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# Maximal solutions of nonlinear parabolic equations with absorption

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**Abstract** We study the existence and the uniqueness of the solution of the problem (P):  $\partial_t u - \Delta u + f(u) = 0$  in  $Q := \Omega \times (0, \infty)$ ,  $u = \infty$  on the parabolic boundary  $\partial_b Q$  when  $\Omega$  is a domain in  $\mathbb{R}^N$  with a compact boundary and  $f$  a continuous increasing function satisfying super linear growth condition. We prove that in most cases, the existence and uniqueness is reduced to the same property for the stationary equation.

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*Key words.* Parabolic equations, singular solutions, self-similarity, removable singularities

## 1 Introduction

Let  $\Omega$  be a bounded domain in  $\mathbb{R}^N$  with boundary  $\partial\Omega := \Gamma$ ,  $Q_T^\Omega := \Omega \times (0, T)$  ( $0 < T \leq \infty$ ) and  $\partial_p Q = \bar{\Omega} \times 0 \cup \partial\Omega \times (0, T]$ . We denote by  $\rho_{\partial\Omega}(x)$  the distance from  $x$  to  $\partial\Omega$  and by  $d_p(x, t) = \min\{\rho_{\partial\Omega}(x), t\}$  the distance from  $(x, t) \in Q_\infty^\Omega$  to  $\partial_p Q_\infty^\Omega$ . If  $f \in C(\mathbb{R})$ , we say that a function  $u \in C^{2,1}(Q_\infty^\Omega)$  solution of

$$u_t - \Delta u + f(u) = 0, \quad (1.1)$$

in  $Q_\infty^\Omega$  is a large solution of (1.1) in  $Q_\infty^\Omega$  if it satisfies

$$\lim_{d_p(x,t) \rightarrow 0} u(x, t) = \infty. \quad (1.2)$$

The existence of such a  $u$  is associated to the existence of large solutions to the stationary equation

$$-\Delta w + f(w) = 0, \quad (1.3)$$

in  $\Omega$ , i. e. solutions which satisfy

$$\lim_{\rho_{\partial\Omega}(x) \rightarrow 0} w(x) = \infty, \quad (1.4)$$

and solutions of the ODE

$$\phi' + f(\phi) = 0 \quad \text{in } (0, \infty). \quad (1.5)$$

subject to the initial blow-up condition

$$\lim_{t \rightarrow 0} \phi(t) = \infty. \quad (1.6)$$

A natural assumption on  $f$  is to assume that it is nondecreasing with  $f(0) \geq 0$ . If  $f(a) > 0$ , a necessary and sufficient condition for the existence of a maximal solution  $\bar{w}_\Omega$  to (1.3) is the Keller-Osserman condition,

$$\int_a^\infty \frac{ds}{\sqrt{F(s)}} < \infty, \quad (1.7)$$

where  $F(s) = \int_0^s f(\tau) d\tau$ . A necessary and sufficient condition for the existence of a solution  $\phi$  of (1.6) with initial blow-up is

$$\int_a^\infty \frac{ds}{f(s)} < \infty. \quad (1.8)$$

Furthermore the unique maximal solution  $\bar{\phi}$  is obtained by inversion from the formula

$$\int_{\bar{\phi}(t)}^\infty \frac{ds}{f(s)} = t \quad \forall t > 0. \quad (1.9)$$

It is known that, if  $f$  is convex, (1.7) implies (1.8). If (1.7) holds and there exists a maximal solution to (1.3), it is not always true that this maximal solution is a large solution. In the case of a general nonlinearity, only sufficient conditions are known independent of the regularity of  $\partial\Omega$ , and we recall some of them.

*If  $N \geq 3$  and  $f$  satisfies the weak singularity assumption*

$$\int_a^\infty s^{-2(N-1)/(N-2)} f(s) ds < \infty \quad \forall a > 0. \quad (1.10)$$

*If  $N = 2$  and the exponential order of growth of  $f$  defined by*

$$a_f^+ = \inf \left\{ a \geq 0 : \int_0^\infty f(s) e^{-as} ds < \infty \right\}, \quad (1.11)$$

*is finite.*

When  $f(u) = u^q$  with  $q > 1$ , (1.10) means that  $q < N/(N-2)$ . When  $q \geq N/(N-2)$  the regularity of  $\partial\Omega$  plays a crucial role in the existence of large solutions. A necessary and sufficient condition involving a Wiener type test which uses the  $C_{2,q}^{\mathbb{R}^N}$ -Bessel capacity has been obtained by probabilistic methods by Dhersin and Le Gall [4] in the case  $q = 2$  and extended to the general case by Labutin [6]. Precise pointwise estimates are obtained by Marcus and Véron [11].

Uniqueness of the large solution of (1.3) has been obtained under three types of assumptions (see [7], [10] and [11]):

If  $\partial\Omega = \partial\bar{\Omega}^c$  and  $f(u) = u^q$  with  $1 < q < N/(N-2)$  or if  $N = 2$  and  $f(u) = e^{au}$ .

If  $\partial\Omega$  is locally a continuous graph and  $f(u) = u^q$  with  $q > 1$  or  $f(u) = e^{au}$ .

If  $f(u) = u^q$  with  $q \geq N/(N-2)$  and  $C_{2,q'}^{\mathbb{R}^N}(\partial\Omega \setminus \bar{\Omega}^c) = 0$ , where  $\tilde{E}$  denotes the closure of a set in the fine topology associated to the Bessel capacity  $C_{2,q'}^{\mathbb{R}^N}$ .

