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► **To cite this version:**

Takashi Kishimoto, Yuri Prokhorov, Mikhail Zaidenberg. Group actions on affine cones. Transformation Groups, 2013, 18 (4), pp.1137-1153. 10.1007/s00031-013-9246-5 . hal-00389309v2

**HAL Id: hal-00389309**

**<https://hal.science/hal-00389309v2>**

Submitted on 30 Jan 2010

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# GROUP ACTIONS ON AFFINE CONES

TAKASHI KISHIMOTO, YURI PROKHOROV, AND MIKHAIL ZAIDENBERG

*To Peter Russell on the occasion of his 70th birthday*

ABSTRACT. We address the following question:

*For which smooth projective varieties, the corresponding affine cone admits an action of a connected algebraic group different from the standard  $\mathbb{C}^*$ -action by scalar matrices and its inverse action?*

We show in particular that the affine cones over anticanonically embedded smooth del Pezzo surfaces of degree  $\geq 4$  possess such an action. Besides, we give some examples of rational Fano threefolds which have this property. A question in [FZ<sub>1</sub>] whether this property holds also for smooth cubic surfaces, occurs to be out of reach for our methods. Nevertheless, we provide a general geometric criterion that could be helpful in this case as well.

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The first author was supported by a Grant-in-Aid for Scientific Research of JSPS No. 20740004. The second author was partially supported by RFBR, No. 08-01-00395-a and Leading Scientific Schools (grants No. NSh-1983.2008.1, NSh-1987.2008.1). This work was done during a stay of the second and third authors at the Max Planck Institut für Mathematik at Bonn and a stay of the first and the second authors at the Institut Fourier, Grenoble. The authors thank these institutions for hospitality.

## INTRODUCTION

All varieties in this paper are defined over  $\mathbb{C}$ . By Corollary 1.13 in [FZ<sub>1</sub>], an isolated Cohen-Macaulay singularity  $(X, x)$  of a normal quasiprojective variety  $X$  is rational provided that  $X$  admits an effective action of the additive group  $\mathbb{C}_+$ , in particular of a connected non-abelian algebraic group. In the opposite direction, let us observe that, for instance, the singularity at the origin of the affine Fermat cubic in  $\mathbb{A}^4$

$$x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0$$

is rational. The question was raised [FZ<sub>1</sub>, Question 2.22] whether it also admits a non-diagonal action of a connected algebraic group, in particular, a  $\mathbb{C}_+$ -action. So far, we do not know the answer. However, we answer in affirmative a similar question for all del Pezzo surfaces of degree  $d \geq 4$ .

**Theorem 0.1.** *Let  $Y_d$  be a smooth del Pezzo surface of degree  $d$  anticanonically embedded into  $\mathbb{P}^d$ , and let  $X_d = \text{AffCone}(Y_d) \subseteq \mathbb{A}^{d+1}$  be the affine cone over  $Y_d$ . If  $4 \leq d \leq 9$  then  $X_d$  admits a nontrivial  $\mathbb{C}_+$ -action. Consequently, the automorphism group  $\text{Aut}(X_d)$  is infinite dimensional. Moreover, the Makar-Limanov invariant of  $X_d$  is trivial.*

Recall [Dol<sub>1</sub>, 10.1.1] that for  $d \leq 5$  the group  $\text{Aut}(Y_d)$  is finite. By definition, the Makar-Limanov invariant of an affine variety  $X$  is the subring of  $\mathcal{O}(X)$  of common invariants of all  $\mathbb{C}_+$ -actions on  $X$ . It is trivial when it consists of the constants.

One of our main results (Theorem 3.9) provides a necessary and sufficient condition for the existence of a nontrivial  $\mathbb{C}_+$ -action on an affine cone. As a corollary, for affine cones of dimension 3 we obtain the following geometric criterion.

**Theorem 0.2.** *Let  $Y$  be a smooth projective rational surface with a polarization  $\varphi_{|H|} : Y \hookrightarrow \mathbb{P}^n$ , and let  $X = \text{AffCone}_H(Y) \subseteq \mathbb{A}^{n+1}$  be the affine cone over  $Y \subseteq \mathbb{P}^n$ . Then  $X$  admits a nontrivial  $\mathbb{C}_+$ -action if and only if  $Y$  contains an  $H$ -polar cylinder i.e., a cylindrical Zariski open set*

$$U = Y \setminus \text{supp}(D) \simeq Z \times \mathbb{A}^1,$$

where  $Z$  is an affine curve and  $D \in |dH|$  is an effective divisor on  $\mathbb{P}^n$ .

