yields to

$$\max\{\underline{w}_\Omega(x), \bar{\phi}(t)\} \leq \underline{u}_{Q^\Omega}(x, t) \quad \forall (x, t) \in Q_\infty^\Omega. \quad (3.7)$$

In order to prove the upper estimate we consider the same m as in the proof of Theorem 2.1 such that $\min\{\min\{w_k(x) : x \in \Omega\}, \bar{\phi}(t)\} \geq m$, and for $k' > k$, there holds

$$w_{k'} + \bar{\phi} \geq k = w_k|_{\partial_p Q_T^\Omega}.$$

Since $w_{k'}(x) + \bar{\phi}(t) + tL$ is a supersolution for (1.1) in Q_T^Ω it follows $w_{k'} + \bar{\phi} + tL \geq w_k$ in Q_T^Ω . Letting successively $k' \rightarrow \infty$ and $k' \rightarrow \infty$, we derive (3.2). \square

From this result we can deduce uniqueness results for solution of

Corollary 3.2 *Under the assumptions of Theorem 3.1, if we assume moreover that f is convex and, for any $\theta \in (0, 1)$, there exists r_θ such that*

$$r \geq r_\theta \implies f(\theta r) \leq \theta f(r). \quad (3.8)$$

Then

$$\underline{w}_\Omega = \bar{w}_\Omega \implies \underline{u}_{Q^\Omega} = \bar{u}_{Q^\Omega}. \quad (3.9)$$

Proof. We fix $T \in (0, 1]$ such that

$$tL(\bar{\phi}(1)) \leq \bar{\phi}(t) \quad \forall t \in (0, T],$$

(remember that L is always positive) and

$$2\underline{w}_\Omega(x) + \bar{\phi}(t) \geq 0 \quad \forall (x, t) \in Q_T^\Omega.$$

Then $\underline{w}_\Omega(x) + \bar{\phi}(t) \geq 0$ and

$$\underline{w}_\Omega(x) + \bar{\phi}(t) + tL(\bar{\phi}(1)) \leq \underline{w}_\Omega(x) + 2\bar{\phi}(t) \leq \underline{w}_\Omega(x) + 2\bar{\phi}(t) \leq 3(\underline{w}_\Omega(x) + \bar{\phi}(t)),$$

from which inequality follows

$$2^{-1}(\underline{w}_\Omega(x) + \bar{\phi}(t)) \leq \underline{u}_{Q^\Omega}(x, t) \leq 3(\underline{w}_\Omega(x) + \bar{\phi}(t)) \quad \forall (x, t) \in Q_T^\Omega.$$

Therefore, if $\underline{w}_\Omega = \bar{w}_\Omega$, it follows

$$\underline{u}_{Q^\Omega} \leq \bar{u}_{Q^\Omega} \leq 6\underline{u}_{Q^\Omega} \quad \text{in } Q_T^\Omega. \quad (3.10)$$

Next we assume $\underline{u}_{Q^\Omega} < \overline{u}_{Q^\Omega}$ and set

$$u^* = \underline{u}_{Q^\Omega} - \frac{1}{6} (\overline{u}_{Q^\Omega} - \underline{u}_{Q^\Omega}).$$

Since f is convex, u^* is a supersolution of (1.1) in Q_T^Ω (see [8], [10]) and $u^* < \underline{u}_{Q^\Omega}$. Up to take a smaller T , we can also assume from (3.8) that $\min\{\underline{u}_{Q^\Omega}(x, t) : (x, t) \in Q_T^\Omega\} \geq r_{1/12}$, thus

$$f(\underline{u}_{Q^\Omega}/12) \leq \frac{1}{12} f(\underline{u}_{Q^\Omega}) \quad \text{in } Q_T^\Omega.$$

Therefore $\underline{u}_{Q^\Omega}/12$ is a subsolution for (1.1) in Q_T^Ω and $12^{-1}\underline{u}_{Q^\Omega} < u^*$. Using a standard result of sub and super solutions and the fact that f is locally Lipschitz continuous, we see that there exists some $u^\#$ solution of (1.1) in Q_T^Ω such that

$$\frac{1}{12}\underline{u}_{Q^\Omega} \leq u^\# \leq u^* < \underline{u}_{Q^\Omega} \quad \text{in } Q_T^\Omega. \quad (3.11)$$

