The quadratic Fock functor

Luigi Accardi, Ameur Dhahri

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Luigi Accardi & Ameur Dhahri

Volterra Center, University of Roma Tor Vergata
Via Columbia 2, 00133 Roma, Italy
e-mail:accardi@volterra.uniroma2.it
ameur@volterra.uniroma2.it

Abstract

We construct the quadratic analogue of the boson Fock functor. While in the first order (linear) case all contractions on the 1–particle space can be second quantized, the semigroup of contractions that admit a quadratic second quantization is much smaller due to the nonlinearity. The encouraging fact is that it contains, as proper subgroups (i.e. the contractions), all the gauge transformations of second kind and all the a.e. invertible maps of $\mathbb{R}^d$ into itself leaving the Lebesgue measure quasi-invariant (in particular **all diffeomorphism of** $\mathbb{R}^d$). This allows quadratic 2-d quantization of gauge theories, of representations of the Witt group (in fact it continuous analogue), of the Zamolodchikov hierarchy, and much more.... Within this semigroup we characterize the unitary and the isometric elements and we single out a class of natural contractions.

1 Introduction

The boson (this specification will be omitted in the following) Fock functor has its origins in Heisenberg commutation relations. If H is a complex Hilbert space the Heisenberg $*$–Lie algebra $Heis(H)$ is defined by generators.

$$\{A_g, A_f^+, 1 \text{ (central element)} \ : \ f \in H\}$$

commutation relations

$$[A_f, A_g^+] = \langle f, g\rangle \cdot 1 \qquad ; \qquad f, g \in H$$

(the omitted commutation relations are zero) and involution

$$(A_f)^* = A_f^+ \qquad ; \qquad f \in H$$

On the universal enveloping algebra of $Heis(H)$, denoted $U(Heis(H))$, there is a unique state satisfying

$$\varphi(1) = 1$$

$$\varphi(xA_g) = 0 \qquad ; \qquad \forall x \in U(Heis(H)) \; ; \; \forall g \in H$$

Denoting $\Gamma(H)$ the GNS space of $U(Heis(H))$ with respect to φ, the map $H \mapsto \Gamma(H)$ is a functor defined on the category of Hilbert spaces, with morphisms given by contractions to the category of infinite dimensional Hilbert spaces with the same morphisms.
$\Gamma(H)$ is called the Fock space over H and, if V is a contraction on H its image $\Gamma(V)$ is called the Fock second quantization of V.
The domain of Γ is maximal in the sense that, if V is not a contraction on H, then $\Gamma(V)$ cannot be a bounded operator on $\Gamma(H)$.

Our goal in this paper is to extend the picture described above, from the Heisenberg algebra, describing the white noise commutation relations, to the algebra describing the commutation relations of the renormalized square of white noise.
The algebra of the renormalized square of white noise (RSWN) with test function algebra

$$\mathcal{A} := L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$$

is the $*$-Lie-algebra, with central element denoted 1, generators

$$\{B_f^+, B_h, N_g \; : \; f, g, h \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)\}$$

involution

$$(B_f^+)^* = B_f \qquad , \qquad N_f^* = N_{\bar{f}}$$

and commutation relations

$$[B_f, B_g^+] = 2c\langle f, g\rangle + 4N_{\bar{f}g}, \;\; [N_a, B_f^+] = 2B_{af}^+, \;\; c > 0$$

$$[B_f^+, B_g^+] = [B_f, B_g] = [N_a, N_{a'}] = 0$$

for all a, a', f, $g \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ (the theory can be developed for more general Hilbert algebras, but we will deal only with this case). This is a current algebra over $sl(2, \mathbb{R})$ with test function algebra $\mathcal{A}$. One can prove

that, on the universal enveloping algebra $U(RSWN)$ of the $RSWN$ algebra, there exists a unique state φ_F such that

$$\varphi_F(1) = 1$$

$$\varphi_F(xB_g) = \varphi_F(xN_f) = 0 \qquad ; \qquad \forall f, g \in \mathcal{A} \; ; \; \forall x \in U(RSWN)$$

By analogy with the Heisenberg algebra, it is natural to call this state *the quadratic Fock state* and the associated GNS space, denoted $\Gamma_2(\mathcal{A})$, *the quadratic Fock space*. The Fock representation of the RSWN is characterized by a cyclic vector Φ, also called *vacuum* as in the first order case, satisfying

$$B_f\Phi = N_g\Phi = 0$$

for all $f, g \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$.
We refer the interested reader to [4], [5] for more details.
The extensions, to the quadratic case, of the second quantization procedure for linear operators on $\mathcal{A}$ requires the solution of the following two problems:
(1) *when does a linear operator on $\mathcal{A}$ induce a linear operator on $\Gamma_2(\mathcal{A})$*?
(2) In the cases in which the answer to problem (1) is positive, when is the induced operator bounded (a contraction, unitary, isometric, ...)?
By inspection on the explicit form of the scalar product of the quadratic Fock space (see Lemma 2 below) one is led to conjecture that two classes of linear transformations of $\mathcal{A}$ should induce contractions on $\Gamma_2(\mathcal{A})$:

(i) $*$–endomorphisms of the Hilbert algebra $\mathcal{A}$

(ii) generalized gauge transformations of the form

$$f \mapsto e^\alpha f \qquad ; \qquad e^\alpha f(x) := e^{\alpha(x)} f(x) \; ; \; x \in \mathbb{R}^d$$

where $\alpha \in \mathbb{R}^d \to \mathbb{C}$ is a complex valued Borel function with negative real part (the $-\infty$ value is allowed to include functions with non full support).

One of our main results is that these are essentially all the linear operators on $\mathcal{A}$ which admit a contractive second quantization on the quadratic Fock space.
The scheme of the present paper is the following. In section 2, we recall some properties on the quadratic exponential vectors. Moreover, we prove that the quadratic Fock space is an interacting Fock space with scalar product

given explicitly. In section 3, we characterize those operator on the one–particle Hilbert algebra whose quadratic second quantization is isometric (resp. unitary). In section 4, we show with a counter–example that even very simple contractions have a second quantization that is not a contraction and we give a sufficient condition for this to happen. We also introduce the natural candidates for the role of quadratic analogue of the free Hamiltonian evolution and of the Ornstein–Uhlenbeck semigroup.

2 The quadratic Fock space

For $n \in \mathbb{N}$ the quadratic n–particle space is the closed linear span of the set

$$\{B_f^{+n}\Phi \ : \ f \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)\}$$

where by definition $B_f^{+0}\Phi = \Phi$, for all $f \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. The quadratic Fock space $\Gamma_2(L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d))$ is the orthogonal sum of all the quadratic n–particle spaces. The quadratic exponential vector with test function $f \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, if it exists, is defined by

$$\Psi(f) = \sum_{n \geq 0} \frac{B_f^{+n}\Phi}{n!} \tag{1}$$

where by definition

$$\Psi(0) = B_f^{+0}\Phi = \Phi \tag{2}$$

The following theorem was proved in [2].