Using this criterion, we show in Proposition 3.13 that for every smooth projective rational surface  $Y$  there exists a polarization  $\varphi_{|H|} : Y \hookrightarrow \mathbb{P}^n$  such that  $Y$  contains an  $H$ -polar cylinder and so the corresponding affine cone possesses an effective action of  $\mathbb{C}_+$ . It would be interesting to classify in any dimension all pairs  $(Y, H)$ , where  $Y$  is a smooth projective variety and  $H$  an ample divisor on  $Y$ , such that the affine cone  $X = \text{AffCone}_H(Y)$  admits an effective  $\mathbb{C}_+$ -action. We recover this classification for  $\dim_{\mathbb{C}}(Y) = 1$  and give some concrete examples in higher dimensions, especially in dimensions 2 and 3.

A theorem due to Matsumura, Monsky and Andreotti (see [MM], or [GH, §I.4], or Section 1 below) claims that any automorphism of a smooth hypersurface  $Y$  in  $\mathbb{P}^n$  of degree  $d$ , where  $d, n \geq 3$  and  $(d, n) \neq (4, 3)$ , is restriction of a unique projective linear transformation, and  $\text{Aut}(Y)$  is a finite group. In Corollary 2.4 we show that the automorphism group  $\text{Aut}(X)$  of the affine cone  $X$  over a smooth, non-birationally ruled projective variety  $Y$  is a linear group, and actually a central extension of a finite group by  $\mathbb{C}^*$ . Consequently, among the affine cones over smooth projective surfaces

in  $\mathbb{P}^3$ , only those of degree  $\leq 3$  can admit a nontrivial action of a connected algebraic group, and their automorphism groups can be infinite-dimensional. Actually, the 3-fold affine quadric cone possesses an effective linear action of the additive group  $\mathbb{C}_+^2$ , see Example 3.3 below or [AS], [Sh].

In Section 1 we give a short overview of the known results on the automorphism groups. In Section 2 we collect generalities on automorphisms of affine cones. Theorems 0.1 and 0.2 are proven in Section 3. In Section 4 we summarize some geometric facts that could be useful (in view of the criterion of Theorem 0.2) in order to answer Question 2.22 in [FZ<sub>1</sub>] cited above. In the final section 5 we describe two families of rational Fano threefolds such that the affine cones over their anti-canonical embeddings possess effective  $\mathbb{C}_+$ -actions.

**Acknowledgements:** We thank Alvaro Liendo, who participated in discussions on early stages of this work, for his attention and comments. Our thanks also to Dmitry Akhiezer, Michel Brion, and Dimitri Timashev for providing useful information on homogeneous varieties, to Dmitry Akhiezer and Ivan Arzhantsev for reading some chapters and valuable remarks.

## 1. GROUP ACTIONS ON PROJECTIVE VARIETIES

In this section we recall some well known facts about the automorphism groups of projective or quasiprojective varieties; see e.g., [GH, §I.4], [LZ, §II.3]. For an algebraic variety  $Y$ , we let  $\text{Aut}(Y)$  denote the group of all biregular automorphisms of  $Y$  and  $\text{Bir}(Y)$  the group of all birational transformations of  $Y$  into itself. For a projective or an affine embedding  $Y \hookrightarrow \mathbb{P}^n$  ( $Y \hookrightarrow \mathbb{A}^n$ , respectively) we let  $\text{Lin}(Y)$  denote the group of all automorphisms of  $Y$  which extend linearly to the ambient space.

**1.1. Automorphisms of smooth projective hypersurfaces.** In the following theorems we gather some results concerning the groups  $\text{Lin}$ ,  $\text{Aut}$ , and  $\text{Bir}$  for projective hypersurfaces; see Matsumura and Monsky [MM], Iskovskikh and Manin [IM], Pukhlikov [Pu1]-[Pu3], Cheltsov [Chel], de Fernex, Ein and Mustăța [DEM].