Then $u^\#$ is a large solution, which contradicts the minimality of \underline{u}_{Q^Ω} on Q_T^Ω . Finally $\underline{u}_{Q^\Omega} = \overline{u}_{Q^\Omega}$ in Q_∞^Ω . \square

Lemma 3.3 *Let Ω be a bounded domain in \mathbb{R}^N and, for $\epsilon > 0$, $\Omega_\epsilon := \{x \in \mathbb{R}^N : \text{dist}(x, \overline{\Omega}) < \epsilon\}$. The four following assertions are equivalent:*

- (i) $\partial\Omega = \partial\overline{\Omega}^c$.
- (ii) For any $x \in \partial\Omega$, there exists a sequence $\{x_n\} \subset \overline{\Omega}^c$ such that $x_n \rightarrow x$.
- (iii) For any $x \in \partial\Omega$ and any $\epsilon > 0$, $B_\epsilon(x) \cap \overline{\Omega}^c \neq \emptyset$.
- (iv) For any $x \in \partial\Omega$, $\lim_{\epsilon \rightarrow 0} \text{dist}(x, \Omega_\epsilon^c) = 0$.
- (v) $\Omega = \overline{\overline{\Omega}}^c$.

Proof. There always holds $\partial\overline{\Omega}^c = \overline{\overline{\Omega}^c} \cap \overline{\Omega} \subset \Omega^c \cap \overline{\Omega} = \partial\Omega$.

(i) \implies (iii). Assume (iii) does not hold, there exist $x_0 \in \partial\Omega$ and $\epsilon_0 > 0$ such that $B_{\epsilon_0}(x_0) \cap \overline{\Omega}^c = \emptyset$. Thus $x_0 \notin \overline{\overline{\Omega}^c}$, and $x_0 \notin \partial\overline{\Omega}^c$. Therefore (i) does not hold.

(iii) \implies (i). Let $x_0 \in \partial\Omega$. If, for any $\epsilon > 0$, $B_\epsilon(x_0) \cap \overline{\Omega}^c \neq \emptyset$, then $x_0 \in \overline{\overline{\Omega}^c}$. Because $x_0 \in \Omega^c \cap \overline{\Omega}$, it implies that $x_0 \in \overline{\Omega} \cap \overline{\overline{\Omega}^c} = \partial\overline{\Omega}^c$.

The equivalence between (iii) and (ii) is obvious.

(ii) \implies (iv). We assume (iv) does not hold. There exist $x_0 \in \partial\Omega$, $\alpha > 0$ and a sequence of positive real numbers $\{\epsilon_n\}$ converging to 0 such that $\text{dist}(x_0, \Omega_{\epsilon_n}^c) \geq \alpha$. Since for $\epsilon \geq \epsilon_n$, $\Omega_\epsilon^c \subset \Omega_{\epsilon_n}^c$, there holds $\text{dist}(x_0, \Omega_\epsilon^c) \geq \alpha$. Furthermore, this inequality holds for any $\epsilon > 0$. If there exist a sequence $\{x_n\} \subset \overline{\overline{\Omega}^c}$ such that $x_n \rightarrow x_0$, then $\text{dist}(x_n, \overline{\Omega}) = \delta_n > 0$, thus $x_n \in \Omega_{\delta_n}^c$. Consequently $|x_n - x_0| \geq \alpha$, which is impossible. Therefore (ii) does not hold.

(iv) \implies (iii). Let $x \in \partial\Omega$ and $x_n \in \Omega_{1/n}^c$ such that $|x - x_n| = \text{dist}(x, \Omega_{1/n}^c) \rightarrow 0$. Since $\Omega_{1/n}^c \subset \overline{\Omega}$, $x_n \in \overline{\Omega}^c$ and $x_n \rightarrow x$.