Theorem 1 *The quadratic exponential vector $\Psi(f)$ exists if and only if $\|f\|_\infty < \frac{1}{2}$. The set of these vectors is linearly independent and total in $\Gamma_2(L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d))$. Furthermore, the scalar product between two exponential vectors, $\Psi(f)$ and $\Psi(g)$, is given by*

$$\langle \Psi(f), \Psi(g) \rangle = e^{-\frac{c}{2}\int_{\mathbb{R}^d} \ln(1-4\bar{f}(s)g(s))ds} \tag{3}$$

The explicit form of the scalar product between two quadratic n–particle vectors is due to Barhoumi, Ouerdiane, Riahi [6]. Its proof, which we include for completeness, one needs the following preliminary result which uses the

identity, proved in Proposition 1 of [2]. This identity will be frequently used in the following:

$$\begin{aligned}
||B_f^{+m}\Phi||^2 &= c\sum_{k=0}^{m-1} 2^{2k+1}\frac{m!(m-1)!}{((m-k-1)!)^2}|||f^{k+1}||_2^2\|B_f^{+(m-k-1)}\Phi\|^2 \\
&= c\sum_{k=1}^{m-1} 2^{2k+1}\frac{m!(m-1)!}{((m-k-1)!)^2}|||f^{k+1}||_2^2\|B_f^{+(m-k-1)}\Phi\|^2 \\
&\quad +2mc\|f\|_2^2\|B_f^{+(m-1)}\Phi\|^2 \\
&= c\sum_{k=0}^{m-2} 2^{2k+3}\frac{m!(m-1)!}{(((m-1)-k-1)!)^2}\|f^{k+2}\|_2^2\|B_f^{+((m-1)-k-1)}\Phi\|^2 \\
&\quad +2mc\|f\|_2^2\|B_f^{+(m-1)}\Phi\|^2 \qquad (4)
\end{aligned}$$

Lemma 1 *For all f, $g \in L^2(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$ such that $\|f\|_\infty < \frac{1}{2}$, $\|g\|_\infty < \frac{1}{2}$, one has*

$$\langle B_f^{+n}\Phi, B_g^{+n}\Phi\rangle = n!\frac{d^n}{dt^n}\Big|_{t=0}\langle\Psi(tf),\Psi(g)\rangle \qquad (5)$$

Proof. Let f, $g \in L^2(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$ such that $\|f\|_\infty < \frac{1}{2}$, $\|g\|_\infty < \frac{1}{2}$. For all $0 \le t \le 1$, one has

$$\langle\Psi(tf),\Psi(g)\rangle = \sum_{m\ge 0}\frac{t^m}{(m!)^2}\langle B_f^{+m}\Phi, B_g^{+m}\Phi\rangle$$

We now prove that, for $0 \le t \le 1$, the above series can be differentiated (in t) term by term. For all $m \ge n$, one has

$$\begin{aligned}
\frac{d^n}{dt^n}\Big(\frac{t^m}{(m!)^2}\langle B_f^{+m}\Phi, B_g^{+m}\Phi\rangle\Big) &= \frac{m!t^{m-n}}{(m!)^2(m-n)!}\langle B_f^{+m}\Phi, B_g^{+m}\Phi\rangle \\
&= \frac{t^{m-n}}{m!(m-n)!}\langle B_f^{+m}\Phi, B_g^{+m}\Phi\rangle
\end{aligned}$$

So that, for $0 \le t \le 1$

$$\Big|\frac{d^n}{dt^n}\Big(\frac{t^m}{(m!)^2}\langle B_f^{+m}\Phi, B_g^{+m}\Phi\rangle\Big)\Big| \le U_m := \frac{1}{m!(m-n)!}\|B_f^{+m}\Phi\|\|B_g^{+m}\Phi\|$$

From the identity (4) it follows that

$$c\sum_{k=0}^{m-2} 2^{2k+3}\frac{m!(m-1)!}{(((m-1)-k-1)!)^2}\|f^{k+2}\|_2^2\|B_f^{+((m-1)-k-1)}\Phi\|^2$$
$$\leq \Big(4m(m-1)\|f\|_\infty^2\Big)\Big[c\sum_{k=0}^{m-2} 2^{2k+1}\frac{(m-1)!(m-2)!}{(((m-1)-k-1)!)^2}\|f^{k+1}\|_2^2$$
$$\|B_f^{+((m-1)-k-1)}\Phi\|^2\Big] = \Big(4m(m-1)\|f\|_\infty^2\Big)\|B_f^{+m}\Phi\|^2$$

In conclusion

$$||B_f^{+m}\Phi||^2 \leq \Big[4m(m-1)\|f\|_\infty^2 + 2m\|f\|^2\Big]\|B_f^{+(m-1)}\Phi\|^2$$

Therefore

$$\begin{aligned}\|B_f^{+m}\Phi\|\|B_g^{+m}\Phi\| \;\leq\; & \sqrt{4m(m-1)\|f\|_\infty^2 + 2m\|f\|_2^2} \\ & \sqrt{4m(m-1)\|g\|_\infty^2 + 2m\|g\|_2^2}\|B_f^{+(m-1)}\Phi\|\|B_g^{+(m-1)}\Phi\|\end{aligned}$$

The definition of U_m then implies that

$$U_m \leq \frac{\sqrt{4m(m-1)\|f\|_\infty^2 + 2m\|f\|_2^2}\sqrt{4m(m-1)\|g\|_\infty^2 + 2m\|g\|_2^2}}{m(m-n)}\; U_{m-1}$$

If f and g are non-vanishing functions, then

$$\lim_{m\to\infty}\frac{U_m}{U_{m-1}} \leq 4\|f\|_\infty\|g\|_\infty < 1$$

because $\|f\|_\infty < \frac{1}{2}$, $\|g\|_\infty < \frac{1}{2}$. Hence, the series $\sum_m U_m$ converges. This implies that

$$\frac{d^n}{dt^n}\langle\Psi(tf),\Psi(g)\rangle = \sum_{m\geq n}\frac{t^{m-n}}{m!(m-n)!}\langle B_f^{+m}\Phi, B_g^{+m}\Phi\rangle$$

Evaluating the derivative at $t=0$, one obtains (5). □

Lemma 2 *For all f, $g \in L^2(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$ the following identity holds*

$$\langle B_f^{+n}\Phi, B_g^{+n}\Phi\rangle = \sum_{i_1+2i_2+\ldots+ki_k=n}\frac{(n!)^2 2^{2n-1}c^{i_1+\ldots+i_k}}{i_1!\ldots i_k!2^{i_2}\ldots k^{i_k}}\langle f,g\rangle^{i_1}\langle f^2,g^2\rangle^{i_2}\ldots\langle f^k,g^k\rangle^{i_k} \quad (6)$$

Proof. The complex linearity of the map $f \mapsto B_f^+$ implies that, for all $\lambda_1,\ \lambda_2 \in \mathbb{C}$,

$$\langle B^{+n}_{\lambda_1 f}\Phi, B^{+n}_{\lambda_2 g}\Phi\rangle = \bar{\lambda}_1^n \lambda_2^n \langle B_f^{+n}\Phi, B_g^{+n}\Phi\rangle$$

Therefore it will be sufficient to prove the identity (6) for all $f,\ g \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ such that $\|f\|_\infty, \|g\|_\infty < \frac{1}{2}$. In this case one has

$$\begin{aligned}\langle B_f^{+n}\Phi, B_g^{+n}\Phi\rangle &= n!\frac{d^n}{dt^n}\Big|_{t=0}\langle \Psi(tf), \Psi(g)\rangle \\ &= n!\frac{d^n}{dt^n}\Big|_{t=0}\Big(\exp\big(-\langle \log(1-4t\bar{f}g)\rangle\big)\Big) \qquad (7)\end{aligned}$$

where

$$\langle \log(1-4t\bar{f}g)\rangle := \frac{c}{2}\int_{\mathbb{R}^d} \log\left(1-4t\bar{f}(s)g(s)\right)ds$$

