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SHORTENING ALL THE SIMPLE CLOSED GEODESICS ON SURFACES WITH BOUNDARY

ATHANASE PAPADOPOULOS AND GUILLAUME THÉRET

ABSTRACT. We give a proof of an unpublished result of Thurston showing that given any hyperbolic metric on a surface of finite type with nonempty boundary, there exists another hyperbolic metric on the same surface for which the lengths of all simple closed geodesics have are shorter. Furthermore, we show that we can do the shortening in such a way that it is bounded below by a positive constant. This improves a recent result obtained by Parlier in [2]. We include this result in a study of the weak metric theory of the Teichmüller space of surfaces with nonempty boundary. The weak metrics that we consider are defined using lengths of closed geodesics and lengths of geodesic arcs. We prove an equality between two such weak metrics.

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1. INTRODUCTION

Let S be a connected orientable surface of finite topological type and of negative characteristic. All the hyperbolic structures that we shall consider on S are metrically complete, of finite area with totally geodesic boundary. Unless explicitly specified, we shall assume that the boundary ∂S of S is nonempty. From the assumptions, it follows that the boundary components of S are closed geodesics.

Let $\mathcal{T}(S)$ denote the Teichmüller space of S , that is, the space of hyperbolic structures on S (of the required type) up to homeomorphisms homotopic to the identity. (In this paper, homotopies of a surface fix setwise the boundary components but not necessarily pointwise.)

Let $\mathcal{C} = \mathcal{C}(S)$ be the set of simple closed geodesics in S , boundary components included. This set is defined relative to a hyperbolic structure which is understood, and it is known that there exists a natural correspondence

between any two such sets relative to different underlying hyperbolic structures.

A *weak metric* on a set is a structure that satisfies all the axioms of a metric except the symmetry axiom.

We consider the following function on $\mathcal{T}(S) \times \mathcal{T}(S)$:

$$(1) \quad k(X, Y) = \log \sup_{\gamma \in \mathcal{C}} \frac{l_Y(\gamma)}{l_X(\gamma)}.$$

A result of Thurston (Proposition 2.1 of [4]) says that in the case where ∂S is empty, the function k defines a weak metric on the Teichmüller space of S . We shall see in the following section that for any surface S with boundary, in contrast with the case of surfaces without boundary, there exist hyperbolic structures X and Y on S satisfying $k(X, Y) < 0$. In other words, it is possible, on hyperbolic surfaces with boundary, to contract the length of all simple closed geodesics by a uniform amount. In particular, the function k is not a weak metric.

We shall call a *simple geodesic arc* in S a geodesic segment which is properly embedded in that surface, that is, the arc has no self-intersection, the interior of the arc is in the interior of S and the endpoints of the arc are on ∂S . Let $\mathcal{B} = \mathcal{B}(S)$ be the union of the set of geodesic boundary components of S with the set of simple geodesic arcs that are perpendicular to the boundary. (Again, the set \mathcal{B} is defined relative to some hyperbolic structure, but there exists a natural correspondence between two such sets relative to different hyperbolic structures.)

We set

$$K(X, Y) = \log \sup_{\gamma \in \mathcal{B}} \frac{l_Y(\gamma)}{l_X(\gamma)}.$$

We proved in [1] that the function K is a weak metric on Teichmüller space.

In the paper [4], Thurston defined a weak metric on the Teichmüller space of a surface of finite type without boundary by the formula

$$L(X, Y) = \log \inf_{\phi} L(\phi),$$

where the infimum is taken over the set of orientation-preserving homeomorphisms ϕ homotopic to the identity, and where $L(\phi)$ denotes the *Lipschitz constant* of ϕ , defined as

$$L(\phi) = \sup_{x \in y} \left(\frac{d(\phi(x), \phi(y))}{d(x, y)} \right)$$

where d is the distance function on S . Thurston's weak metric is asymmetric in the sense that there exist two elements X and Y satisfying $L(X, Y) \neq L(Y, X)$. We shall see that L is also a weak metric in the context of surfaces with boundary, and we shall prove an analogue of a result of Thurston (Theorem 8.5 of [4]), namely that

$$K = L.$$

2. SHRINKING ALL SIMPLE CLOSED GEODESICS

Consider a foliation \mathcal{E} of the hyperbolic plane \mathbb{H}^2 by the set of curves that are equidistant from a given geodesic, and consider the foliation \mathcal{G} of \mathbb{H}^2 by the curves that are orthogonal to the leaves of \mathcal{E} (Figure 1). The leaves of \mathcal{G} are geodesics. We start with the following:

Lemma 2.1 (Projection along equidistant curves). *The projection map from \mathbb{H}^2 to some leaf of \mathcal{G} along the leaves of \mathcal{E} is distance non-increasing. Furthermore, the distance between any two points is equal to the distance between their projections if and only if the two points are on some leaf of \mathcal{G} .*

Proof. Let γ be the geodesic line in \mathbb{H}^2 of which \mathcal{E} is the set of equidistant curves. The curves that are orthogonal to the curves of the family \mathcal{E} are precisely the geodesic lines that are orthogonal to γ .

Let x, y be two points in the hyperbolic plane \mathbb{H}^2 . If these points lie on the same leaf of \mathcal{E} , their projection is a point, and the result follows in this case. Thus we can assume that the points x and y lie on distinct leaves of the foliation \mathcal{E} . Consider the geodesic segment, σ , joining x to y . By assumption, σ is transverse to the leaves of \mathcal{E} . The goal is to compare the length of σ with the length of any geodesic arc which is perpendicular to \mathcal{E} and whose endpoints lie on the same equidistant curves as the endpoints of σ . If the segment σ is itself contained in a leaf of \mathcal{G} , then the projection of σ keeps the length of σ constant. Thus, we can assume that σ is not contained in a leaf of \mathcal{G} .

Up to dividing σ in two geodesic segments, we can assume that the interior of σ lies in a single component of $\mathbb{H}^2 \setminus \gamma$. Furthermore, the geodesic arcs on which we project σ have all the same length, we can assume that the geodesic arc, k , on which we project σ has a unique endpoint in common with σ .

There are two possibilities for choosing the arc k , which correspond to the two possibilities for the common endpoint between k and σ . Let us specify a choice for this common endpoint that we call A . Consider the two leaves of \mathcal{E} passing through x and y . Since the segment σ lies in a single component

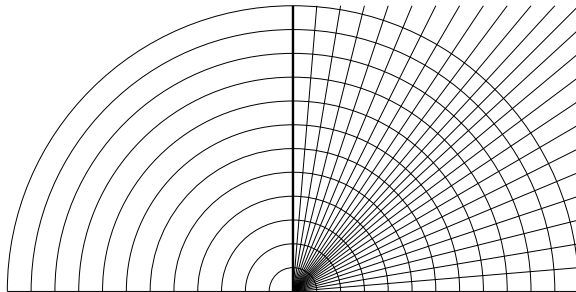


FIGURE 1. In the upper-half plane model of the hyperbolic plane, the foliation by Euclidean circles is the foliation \mathcal{G} by geodesic lines, and the orthogonal foliation in the quarter plane to the right is the orthogonal foliation \mathcal{E} by equidistant lines to the vertical geodesic line. (Only the part of \mathcal{E} in the right-quarter plane is drawn.)

of $\mathbb{H}^2 \setminus \gamma$, one of these two leaves is farther from γ than the other. Let us choose the arc k so that the common point A lies on this farthest leaf.