(iii) \implies (v). We first notice that $\overline{\Omega} = \bigcap_{\epsilon > 0} \Omega_\epsilon = \bigcap_{\epsilon > 0} \overline{\Omega}_\epsilon$ and $\Omega \subset \overset{\circ}{\overline{\Omega}}$. If there exists some $x \in \overset{\circ}{\overline{\Omega}} \setminus \Omega$, then for some $\epsilon > 0$, $B_\epsilon(x) \subset \overline{\Omega}$ which implies $B_\epsilon(x) \cap \overline{\Omega}^c = \emptyset$. But $x \notin \Omega$ implies $x \in \partial\Omega$. Thus (iii) does not hold.

(v) \implies (iii). If (iii) does not hold, there exists $x \in \partial\Omega$ and $\epsilon > 0$ such that $B_\epsilon(x) \cap \overline{\Omega}^c = \emptyset \iff B_\epsilon(x) \subset \overline{\Omega}$. Therefore $x \in \overset{\circ}{\overline{\Omega}} \setminus \Omega$. \square

Definition 3.4 A solution U (resp. W to problem (1.1) in Q_∞^Ω (resp. (1.3) in Ω) is called an exterior maximal solution if it is larger than the restriction to Q_∞^Ω (resp. Ω) of any solution of (1.1) (resp. (1.3)) defined in an open neighborhood of Q_∞^Ω (resp. Ω).

Proposition 3.5 Assume Ω is a bounded domain in \mathbb{R}^N such that $\partial\Omega = \partial\overline{\Omega}^c$ and $f \in C(\mathbb{R})$ is nondecreasing and satisfies (1.7). Then there exists an exterior maximal solution \underline{w}_Ω^* to problem (1.3) in Ω .

Proof. Since $\partial\Omega = \partial\overline{\Omega}^c$ we can consider the decreasing sequence of the $\Omega_{1/n}$ defined in Lemma 3.3 with $\epsilon = 1/n$ and, for each n , the minimal large solutions \underline{w}_n of (1.3) in $\Omega_{1/n}$: this is possible since $\partial\Omega_{1/n}$ is Lipschitz. The sequence $\{\underline{w}_n\}$ is increasing. Its restriction to Ω is bounded from above by the maximal solution \overline{w}_Ω . It converges to some function \underline{w}_Ω^* . By Lemma 3.3-(v), \underline{w}_Ω^* is a solution of (1.3) in the interior of $\bigcap_n \Omega_{1/n}$ which is Ω . If w is any solution of (1.3) defined in an open neighborhood of $\overline{\Omega}$, it is defined in $\Omega_{1/n}$ for n large enough and therefore smaller than \underline{w}_n . Thus $w|_\Omega \leq \underline{w}_\Omega^*$. Consequently, \underline{w}_Ω^* coincides with the supremum of the restrictions to Ω of solutions of (1.3) defined in an open neighborhood of $\overline{\Omega}$. \square

Proposition 3.6 Let $f \in C(\mathbb{R})$ be a nondecreasing function for which (1.7) holds and Ω a bounded domain in \mathbb{R}^N such that $\partial\Omega = \partial\overline{\Omega}^c$. Then \underline{w}_Ω^* is smaller than any large solution. Furthermore, if $\partial\Omega$ satisfies the Wiener regularity criterion and is locally the graph of a continuous function, then $\underline{w}_\Omega = \underline{w}_\Omega^*$.

Proof. We first notice that Wiener criterion implies statement (iii) in Lemma 3.3, hence $\partial\Omega = \partial\overline{\Omega}^c$. If w_Ω is a large solution, it dominates on $\partial\Omega$, and therefore in Ω by the maximum principle, the restriction to Ω of any function w solution of (1.3) in an open neighborhood of $\overline{\Omega}$. Then

$$\underline{w}_\Omega^* \leq w_\Omega.$$

Consequently, if \underline{w}_Ω^* is a large solution, it coincides with the minimal large solution \underline{w}_Ω . Because $\partial\Omega$ is compact, there exists a finite number of bounded open subset \mathcal{O}_j , hyperplanes H_j and continuous functions h_j from $H_j \cap \overline{\mathcal{O}_j}$ into \mathbb{R}_+ such that

$$\partial\Omega \cap \overline{\mathcal{O}_j} = \{x = x' + h_j(x')\nu_j : \forall x' \in H_j \cap \overline{\mathcal{O}_j}\}$$

