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Generalization of a going-down theorem in the category of Chow-Grothendieck motives due to N. Karpenko

Charles De Clercq

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Abstract

Let $\mathbb{M} := (M(X), p)$ be a direct summand of the motive associated with a geometrically split, geometrically variety over a field F satisfying the nilpotence principle. We show that under some conditions on an extension E/F , if \mathbb{M} is a direct summand of another motive M over an extension E , then \mathbb{M} is a direct summand of M over F .

I Introduction

Let Λ be a finite commutative ring. Our main reference on the category $CM(F; \Lambda)$ of Chow-Grothendieck motives with coefficients in Λ is [1].

The purpose of this note is to generalize the following theorem due to N. Karpenko ([2], proposition 4.5). Throughout this paper we understand a F -variety over a field F as a separated scheme of finite type over F .

Theorem I.1. *Let Λ be a finite commutative ring. Let X be a geometrically split, geometrically irreducible F -variety satisfying the nilpotence principle. Let $M \in CM(F; \Lambda)$ be another motive. Suppose that an extension E/F satisfies*

1. *the E -motive $M(X)_E \in CM(E; \Lambda)$ of the E -variety X_E is indecomposable;*
2. *the extension $E(X)/F(X)$ is purely transcendental;*
3. *the motive $M(X)_E$ is a direct summand of the motive M .*

Then the motive $M(X)$ is a direct summand of the motive M .

We generalize this theorem when the motive $M(X) \in CM(F; \Lambda)$ is replaced by a direct summand $(M(X), p)$ associated with a projector $p \in \text{End}_{CM(F; \Lambda)}(M(X))$. The proof given by N. Karpenko in [2] cannot be used in the case where $M(X)$ is replaced by a direct summand because of the use on the *multiplicity* ([1], §75) as the multiplicity of a projector in the category $CM(F; \Lambda)$ is not always equal to 1 (and it can even be 0). The proof given here for its generalization gives also another proof of theorem I.1.

II Suitable basis of the dual module of a geometrically split F -variety

Let X be a geometrically split, geometrically irreducible F -variety satisfying the nilpotence principle. We note $CH(\overline{X}; \Lambda)$ as the colimit of the $CH(X_K; \Lambda)$ over all extensions K of F . By assumption there is a integer $n = rk(X)$ such that

$$CH(\overline{X}; \Lambda) \simeq \bigoplus_{i=0}^n \Lambda.$$

Let $(x_i)_{i=0}^n$ be a base of the Λ -module $CH(\overline{X}; \Lambda)$. Each element x_i of the basis is associated with a subvariety of X_E , where E is a splitting field of X . We note $\varphi(i)$ for the dimension of the E -variety associated to x_i .

Proposition II.1. *Let X be a geometrically split F -variety. Then the pairing*

$$\Psi : \begin{array}{ccc} CH(\overline{X}; \Lambda) \times CH(\overline{X}; \Lambda) & \longrightarrow & \Lambda \\ (\alpha, \beta) & \longmapsto & \deg(\alpha \cdot \beta) \end{array}$$

is bilinear, symmetric and non-degenerate.

The pairing Ψ induces an isomorphism between $CH(\overline{X}; \Lambda)$ and its dual module $Hom_{\Lambda}(CH(\overline{X}; \Lambda), \Lambda)$. This isomorphism is given by

$$\begin{array}{ccc} CH(\overline{X}; \Lambda) & \longrightarrow & Hom_{\Lambda}(CH(\overline{X}; \Lambda), \Lambda) \\ x & \longmapsto & \Psi(x, \cdot) \end{array}$$

Considering the inverse images of the dual basis of $Hom_{\Lambda}(CH(\overline{X}; \Lambda), \Lambda)$ associated with the basis x_i , we get another basis $(x_i^*)_{i=0}^n$ of $CH(\overline{X}; \Lambda)$ such that

$$\Psi(x_i, x_j^*) = \delta_{ij}$$

where δ_{ij} is the usual Kronecker symbol.

Proposition II.2. *Let M and N be two motives in $CM(F; \Lambda)$ such that M is split. Then there is an isomorphism*

$$CH^*(M; \Lambda) \otimes CH^*(N; \Lambda) \longrightarrow CH^*(M \otimes N; \Lambda)$$

Proof. c.f. [1] proposition 64.3. □

Let Y be a smooth complete irreducible F -variety. We note M for the motive $(M(Y), q)$ associated with a projector $q \in End(M(Y))$. Then we have the following computations.