Denoting $h(t,s) := \log\left(1-4t\bar{f}(s)g(s)\right)$, its k–th derivative (in t) is

$$h^{(k)}(t,s) = 2^{2k}(k-1)!(\bar{f}(s))^k(g(s))^k(1-4t\bar{f}(s)g(s))^{-k}$$

Hence, uniformly for $t \leq 1$

$$|h^{(k)}(t,s)| \leq \frac{2^{2k}(k-1)!|f(s)|^k|g(s)|^k}{(1-4\|f\|_\infty\|g\|_\infty)^k} \qquad (8)$$

Thus, the left hand side of (8) is integrable in s and

$$\langle h^{(k)}(t)\rangle = 2^{2k}(k-1)!\int_{\mathbb{R}^d}\frac{(\bar{f}(s))^k(g(s))^k}{(1-4t\bar{f}(s)g(s))^k}ds$$

Putting $t=0$ one finds

$$\langle h^{(k)}(0)\rangle = 2^{2k}(k-1)!\langle f^k, g^k\rangle \qquad (9)$$

Combining the identity (cf. Refs [6], [7])

$$\frac{d^n}{dt^n}e^{\varphi(t)} = \sum_{i_1+2i_2+\ldots+ki_k=n}\frac{2^{2n}n!}{i_1!\ldots i_k!}\Big(\frac{\varphi^{(1)}(t)}{1!}\Big)^{i_1}\cdots\Big(\frac{\varphi^{(k)}(t)}{k!}\Big)^{i_k}e^{\varphi(t)} \qquad (10)$$

with (7), (9) and (10) one obtains

$$\begin{aligned}\langle B_f^{+n}\Phi, B_g^{+n}\Phi\rangle &= n!\frac{d^n}{dt^n}\Big|_{t=0}\langle \Psi(tf), \Psi(g)\rangle \\ &= \sum_{i_1+2i_2+\ldots+ki_k=n}\frac{n!2^{2n-1}n!c^{i_1+\ldots+i_k}}{i_1!\ldots i_k!2^{i_2}\ldots k^{i_k}}\langle f,g\rangle^{i_1}\langle f^2,g^2\rangle^{i_2}\cdots\langle f^k,g^k\rangle^{i_k}\end{aligned}$$

from which (6) follows. □

The following theorem is an immediate consequence of Lemma 2.

Theorem 2 *There is a natural ismorphism between the quadratic Fock space* $\Gamma_2(L^2(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d))$ *and the interacting Fock space* $\oplus_{n=0}^\infty \otimes_{symm}^n \{L^2(\mathbb{R}^d), \langle\cdot,\cdot\rangle_n\}$, *with scalar products:*

$$\langle f^{\otimes n}, g^{\otimes n}\rangle_n = \sum_{i_1+2i_2+\ldots+ki_k=n} \frac{2^{2n-1}(n!)^2 c^{i_1+\ldots+i_k}}{i_1!\ldots i_k! 2^{i_2}\ldots k^{i_k}} \langle f, g\rangle^{i_1}\langle f^2, g^2\rangle^{i_2}\ldots\langle f^k, g^k\rangle^{i_k}$$

3 Quadratic second quantization of contractions

Let T be a linear operator on $L^2(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$. If the map

$$\Psi(f) \mapsto \Psi(Tf) \tag{11}$$

is well defined for all quadratic exponential vectors then, by the linear independence of these vectors, it admits a linear extension to a dense subspace of $\Gamma_2(L^2(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d))$, denoted $\Gamma_2(T)$ and called *the quadratic second quantization of* T.
From (2) and (11) it follows that, if $\Gamma_2(T)$ exists then, whatever T is, it leaves the quadratic vacuum invariant:

$$\Gamma_2(T)\Phi = \Phi$$

Lemma 3 *Let* T *be a linear operator on* $L^2(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$. *Then* $\Gamma_2(T)$ *is well defined on the set of all the exponential vectors if and only if* T *is a contraction on* $L^2(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$ *equipped with the norm* $\|.\|_\infty$.

Proof. Sufficiency. If $T : L^\infty(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d)$ is a contraction, then $\|Tf\|_\infty \leq \|f\|_\infty < 1/2$ for any test function $f \in L^2(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$ such that $\|f\|_\infty < 1/2$. Therefore $\Gamma_2(T)\Psi(f)$ is well defined.
Necessity. If $\Gamma_2(T)$ is well defined, then one has $\|Tg\|_\infty < \frac{1}{2}$, for any $g \in L^2(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$ such that $\|g\|_\infty < \frac{1}{2}$. By linearity T maps the open unit $\|.\|_\infty$–ball of $L^2(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$ into itself, i.e. it is a contraction. □

3.1 Isometric and unitarity characterization of the quadratic second quantization

Let us start by giving a sufficient condition on T, which ensures that $\Gamma_2(T)$ is an isometry (resp. unitary operator).

A Hilbert algebra endomorphism (resp. automorphism) T of $L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ is said to be a *$*$-endomorphism* (resp. *$*$-automorphism*) if T is an isometry (resp. a unitary operator) with respect to the pre-Hilbert structure of $L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, which satisfies

$$T(fg) = T(f)T(g), \;\; (T(f))^* = T(\bar{f}).$$

The following proposition is an immediate consequence of Lemma 2.

Proposition 1 *If $\alpha : \mathbb{R}^d \to \mathbb{R}$ is a Borel function, T_1 is a $*$-endomorphism of $L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ and*

$$T := e^{i\alpha} T_1$$

then $\Gamma_2(T)$ is an isometry. Moreover, if T_1 is a $*$-automorphism of $L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, then $\Gamma_2(T)$ is unitary.

Proof. To prove that $\Gamma_2(T)$ is an isometry it is sufficient to prove that it preserves the scalar product of two arbitray quadratic exponential vectors. From (1) and the mutual orthogonality of different n–particle spaces, it will be sufficient to prove that, for each $n \in \mathbb{N}$ and $f, g \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ one has:

$$\langle B^{+n}_{Tf}\Phi, B^{+n}_{Tg}\Phi\rangle = \langle B^{+n}_{f}\Phi, B^{+n}_{g}\Phi\rangle$$

and, because of Lemma 2, this identity follows from

$$\langle (Tf)^k, (Tg)^k\rangle = \langle f^k, g^k\rangle \qquad ; \qquad \forall k \in \mathbb{N} \qquad ; \qquad \forall f, g \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$$

But this identity holds because our assumptions on T imply that

$$\langle (Tf)^k, (Tg)^k\rangle = \langle e^{ik\alpha}(T_1 f)^k, e^{ik\alpha}(T_1 g)^k\rangle = \langle T_1(f^k), T_1(g^k)\rangle = \langle f^k, g^k\rangle$$

Thus $\Gamma_2(T)$ is an isometry. If, in addition, T_1 is a $*$-automorphism of $L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, then T is surjective. Hence the range of $\Gamma_2(T)$, containing all the quadratic exponential vectors, is the whole quadratic Fock space. The thesis then follows because an isometry with full range is unitary. □

In the following our goal is to prove the converse of the above proposition.