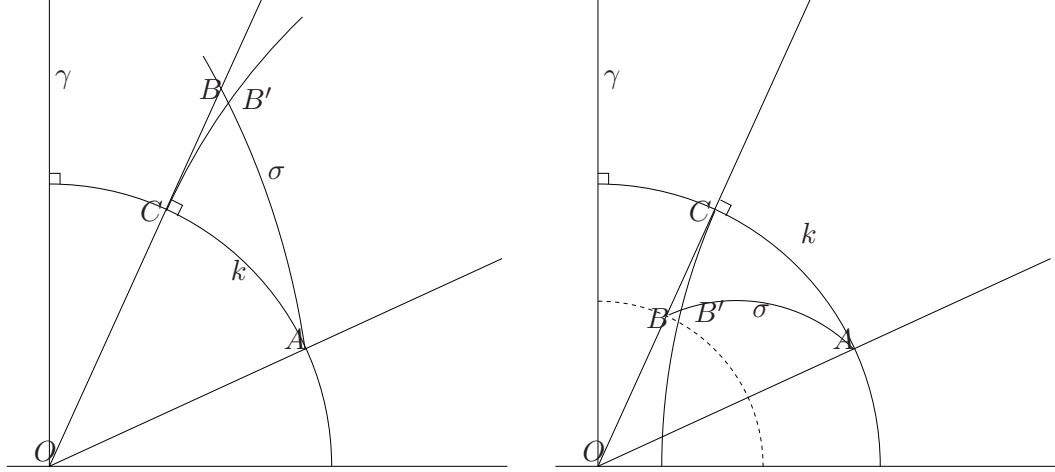


FIGURE 2. In these two figures we have represented a geodesic segment σ which is transverse to the foliation \mathcal{E} whose leaves are equidistant curves from the geodesic γ . There are two natural candidates for the geodesic segment k onto which one can project σ . We consider the one for which the geodesic segment perpendicular to k through C cuts σ in its interior. In the left-hand side picture, k lies below σ whereas it lies above σ in the right-hand side picture.

We are led to consider the curvilinear triangle (ABC) having two geodesic edges, namely $[AB] = \sigma$ and the geodesic segment $[AC] = k$ on which we project σ , and whose third edge $[BC]$ is the arc of an equidistant curve that connects the endpoint B of σ to the endpoint C of k (see Figure 2). Note that the angle \widehat{BCA} is a right angle.

Consider the geodesic passing through C perpendicularly to k . By convexity and thanks to our choice for k (see Figure 2), this geodesic intersects the segment σ in an interior point B' . Hence,

$$AB \geq AB'.$$

Now consider the geodesic triangle $AB'C$. Since the angle $\widehat{B'CA}$ is a right angle, we have, by hyperbolic trigonometry,

$$\cosh(AB') = \cosh(B'C) \cosh(AC).$$

Hence,

$$AB' > AC,$$

since $B'C > 0$ by assumption. Thus, we have

$$AB > AC,$$

that is, the length of σ is strictly greater than the length of k . All the cases have been dealt with. This concludes the proof. \square

We recall a few facts about Nielsen extensions of hyperbolic surfaces with boundary. Let X be a hyperbolic structure on S . With the above requirements on S , the *Nielsen extension* \hat{X} of X can be defined as the unique

complete hyperbolic surface which contains X and which retracts on X . Another description of \hat{X} is that this surface is obtained from X by gluing a semi-infinite cylinder with geodesic boundary along each boundary component of X . Note that the isometry type of each semi-infinite cylinder we glue is completely determined by the length of its unique boundary component, and that the hyperbolic structure \hat{X} does not depend upon the way these cylinders are glued to ∂S (that is, the twist parameters have no contribution). Note also that the hyperbolic surface \hat{X} has infinite area.

Let us remark that the Nielsen extension \hat{X} , although it is a natural complete hyperbolic structure on a surface homeomorphic to the interior to S , is distinct from the unique (Poincaré) complete hyperbolic structure on the interior of S that is in the conformal class of the restriction of the metric X to that interior.

The *convex core* of a complete infinite-area hyperbolic structure on a surface of finite type is the hyperbolic surface with boundary obtained by cutting out each infinite half-cylinder along the unique geodesic on which it retracts. The convex core of the Nielsen extension \hat{X} of X is the hyperbolic surface X that we started with.

At the level of the universal coverings, we have the following picture: The universal covering of the hyperbolic surface X with boundary is a subset of the hyperbolic plane bounded by the preimage of the boundary ∂S . This preimage consists in infinitely many disjoint geodesic lines. (If one identifies the hyperbolic plane with the unit disk, the limit set of the corresponding Fuchsian group is a Cantor set of the unit circle.) The universal covering of the Nielsen extension \hat{X} of X is the hyperbolic plane \mathbb{H}^2 , and it naturally contains the universal covering of X . The infinite half-cylinders in \hat{X} lift to the closed half-planes in the complement of the universal covering of X .

Consider two hyperparallel geodesic lines in \mathbb{H}^2 and let $\tilde{\alpha}$ be their common perpendicular geodesic segment. Let ϵ be a positive number. An ϵ -strip S_ϵ around $\tilde{\alpha}$ is a strip containing α and bounded by two hyperparallel geodesics which are at a distance at most ϵ apart (See Figure 3). The *core* of the ϵ -strip S_ϵ is the geodesic segment joining the boundary components of S_ϵ perpendicularly. Note that the core, c_ϵ of S_ϵ has length ϵ . We shall equip an ϵ -strip with the foliation by arcs that are equidistant from the core. This foliation induced an isometric correspondence between the boundary geodesics of the ϵ -strip, which we shall refer to as the *canonical* isometry between these geodesics.