In this article we extend most of the above mentioned results to the parabolic equation (1.1). We first prove that, if  $f$  is super-additive, i. e.

$$f(x+y) \geq f(x) + f(y) \quad \forall (x, y) \in \mathbb{R} \times \mathbb{R}, \quad (1.12)$$

and satisfies (1.7) and (1.8), there exists a maximal solution  $\bar{u}_{Q^\Omega}$  to (1.1) in  $Q^\Omega$ , and it satisfies

$$\bar{u}_{Q^\Omega}(x, t) \leq \bar{w}_\Omega(x) + \bar{\phi}(t) \quad \forall (x, t) \in Q^\Omega. \quad (1.13)$$

If we assume also that  $\partial\Omega = \partial\bar{\Omega}^c$ , there holds

$$\max\{\bar{w}_\Omega(x), \bar{\phi}(t)\} \leq \bar{u}_{Q^\Omega}(x, t) \quad \forall (x, t) \in Q^\Omega. \quad (1.14)$$

Under the assumption  $\partial\Omega = \partial\bar{\Omega}^c$ , it is possible to consider a decreasing sequence of smooth bounded domains  $\Omega^n$  such that  $\bar{\Omega}^n \subset \Omega^{n-1}$  and prove that the increasing sequence of large solutions  $\bar{u}_{Q^{\Omega^n}}$  of (1.1) in  $Q^{\Omega^n} := \Omega^n \times (0, \infty)$ , converges to the exterior maximal solution  $\underline{u}_{Q^\Omega}$  of (1.1) in  $Q^\Omega$ . If we procede similarly with the large solutions  $\bar{w}_{\Omega^n}$  of (1.3) in  $\Omega^n$  and denote by  $\underline{w}_\Omega$  their limit, then we prove that

$$\max\{\underline{w}_\Omega(x), \bar{\phi}(t)\} \leq \underline{u}_{Q^\Omega}(x, t) \quad \forall (x, t) \in Q^\Omega. \quad (1.15)$$

The main result of this article is the following

**Theorem 1.** Assume  $\Omega$  is a bounded domain such that  $\partial\Omega = \partial\bar{\Omega}^c$ ,  $f \in C(\mathbb{R})$  is nondecreasing and satisfies (1.7), (1.8) and (1.12). Then, if  $\underline{w}_\Omega = \bar{w}_\Omega$ , there holds  $\underline{u}_{Q^\Omega} = \bar{u}_{Q^\Omega}$ .

Consequently, if (1.3) admits a unique large solution in  $\Omega$ , the same holds for (1.1) in  $Q_\infty^\Omega$ .

## 2 The maximal solution

In this section  $\Omega$  is a bounded domain in  $\mathbb{R}^N$  and  $f \in C(\mathbb{R})$  is nondecreasing and satisfies (1.7) and (1.8). We set  $k_0 = \inf\{\ell \geq 0 : f(\ell) > 0\}$  and assume also that, for any  $m \in \mathbb{R}$  there exists  $L \in \mathbb{R}_+$  such that

$$\forall (x, y) \in \mathbb{R}^2, x \geq m, y \geq m \implies f(x+y) \geq f(x) + f(y) - L. \quad (2.1)$$

**Theorem 2.1** Under the previous assumptions there exists a maximal solution  $\bar{u}_{Q^\Omega}$  in  $Q_\infty^\Omega$ .

*Proof. Step 1- Approximation and estimates.* Let  $\Omega_n$  be an increasing sequence of smooth domains such that  $\overline{\Omega}_n \subset \Omega_{n+1}$  and  $\cup \Omega_n = \Omega$ . For each of these domains and  $(n, k) \in \mathbb{N}_*^2$  we denote by  $w = w_{n,k}$  the solutions of

$$\begin{cases} -\Delta w + w^q = 0 & \text{in } \Omega_n \\ w = k & \text{in } \partial\Omega_n. \end{cases} \quad (2.2)$$

where  $\partial_p Q_\infty^{\Omega_n} := \partial\Omega_n \times (0, \infty) \cup \overline{\Omega}_n \times \{0\}$ . By [5] there exists a decreasing function  $g$  from  $\mathbb{R}_+$  to  $\mathbb{R}$ , with limit  $\infty$  at zero, such that

$$w_{n,k}(x) \leq g(\rho_{\partial\Omega_n}(x)) \quad \forall x \in \Omega_n. \quad (2.3)$$

The mapping  $k \rightarrow w_{n,k}$  is increasing, while  $n \rightarrow w_{n,k}$  is decreasing. If we set

$$\overline{w}_\Omega = \lim_{n \rightarrow \infty} \lim_{k \rightarrow \infty} w_{n,k}, \quad (2.4)$$

it is classical that  $\overline{w}_\Omega$  is the maximal solution of (1.3) in  $\Omega$ , and it satisfies

$$w(x) \leq g(\rho_{\partial\Omega}(x)) \quad \forall x \in \Omega. \quad (2.5)$$

We denote also by  $u = u_{n,k}$  the solution of

$$\begin{cases} u_t - \Delta u + f(u) = 0 & \text{in } Q_\infty^{\Omega_n} \\ u = k & \text{in } \partial_p Q_\infty^{\Omega_n}. \end{cases} \quad (2.6)$$

By the maximum principle  $k \rightarrow u_{n,k}$  is increasing and  $n \rightarrow u_{n,k}$  decreasing. If we denote by  $\bar{\phi}$  the maximal solution of the ODE (1.5), then  $\bar{\phi}(t)$  is expressed by inversion by (1.9). If  $t_k = \bar{\phi}^{-1}(k)$ , there holds, since  $\bar{\phi}$  is decreasing,

$$\bar{\phi}(t + t_k) \leq u_{n,k}(x, t) \quad \text{in } Q_\infty^{\Omega_n}. \quad (2.7)$$