**Theorem 1.1.** *Let  $Y$  be a smooth hypersurface in  $\mathbb{P}^n$  of degree  $d$ . Then for all  $d, n \geq 3$  except for  $(d, n) = (4, 3)$ ,*

$$\text{Aut}(Y) = \text{Lin}(Y)$$

*and this group is finite. It is trivial for a general hypersurface of degree  $d \geq 3$ .*

There is a similar result for Schubert hypersurfaces in flag varieties, see Theorem 8.8 in [Te]. For the group of birational transformations, the following hold.

**Theorem 1.2.** *For  $Y \subseteq \mathbb{P}^n$  as above and for all  $d > n \geq 2$  except for  $(d, n) = (3, 2)$ ,*

$$\text{Bir}(Y) = \text{Aut}(Y).$$

*This group is finite except in the case  $(d, n) = (4, 3)$  of a smooth quartic surface  $Y \subseteq \mathbb{P}^3$ , where it is discrete, but can be infinite and different from  $\text{Lin}(Y)$  which is finite. The group  $\text{Bir}(Y)$  of a very general quartic surface  $Y \subseteq \mathbb{P}^3$  is trivial.*

The case  $d \leq n$  is much more complicated. However, in this case there are deep partial results, see e.g., [IM, Pu1, DEM].

Let us indicate briefly some ideas used in the proofs. In case  $d \neq n + 1$  the proof is easy and exploits the fact that the canonical divisor  $K_Y = \mathcal{O}_Y(d - n - 1)$  is  $\text{Aut}(Y)$ -stable. In case  $d = n + 1$  the equalities  $\text{Bir}(Y) = \text{Aut}(Y) = \text{Lin}(Y)$  follow since such hypersurfaces represent Mori minimal models. Indeed a birational map between minimal models is an isomorphism in codimension 1, see e.g., [KM], hence it induces an isomorphism of the corresponding Picard groups. If  $n \geq 4$  then  $\text{Pic}(Y) \simeq \mathbb{Z}$  by the Lefschetz Hyperplane Section Theorem. Therefore any birational transformation  $\varphi$  of  $Y$  acts trivially on  $\text{Pic}(Y)$  and so preserves the complete linear system of hyperplane sections  $|\mathcal{O}_Y(1)|$ . Since  $Y$  is linearly normal<sup>1</sup>,  $\varphi$  is induced by a projective linear transformation of the ambient projective space  $\mathbb{P}^n$ . For the proof of finiteness of the group  $\text{Lin}(Y)$  and its triviality for general hypersurfaces, we refer to the classical paper of Matsumura and Monsky [MM].

By virtue of the Noether-Lefschetz Theorem, these arguments can be equally applied to very general smooth surfaces in  $\mathbb{P}^3$  of degree  $d \geq 4$ . For an arbitrary smooth surface  $Y$  in  $\mathbb{P}^3$ , the minimality of  $Y$  should be combined with the fact that  $\text{Pic}(Y)$  is torsion free. Indeed, any smooth surface in  $\mathbb{P}^3$  of degree  $d \geq 4$  represents a minimal model and so is not birationally ruled, hence its birational automorphisms are biregular; see e.g., [Mat, Theorem 1-8-6]. For  $d > 4$  the canonical class  $K_Y$  defines an equivariant polarization of  $Y$ , and  $\mathcal{O}_Y(1) \sim \frac{1}{d-4}K_Y$ . Since  $\text{Pic}(Y)$  is torsion free and  $H^0(\mathcal{O}_{\mathbb{P}^3}(1)) \rightarrow H^0(\mathcal{O}_Y(1))$  is a surjection, all automorphisms of  $Y$  are linear. This is not true, in general, in the case of a smooth quartic surface in  $\mathbb{P}^3$ . An example of such a surface with infinite automorphism group due to Fano and Severi is discussed in [MM, Theorem 4]. A non-linear biregular involution exists on any smooth quartic in  $\mathbb{P}^3$  containing skew lines, for instance, on the Fermat quartic  $x^4 + y^4 + z^4 + u^4 = 0$ ; see Takahashi [Ta].

For a quadric hypersurface  $X \subseteq \mathbb{A}^{n+1}$  of dimension  $n \geq 2$ , the group  $\text{Aut}(X)$  is infinite-dimensional [To, Lemma 1.1], cf. also example 3.3 below. For  $n = 2$  this group has an amalgamated product structure [DG]; cf. also [ML].