Let α be a simple geodesic arc joining perpendicularly a boundary component γ_1 of the hyperbolic surface X to a boundary component γ_2 . (We may have $\gamma_1 = \gamma_2$. Consider lifts $\tilde{\gamma}_1, \tilde{\gamma}_2$ of γ_1, γ_2 to the universal covering. The lifts $\tilde{\gamma}_1, \tilde{\gamma}_2$ are hyperparallel geodesic lines and there is a unique lift $\tilde{\alpha}$ of α that joins them perpendicularly. For any small enough $\epsilon > 0$, the ϵ -strip around $\tilde{\alpha}$ projects to an embedded strip containing α in the Nielsen extension \hat{X} of X . We call such a strip an ϵ -strip in \hat{X} .

We now define a construction that we call *peeling an ϵ -strip from a hyperbolic surface with boundary*. Start from a hyperbolic structure X on the

surface with boundary S and consider X as embedded in its Nielsen extension \hat{X} . Consider an ϵ -strip, B , in \hat{X} . Consider the hyperbolic structure \hat{Y}_B on \hat{X} obtained by cutting out the strip B from \hat{X} and by gluing back the geodesic sides of the closure of $\hat{X} \setminus B$ by the canonical isometry that identifies the endpoints of the core of B . Another way of obtaining \hat{Y}_B is by collapsing the strip B along the leaves of the foliation of this strip by equidistant arcs defined above. Let $\hat{f}_B : \hat{X} \rightarrow \hat{Y}_B$ be the collapsing map. Let Y_B be the hyperbolic structure on S obtained by restricting the hyperbolic structure \hat{Y}_B to its convex core.

The image, $\hat{\alpha}_B$ of the strip B by \hat{f}_B is an infinite geodesic.

Proposition 2.2. *The map $\hat{f}_B : \hat{X} \rightarrow \hat{Y}_B$ is 1-Lipschitz and it is homotopic to the identity map of S . More precisely, \hat{f}_B is length-preserving in the complement of B and it strictly decreases, by a uniform amount, distances between points that are “separated by” B and contained in X , that is, points that are joined by a geodesic of shortest length which intersects the strip B transversely and is contained in X .*

Proof. The assertion regarding the homotopy between \hat{f}_B and the identity map is clear. Also, it is clear that the map \hat{f}_B restricted to $\hat{X} \setminus B$ is length-preserving. Thus, it remains to show that the map strictly decreases the distances between points separated by B by a uniform amount.

Let $x, y \in X \subset \hat{X}$ be two points separated by B and let $[xy]$ be a shortest geodesic segment joining x to y . The length xy of this segment equals $d_{\hat{X}}(x, y)$. By the assumptions made on x and y , the intersection $[xy] \cap B$ has only one component, and we denote it by $[x'y']$. The image of $[xy]$ by \hat{f}_B is a piecewise geodesic curve, namely, $[\hat{f}_B(x)\hat{f}_B(x')] \cup [\hat{f}_B(x')\hat{f}_B(y')] \cup [\hat{f}_B(y')\hat{f}_B(y)]$. Hence,

$$d_{\hat{Y}_B}(\hat{f}_B(x), \hat{f}_B(y)) \leq \hat{f}_B(x)\hat{f}_B(x') + \hat{f}_B(x')\hat{f}_B(y') + \hat{f}_B(y')\hat{f}_B(y).$$

We already noted that the lengths of the segments outside B are preserved by \hat{f}_B . Therefore, it suffices to show that the length $\hat{f}_B(x')\hat{f}_B(y')$ is strictly

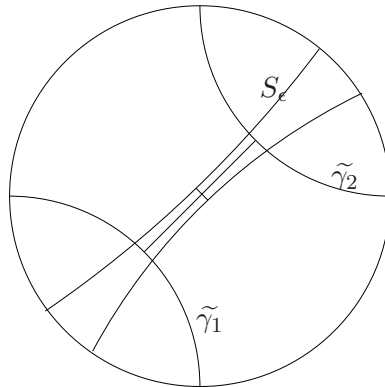


FIGURE 3. The segment $\tilde{\alpha}$ is the geodesic segment joining perpendicularly the two hyperparallel geodesics $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$. The ϵ -strip S_ϵ is bounded by two other hyperparallel geodesics drawn.

smaller than $x'y'$ by a uniform amount. Thus, we can assume that the points x and y lie on distinct boundary components of B .