Furthermore, if  $f(k) \geq 0$  (which holds if  $k \geq k_0$ ),  $w_{n,k} \leq k$ . Therefore

$$w_{n,k}(x) \leq u_{n,k}(x, t) \quad \text{in } Q_\infty^{\Omega_n}. \quad (2.8)$$

Combining (2.7) and (2.8), we derive

$$\max\{w_{n,k}(x), \bar{\phi}(t + t_k)\} \leq u_{n,k}(x, t) \quad \forall (x, t) \in Q_\infty^{\Omega_n}. \quad (2.9)$$

Next we obtain an upper estimate: For fixed  $T > 0$ , we consider  $m \geq \bar{\phi}(T)$  such that  $\min\{\overline{w}_\Omega(x) : x \in \Omega\} > m$ . For  $n \geq n_1$  and  $k \geq k_1$  there holds  $\min\{w_{n,k}(x) : x \in \Omega\} \geq m$ . Let  $L = L(m) \geq 0$  be the corresponding damping term from (2.1). If  $v_{n,k} = w_{n,k}(x) + \bar{\phi}(t + t_k)$ , then it satisfies

$$v_t - \Delta v + f(v) = f(v) - f(\bar{\phi}(\cdot + t_k)) - f(w_{n,k}) \geq -L \quad \text{if } (x, t) \in \Omega_n \times [0, T - t_k]. \quad (2.10)$$

Since  $L \geq 0$ , the function  $\tilde{v}_{n,k} := v_{n,k} + Lt$  is a supersolution for (1.1) in  $Q_{T-t_k}^{\Omega_n} := \Omega_n \times (0, T - t_k)$  which dominates  $u_{n,k}$  on  $\partial_p Q_{T-t_k}^{\Omega_n}$ , thus in  $Q_{T-t_k}^{\Omega_n}$  by the maximum principle. Therefore

$$u_{n,k}(x, t) \leq w_{n,k}(x) + \bar{\phi}(t + t_k) + Lt \quad \forall (x, t) \in Q_{T-t_k}^{\Omega_n}. \quad (2.11)$$

*Step 2- Final estimates and maximality.* Using the different monotonicity properties of the mapping  $(k, n) \mapsto w_{n,k}$  and the estimates (2.9) and (2.11), it follows that the function defined by

$$\bar{u}_{Q^\Omega} := \lim_{n \rightarrow \infty} \lim_{k \rightarrow \infty} u_{n,k} \quad (2.12)$$

is a solution of (1.1) in  $Q_\infty^\Omega$ . Furthermore

$$\max\{w_\Omega(x), \bar{\phi}(t)\} \leq \bar{u}_{Q^\Omega}(x, t) \quad \forall (x, t) \in Q_\infty^\Omega, \quad (2.13)$$

and, for all  $T > 0$  such that  $\phi(T) \leq \min\{w_\Omega(x) : x \in \Omega\}$ ,

$$\bar{u}_{Q^\Omega}(x, t) \leq w_\Omega(x) + \bar{\phi}(t) + tL(\phi(T)) \quad \forall (x, t) \in Q_T^\Omega. \quad (2.14)$$

Next, we consider  $u \in C^{2,1}(Q_\infty^\Omega)$ , solution of (1.1) in  $Q_\infty^\Omega$ . Then, for  $\epsilon > 0$  and  $n \in \mathbb{N}$ , there exists  $k^* > 0$  such that for  $k \geq k^*$ ,

$$u_{n,k}(x, t - \epsilon) \geq u(x, t) \quad \forall (x, t) \in \Omega_n \times (\epsilon, \infty).$$

Letting successively  $k \rightarrow \infty$ ,  $n \rightarrow \infty$  and  $\epsilon \rightarrow 0$ , yields to  $\bar{u}_{Q^\Omega} \geq u$  in  $Q_\infty^\Omega$ .  $\square$

Since  $\bar{w}_\Omega$  be a large solution in  $\Omega$  implies the same boundary blow-up for  $\bar{u}_{Q^\Omega}$  on  $\partial\Omega \times (0, \infty)$ , we give below some conditions which implies that  $\bar{u}_{Q^\Omega}$  is a large solution.

**Corollary 2.2** *Assume the assumptions of Theorem 2.1 are fulfilled. Then  $\bar{u}_{Q^\Omega}$  is a large solution if one of the following additional conditions is satisfied:*

- (i)  $N \geq 3$  and  $f$  satisfies the weak singularity condition (1.10).
- (ii)  $N = 2$  and the exponential order of growth of  $f$  defined by (1.11) is positive.
- (iii)  $N \geq 3$  and  $\partial\Omega$  satisfies the Wiener regularity criterion.