Lemma II.3. *For any integers i, j, k and s less than $rk(X) = n$, and for any cycles y and y' in $CH(\overline{Y}; \Lambda)$, with 1 being the identity class in either $CH(\overline{X}; \Lambda)$ or $CH(\overline{Y}; \Lambda)$ we have*

1. $(x_i \times x_j^*) \circ (x_k \times x_s^*) = \delta_{is}(x_k \times x_j^*)$
2. $(x_i \times y \times 1) \circ (x_k \times x_s^*) = \delta_{is}(x_k \times y \times 1)$
3. $(y' \times x_j^*) \circ (x_i \times y) = \deg(y' \cdot y)(x_i \times x_j^*)$

Proof. We only compute (2) (other cases are similar).

$$(x_i \times y \times 1) \circ (x_k \times x_s^*) = (\overline{X} \overline{Y} \times \overline{X})_* ((\overline{X} \times \overline{X})_{p_{\overline{Y} \times \overline{X}}}^*(x_k \times x_s^*) \cdot (p_{\overline{X} \times \overline{Y} \times \overline{X}}^*)^*(x_i \times y \times 1)) \quad (\text{II.1})$$

$$= (\overline{X} \overline{Y} \times \overline{X})_* ((x_k \times x_s^* \times 1 \times 1) \cdot (1 \times x_i \times y \times 1)) \quad (\text{II.2})$$

$$= (\overline{X} \overline{Y} \times \overline{X})_* (x_k \times (x_s^* \cdot x_i) \times y \times 1) \quad (\text{II.3})$$

$$= \delta_{is}(x_k \times y \times 1) \quad (\text{II.4})$$

□

III Rational cycles of a geometrically split F -variety

Let X be a geometrically split F -variety. We note $(M(X), p)$ the direct summand of $M(X)$ associated with a projector $p \in CH_{\dim(X)}(X \times X; \Lambda)$. Considering the motive M defined in the previous section, if $(M(X_E), p_E)$ is a direct summand of M_E for some extension E/F , then there exists cycles $f \in CH(X_E \times Y_E; \Lambda)$ and $g \in CH(Y_E \times X_E; \Lambda)$ such that $f \circ g = p_E$. We can write these cycles in suitable basis of $CH(\overline{X} \times \overline{Y}; \Lambda)$, $CH(\overline{Y} \times \overline{X}; \Lambda)$ and $CH(\overline{X} \times \overline{X}; \Lambda)$ by proposition II.2. Thus there are two subsets F and G of $\{0, \dots, n\}$, scalars f_i, g_j, p_{ij} and cycles y_i, y'_j in $CH(\overline{Y}; \Lambda)$ such that

1. $\overline{f} = \sum_{i \in F} f_i(x_i \times y_i)$
2. $\overline{g} = \sum_{j \in G} g_j(y'_j \times x_j^*)$
3. $\overline{p} = \sum_{i \in F} \sum_{j \in G} p_{ij}(x_i \times x_j^*)$

With $p_{ij} = f_i g_j \deg(y'_j \cdot y_i)$ by lemma II.3 as $g \circ f = p_E$.

Notation III.1. Let $p \in CH_{\dim(X)}(X \times X)$ be a non-zero projector. Considering \bar{p} , the image of p in a splitting field of the F -variety X , we can write $\bar{p} = \sum_{i \in P_1} \sum_{j \in P_2} p_{ij}(x_i \times x_j^*)$. We define the least codimension of p (denoted $\text{cadmin}(p)$) by

$$\text{cadmin}(p) := \min_{(i,j), p_{ij} \neq 0} (\dim(\bar{X}) - \varphi(i))$$

Proposition III.2. Let $p \in CH_{\dim(X)}(X \times X)$ be a non-zero projector. We consider its decomposition $\bar{p} = \sum_{i \in P_1} \sum_{j \in P_2} p_{ij}(x_i \times x_j^*)$ in a splitting field of X . Then for any $i \in P_1$ and $j \in P_2$ we have

$$p_{ij} = \sum_{k \in P_1 \cap P_2} p_{kj} p_{ik}$$

Proof. We can assume that $\varphi(i)$ is constant on P_1 . Then a straightforward computation gives

$$\bar{p} \circ \bar{p} = \left(\sum_{i \in P_1} \sum_{j \in P_2} p_{ij}(x_i \times x_j^*) \right) \circ \left(\sum_{k \in P_1} \sum_{s \in P_2} p_{ks}(x_k \times x_s^*) \right) \quad (\text{III.1})$$

$$= \sum_{i \in P_1} \sum_{j \in P_2} \sum_{k \in P_1} \sum_{s \in P_2} p_{ij} p_{ks}(x_i \times x_j^*) \circ (x_k \times x_s^*) \quad (\text{III.2})$$

$$= \sum_{i \in P_1} \sum_{j \in P_2} \sum_{k \in P_1} \sum_{s \in P_2} p_{ij} p_{ks} \delta_{is}(x_k \times x_j^*) \quad (\text{III.3})$$

$$= \sum_{k \in P_1} \sum_{s \in P_2} \left(\sum_{i \in P_1 \cap P_2} p_{ij} p_{ki}(x_k \times x_s^*) \right) \quad (\text{III.4})$$