Lemma 4 *i) If $\Gamma_2(T)$ is a unitary operator, then*

$$\langle (Tf)^n, (Tg)^n \rangle = \langle f^n, g^n \rangle \tag{12}$$

for all $n \in \mathbb{N}^*$ and $f,\, g \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$.

ii) If $\Gamma_2(T)$ is an isometry, then for all $n \in \mathbb{N}^*$ and $f \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$

$$\|(Tf)^n\|_2 = \|f^n\|_2$$

Proof. Suppose that $\Gamma_2(T)$ is a unitary operator. Let us fix two functions $f,\, g \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ such that $\|f\|_\infty < \frac{1}{2}$, $\|g\|_\infty < \frac{1}{2}$. Then, one has

$$\langle \Psi(Tf), \Psi(Tg) \rangle = \langle \Psi(f), \Psi(g) \rangle$$

It follows that

$$\langle \Psi(tTf), \Psi(Tg) \rangle = \langle \Psi(tf), \Psi(g) \rangle$$

for all t such that $|t| < 1$. Therefore, Lemma 1 implies that

$$\langle B_{Tf}^{+n}\Phi, B_{Tg}^{+n}\Phi \rangle = \langle B_f^{+n}\Phi, B_g^{+n}\Phi \rangle \tag{13}$$

for all $n \in \mathbb{N}$. Let us prove the statement i) by induction.

- For $n = 1$, we have

$$\langle B_{Tf}^{+}\Phi, B_{Tg}^{+}\Phi \rangle = \langle B_f^{+}\Phi, B_g^{+}\Phi \rangle$$

This gives

$$\langle Tf, Tg \rangle = \langle f, g \rangle$$

- Suppose that (12) holds for $k \leq n$. Then, from (13) and the identity (4), one obtains

$$\begin{aligned}
&\langle B_{Tf}^{+(n+1)}\Phi, B_{Tg}^{+(n+1)}\Phi \rangle \\
&= c \sum_{k=0}^{n} 2^{2k+1} \frac{n!(n+1)!}{((n-k)!)^2} \langle (Tf)^{k+1}, (Tg)^{k+1} \rangle \langle B_{Tf}^{+(n-k)}\Phi, B_{Tg}^{+(n-k)}\Phi \rangle \\
&= 2^{2n+1} n!(n+1)! c \, \langle (Tf)^{n+1}, (Tg)^{n+1} \rangle \\
&\quad + c \sum_{k=0}^{n-1} 2^{2k+1} \frac{n!(n+1)!}{((n-k)!)^2} \langle (Tf)^{k+1}, (Tg)^{k+1} \rangle \langle B_{Tf}^{+(n-k)}\Phi, B_{Tg}^{+(n-k)}\Phi \rangle \\
&= 2^{2n+1} n!(n+1)! c \, \langle f^{n+1}, g^{n+1} \rangle \\
&\quad + c \sum_{k=0}^{n-1} 2^{2k+1} \frac{n!(n+1)!}{((n-k)!)^2} \, \langle f^{k+1}, g^{k+1} \rangle \langle B_{f}^{+(n-k)}\Phi, B_{g}^{+(n-k)}\Phi \rangle
\end{aligned}$$

By the induction assumption, one has

$$c\sum_{k=0}^{n-1} 2^{2k+1}\frac{n!(n+1)!}{((n-k)!)^2}\,\langle (Tf)^{k+1},(Tg)^{k+1}\rangle\langle B_{Tf}^{+(n-k)}\Phi, B_{Tg}^{+(n-k)}\Phi\rangle$$
$$= c\sum_{k=0}^{n-1} 2^{2k+1}\frac{n!(n+1)!}{((n-k)!)^2}\,\langle f^{k+1},g^{k+1}\rangle\langle B_{f}^{+(n-k)}\Phi, B_{g}^{+(n-k)}\Phi\rangle$$

which implies that

$$\langle (Tf)^{n+1},(Tg)^{n+1}\rangle = \langle f^{n+1},g^{n+1}\rangle \qquad ; \qquad \forall n\in\mathbb{N}^*$$

Thus (12) holds for all $n\in\mathbb{N}^*$.
The proof of statement ii) is obtained by replacing, in the above argument, the test function g by f. □

Lemma 5 *Suppose that $\Gamma_2(T)$ is an isometry. Then, for any $I\subset\mathbb{R}^d$ such that $|I|<\infty$, one has*

$$|T(\chi_I)(x)| = 1$$

on supp$(T(\chi_I))$ a.e.

Proof. By assumption $\Gamma_2(T)$ is an isometry, hence from Lemma 4, $\forall n\in\mathbb{N}$:

$$\langle (T(\chi_I))^n,(T(\chi_I))^n\rangle = \langle (\chi_I)^n,(\chi_I)^n\rangle = \langle \chi_I,\chi_I\rangle = |I| \tag{14}$$

for any subset $I\subset\mathbb{R}^d$ such that $|I|<\infty$. But, one has

$$\langle (T(\chi_I))^n,(T(\chi_I))^n\rangle = |\{x\in\mathbb{R}^d,\ |T(\chi_I)(x)|=1\}| + \int_J |T(\chi_I)(x)|^{2n}dx \tag{15}$$

where $|\ \cdot\ |$ denotes Lebesgue measure and

$$J := \{x\in\mathbb{R}^d,\ |T(\chi_I)(x)|\neq 1 \ \text{ and } \ |T(\chi_I)(x)|>0\}$$

Since the identity (15) holds $\forall n\in\mathbb{N}$, it follows that

$$\int_J |T(\chi_I(x)|^{2n}dx = \int_J |T(\chi_I(x)|^{2(n+1)}dx \qquad ; \qquad \forall n\in\mathbb{N}$$

But it is not difficult to prove that this is impossible if $|J|>0$. □

Lemma 6 *If $I \subset \mathbb{R}^d$ such that $|I| < \infty$ and $\Gamma_2(T)$ is an isometry, then there exist a function $\alpha_I : \mathbb{R}^d \to \mathbb{R}$ and a subset $\tau(I) \subset \mathbb{R}^d$ such that*

$$T(\chi_I) = e^{i\alpha_I}\chi_{\tau(I)}$$

and $|I| = |\tau(I)|$. Moreover, if I_1, I_2 is an arbitrary partition of I, then

$$\tau(I) = \tau(I_1) \cup \tau(I_2) \qquad , \qquad a.e. \tag{16}$$

In particular, if $I_1 \subset I$, then a.e. $\tau(I_1) \subset \tau(I)$.

Proof. Lemma 5 implies that there exist a function $\alpha_I : \mathbb{R}^d \to \mathbb{R}$ and a subset $\tau(I) \subset \mathbb{R}^d$ such that $T(\chi_I) = e^{i\alpha_I}\chi_{\tau(I)}$. From (14) one has

$$|\tau(I)| = \langle T(\chi_I), T(\chi_I)\rangle = \langle \chi_I, \chi_I\rangle = |I|$$

Let I_1, I_2 be a partition of I. From $\chi_I = \chi_{I_1 \cup I_2} = \chi_{I_1} + \chi_{I_2}$, it follows that

$$T(\chi_I) = T(\chi_{I_1}) + T(\chi_{I_2})$$

i.e.

$$e^{i\alpha_I}\chi_{\tau(I)} = e^{i\alpha_{I_1}}\chi_{\tau(I_1)} + e^{i\alpha_{I_2}}\chi_{\tau(I_2)}$$

Multiplying both sides by $\chi_{\tau(I_1)\cup\tau(I_2)}$, one finds

$$e^{i\alpha_I}\chi_{\tau(I)\cap[\tau(I_1)\cup\tau(I_2)]} = e^{i\alpha_{I_1}}\chi_{\tau(I_1)} + e^{i\alpha_{I_2}}\chi_{\tau(I_2)} = e^{i\alpha_I}\chi_{\tau(I)}$$