First, suppose that the images of the points x and y under \hat{f}_B coincide, that is, suppose that x and y are on the same equidistant curve from the core of B . Then,

$$d_{\hat{X}}(x, y) - d_{\hat{Y}_B}(\hat{f}_B(x), \hat{f}_B(y)) = d_{\hat{X}}(x, y) \geq \epsilon.$$

This proves the lemma in this case, and we are left to consider a pair of points x and y that do not lie on the same equidistant curve from the core of B , that is, curves whose images under \hat{Y}_B are non-trivial geodesic segments.

Both projections of $[xy]$ on any side of B along the arcs equidistant to the core give rise to a geodesic segment with exactly one endpoint in common with $[xy]$, and whose length equals that of $[\hat{f}_B(x)\hat{f}_B(y)]$. We now follow the same proof as that of Lemma 2.1, but taking care this time of the difference between the length of the curve $[xy]$ and the length of its projection. In order to lighten the reading, we keep the notation used in the proof of 2.1. Thus, we denote the segment $[xy]$ by σ and we shall specify a choice for the projection k of σ .

Before explaining how to choose k , we remark that there is a uniform upper bound $M = M(X) > 0$ to the length of such a projection k . To see this, refer to Figure 3. Any geodesic contained in X and intersecting B is contained in the big strip in S_ϵ bounded by $\tilde{\gamma}_1$ and $\tilde{\gamma}_1$. This implies that there is an upper bound for the length of σ which only depends upon X and the choice of the ϵ -strip B in \hat{X} . Since the projection is 1-Lipschitz, this implies the same result for the length of k .

If σ intersects the core c_ϵ of S_ϵ , then we subdivide σ into two segments such that they both lie in different components of $S_\epsilon \setminus c_\epsilon$. It then suffices to show the property for each of these segments. We can therefore assume that σ is contained in S_ϵ and that it does not intersect the core c_ϵ of S_ϵ . We are now back to the situation studied in the proof of Lemma 2.1, but with the constraint on σ of being contained in a strip of width ϵ . We keep the same choice for k as the one settled in that proof and we refer the reader to Figure 2 for what follows.

Consider the triangle ABC as shown in Figure 2, but where now the three edges are now taken to be geodesics, namely, the edge $[AB]$ is the geodesic segment σ , the edge $[AC]$ is the geodesic segment k , and the third edge is the geodesic segment joining the points B and C . Note that since the angle at C of the triangle $\widehat{ACB'}$ of Figure 2 is equal to $\pi/2$, the angle $\widehat{ACB'}$ of the triangle that we consider now is greater than or equal to $\pi/2$. We must show that the difference $AB - AC$ is bounded from below by a uniform positive constant. Since the angle \widehat{ACB} is greater or equal to $\pi/2$, we have

$$\begin{aligned} \cosh(AB) &= \cosh(AC) \cosh(CB) - \sinh(AC) \sinh(CB) \cos(\widehat{ACB}) \\ &\geq \cosh(AC) \cosh(CB). \end{aligned}$$

Since $CB \geq \epsilon$, we get

$$\cosh(AB) \geq \cosh(AC) \cosh(\epsilon).$$

Now

$$\begin{aligned} \cosh(AB) &\geq \cosh(AC) \cosh(\epsilon) \\ &\geq \cosh(AC)(1 + \epsilon^2/2). \end{aligned}$$

Hence,

$$\begin{aligned} \cosh(AB) - \cosh(AC) &\geq \cosh(AC)\epsilon^2/2 \\ &\geq \epsilon^2/2. \end{aligned}$$

Multiplying the inequality by 2 and expanding cosh with exponentials, we get

$$\begin{aligned} e^{AB} - e^{AC} &\geq \epsilon^2 - (e^{-AB} - e^{-AC}) \\ &\geq \epsilon^2. \end{aligned}$$

The last inequality comes from the fact that $AB \geq AC$, that is, $e^{-AB} - e^{-AC} \leq 0$. We get

$$e^{AB-AC} \geq 1 + e^{-AC}\epsilon^2,$$

or,

$$AB - AC \geq \log(1 + e^{-AC}\epsilon^2).$$

We saw that there exists a positive number $M = M(X) > 0$ such that $AC \leq M$. Finally we get

$$AB - AC \geq \log(1 + e^{-M}\epsilon^2) > 0.$$

This concludes the proof. \square

The method used in the proof of Proposition 2.2 is due to Thurston [4].