*Proof.* Under condition (i) or (ii), for any  $x_0 \in \partial\Omega$ , there exists a solution  $w_{c,x_0}$  of

$$\begin{cases} -\Delta w + f(w) = c\delta_{x_0} & \text{in } B_R(x_0) \\ w = 0 & \text{in } \partial B_R(x_0), \end{cases} \quad (2.15)$$

where  $R > 0$  is chosen such that  $\bar{\Omega} \subset B_R(x_0)$  and  $c > 0$  is arbitrary under condition (i) and smaller than  $2/a_f^+$  in case (ii). The function  $w_{c,x_0}$  is radial with respect to  $x_0$  and

$$\lim_{x \rightarrow x_0} w_{c,x_0}(x) = \infty.$$

If  $x \in \Omega$ , we denote by  $x_0$  a projection of  $x$  on  $\partial\Omega$ . Since

$$w_n(x) \geq w_{c,x_0}(x) \implies \bar{w}_\Omega(x) \geq w_{c,x_0}(x),$$

we derive from (2.13),

$$\lim_{\rho_{\partial\Omega}(x) \rightarrow 0} \bar{u}_{Q^\Omega}(x, t) = \infty,$$

uniformly with respect to  $t > 0$ . In case (iii) we see that, for any  $k > 0$

$$\bar{w}_\Omega(x) \geq w_{k,\infty}(x) \quad \forall x \in \Omega, \quad (2.16)$$

where  $w_{k,\infty}$  is the solution of (2.2), with  $\Omega_n$  replaced by  $\Omega$ . This again implies (2.16).  $\square$

Using estimate (2.13) leads to the asymptotic behavior of  $\bar{u}_{Q^\Omega}(x, t)$  when  $t \rightarrow \infty$ .

**Corollary 2.3** *Assume the assumptions of Theorem 2.1 are fulfilled. Then  $\bar{u}_{Q^\Omega}(x, t) \rightarrow \bar{w}_\Omega(x)$  locally uniformly on  $\Omega$  when  $t \rightarrow \infty$ .*

*Proof.* For any  $k > k_0$  and  $n \in \mathbb{N}_*$  and any  $s > 0$ , there holds by the maximum principle,

$$u_{n,k}(x, s) \leq k = u_{n,k}(x, 0) \quad \forall x \in \Omega_n.$$

Using the monotonicity of  $f$ , we derive  $u_{n,k}(x, t+s) \leq u_{n,k}(x, t)$  for any  $(x, t) \in Q_\infty^\Omega$ . Letting  $k \rightarrow \infty$  and then  $n \rightarrow \infty$  yields to

$$\bar{u}_{Q^\Omega}(x, t+s) \leq \bar{u}_{Q^\Omega}(x, t) \quad \forall (x, t) \in Q_\infty^\Omega. \quad (2.17)$$

It follows that  $\bar{u}_{Q^\Omega}(x, t)$  converges to some  $W(x)$  as  $t \rightarrow \infty$  and  $\bar{w}_\Omega \leq W$  from (2.13). Using the parabolic equation regularity theory, we derive that the trajectory  $\mathcal{T} := \bigcup_{t \geq 0} \{\bar{u}_{Q^\Omega}(\cdot, t)\}$  is compact in the  $C_{loc}^1(\Omega)$ -topology. Therefore  $W$  is a solution of (1.3) in  $\Omega$ . It coincides with  $\bar{w}_\Omega$  because of the maximality.  $\square$

### 3 Large solutions

In this section we construct a minimal-maximal solution of (1.1) which is the minimal large solution whenever it exists. If  $\partial\Omega$  is regular enough, the construction of the minimal large solution is easy.

**Theorem 3.1** *Let  $\Omega$  be a bounded domain in  $\mathbb{R}^N$  the boundary of which satisfies the Wiener regularity condition. If  $f \in C(\mathbb{R})$  is nondecreasing and satisfies (1.7), (1.8) and (2.1), then there exists a minimal large solution  $\underline{u}_{Q^\Omega}$  to (1.1) in  $Q_\infty^\Omega$ . Furthermore*

$$\max\{\underline{u}_\Omega(x), \bar{\phi}(t)\} \leq \underline{u}_{Q^\Omega}(x, t) \quad \forall (x, t) \in Q_\infty^\Omega, \quad (3.1)$$

and, for any  $T > 0$ ,

$$\underline{u}_{Q^\Omega}(x, t) \leq \underline{u}_\Omega(x) + \bar{\phi}(t) + tL(\bar{\phi}(T)) \quad \forall (x, t) \in Q_T^\Omega, \quad (3.2)$$

where  $L(\bar{\phi}(T))$  is as in (2.16), and  $\underline{u}_\Omega$  denotes the minimal large solution of (1.3) in  $\Omega$ .

*Proof.* For  $k \geq k_0$  (see Section 2), we denote by  $\underline{u}_k$  the solution of

$$\begin{cases} u_t - \Delta u + f(u) = 0 & \text{in } Q_\infty^\Omega \\ u = k & \text{in } \partial_p Q_\infty^\Omega. \end{cases} \quad (3.3)$$

When  $k$  increases,  $\underline{u}_k$  increases and converges to some large solution  $\underline{u}_{Q^\Omega}$  of (1.1) in  $Q_\infty^\Omega$ . If  $u$  is any large solution of (1.1) in  $Q_\infty^\Omega$ , then the maximum principle and (1.2) implies  $u \geq \underline{u}_k$ . Therefore  $u \geq \underline{u}_{Q^\Omega}$ . The same assumption allows to construct the solution  $w_k$  of

$$\begin{cases} -\Delta w + f(w) = 0 & \text{in } \Omega \\ w = k & \text{in } \partial\Omega, \end{cases} \quad (3.4)$$

and, by letting  $k \rightarrow \infty$ , to obtain the minimal large solution  $\underline{w}_\Omega$  of (1.3) in  $\Omega$ . Next we first observe, that, as in the proof of Theorem 2.1, (2.10) applies under the form

$$\bar{\phi}(t + t_k) \leq u_k(x, t) \quad \text{in } Q_\infty^\Omega, \quad (3.5)$$

where, we recall it,  $t_k = \bar{\phi}^{-1}(k)$ . In the same way, for  $k \geq k_0$  (with  $f(k) \geq 0$ ), (2.11) holds under the form

$$w_k(x) \leq u_k(x, t) \quad \text{in } Q_\infty^\Omega. \quad (3.6)$$

Letting  $k \rightarrow \infty$  yields to

$$\max\{\underline{w}_\Omega(x), \bar{\phi}(t)\} \leq \underline{u}_{Q^\Omega}(x, t) \quad \forall (x, t) \in Q_\infty^\Omega. \quad (3.7)$$