Therefore, one has $\tau(I) = \tau(I_1) \cup \tau(I_2)$ a.e. Since the partition I_1, I_2 of I is arbitrary, it follows that $I_1 \subset I$ implies that $\tau(I_1) \subset \tau(I)$. □

Lemma 7 *If $\Gamma_2(T)$ is an isometry and I_1, $I_2 \subset \mathbb{R}^d$ are such that $|I_1| < \infty$, $|I_2| < \infty$ and $|I_1 \cap I_2| = 0$, then $|\tau(I_1) \cap \tau(I_2)| = 0$.*

Proof. Suppose that $|I_1 \cap I_2| = 0$. Then, from the identity

$$\chi_{I_1 \cup I_2} = \chi_{I_1} + \chi_{I_2} - \chi_{I_1 \cap I_2}$$

it follows that, a.e.

$$\chi_{I_1 \cup I_2} = \chi_{I_1} + \chi_{I_2}$$

and therefore also

$$T(\chi_{I_1 \cup I_2}) = T(\chi_{I_1}) + T(\chi_{I_2}) \qquad ; \qquad a.e$$

Applying (14) one then gets

$$\begin{aligned}
|I_1| + |I_2| &= \langle \chi_{I_1 \cup I_2}, \chi_{I_1 \cup I_2} \rangle \\
&= \langle T(\chi_{I_1 \cup I_2}), T(\chi_{I_1 \cup I_2}) \rangle \\
&= \langle T(\chi_{I_1}), T(\chi_{I_1}) \rangle + \langle T(\chi_{I_2}), T(\chi_{I_2}) \rangle \\
&\quad + \langle T(\chi_{I_1}), T(\chi_{I_2}) \rangle + \langle T(\chi_{I_2}), T(\chi_{I_1}) \rangle \\
&= |I_1| + |I_2| + \int_{\tau(I_1) \cap \tau(I_2)} e^{i(\alpha_{I_2} - \alpha_{I_1})}(x) dx \\
&\quad + \int_{\tau(I_1) \cap \tau(I_2)} e^{-i(\alpha_{I_2} - \alpha_{I_1})}(x) dx \\
&= |I_1| + |I_2| + 2 \int_{\tau(I_1) \cap \tau(I_2)} cos((\alpha_{I_2} - \alpha_{I_1})(x)) dx \qquad (17)
\end{aligned}$$

which implies that

$$\int_{\tau(I_1) \cap \tau(I_2)} cos((\alpha_{I_2} - \alpha_{I_1})(x)) dx = 0 \qquad (18)$$

Put $I = I_1 \cup I_2$. From the identities

$$\begin{aligned}
e^{i\alpha_I} \chi_{\tau(I)} &= e^{i\alpha_{I_1}} \chi_{\tau(I_1)} + e^{i\alpha_{I_2}} \chi_{\tau(I_2)} \\
\tau(I) &= \tau(I_1) \cup \tau(I_2) \qquad a.e
\end{aligned}$$

it follows that if $x \in \tau(I_1) \cap \tau(I_2)$, then

$$e^{i\alpha_I}(x) = e^{i\alpha_{I_1}}(x) + e^{i\alpha_{I_2}}(x)$$

Thus, one obtains

$$e^{i(\alpha_I(x) - \alpha_{I_1}(x)} = 1 + e^{i(\alpha_{I_2}(x) - \alpha_{I_1}(x))}$$

This gives

$$1 = |1 + e^{i(\alpha_{I_2}(x) - \alpha_{I_1}(x))}|^2 = 2 + 2cos(\alpha_{I_2}(x) - \alpha_{I_1}(x))$$

which yields that

$$cos(\alpha_{I_2}(x) - \alpha_{I_1}(x)) = -\frac{1}{2}$$

This, together with (18) implies that $|\tau(I_1) \cap \tau(I_2)| = 0$. □

Lemma 8 *In the notations and assumptions of Lemma 6, for any $I \subset \mathbb{R}^d$ such that $|I| < \infty$ and any $I_1 \subset I$ one has*

$$e^{i\alpha_{I_1}}\chi_{\tau(I_1)} = e^{i\alpha_I}\chi_{\tau(I_1)}$$

for almost any $x \in \tau(I_1)$.

Proof. Let $I_2 = I \setminus I_1$. Arguing as in the proof of of Lemma 6 one finds that

$$e^{i\alpha_I}\chi_{\tau(I)} = e^{i\alpha_{I_1}}\chi_{\tau(I_1)} + e^{i\alpha_{I_2}}\chi_{\tau(I_2)}$$

Thus, if we multiply the two sides in the above identity by $\chi_{\tau(I_1)}$, then from Lemmas 6, 7, it follows that

$$e^{i\alpha_I}\chi_{\tau(I_1)} = e^{i\alpha_{I_1}}\chi_{\tau(I_1)} \qquad , \qquad a.e$$

□

Lemma 9 *In the notations and assumptions of Lemma 6 there exists a function $\alpha : \mathbb{R}^d \to \mathbb{R}$ such that for any $I \subset \mathbb{R}^d$, with $|I| < \infty$*

$$T(\chi_I) = e^{i\alpha}\chi_{\tau(I)}$$

where $\tau(I) \subset \mathbb{R}^d$ and $|\tau(I)| = |I|$.

Proof. Let $(I_n)_n$ be an increasing sequence of subsets of $\mathbb{R}^d$ such that $|I_n| < \infty$, $\forall n \in \mathbb{N}$ and $\bigcup_{n\in\mathbb{N}} I_n = \mathbb{R}^d$. Define the function $\alpha : \mathbb{R}^d \to \mathbb{R}$ by $\alpha(x) = \alpha_{I_n}(x)$, for any $n \in \mathbb{N}$ such that $x \in I_n$, where α_{I_n} is defined as in Lemma (6). Then α is well defined because, denoting

$$n(x) := \min\{n \in \mathbb{N},\, x \in I_n\} \qquad ; \qquad x \in \mathbb{R}^d$$

Lemma 8 implies that, for any m, $n \in \mathbb{N}$ such that $n(x) \leq m \leq n$,

$$e^{i\alpha_{I_m}}\chi_{\tau(I_m)} = e^{i\alpha_{I_n}}\chi_{\tau(I_m)}$$

In particular, for any $n \geq n(x)$, one has

$$e^{i\alpha_{I_n}}\chi_{\tau(I_{n(x)})} = e^{i\alpha_{I_{n(x)}}}\chi_{\tau(I_{n(x)})}$$

which implies that

$$\alpha_{I_n}(x) = \alpha_{I_{n(x)}},\ \forall n \geq n(x)$$

This ends the proof of the above lemma. □

Using all together Proposition 1, Lemmas 4, 6 and 9, we prove the following.