We need the following corollary in order to obtain the main result of this section (Theorem 2.4 below).

Corollary 2.3. *Let λ be a measured geodesic lamination of S . Then $l_{Y_B}(\lambda) \leq l_X(\lambda)$, with strict inequality if and only if $\lambda \cap B \neq \emptyset$.*

Proof. Embed X and Y_B isometrically in their respective Nielsen extensions. Note that all the leaves of the lamination λ that intersect B do it transversely. Cover the support of the lamination λ by finitely many rectangles, R_1, \dots, R_n , small enough so that every intersection of B with one of them is either empty or a quadrilateral. Up to reordering the rectangles, we can assume that the rectangles R_1, \dots, R_m , $m \leq n$, intersect B effectively and that the other rectangles do not intersect B . The length of λ is given by

$$l_X(\lambda) = \sum_{i=1}^n \int_{\lambda_i} l_X(x) d\lambda(x) = \sum_{i=1}^n \int_{\lambda_i} l_{\hat{X}}(x) d\lambda(x),$$

where for each $i = 1, \dots, n$, λ_i denotes the set of leaves of $\lambda \cap R_i$ and where for $x \in \lambda_i$, $l_X(x) = l_{\hat{X}}(x)$ denotes the length of the leaf of $\lambda \cap R_i$ containing x .

We have

$$l_{Y_B}(\lambda) = l_{\hat{Y}_B}(\lambda) = \sum_{i=1}^n \int_{\lambda_i} l_{\hat{Y}_B}(\hat{f}_B(x)) d\lambda(x).$$

By Proposition 2.2, there exists a constant $C = C(B, X) > 0$ such that, for all $x \in \lambda_i$ with $R_i \cap B \neq \emptyset$,

$$l_{\hat{X}}(x) - l_{\hat{Y}_B}(\hat{f}_B(x)) \geq C.$$

Therefore,

$$\begin{aligned} l_{Y_B}(\lambda) &= \sum_{i=1}^n \int_{\lambda_i} l_{\hat{Y}_B}(\hat{f}_B(x)) d\lambda(x) \\ &\leq \sum_{i=1}^n \int_{\lambda_i} l_{\hat{X}}(x) d\lambda(x) - C \sum_{i=1}^m \int_{\lambda_i} d\lambda(x) \\ &\leq l_X(\lambda) - Ci(\lambda, B). \end{aligned}$$

This concludes the proof. \square

Theorem 2.4. *For any point X in Teichmüller space $\mathcal{T}(S)$, there exists a point Y in $\mathcal{T}(S)$ such that $k(X, Y) < 0$.*

Proof. Choose a finite collection of geodesic arcs, \mathcal{A} , joining the boundary of S to itself such that any simple closed geodesic is intersected by one of these arcs. Choose a collection of ϵ -strip, one around each arc of \mathcal{A} . Peel the ϵ -strips, one after the other. We thus get a new hyperbolic structure Y on S and a 1-Lipschitz map from the Nielsen extension of X to that of Y .

Since any measured geodesic lamination of X is intersected by an arc of \mathcal{A} , the length of a measured geodesic lamination decreases when we pass from X to Y .

Since $\mathcal{PML}(S)$ is compact, the supremum

$$\sup_{\alpha \in \mathcal{PML}(S)} \frac{l_Y(\alpha)}{l_X(\alpha)}$$

is attained by a measured geodesic lamination. Since the length of such a geodesic lamination has been strictly decreased, this shows that $k(X, Y) < 0$. This concludes the proof. \square

Remark 2.5. The preceding result improves a theorem by Parlier [2] which says that for any surface S of finite type with non-empty boundary and for any hyperbolic structure X on S , there exists a hyperbolic structure Y on S such that for every γ in \mathcal{C} , we have $\frac{l_X(\gamma)}{l_Y(\gamma)} < 1$. (Parlier's result only proves that $k(X, Y) \leq 0$). Note that whereas pariler's result shows that the function k defined in (1), in the case of a surface with nonempty boundary, is not a weak metric because it does not separate points, Theorem 2.4 shows that for any surface with boundary, this function can even take negative values.