In order to prove the upper estimate we consider the same  $m$  as in the proof of Theorem 2.1 such that  $\min\{\min\{w_k(x) : x \in \Omega\}, \bar{\phi}(t)\} \geq m$ , and for  $k' > k$ , there holds

$$w_{k'} + \bar{\phi} \geq k = w_k|_{\partial_p Q_T^\Omega}.$$

Since  $w_{k'}(x) + \bar{\phi}(t) + tL$  is a supersolution for (1.1) in  $Q_T^\Omega$  it follows  $w_{k'} + \bar{\phi} + tL \geq w_k$  in  $Q_T^\Omega$ . Letting successively  $k' \rightarrow \infty$  and  $k' \rightarrow \infty$ , we derive (3.2).  $\square$

From this result we can deduce uniqueness results for solution of

**Corollary 3.2** *Under the assumptions of Theorem 3.1, if we assume moreover that  $f$  is convex and, for any  $\theta \in (0, 1)$ , there exists  $r_\theta$  such that*

$$r \geq r_\theta \implies f(\theta r) \leq \theta f(r). \quad (3.8)$$

Then

$$\underline{w}_\Omega = \bar{w}_\Omega \implies \underline{u}_{Q^\Omega} = \bar{u}_{Q^\Omega}. \quad (3.9)$$

*Proof.* We fix  $T \in (0, 1]$  such that

$$tL(\bar{\phi}(1)) \leq \bar{\phi}(t) \quad \forall t \in (0, T],$$

(remember that  $L$  is always positive) and

$$2\underline{w}_\Omega(x) + \bar{\phi}(t) \geq 0 \quad \forall (x, t) \in Q_T^\Omega.$$

Then  $\underline{w}_\Omega(x) + \bar{\phi}(t) \geq 0$  and

$$\underline{w}_\Omega(x) + \bar{\phi}(t) + tL(\bar{\phi}(1)) \leq \underline{w}_\Omega(x) + 2\bar{\phi}(t) \leq \underline{w}_\Omega(x) + 2\bar{\phi}(t) \leq 3(\underline{w}_\Omega(x) + \bar{\phi}(t)),$$

from which inequality follows

$$2^{-1}(\underline{w}_\Omega(x) + \bar{\phi}(t)) \leq \underline{u}_{Q^\Omega}(x, t) \leq 3(\underline{w}_\Omega(x) + \bar{\phi}(t)) \quad \forall (x, t) \in Q_T^\Omega.$$

Therefore, if  $\underline{w}_\Omega = \bar{w}_\Omega$ , it follows

$$\underline{u}_{Q^\Omega} \leq \bar{u}_{Q^\Omega} \leq 6\underline{u}_{Q^\Omega} \quad \text{in } Q_T^\Omega. \quad (3.10)$$



Next we assume  $\underline{u}_{Q^\Omega} < \overline{u}_{Q^\Omega}$  and set

$$u^* = \underline{u}_{Q^\Omega} - \frac{1}{6} (\overline{u}_{Q^\Omega} - \underline{u}_{Q^\Omega}).$$

Since  $f$  is convex,  $u^*$  is a supersolution of (1.1) in  $Q_T^\Omega$  (see [8], [10]) and  $u^* < \underline{u}_{Q^\Omega}$ . Up to take a smaller  $T$ , we can also assume from (3.8) that  $\min\{\underline{u}_{Q^\Omega}(x, t) : (x, t) \in Q_T^\Omega\} \geq r_{1/12}$ , thus

$$f(\underline{u}_{Q^\Omega}/12) \leq \frac{1}{12} f(\underline{u}_{Q^\Omega}) \quad \text{in } Q_T^\Omega.$$

Therefore  $\underline{u}_{Q^\Omega}/12$  is a subsolution for (1.1) in  $Q_T^\Omega$  and  $12^{-1}\underline{u}_{Q^\Omega} < u^*$ . Using a standard result of sub and super solutions and the fact that  $f$  is locally Lipschitz continuous, we see that there exists some  $u^\#$  solution of (1.1) in  $Q_T^\Omega$  such that

$$u^\# \leq u^* < \underline{u}_{Q^\Omega} \quad \text{in } Q_T^\Omega. \quad (3.11)$$

Then  $u^\#$  is a large solution, which contradicts the minimality of  $\underline{u}_{Q^\Omega}$  on  $Q_T^\Omega$ . Finally  $\underline{u}_{Q^\Omega} = \overline{u}_{Q^\Omega}$  in  $Q_\infty^\Omega$ .  $\square$

**Lemma 3.3** *Let  $\Omega$  be a bounded domain in  $\mathbb{R}^N$  and, for  $\epsilon > 0$ ,  $\Omega_\epsilon := \{x \in \mathbb{R}^N : \text{dist}(x, \overline{\Omega}) < \epsilon\}$ . The four following assertions are equivalent:*

- (i)  $\partial\Omega = \partial\overline{\Omega}^c$ .
- (ii) For any  $x \in \partial\Omega$ , there exists a sequence  $\{x_n\} \subset \overline{\Omega}^c$  such that  $x_n \rightarrow x$ .
- (iii) For any  $x \in \partial\Omega$  and any  $\epsilon > 0$ ,  $B_\epsilon(x) \cap \overline{\Omega}^c \neq \emptyset$ .
- (iv) For any  $x \in \partial\Omega$ ,  $\lim_{\epsilon \rightarrow 0} \text{dist}(x, \Omega_\epsilon^c) = 0$ .
- (v)  $\Omega = \overline{\Omega}^o$ .