Theorem 3 *$\Gamma_2(T)$ is an isometry (resp. unitary) if and only if there exist a function α from $\mathbb{R}^d$ to $\mathbb{R}$ and a $*$-endomorphism (resp. $*$-automorphism) T_1 of $L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ such that*

$$T = e^{i\alpha} T_1$$

Proof. Sufficiency has been proved in Proposition 1.
Necessity. Suppose that $\Gamma_2(T)$ is an isometry. Then, from Lemma 4, T is an isometry. Moreover, Lemma 9 implies that there exists a function $\alpha : \mathbb{R}^d \to \mathbb{R}$ such that for any $I \subset \mathbb{R}^d$, $|I| < \infty$

$$T(\chi_I) = e^{i\alpha} \chi_{\tau(I)}$$

where $\tau(I) \subset \mathbb{R}^d$ and $|\tau(I)| = |I|$. Define the map T_1 by:

$$T_1 : \chi_I \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d) \to T_1(\chi_I) := \chi_{\tau(I)} \tag{19}$$

for all $I \subset \mathbb{R}^d$ such that $|I| < \infty$. In order to prove that T_1 extends, by linearity and continuity, to a $*$-endomorphism of $L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, it is sufficient to prove that for all $I, J \subset \mathbb{R}$ with $|I| < \infty$, $|J| < \infty$

$$T_1(\chi_I \chi_J) = T_1(\chi_I) T_1(\chi_J) = \chi_{\tau(I)} \chi_{\tau(J)} = \chi_{\tau(I) \cap \tau(J)} \qquad , \qquad a.e \tag{20}$$

But, by definition of T_1 one has

$$T_1(\chi_I \chi_J) = T_1(\chi_{I \cap J}) = \chi_{\tau(I \cap J)}$$

therefore our thesis is equivalent to

$$\tau(I) \cap \tau(I) = \tau(I \cap J) \qquad , \qquad a.e \tag{21}$$

Finally, since from Lemma 6 we know that $\tau(I \cap J) \subset \tau(I) \cap \tau(J)$, (21) will follow if we prove that

$$|\tau(I) \cap \tau(J)| = |\tau(I \cap J)| \tag{22}$$

To prove (22) notice that, since T, hence T_1, is an isometry, one has

$$\langle T_1(\chi_{I \cup J}), T_1(\chi_{I \cup J}) \rangle = \langle \chi_{I \cup J}, \chi_{I \cup J} \rangle = |I| + |J| - |I \cap J| \tag{23}$$

On the other hand, from Lemma 6 we know that the map $I \mapsto \tau(I)$ is finitely addditive, hence monotone. Therefore, using linearity, (19) and the identity $\chi_{I\cup J} = \chi_I + \chi_J - \chi_{I\cap J}$, we find

$$\begin{aligned}
\langle T_1(\chi_{I\cup J}), T_1(\chi_{I\cup J})\rangle &= \langle T_1(\chi_I) + T_1(\chi_J) - T_1(\chi_{I\cap J}), T_1(\chi_I) \\
&\qquad + T_1(\chi_J) - T_1(\chi_{I\cap J})\rangle \\
&= \langle T_1(\chi_I), T_1(\chi_I)\rangle + \langle T_1(\chi_I), T_1(\chi_J)\rangle \\
&\quad - \langle T_1(\chi_I), T_1(\chi_{I\cap J})\rangle + \langle T_1(\chi_J), T_1(\chi_I)\rangle \\
&\quad + \langle T_1(\chi_J), T_1(\chi_J)\rangle - \langle T_1(\chi_J), T_1(\chi_{I\cap J})\rangle \\
&\quad - \langle T_1(\chi_{I\cap J}), T_1(\chi_I)\rangle - \langle T_1(\chi_{I\cap J}), T_1(\chi_J)\rangle \\
&\quad + \langle T_1(\chi_{I\cap J}), T_1(\chi_{I\cap J})\rangle \\
&= \langle \chi_{\tau(I)}, \chi_{\tau(I)}\rangle + \langle \chi_{\tau(I)}, \chi_{\tau(J)}\rangle - \langle \chi_{\tau(I)}, \chi_{\tau(I\cap J)}\rangle \\
&\quad + \langle \chi_{\tau(J)}, \chi_{\tau(I)}\rangle + \langle \chi_{\tau(J)}, \chi_{\tau(J)}\rangle - \langle \chi_{\tau(J)}, \chi_{\tau(I\cap J)}\rangle \\
&\quad - \langle \chi_{\tau(I\cap J)}, \chi_{\tau(I)}\rangle - \langle \chi_{\tau(I\cap J)}, \chi_{\tau(J)}\rangle \\
&\quad + \langle \chi_{\tau(I\cap J)}, \chi_{\tau(I\cap J)}\rangle
\end{aligned}$$

Using the isometry property and the fact that $\tau(I\cap J) \subseteq \tau(I)\cap\tau(J)$, we see that this expression is equal to

$$\begin{aligned}
&|I| + |\tau(I)\cap\tau(J)| - |\tau(I\cap J)| + |\tau(I)\cap\tau(J)| + |J| - |\tau(I\cap J)| \\
&\quad - |\tau(I\cap J)| - |\tau(I\cap J)| + |\tau(I\cap J)| \\
&= |I| + |J| + 2|\tau(I)\cap\tau(J)| - 3|\tau(I\cap J)| \\
&= |I| + |J| + 2|\tau(I)\cap\tau(J)| - 3|I\cap J|
\end{aligned}$$

Since this is equal to the right hand side of (23), we conclude that

$$-|I\cap J| = 2|\tau(I)\cap\tau(J)| - 3|I\cap J| \Leftrightarrow |\tau(I)\cap\tau(J)| = |I\cap J| = |\tau(I\cap J)|$$

which is equivalent to (22) and therefore to (20).
Since a unitary operator is an isometry we conclude that, if $\Gamma_2(T)$ is unitary, then T_1, defined by (19), is an invertible $*$-endomorphism of $L^2(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$, i.e. a $*$-automorphism. □

4 Quadratic second quantization of contractions

We will use the following remark.
Remark Let $A = (a_{ij})_{i,j}$, $B = (b_{ij})_{i,j}$, $C = (c_{ij})_{i,j}$ and $D = (d_{ij})_{i,j}$ be

matrices such that, in the operator order:

$$0 \leq A \leq B, \text{ and } 0 \leq C \leq D$$

Then by Schur's Lemma

$$0 \leq ((b_{ij} - a_{ij})c_{ij})_{i,j} \Leftrightarrow (a_{ij}c_{ij})_{i,j} \leq (b_{ij}c_{ij})_{i,j}$$

$$0 \leq (b_{ij}(d_{ij} - c_{ij}))_{i,j} \Leftrightarrow (b_{ij}c_{ij})_{i,j} \leq (b_{ij}d_{ij})_{i,j}$$

Consequently one has

$$0 \leq (a_{ij}c_{ij})_{i,j} \leq (b_{ij}d_{ij})_{i,j} \tag{24}$$

Theorem 4 *The set of all operators T such that $\Gamma_2(T)$ is a contraction is a multiplicative semigroups denoted $Contr_2(L^2 \cap L^\infty)$. Moreover*

$$\Gamma_2(S)\Gamma_2(T) = \Gamma_2(ST) \qquad ; \ \forall S, T \in Contr_2(L^2 \cap L^\infty) \tag{25}$$

Proof. Let $S, T \in \mathrm{Contr}_2(L^2 \cap L^\infty)$. Then $\Gamma_2(S)$, $\Gamma_2(T)$ and hence $\Gamma_2(S)\Gamma_2(T)$ is a contraction on $\Gamma_2(L^2 \cap L^\infty)$. Therefore it is uniquely determined by its value on the quadratic exponential vectors. If $\Psi(f)$ is such a vector, then

$$\Gamma_2(S)\Gamma_2(T)\Psi(f) = \Gamma_2(S)\Psi(Tf) = \Psi(STf) = \Gamma_2(ST)\Psi(f)$$

Thus $\Gamma_2(ST)$ is a contraction and (25) holds. □

Now, we prove the following.