Moreover $p \circ p = p$, thus if $(k, s) \in P_1 \times P_2$ we have $p_{ks} = \sum_{i \in P_1 \cap P_2} p_{is} p_{ki}$. □

IV General properties of Chow groups

Embedding the Chow group of the F -variety X is quite useful for computations, but the generalization of the theorem I.1 needs a direct construction of some F -rational cycles f and g . We study in this section some properties of rational elements in Chow groups and how they behave when the extension $E(X)/F(X)$ is purely transcendental.

Proposition IV.1. Let Y be an F -variety. Let E/F be a purely transcendental extension. Then the morphism

$$\text{res}_{E/F} : CH(Y; \Lambda) \longrightarrow CH(Y_E; \Lambda)$$

is an epimorphism.

Proof. Indeed the morphism $\text{res}_{E/F}$ coincides with the composition

$$CH(Y; \Lambda) \longrightarrow CH(Y \times \mathbb{A}_F^n; \Lambda) \longrightarrow CH(Y_E; \Lambda).$$

As the extension E/F is purely transcendental, there is an isomorphism between E and the function field of an affine space \mathbb{A}_F^n for some integer n . The first map is an epimorphism by the homotopy invariance of Chow groups ([1], theorem 57.13) and the second map is an epimorphism as well ([1], corollary 57.11). □

V Generalization of the going-down theorem in the category of Chow-Grothendieck motives

We now have all the material needed to prove the generalization of theorem I.1.

Theorem V.1. Let Λ be a finite commutative ring. Let X be a geometrically split, geometrically irreducible F -variety satisfying the nilpotence principle. Let also $M \in CM(F; \Lambda)$ be a motive. Suppose that an extension E/F satisfies

1. the E -motive $(M(X)_{E,p_E})$ associated with the E -variety X_E is indecomposable;
2. the extension $E(X)/F(X)$ is purely transcendental;
3. the motive $(M(X_E), p_E)$ is a direct summand of the E -motive M_E .

Then the motive $(M(X), p)$ is a direct summand of the motive M .

Proof. We can consider that $M = (Y, q)$ for some smooth complete F -variety Y and a projector $q \in CH_{\dim(Y)}(Y \times Y; \Lambda)$. If p is equal to 0 then the motive $(M(X), p)$ is the 0 motive and $(M(X), p)$ is a direct summand of M . Now suppose that p is not equal to 0.

As $(M(X)_{E,p_E})$ is a direct summand of M_E , there are E -rational cycles $f \in CH_{\dim(X_E)}(X_E \times Y_E; \Lambda)$ and $g \in CH_{\dim(Y_E)}(Y_E \times X_E; \Lambda)$ such that $g \circ f = p_E$. We can decompose the images of these cycles in a splitting field of X in suitable basis for computations :

1. $\bar{f} = \sum_{i \in F} f_i(x_i \times y_i)$
2. $\bar{g} = \sum_{j \in G} g_j(y'_j \times x_j^*)$
3. $\bar{p} = \sum_{i \in F} \sum_{j \in G} p_{ij}(x_i \times x_j^*)$

with $p_{ij} = f_i g_j \deg(y'_j \cdot y_i)$.

Splitting terms whose first codimension is minimal in \bar{f} and \bar{p} by introducing

$$F_1 := \{i \in F, \varphi(i) = \text{cdmin}(p)\}$$

we get

1. $\bar{f} = \sum_{i \in F_1} f_i(x_i \times y_i) + \sum_{i \in F \setminus F_1} f_i(x_i \times y_i)$
2. $\bar{p} = \sum_{i \in F_1} \sum_{j \in G} p_{ij}(x_i \times x_j^*) + \sum_{i \in F \setminus F_1} \sum_{j \in G} p_{ij}(x_i \times x_j^*)$

As $E(X)$ is an extension of E , the cycle \bar{f} is $E(X)$ -rational. Proposition IV.1 implies that the change of field $\text{res}_{E(X)/F(X)}$ is an epimorphism, hence \bar{f} is an $F(X)$ -rational cycle.