*Proof.* There always holds  $\partial\overline{\Omega}^c = \overline{\Omega}^c \cap \overline{\Omega} \subset \Omega^c \cap \overline{\Omega} = \partial\Omega$ .

(i)  $\implies$  (iii). Assume (iii) does not hold, there exist  $x_0 \in \partial\Omega$  and  $\epsilon_0 > 0$  such that  $B_{\epsilon_0}(x_0) \cap \overline{\Omega}^c = \emptyset$ . Thus  $x_0 \notin \overline{\Omega}^c$ , and  $x_0 \notin \partial\overline{\Omega}^c$ . Therefore (i) does not hold.

(iii)  $\implies$  (i). Let  $x_0 \in \partial\Omega$ . If, for any  $\epsilon > 0$ ,  $B_\epsilon(x) \cap \overline{\Omega}^c \neq \emptyset$ , then  $x \in \overline{\Omega}^c$ . Because  $x \in \Omega^c \cap \overline{\Omega}$ , it implies that  $x \in \overline{\Omega} \cap \overline{\Omega}^c = \partial\overline{\Omega}^c$ .

The equivalence between (iii) and (ii) is obvious.

(ii)  $\implies$  (iv). We assume (iv) does not hold. There exist  $x_0 \in \partial\Omega$ ,  $\alpha > 0$  and a sequence of positive real numbers  $\{\epsilon_n\}$  converging to 0 such that  $\text{dist}(x_0, \Omega_{\epsilon_n}^c) \geq \alpha$ . Since for  $\epsilon \geq \epsilon_n$ ,  $\Omega_\epsilon^c \subset \Omega_{\epsilon_n}^c$ , there holds  $\text{dist}(x_0, \Omega_\epsilon^c) \geq \alpha$ . Furthermore, this inequality holds for any  $\epsilon > 0$ . If there exist a sequence  $\{x_n\} \subset \overline{\Omega}^c$  such that  $x_n \rightarrow x_0$ , then  $\text{dist}(x_n, \overline{\Omega}) = \delta_n > 0$ , thus  $x_n \in \Omega_{\delta_n}^c$ . Consequently  $|x_n - x_0| \geq \alpha$ , which is impossible. Therefore (ii) does not hold.

(iv)  $\implies$  (iii). Let  $x \in \partial\Omega$  and  $x_n \in \Omega_{1/n}^c$  such that  $|x - x_n| = \text{dist}(x, \Omega_{1/n}^c) \rightarrow 0$ . Since  $\Omega_{1/n}^c \subset \overline{\Omega}$ ,  $x_n \in \overline{\Omega}^c$  and  $x_n \rightarrow x$ .

(iii)  $\implies$  (v). We first notice that  $\overline{\Omega} = \bigcap_{\epsilon > 0} \Omega_\epsilon = \bigcap_{\epsilon > 0} \overline{\Omega}_\epsilon$  and  $\Omega \subset \overline{\Omega}^o$ . If there exists some  $x \in \overline{\Omega} \setminus \Omega$ , then for some  $\epsilon > 0$ ,  $B_\epsilon(x) \subset \overline{\Omega}$  which implies  $B_\epsilon(x) \cap \overline{\Omega}^c = \emptyset$ . But  $x \notin \Omega$  implies  $x \in \partial\Omega$ . Thus (iii) does not hold.

(v)  $\implies$  (iii). If (iii) does not hold, there exists  $x \in \partial\Omega$  and  $\epsilon > 0$  such that  $B_\epsilon(x) \cap \overline{\Omega}^c = \emptyset \iff B_\epsilon(x) \subset \overline{\Omega}$ . Therefore  $x \in \overline{\Omega} \setminus \Omega$ .  $\square$

**Definition 3.4** A solution  $U$  (resp.  $W$  to problem (1.1) in  $Q_\infty^\Omega$  (resp. (1.3) in  $\Omega$ ) is called an exterior maximal solution if it is larger than the restriction to  $Q_\infty^\Omega$  (resp.  $\Omega$ ) of any solution of (1.1) (resp. (1.3)) defined in an open neighborhood of  $Q_\infty^\Omega$  (resp.  $\Omega$ ).

**Proposition 3.5** Assume  $\Omega$  is a bounded domain in  $\mathbb{R}^N$  such that  $\partial\Omega = \partial\overline{\Omega}^c$  and  $f \in C(\mathbb{R})$  is nondecreasing and satisfies (1.7). Then there exists a exterior maximal solution  $\underline{w}_\Omega^*$  to problem (1.3) in  $\Omega$ .