Proposition 2 *If $T = \mathcal{M}_\varphi T_1$ is a contraction for $L^2(\mathbb{R}^d)$ and $L^\infty(\mathbb{R}^d)$, where $\varphi \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ such that $\|\varphi\|_\infty \leq 1$ and T_1 is an homomorphism of $L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, then $\Gamma_2(T)$ is a contraction.*

Proof. We have

$$\begin{aligned} \|\Gamma_2(T)(\alpha_1\Psi(f_1) + \ldots + \alpha_l\Psi(f_l))\|^2 &= \|\alpha_1\Psi(Tf_1) + \ldots + \alpha_l\Psi(Tf_l)\|^2 \\ &= \sum_{i,j=1}^{l} \bar{\alpha}_i\alpha_j\langle\Psi(Tf_i), \Psi(Tf_j)\rangle \qquad (26) \\ &= \sum_{n\geq 0} \frac{1}{(n!)^2}\Big[\sum_{i,j=1}^{l} \bar{\alpha}_i\alpha_j\langle B^{+n}_{Tf_i}\Phi, B^{+n}_{Tf_j}\Phi\rangle\Big]. \end{aligned}$$

Put

$$A_{n,T} = \Big(\langle B_{Tf_i}^{+n}\Phi, B_{Tf_j}^{+n}\Phi\rangle\Big)_{i,j},\ A_n = \Big(\langle B_{f_i}^{+n}\Phi, B_{f_j}^{+n}\Phi\rangle\Big)_{i,j}.$$

Now, our purpose is to prove, under the assumptions of the above proposition, that

$$0 \leq A_{n,T} \leq A_n, \tag{27}$$

for all $n \in \mathbb{N}$.

Note that, for $v = (\alpha_1, \ldots, \alpha_l)$, one has

$$\begin{aligned}\langle v, A_{n,T}v\rangle &= \sum_{i,j=1}^{l} \bar{\alpha}_i\alpha_j\langle B_{Tf_i}^{+n}\Phi, B_{Tf_j}^{+n}\Phi\rangle \\ &= \|\alpha_1 B_{Tf_1}^{+n}\Phi + \ldots + \alpha_l B_{Tf_l}^{+n}\Phi\|^2.\end{aligned}$$

This implies that $A_{n,T}$ is a positive matrix. Now, let us prove the second inequality in (27) by induction on n.

- For $n = 1$, one has

$$\begin{aligned}\langle v, A_{1,T}v\rangle &= \sum_{i,j=1}^{l} \bar{\alpha}_i\alpha_j\langle B_{Tf_i}^{+}\Phi, B_{Tf_j}^{+}\Phi\rangle \\ &= 2c\sum_{i,j=1}^{l} \bar{\alpha}_i\alpha_j\langle Tf_i, Tf_j\rangle \\ &= 2c\|T(\alpha_1 f_1 + \ldots + \alpha_l f_l)\|_2^2 \\ &\leq 2c\|\alpha_1 f_1 + \ldots + \alpha_l f_l\|_2^2.\end{aligned}$$

Because

$$2c\|\alpha_1 f_1 + \ldots + \alpha_l f_l\|_2^2 = \sum_{i,j=1}^{l} \bar{\alpha}_i\alpha_j\langle B_{f_i}^{+}\Phi, B_{f_j}^{+}\Phi\rangle = \langle v, A_1 v\rangle,$$

one obtains that $A_{1,T} \leq A_1$.

- Let $n \geq 1$ and suppose that $A_{n,T} \leq A_n$. Note that for any $f, g \in L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, Proposition 1 of [2] implies that

$$\begin{aligned}\langle B_f^{+(n+1)}\Phi, B_g^{+(n+1)}\Phi\rangle &= c\sum_{k=0}^{n} 2^{2k+1}\frac{n!(n+1)!}{((n-k)!)^2}\,\langle f^{k+1}, g^{k+1}\rangle \\ &\quad \langle B_f^{+(n-k)}\Phi, B_g^{+(n-k)}\Phi\rangle.\end{aligned}$$

Then, one gets

$$\begin{aligned}\langle v, A_{n+1,T}v\rangle &= \sum_{i,j=1}^{l}\bar{\alpha}_i\alpha_j\langle B^{+(n+1)}_{Tf_i}\Phi, B^{+(n+1)}_{Tf_j}\Phi\rangle \\ &= c\sum_{k=0}^{n}2^{2k+1}\frac{(n+1)!n!}{((n-k)!)^2} \\ &\qquad \Big[\sum_{i,j=1}^{l}\bar{\alpha}_i\alpha_j\langle (Tf_i)^{k+1},(Tf_j)^{k+1}\rangle\langle B^{+(n-k)}_{Tf_i}\Phi, B^{+(n-k)}_{Tf_j}\Phi\rangle\Big].\end{aligned}$$

Put

$$M_k = (\langle f_i^{k+1}, f_j^{k+1}\rangle)_{i,j},\ M_{k,T} = (\langle (Tf_i)^{k+1},(Tf_j)^{k+1}\rangle)_{i,j}.$$

This gives

$$\begin{aligned}\langle v, M_{k,T}v\rangle &= \sum_{i,j=1}^{l}\bar{\alpha}_i\alpha_j\langle (Tf_i)^{k+1},(Tf_j)^{k+1}\rangle \\ &= \|\alpha_1(Tf_1)^{k+1}+\ldots+\alpha_l(Tf_l)^{k+1}\|_2^2 \\ &= \|\varphi^{k+1}T_1(\alpha_1 f_1^{k+1}+\ldots+\alpha_l f_l^{k+1})\|_2^2 \\ &\le \|\varphi\|_\infty^k\|T(\alpha_1 f_1^{k+1}+\ldots+\alpha_l f_l^{k+1})\|_2^2 \\ &\le \|\alpha_1 f_1^{k+1}+\ldots+\alpha_l f_l^{k+1}\|_2^2 = \langle v, M_k v\rangle.\end{aligned}$$

This proves that

$$0 \le M_{k,T} \le M_k. \tag{28}$$

Note that by induction assumption

$$0 \le A_{n-k,T} \le A_{n-k}, \tag{29}$$

for all $k = 0,\ldots,n$. Therefore, Lemma 24 and identies (28), (29) implies that

$$A_{n+1,T} \le A_{n+1}.$$

Hence, we have proved that

$$\langle v, A_{n,T}v\rangle = \sum_{i,j=1}^{l}\bar{\alpha}_i\alpha_j\langle B^{+n}_{Tf_i}\Phi, B^{+n}_{Tf_j}\Phi\rangle \le \langle v, A_n v\rangle = \sum_{i,j=1}^{l}\bar{\alpha}_i\alpha_j\langle B^{+n}_{f_i}\Phi, B^{+n}_{f_j}\Phi\rangle,$$

for all n. After using (26), it is easy to conclude that $\Gamma_2(T)$ is a contraction. □

Remark

The contractions considered in Proposition 2 are very special, however they are sufficient to prove the existence of the quadratic free Hamiltonian and the quadratic Ornstein–Uhlenbeck semigroup. In fact taking

$$T = T = e^{z1_{\mathcal{A}}}$$

with $Re(z) \leq 0$ where $\mathcal{A} = L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, Proposition 2 implies that $\Gamma_2(e^{z1_{\mathcal{A}}})$ is a holomorphic semigroup which, by the remark done at the beginning of section (3), leaves the vacuum vector Φ invariant. In particular, for $z = it$, $t \in \mathbb{R}$, the generator H_0 of the strongly continuous 1–parameter unitary group

$$\Gamma_2(e^{it1_{\mathcal{A}}}) = e^{itH_0}$$

is the quadratic analogue of the free Hamiltonian. By analytic continuation one has

$$\Gamma_2(e^{z1_{\mathcal{A}}}) = e^{zH_0}$$

Moreover Lemma 2 shows that its action on the n–particle space is the same as the action of the number operator in the usual Fock space, i.e. it is reduced to multiplication by

$$e^{zn}$$

Thus H_0 is the positive self–adjoint operator characterized by the property that, for any $n \in \mathbb{N}$, the n–particle space is the eigenspace of H_0 corresponding to the the eigenvalue n.