Considering the morphism $\text{Spec}(F(X)) \rightarrow X$ associated with the generic point of the geometrically irreducible variety X , we get a morphism

$$\epsilon : (X \times Y)_{F(X)} \rightarrow X \times Y \times X$$

This morphism induces a pull-back $\epsilon^* : CH_{\dim(X)}(\bar{X} \times \bar{Y} \times \bar{X}; \Lambda) \rightarrow CH_{\dim(X)}(\bar{X} \times \bar{Y}; \Lambda)$ mapping any cycle of the form $\alpha \times \beta \times 1$ on $\alpha \times \beta$ and vanishing on elements $\alpha \times \beta \times \gamma$ if $\text{codim}(\gamma) > 0$. Moreover ϵ^* induces an epimorphism of F -rational cycles onto $F(X)$ -rational cycles ([1], corollary 57.11). We can thus choose a F -rational cycle $f_1 \in CH_{\dim(X)}(\bar{X} \times \bar{Y} \times \bar{X}; \Lambda)$ such that $\epsilon^*(f_1) = \bar{f}$.

By the expression of the pull-back ϵ^* we can assume

$$\bar{f}_1 = \sum_{i \in F_1} f_i(x_i \times y_i \times 1) + \sum_{i \in F \setminus F_1} f_i(x_i \times y_i \times 1) + \sum (\alpha \times \beta \times \gamma)$$

where the codimension of the cycles γ is non-zero.

Considering f_1 as a correspondance from \bar{X} to $\bar{X} \times \bar{Y}$, we consider $f_2 := f_1 \circ p$ which is also a F -rational cycle. We have

$$\bar{f}_2 = \left(\sum_{i \in F_1} f_i(x_i \times y_i \times 1) \right) \circ \left(\sum_{i \in F_1} \sum_{j \in G} p_{ij}(x_i \times x_j^*) \right) + \sum_{i \in F \setminus F_1} \sum_{j \in G} \lambda_{ij}(x_i \times y_j \times 1) + \sum \tilde{\alpha} \times \tilde{\beta} \times \tilde{\gamma} \quad (\text{V.1})$$

$$= \sum_{i \in F_1} \sum_{j \in F_1 \cap G} f_j p_{ij}(x_i \times y_j \times 1) + \sum_{i \in F \setminus F_1} \sum_{j \in G} \lambda_{ij}(x_i \times y_j \times 1) + \sum \tilde{\alpha} \times \tilde{\beta} \times \tilde{\gamma} \quad (\text{V.2})$$

where the cycles $\tilde{\gamma}$ are of non-zero codimension, the cycles $\tilde{\alpha}$ are such that $\text{codim}(\tilde{\alpha}) \geq \text{cdmin}(p)$ and where elements λ_{ij} are scalars.

We now consider the diagonal embedding

$$\Delta : \begin{array}{ccc} \bar{X} \times \bar{Y} & \longrightarrow & \bar{X} \times \bar{Y} \times \bar{X} \\ (x, y) & \longmapsto & (x, y, x) \end{array}$$

The morphism Δ induces a pull-back $\Delta^* : CH_{\dim(X)}(\overline{X} \times \overline{Y} \times \overline{X}; \Lambda) \longrightarrow CH_{\dim(X)}(\overline{X} \times \overline{Y}; \Lambda)$

We note $f_3 := \Delta^*(f_2)$, which is also a F -rational cycle and whose expression in a splitting field of X is

$$f_3 = \sum_{i \in F_1} \sum_{j \in F_1 \cap G} f_j p_{ij}(x_i \times y_j) + \sum_{i \in F \setminus F_1} \sum_{j \in G} \lambda_{ij}(x_i \times y_j) + \sum (\tilde{\alpha} \cdot \tilde{\gamma}) \times \tilde{\beta}$$

where $\text{codim}(\tilde{\alpha} \cdot \tilde{\gamma}) > \text{cdmin}(p)$ as $\text{codim}(\tilde{\alpha}) \geq \text{cdmin}(p)$ and $\text{codim}(\tilde{\gamma}) > 0$.