Remark 2.6. Consider the peeling map \hat{f}_B described above. This map strictly decreases any elements of $\mathcal{C} \cup \mathcal{B}$ which intersects the strip B , and it leaves the lengths of the elements that are disjoint from B unchanged. In the paper [1], we defined the following function on the space $\mathcal{T}(S) \times \mathcal{T}(S)$ associated to a surface with boundary S :

$$d(X, Y) = \log \sup_{\gamma \in \mathcal{B} \cup \mathcal{C}} \frac{l_X(\gamma)}{l_Y(\gamma)}.$$

and we showed that this function defines a weak metric. Thus, using Theorem 2.4 above, there necessarily exists an arc on S whose length increases when we pass from X to Y_B . This arc is necessarily the arc α contained in the strip B . It is therefore possible to compute the distance $d(X, Y_B)$ explicitly.

3. LIPSCHITZ METRIC

We consider the function defined by

$$L(X, Y) = \log \inf_{\phi} L(\phi),$$

where the supremum is taken over all orientation-preserving homeomorphisms ϕ homotopic to the identity and where $L(\phi)$ denotes the Lipschitz constant of ϕ . This definition is due to Thurston, who proved that this function is a weak metric in the case of surfaces of finite type without boundary.

Theorem 3.1. *The function L is a weak metric on Teichmüller space.*

Proof. The point is to prove that $L(X, Y) = 0 \Rightarrow X = Y$. It should be possible to use Thurston's argument for the analogous result on surfaces without boundary (this is Proposition 2.1 in [4]), but the result also follows from Theorem 3.3 below. \square

The following is then a consequence of Theorem 2.4.

Corollary 3.2. *For surfaces with boundary, we have $L \neq k$.*

On the other hand, we have

Theorem 3.3. *For any surface of finite type S , we have*

$$L = K$$

Proof. Let ϕ be a homeomorphism of S homotopic to the identity. Consider the doubled homeomorphism, ϕ^d , of S^d . By definition, we have

$$L(\phi^d) = \sup_{x \neq y \in S^d} \frac{d_{Y^d}(\phi(x), \phi(y))}{d_{X^d}(x, y)} \geq L(\phi) = \sup_{x \neq y \in S} \frac{d_Y(\phi(x), \phi(y))}{d_X(x, y)}.$$

Hence, by taking the infimum in the right hand side over all homeomorphisms of S and then in the left hand side over all homeomorphisms of S^d , we get

$$L(X^d, Y^d) \geq L(X, Y).$$

Now we have the following sequence of inequalities

$$L(X^d, Y^d) = K(X^d, Y^d) = K(X, Y) \leq L(X, Y),$$

where the first equality is Thurston's Theorem 8.5 in [4], where the second equality is established in [1] (Corollary 2.8) and where the last inequality follows easily from the definitions. Putting everything together, we obtain the result. \square

Remark 3.4. It is not clear whether the infimum over the Lipschitz constants is realized by a homeomorphism of S .

We conclude with the following questions:

Question 3.5. Given a hyperbolic metric X on a surface S with nonempty boundary, can we always find another hyperbolic metric Y such that every geodesic arc in X is contracted when we pass from X to Y ? Of course, the boundary curves cannot all be contracted, by a result in [1] that we already quoted above.

Question 3.6. We can ask the same question above, concerning geodesic arcs and interior geodesic closed curves, instead of only geodesic arcs.

REFERENCES

- [1] L. Liu, A. Papadopoulos, W. Su, G. Th  ret, On length spectrum metrics and weak metrics on Teichm  ller spaces of surfaces with boundary, preprint 2009, arXiv:0903.0744v1 [math.GT].
- [2] H. Parlier, Lengths of geodesics on Riemann surfaces with boundary. Ann. Acad. Sci. Fenn. Math. 30 (2005), no. 2, 227–236.
- [3] W. Thurston, a spine for Teichm  ller space, unpublished manuscript, 3 pages (1986).
- [4] W. Thurston, Minimal stretch maps between hyperbolic surfaces, preprint, 1986, Arxiv:math GT/9801039.

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