By considering the action of the *number operators* N_f, defined at the beginning of section (1), one easily verifies that the definition of N_f can be extended to the case in which f is a multiple of the identity function 1, so that N_1 is well defined. With this notation one has the identity

$$H_0 = \frac{1}{2} N_1$$

Using the functional realization of the quadratic Fock space given by Theorem 2 it is clear that the contraction semigroup

$$\Gamma_2(e^{-t1_{\mathcal{A}}}) = e^{-tH_0}$$

is positivity preserving and its explicit form gives that

$$\Gamma_2(e^{-t1_{\mathcal{A}}})1 = e^{-t}1 \leq 1$$

(here we are extending in the obvious way the action of $\Gamma_2(e^{-t1_{\mathcal{A}}})$ to the multiples of the identity function which is not in $L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$). This means that the semigroup e^{-tH_0} is sub–Markovian. The above discussion shows that e^{-tH_0} is a natural candidate for the role of quadratic analogue of the Ornstein–Uhlenbeck semigroup. A more detailed analysis of this semigroup and of its properties will be discussed elsewhere.

4.1 A counterexample

In this subsection, we discuss the behavior of contractions under quadratic second quantization.

Lemma 10 *Let T be a linear operator on $L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. If T is a contraction on $L^2(\mathbb{R}^d)$ and on $L^\infty(\mathbb{R}^d)$, then for any quadratic exponential vector $\Psi(f)$ one has*

$$\|\Gamma_2(T)\Psi(f)\| \leq \|\Psi(f)\| \tag{30}$$

Proof. Recall that

$$\|\Gamma_2(T)\Psi(f)\|^2 = \|\Psi(Tf)\|^2 = \sum_{n\geq 0} \frac{\|B_{Tf}^{+n}\Phi\|^2}{(n!)^2} \tag{31}$$

and that, because of Lemma 2:

$$\|B_{Tf}^{+n}\Phi\|^2 = \sum_{i_1+2i_2+\ldots+ki_k=n} \frac{2^{2n-1}(n!)^2 c^{i_1+\ldots+i_k}}{i_1!\ldots i_k! 2^{i_2}\ldots k^{i_k}} \|Tf\|_2^{i_1} \|(Tf)^2\|_2^{i_2} \ldots \|(Tf)^k\|_2^{i_k}$$

If T is a contraction on $L^2(\mathbb{R}^d)$ and on $L^\infty(\mathbb{R}^d)$ then by the Riesz–Thorin Theorem, for all $p \geq 2$, T is also a contraction from $L^p(\mathbb{R}^d)$ into itself. Therefore, for any $p \geq 1$ and $i \in \mathbb{N}$:

$$\|(Tf)^p\|_2^i = \left[\left(\int |Tf|^{2p}\right)^{1/2p}\right]^{pi} = \|Tf\|_{2p}^{pi} \leq 1$$

for all $j = 1, \ldots, k$. This proves that for any $n \in \mathbb{N}$

$$\|B_{Tf}^{+n}\Phi\|^2 \leq \|B_f^{+n}\Phi\|^2$$

and, in view of (31), this implies (30). □

From Lemma (10) it follows that the fact that T is a contraction for $L^2(\mathbb{R}^d)$ and for $L^\infty(\mathbb{R}^d)$ is a necessary condition for $\Gamma_2(T)$ to be a contraction. The following counterexample shows that this condition is not sufficient.

Define the linear operator $T : L^2(\mathbb{R}) \cap L^\infty(\mathbb{R}) \to L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ by

$$Tf = \Big(\int_0^1 f(t)dt \Big)\, \chi_{[0,1]}$$

It is easy to verify that T is a contraction in both L^2 and L^∞. Therefore, from Lemma 10, one has

$$\|\Gamma_2(T)\Psi(f)\| \leq \|\Psi(f)\|$$

In the following we will show that some linear combinations of quadratic exponential vectors violate the inequality

$$\|\Gamma_2(T)\Big(\sum_i \alpha_i \Psi(f_i) \Big)\| \leq \| \sum_i \alpha_i \Psi(f_i)\|$$

In fact taking

$$f_1 := \lambda \chi_{[0,\frac{1}{2}]} \quad ; \quad f_2 := \lambda \chi_{[0,1]} \quad ; \quad \lambda \in \mathbb{R} \quad ; \quad |\lambda| < \frac{1}{2}$$

one has

$$Tf_1 = \frac{\lambda}{2}\chi_{[0,1]} \qquad ; \qquad Tf_2 = \lambda \chi_{[0,1]}$$

and

$$\|\Gamma_2(T)\Big(\alpha_1\Psi(f_1) + \alpha_2\Psi(f_2)\Big)\|^2 = \langle \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}, B \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \rangle,$$
$$\|\alpha_1\Psi(f_1) + \alpha_2\Psi(f_2)\|^2 = \langle \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}, A \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \rangle$$

where the matrices A, B are defined by:

$$A := (\langle \Psi(f_i), \Psi(f_j) \rangle)_{1 \leq i,j \leq 2} \qquad ; \qquad B := (\langle \Psi(Tf_i), \Psi(Tf_j) \rangle)_{1 \leq i,j \leq 2}$$

The contraction condition

$$\|\Gamma_2(T)\Big(\alpha_1\Psi(f_1)+\alpha_2\Psi(f_2)\Big)\|^2 \leq \|\alpha_1\Psi(f_1)+\alpha_2\Psi(f_2)\|^2$$

is equivalent to say that $B \leq A$. In the following we prove that this inequality is not true. In fact recalling (3), i.e.

$$\langle\Psi(f),\Psi(g)\rangle = e^{-\frac{c}{2}\int_{\mathbb{R}}\ln(1-4\bar{f}(x)g(x))dx}$$

one finds

$$A=\begin{pmatrix} (\frac{1}{1-4\lambda^2})^{\frac{c}{4}} & (\frac{1}{1-4\lambda^2})^{\frac{c}{4}} \\ (\frac{1}{1-4\lambda^2})^{\frac{c}{4}} & (\frac{1}{1-4\lambda^2})^{\frac{c}{2}} \end{pmatrix},\ B=\begin{pmatrix} (\frac{1}{1-\lambda^2})^{\frac{c}{2}} & (\frac{1}{1-2\lambda})^{\frac{c}{2}} \\ (\frac{1}{1-2\lambda^2})^{\frac{c}{2}} & (\frac{1}{1-4\lambda^2})^{\frac{c}{2}} \end{pmatrix}$$

and a simple calculation proves that $det(A-B) \leq 0$.

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