For a smooth cubic surface  $Y \subseteq \mathbb{P}^3$  the group  $\text{Aut}(Y) = \text{Lin}(Y)$  is finite, while the Cremona group  $\text{Bir}(Y) \simeq \text{Bir}(\mathbb{P}^2)$  is infinite-dimensional. The automorphism groups of such surfaces were listed by Hosoh [Ho<sub>1</sub>] who corrected an earlier classification by Segre [Se]; see also Manin [Man] and Dolgachev [Dol<sub>1</sub>]. The largest order of such a group is 648. This upper bound is achieved only for the Fermat cubic surface, see [Ho<sub>2</sub>]. The least common multiple of the orders of all these automorphism groups is  $3240 = 2^3 \cdot 3^4 \cdot 5$  (Gorinov [Gor]).

The Fermat quartic  $x^4 + y^4 + z^4 + u^4 = 0$  and the smooth quartic  $x^4 + y^4 + z^4 + u^4 + 12xyzu = 0$  in  $\mathbb{P}^3$  can be also characterized in terms of the orders of their automorphism groups, see Mukai [Mu], Kondo [Ko<sub>1</sub>], and Oguiso [Og].

**1.2. Automorphisms of smooth projective varieties.** According to the well known Matsumura Theorem<sup>2</sup> the group  $\text{Bir}(Y)$  of a smooth projective variety  $Y$  of general type is finite, hence also the group  $\text{Aut}(Y)$  is. In particular, this holds if  $c_1(Y) < 0$ . See e.g., Xiao [Xi<sub>1</sub>, Xi<sub>2</sub>] for effective bounds of orders of automorphism groups for the general type surfaces. A vast literature is devoted to automorphism groups of K3 and Enriques surfaces. These groups are discrete, and infinite in many cases. Finite groups

<sup>1</sup>I.e.,  $H^0(\mathcal{O}_{\mathbb{P}^n}(1)) \rightarrow H^0(\mathcal{O}_Y(1))$  is a surjection.

<sup>2</sup>Which generalizes earlier results by Andreotti for surfaces of general type; see Kobayashi-Ochiai [KO] and Noguchi-Sunada [NS] for further generalizations.

of automorphism of  $K3$ -surfaces were classified e.g., in [Dol<sub>2</sub>], [IS], [Ko<sub>2</sub>], [Mas], [Ni], [Xi<sub>2</sub>].

For varieties of non-general type we have the following result due to Kalka-Shiffman-Wong [KSW] and Lin [LZ, Theorem II.3.1.2].

**Theorem 1.3.** *Let  $Y$  be a smooth projective variety of dimension  $n$ . Suppose that not all Chern numbers of  $Y$  vanish and either  $c_1(Y) \leq 0$  or  $H^{n,0}(Y) \neq 0$ . Then  $\text{Aut}(Y)$  is a discrete group.*

The first assumption is fulfilled, for instance, if  $e(Y) = c_n(Y) \neq 0$ , or  $e(\mathcal{O}_Y) \neq 0$ , or  $c_1^n(Y) \neq 0$ , where  $e$  stands for the Euler characteristic. However, this assumption does not hold for an abelian variety  $Y = A$ . For any projective embedding  $A \hookrightarrow \mathbb{P}^N$ , the group  $\text{Lin}(A)$  is finite, see [GH, §II.6], while the group  $\text{Aut}(A) \supseteq A$  is infinite. Conversely, by Théorème Principale I of Blanchard [Bl] for any finite subgroup  $G \subseteq \text{Aut}(A)$  there exists a projective embedding  $A \hookrightarrow \mathbb{P}^N$  which linearizes  $G$ . A general form of Blanchard's Theorem is as follows (cf. [Ak<sub>3</sub>, Theorem 3.2.1]).

**Theorem 1.4.** *Let  $Y$  be a smooth projective variety and  $G \subseteq \text{Aut}(Y)$  a subgroup which acts finitely on  $\text{Pic}(Y)$ . Then there is a  $G$ -equivariant projective embedding  $Y \hookrightarrow \mathbb{P}^N$ .*

Indeed, such an embedding corresponds to a very ample  $G$ -invariant divisor class. However, if  $G$  acts finitely on  $\text{Pic}(Y)$  then the orbit of any ample class is an ample  $G$ -invariant class.