We can compute the $g \circ f_3$:

$$\overline{g} \circ \overline{f_3} = \overline{g} \circ \left(\sum_{i \in F_1} \sum_{j \in G} f_j p_{ij}(x_i \times y_j) \right) + \overline{g} \circ \left(\sum_{i \in F \setminus F_1} \sum_{j \in G} \lambda_{ij}(x_i \times y_j) \right) + \overline{g} \circ \left(\sum (\tilde{\alpha} \cdot \tilde{\gamma}) \times \tilde{\beta} \right) \quad (\text{V.3})$$

$$= \sum_{i \in F_1} \sum_{s \in G} \sum_{j \in F_1 \cap G} g_s f_j p_{ij}(y'_s \times x_s^*) \circ (x_i \times y_j) + \left(\sum \overline{\alpha} \times \overline{\beta} \right) \quad (\text{V.4})$$

With cycles $\overline{\alpha}$ such that $\text{codim}(\overline{\alpha}) > \text{cdmin}(p)$. Computing the component of $g \circ f_3$ for elements of the form $x_k \times x_s^*$ with $\varphi(k) = \text{cdmin}(p)$ we get

$$\overline{g} \circ \overline{f_3} = \sum_{i \in F_1} \sum_{s \in G} \sum_{j \in F_1 \cap G} g_s f_j p_{ij}(y'_s \times x_s^*) \circ (x_i \times y_j) + \left(\sum \overline{\alpha} \times \overline{\beta} \right) \quad (\text{V.5})$$

$$= \sum_{i \in F_1} \sum_{s \in G} \sum_{j \in F_1 \cap G} g_s f_j p_{ij} \deg(y'_s \cdot y_j)(x_i \times x_s^*) \quad (\text{V.6})$$

Now we can see that if $k \in F_1$, then the coefficient of $g \circ f_3$ relatively to an element $x_k \times x_s^*$ is equal to $\sum_{i \in F_1 \cap G} g_s f_i p_{ki} \deg(y_i \cdot y'_s)$. Moreover proposition III.2 says that

$$\sum_{i \in F_1 \cap G} g_s f_i p_{ki} \deg(y_i \cdot y'_s) = \sum_{i \in F_1 \cap G} p_{is} p_{ki} = p_{ks}$$

Since p is non-zero, there exists (k, s) with $k \in F_1$ and $p_{ks} \neq 0$, thus we have shown that the cycle $g \circ f_3$ as a decomposition

$$g \circ f_3 = p_{ks}(x_k \times x_s^*) + \sum_{(i,j) \neq (k,s)} p_{ij}(x_i \times x_j^*) + \sum (\overline{\alpha} \circ \overline{\beta})$$

where $\text{codim}(\overline{\alpha}) > \text{cdmin}(p)$. Since p is a projector, for any integer n the n -th power of $g \circ f_3$ as always a non-zero component relatively to $x_k \times x_s^*$ which is equal to p_{ks} , that is to say

$$\forall n \in \mathbb{N}, (g \circ f_3)^{\circ n} = p_{ks}(x_k \times x_s^*) + \sum_{(i,j) \neq (k,s)} p_{ij}(x_i \times x_j^*) + \sum (\overline{\alpha} \circ \overline{\beta})$$

where $\text{codim}(\overline{\alpha}) > \text{cdmin}(p)$.

As the ring Λ is finite, there is a power of $g \circ (f_3)_E$ which is a non-zero idempotent (cf [2] lemma 3.2). Since the E -motive $(M(X)_E, p_E)$ is indecomposable this power of $g \circ (f_3)_E$ is equal to p_E . Thus we have shown that there exists an integer n_1 such that

$$(g \circ (f_3)_E)^{\circ n_1} = p_E$$

In particular if $g_1 := (g \circ (f_3)_E)^{\circ n_1 - 1} \circ g$ we get $g_1 \circ (f_3)_E = p_E$.

Now we can transpose the last equality and get

$${}^t(f_3)_E \circ {}^t g_1 = {}^t p_E.$$

Repeating the same process as before, we get a F -rational cycle \tilde{g} and an integer n_2 such that

$$({}^t(f_3)_E \circ (\tilde{g})_E)^{\circ n_2} = {}^t p_E$$

Now setting $\hat{g} := (\tilde{g} \circ (f_3))^{\circ n_2 - 1} \circ {}^t \tilde{g}$, we have two F -rational cycles \hat{g} and f_3 such that

$$\hat{g}_E \circ (f_3)_E = p_E$$

Using the nilpotence principle again, there is an integer $\overline{n} \in \mathbb{N}$ such that

$$(\hat{g} \circ f_3)^{\circ \overline{n}} = p$$

Hence if $\hat{f} := f_3 \circ (\hat{g} \circ f_3)^{\circ \overline{n} - 1}$, \hat{f} is a F -rational cycle satisfying

$$\hat{g} \circ \hat{f} = p$$

Thus we have shown that the motive $(M(X), p)$ is a direct summand of the motive M . \square

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