*Proof.* Since  $\partial\Omega = \partial\overline{\Omega}^c$  we can consider the decreasing sequence of the  $\Omega_{1/n}$  defined in Lemma 3.3 with  $\epsilon = 1/n$  and, for each  $n$ , the minimal large solutions  $\underline{w}_n$  of (1.3) in  $\Omega_{1/n}$ : this is possible since  $\partial\Omega_{1/n}$  is Lipschitz. The sequence  $\{\underline{w}_n\}$  is increasing. Its restriction to  $\Omega$  is bounded from above by the maximal solution  $\overline{w}_\Omega$ . It converges to some function  $\underline{w}_\Omega^*$ . By Lemma 3.3-(v),  $\underline{w}_\Omega^*$  is a solution of (1.3) in the interior of  $\cap_n \Omega_{1/n}$  which is  $\Omega$ . If  $w$  is any solution of (1.3) defined in an open neighborhood of  $\overline{\Omega}$ , it is defined in  $\Omega_{1/n}$  for  $n$  large enough and therefore smaller than  $\underline{w}_n$ . Thus  $w|_\Omega \leq \underline{w}_\Omega^*$ . Consequently,  $\underline{w}_\Omega^*$  coincides with the supremum of the restrictions to  $\Omega$  of solutions of (1.3) defined in an open neighborhood of  $\overline{\Omega}$ .  $\square$

**Proposition 3.6** Let  $f \in C(\mathbb{R})$  be a nondecreasing function for which (1.7) holds and  $\Omega$  a bounded domain in  $\mathbb{R}^N$  such that  $\partial\Omega = \partial\overline{\Omega}^c$ . Then  $\underline{w}_\Omega^*$  is smaller than any large solution. Furthermore, if  $\partial\Omega$  satisfies the Wiener regularity criterion and is locally the graph of a continuous function, then  $\underline{w}_\Omega = \underline{w}_\Omega^*$ .

*Proof.* We first notice that Wiener criterion implies statement (iii) in Lemma 3.3, hence  $\partial\Omega = \partial\overline{\Omega}^c$ . If  $w_\Omega$  is a large solution, it dominates on  $\partial\Omega$ , and therefore in  $\Omega$  by the maximum principle, the restriction to  $\Omega$  of any function  $w$  solution of (1.3) in an open neighborhood of  $\overline{\Omega}$ . Then

$$\underline{w}_\Omega^* \leq w_\Omega.$$

Consequently, if  $\underline{w}_\Omega^*$  is a large solution, it coincides with the minimal large solution  $\underline{w}_\Omega$ . Because  $\partial\Omega$  is compact, there exists a finite number of bounded open subset  $\mathcal{O}_j$ , hyperplanes  $H_j$  and continuous functions  $h_j$  from  $H_j \cap \overline{\mathcal{O}_j}$  into  $\mathbb{R}_+$  such that

$$\partial\Omega \cap \overline{\mathcal{O}_j} = \{x = x' + h_j(x')\nu_j : \forall x' \in H_j \cap \overline{\mathcal{O}_j}\}$$

where  $\nu_j$  is a fixed unit vector orthogonal to  $H_j$  and  $\partial\Omega \subset \cup_j \mathcal{O}_j$ . We can assume that  $H_j \cap \overline{\mathcal{O}_j} = \overline{B}_j$  is a  $(N-1)$  dimensional closed ball and,

$$G_j := \{x = x' + t\nu_j : x' \in \overline{B}_j, 0 \leq t < h_j(x')\} \subset \Omega,$$

$$G_j^\# := \{x = x' + t\nu_j : x' \in \overline{B}_j, h_j(x') < t \leq a\} \subset \overline{\Omega}^c.,$$

for some  $a > 0$  such that  $a/4 < h_j(x') < 3a/4$  for any  $x' \in \overline{B}_j$ . Finally, we can assume that

$$\mathcal{O}_j = \{x = x' + t\nu_j : x' \in \overline{B}_j, 0 \leq t \leq a\}.$$

Let  $\epsilon \in (0, a/8)$  and

$$G_{j,\epsilon} := \{x = x' + t\nu_j : x' \in \overline{B}_j, \epsilon \leq t < h_j(x') + \epsilon\}.$$

There exists a smooth bounded domain  $\Omega'$  such that  $\overline{\Omega} \subset \Omega'$  and

$$\partial\Omega' \cap \overline{\mathcal{O}}_j = \{x = x' + \ell(x')\nu_j : x' \in \overline{B}_j, h(x') + \epsilon/2 \leq \ell(x') \leq h(x') + 3\epsilon/2\},$$

where  $\ell \in C^\infty(\overline{B}_j)$ . We denote  $G_j := G_{j,0}$ ,

$$\partial_p G_{j,\epsilon} := \{x = x' + t\nu_j : x' \in \partial B_j, \epsilon \leq t \leq h_j(x') + \epsilon\} \cup \{x = x' + \epsilon\nu_j : x' \in B_j\},$$

and

$$\partial_u G_{j,\epsilon} := \{x = x' + (h_j(x') + \epsilon)\nu_j : x' \in B_j\}.$$

Let  $w'$  be the minimal large solution of (1.3) in  $\Omega'$ ,  $\alpha' = \min\{w'(x) : x \in \Omega'\}$  and  $W_\epsilon$  the minimal solution of

$$\begin{cases} -\Delta W + f(W) = 0 & \text{in } G_{j,\epsilon} \\ W = \alpha' & \text{in } \partial_p G_{j,\epsilon} \\ \lim_{t \rightarrow h(x') + \epsilon} W(x' + t\nu_j) = \infty & \forall x' \in B_j. \end{cases} \quad (3.12)$$

Then  $w' \geq W_\epsilon$  in  $G_{j,\epsilon} \cap \Omega'$ . Furthermore  $W_\epsilon(x) = W_\epsilon(x' + t\nu_j) = W_0(x' + (t - \epsilon)\nu_j)$  for any  $x' \in \overline{B}_j$  and  $\epsilon < t < h(x') + \epsilon$ . Therefore, given  $k > 0$ , there exists  $\delta_k > 0$  such that for any

$$x' \in \overline{B}_j \text{ and } h_j(x') - \delta_k \leq t < h_j(x') \implies W_0(x' + t\nu_j) \geq k.$$

As a consequence,  $\liminf_{t \rightarrow h_j(x')} \underline{w}_\Omega^*(x' + t\nu_j) \geq k$ , uniformly with respect to  $x' \in \overline{B}_j$ . This implies that  $\underline{w}_\Omega^*$  is a large solution.  $\square$

*Remark.* We conjecture that the equality  $\underline{w}_\Omega^* = \underline{w}_\Omega$  holds under the mere assumption that the Wiener criterion is satisfied.