where ν_j is a fixed unit vector orthogonal to H_j and $\partial\Omega \subset \bigcup_j \mathcal{O}_j$. We can assume that $H_j \cap \overline{\mathcal{O}_j} = \overline{B}_j$ is a $(N-1)$ dimensional closed ball and,

$$G_j := \{x = x' + t\nu_j : x' \in \overline{B}_j, 0 \leq t < h_j(x')\} \subset \Omega,$$

$$G_j^\# := \{x = x' + t\nu_j : x' \in \overline{B}_j, h_j(x') < t \leq a\} \subset \overline{\Omega}^c.,$$

for some $a > 0$ such that $a/4 < h_j(x') < 3a/4$ for any $x' \in \overline{B}_j$. Finally, we can assume that

$$\mathcal{O}_j = \{x = x' + t\nu_j : x' \in \overline{B}_j, 0 \leq t \leq a\}.$$

Let $\epsilon \in (0, a/8)$ and

$$G_{j,\epsilon} := \{x = x' + t\nu_j : x' \in \overline{B}_j, \epsilon \leq t < h_j(x') + \epsilon\}.$$

There exists a smooth bounded domain Ω' such that $\overline{\Omega} \subset \Omega'$ and

$$\partial\Omega' \cap \overline{\mathcal{O}}_j = \{x = x' + \ell(x')\nu_j : x' \in \overline{B}_j, h(x') + \epsilon/2 \leq \ell(x') \leq h(x') + 3\epsilon/2\},$$

where $\ell \in C^\infty(\overline{B}_j)$. We denote $G_j := G_{j,0}$,

$$\partial_p G_{j,\epsilon} := \{x = x' + t\nu_j : x' \in \partial B_j, \epsilon \leq t \leq h_j(x') + \epsilon\} \cup \{x = x' + \epsilon\nu_j : x' \in B_j\},$$

and

$$\partial_u G_{j,\epsilon} := \{x = x' + (h_j(x') + \epsilon)\nu_j : x' \in B_j\}.$$

Let w' be the minimal large solution of (1.3) in Ω' , $\alpha' = \min\{w'(x) : x \in \Omega'\}$ and W_ϵ the minimal solution of

$$\begin{cases} -\Delta W + f(W) = 0 & \text{in } G_{j,\epsilon} \\ W = \alpha' & \text{in } \partial_p G_{j,\epsilon} \\ \lim_{t \rightarrow h(x') + \epsilon} W(x' + t\nu_j) = \infty & \forall x' \in B_j. \end{cases} \quad (3.12)$$

Then $w' \geq W_\epsilon$ in $G_{j,\epsilon} \cap \Omega'$. Furthermore $W_\epsilon(x) = W_\epsilon(x' + t\nu_j) = W_0(x' + (t - \epsilon)\nu_j)$ for any $x' \in \overline{B}_j$ and $\epsilon < t < h(x') + \epsilon$. Therefore, given $k > 0$, there exists $\delta_k > 0$ such that for any

$$x' \in \overline{B}_j \text{ and } h_j(x') - \delta_k \leq t < h_j(x') \implies W_0(x' + t\nu_j) \geq k.$$

As a consequence, $\liminf_{t \rightarrow h_j(x')} \underline{w}_\Omega^*(x' + t\nu_j) \geq k$, uniformly with respect to $x' \in \overline{B}_j$. This implies that \underline{w}_Ω^* is a large solution. \square

Remark. We conjecture that the equality $\underline{w}_\Omega^* = \underline{w}_\Omega$ holds under the mere assumption that the Wiener criterion is satisfied.