**Theorem 3.7** *Assume  $\Omega$  is a bounded domain in  $\mathbb{R}^N$  such that  $\partial\Omega = \partial\overline{\Omega}^c$  and  $f \in C(\mathbb{R})$  satisfies (1.7), (1.8) and (2.1). Then there exists a exterior maximal solution  $\underline{u}_{Q_\Omega}^*$  to problem (1.1). Furthermore estimates (3.1) and (3.3) hold with  $\underline{w}_\Omega$  replaced by the exterior maximal solution  $\underline{u}_\Omega^*$  to problem (1.3) in  $\Omega$ .*

*Proof.* The construction of  $\underline{u}_{Q_\Omega}^*$  is similar to the one of  $\underline{w}_\Omega$ , since we can restrict to consider open neighborhoods  $Q_{1/n} = \Omega_{1/n} \times (-1/n, \infty)$ . Then  $\underline{u}_{Q_\Omega}^*$  is the increasing limit of the minimal large solutions  $u_n$  of (1.1) in  $Q_{1/n}$ , since  $\overline{Q_\infty}^\Omega = \bigcap_n Q_{1/n}$  and, by Lemma 3.3-(v),  $Q_\infty^\Omega = \overline{Q_\infty}^\Omega$ . We recall that the minimal large solution  $w_n$  of (1.3) in  $\Omega_{1/n}$  is the increasing limit, when  $k \rightarrow \infty$ , of the solution  $w_n^k$  of

$$\begin{cases} -\Delta w + f(w) = 0 & \text{in } \Omega_{1/n} \\ w = k & \text{on } \partial\Omega_{1/n}, \end{cases} \quad (3.13)$$

while the minimal large solution  $u_n$  of (1.1) in  $Q_{1/n}$  is the (always increasing) limit of the solution  $u_n^k$  of

$$\begin{cases} u_t - \Delta u + f(u) = 0 & \text{in } Q_{1/n} \\ u = k & \text{on } \partial_p Q_{1/n}. \end{cases} \quad (3.14)$$

Clearly

$$\max\{w_n^k, \bar{\phi}(\cdot + 1/n)\} \leq u_n(x, t),$$

which implies (3.1). For the other inequality, we see that  $(x, t) \mapsto w_n^k(x) + \bar{\phi}(t) + Lt$  is a supersolution which dominates  $u_n^k$  on  $\partial p$ , where  $L$  corresponds to the minimum of  $w_n^k$  in  $\Omega_{1/n}Q_{1/n}$ . Thus

$$u_n(x, t) \leq w_n^k + \bar{\phi}(\cdot + 1/n),$$

which implies

$$\max\{\underline{w}_\Omega^*(x), \bar{\phi}(t)\} \leq \underline{w}_\Omega^*(x, t) \quad \forall (x, t) \in Q_\infty^\Omega. \quad (3.15)$$

The upper estimate is proved in the following way. If  $k > n$ ,  $\bar{Q}_k \subset Q_n$ . Therefore, choosing  $m$  such that  $\min\{\min\{\underline{w}_{\Omega_{1/k}}(x) : x \in \Omega_{1/k}\}, \min\{\bar{\phi}(t + 1/k) : t \in (0, T]\}\} \geq m$ , we obtain that  $(x, t) \mapsto \underline{w}_{\Omega_{1/k}}(x) + \bar{\phi}(t + 1/k) + Lt$  is a super solution of (1.1) in  $Q_T^{\Omega_{1/k}}$ , thus it dominates the minimal large solution of (1.1) in  $Q_T^{\Omega_{1/n}}$ . Letting successively  $k \rightarrow \infty$  and  $n \rightarrow \infty$ , yields to

$$\underline{w}_\Omega^*(x, t) \leq \underline{w}_\Omega^*(x) + \bar{\phi}(t) \quad \forall (x, t) \in Q_T^\Omega. \quad (3.16)$$

□

The next result extends Corollary 3.2 without the boundary Wiener regularity assumption.

**Theorem 3.8** *Let  $\Omega$  be a bounded domain in  $\mathbb{R}^N$  such that  $\partial\Omega = \partial\bar{\Omega}^c$ . If  $f \in C(\mathbb{R})$  is convex and satisfies (1.7), (1.8), (2.1) and (3.8). Then, if  $\underline{w}_\Omega^\#$  is a large solution, the following implication holds*

$$\underline{w}_\Omega^\# = \bar{w}_\Omega \implies \underline{u}_{Q^\Omega}^\# = \bar{u}_{Q^\Omega}. \quad (3.17)$$

*Proof.* If  $\underline{w}_\Omega^\#$  is a large solution, the same is true for  $\underline{u}_{Q^\Omega}^\#$  because of (3.1). Actually  $\underline{u}_{Q^\Omega}^\#$  is the minimal large solution in  $Q_\infty^\Omega$  for the same reasons as  $\underline{w}_\Omega^\#$ . Therefore the proof of Corollary 3.2 applies and it implies the result. □

*Remark.* We conjecture that (3.17) holds, even if  $\underline{w}_\Omega^\#$  is not a large solution.

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