Theorem 3.7 *Assume Ω is a bounded domain in \mathbb{R}^N such that $\partial\Omega = \partial\overline{\Omega}^c$ and $f \in C(\mathbb{R})$ satisfies (1.7), (1.8) and (2.1). Then there exists a exterior maximal solution $\underline{u}_{Q_\infty}^*$ to problem (1.1). Furthermore estimates (3.1) and (3.2) hold with \underline{w}_Ω replaced by the exterior maximal solution \underline{u}_Ω^* to problem (1.3) in Ω .*

Proof. The construction of $\underline{u}_{Q_\infty}^*$ is similar to the one of \underline{w}_Ω , since we can restrict to consider open neighborhoods $Q_{1/n} = \Omega_{1/n} \times (-1/n, \infty)$. Then $\underline{u}_{Q_\infty}^*$ is the increasing limit of the minimal large solutions u_n of (1.1) in $Q_{1/n}$, since $\overline{Q_\infty}^\Omega = \bigcap_n Q_{1/n}$ and, by Lemma 3.3-(v), $Q_\infty^\Omega = \overline{Q_\infty}^\Omega$. We recall that the minimal large solution w_n of (1.3) in $\Omega_{1/n}$ is the increasing limit, when $k \rightarrow \infty$, of the sequence of solution $\{w_n^k\}$ of

$$\begin{cases} -\Delta w + f(w) = 0 & \text{in } \Omega_{1/n} \\ w = k & \text{on } \partial\Omega_{1/n}, \end{cases} \quad (3.13)$$

while the minimal large solution u_n of (1.1) in $Q_{1/n}$ is the (always increasing) limit of the solutions u_n^k of

$$\begin{cases} u_t - \Delta u + f(u) = 0 & \text{in } Q_{1/n} \\ u = k & \text{on } \partial_p Q_{1/n}. \end{cases} \quad (3.14)$$

Clearly

$$\max\{w_n^k, \bar{\phi}(\cdot + 1/n)\} \leq u_n(x, t),$$

which implies (3.1). For the other inequality, we see that $(x, t) \mapsto w_n^k(x) + \bar{\phi}(t) + Lt$ is a supersolution which dominates u_n^k on ∂_p , where L corresponds to the minimum of w_n^k in $\Omega_{1/n}Q_{1/n}$. Thus

$$u_n(x, t) \leq w_n^k + \bar{\phi}(\cdot + 1/n),$$

which implies

$$\max\{\underline{w}_\Omega^*(x), \bar{\phi}(t)\} \leq \underline{u}_\Omega^*(x, t) \quad \forall (x, t) \in Q_\infty^\Omega. \quad (3.15)$$

The upper estimate is proved in the following way. If $k > n$, $\bar{Q}_k \subset Q_n$. Therefore, choosing m such that $\min\{\min\{\underline{w}_{\Omega_{1/k}}(x) : x \in \Omega_{1/k}\}, \min\{\bar{\phi}(t + 1/k) : t \in (0, T]\}\} \geq m$, we obtain that $(x, t) \mapsto \underline{w}_{\Omega_{1/k}}(x) + \bar{\phi}(t + 1/k) + Lt$ is a super solution of (1.1) in $Q_T^{\Omega_{1/k}}$, thus it dominates the minimal large solution of (1.1) in $Q_T^{\Omega_{1/n}}$. Letting successively $k \rightarrow \infty$ and $n \rightarrow \infty$, yields to

$$\underline{u}_\Omega^*(x, t) \leq \underline{w}_\Omega^*(x) + \bar{\phi}(t) \quad \forall (x, t) \in Q_T^\Omega. \quad (3.16)$$

□

The next result extends Corollary 3.2 without the boundary Wiener regularity assumption.

Theorem 3.8 *Let Ω be a bounded domain in \mathbb{R}^N such that $\partial\Omega = \partial\bar{\Omega}^c$. If $f \in C(\mathbb{R})$ is convex and satisfies (1.7), (1.8), (2.1) and (3.8). Then, if \underline{w}_Ω^* is a large solution, the following implication holds*

$$\underline{w}_\Omega^* = \bar{w}_\Omega \implies \underline{u}_{Q^\Omega}^* = \bar{u}_{Q^\Omega}. \quad (3.17)$$

Proof. If \underline{w}_Ω^* is a large solution, the same is true for $\underline{u}_{Q^\Omega}^*$ because of (3.1). Actually $\underline{u}_{Q^\Omega}^*$ is the minimal large solution in Q_∞^Ω for the same reasons as \underline{w}_Ω^* . Therefore the proof of Corollary 3.2 applies and it implies the result. □

Remark. We conjecture that (3.17) holds, even if \underline{w}_Ω^* is not a large solution.

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