For instance, if  $\text{Pic}(Y)$  is discrete and  $G$  is connected then  $G$  acts trivially on  $\text{Pic}(Y)$ . Hence there exists a  $G$ -equivariant projective embedding  $Y \hookrightarrow \mathbb{P}^N$ .

As another example, consider a smooth Fano variety  $Y$  embedded by a pluri-anticanonical system  $\varphi_{|-mK_Y|} : Y \hookrightarrow \mathbb{P}^N$  for a suitable  $m > 0$ . The canonical bundle  $K_Y$  being stable under the action of the automorphism group  $\text{Aut}(Y)$  on  $Y$ , this embedding is equivariant and realizes  $\text{Aut}(Y)$  as a closed subgroup of  $\text{PGL}_{N+1}(\mathbb{C})$ . In particular, this applies to the anticanonical embeddings of del Pezzo surfaces  $Y_d \hookrightarrow \mathbb{P}^d$  of degree  $d \geq 3$ . A rough description of the automorphisms groups of these surfaces is as follows, see Proposition 10.1.1 in [Dol<sub>1</sub>] (cf. e.g., [De], [dF], [Ho<sub>3</sub>], [DI<sub>1</sub>, DI<sub>2</sub>], [BB], [Bla] for more delicate properties).

**Theorem 1.5.** *Let  $Y = Y_d$  be a del Pezzo surface of degree  $d \geq 3$ . Then the automorphism group  $\text{Aut}(Y)$  acts on the lattice  $Q = (\mathbb{Z}K_Y)^\perp \subseteq \text{Pic}(Y)$  preserving the intersection form. The image of the corresponding homomorphism  $\rho : \text{Aut}(Y) \rightarrow O(Q)$  is contained in the Weyl group  $W(Q)$ . The kernel of  $\rho$  is trivial for  $d \leq 5$  and is a connected linear algebraic group of dimension  $2d - 10$  for  $d \geq 6$ . More precisely, the following hold.*

- (1) *For  $d \leq 5$  the group  $\text{Aut}(Y)$  is finite.*
- (2) *For  $d \geq 6$  the identity component  $\text{Aut}_0(Y) = \ker(\rho)$  contains a 2-torus  $\mathbb{T}_2 \simeq (\mathbb{C}^*)^2$ , and  $\text{Aut}_0(Y) = \mathbb{T}_2$  for  $d = 6$ .*
- (3) *For  $d \geq 7$  besides the 2-torus  $\mathbb{T}_2$  the group  $\text{Aut}_0(Y)$  contains a subgroup isomorphic to  $\mathbb{A}_+^2 = (\mathbb{C}_+)^2$ . In particular, for  $d = 7$  there are a decomposition*

$$\text{Aut}(Y) \simeq (\mathbb{A}_+^2 \rtimes \mathbb{T}_2) \rtimes \mathbb{Z}/2\mathbb{Z}$$

and a faithful presentation  $\mathrm{Aut}_0(Y) \hookrightarrow \mathrm{GL}_3(\mathbb{C})$  with image

$$\begin{pmatrix} 1 & 0 & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}.$$

- (4) For  $d = 8$  either  $Y \rightarrow \mathbb{P}^2$  is a blowup at a point and then  $\mathrm{Aut}(Y) \simeq \mathrm{GL}_2 \rtimes \mathbb{A}_+^2$ , or  $Y \simeq \mathbb{P}^1 \times \mathbb{P}^1$  and then  $\mathrm{Aut}(Y) \simeq (\mathrm{PGL}_2(\mathbb{C}))^2 \rtimes \mathbb{Z}/2\mathbb{Z}$ .
- (5) Finally for  $d = 9$ ,  $Y \simeq \mathbb{P}^2$  and  $\mathrm{Aut}(Y) \simeq \mathrm{PGL}_3(\mathbb{C})$ .

**Remark 1.6.** An effective  $\mathbb{A}_+^2$ -action on a del Pezzo surface  $Y$  of degree  $d = 7$  can be defined via the locally nilpotent derivations

$$(1) \quad \partial_{\alpha,\beta} = \alpha z \frac{\partial}{\partial x} + \beta z \frac{\partial}{\partial y}, \quad (\alpha, \beta) \in \mathbb{A}_+^2.$$

Indeed, the induced  $\mathbb{A}_+^2$ -action on  $\mathbb{A}^3$ :

$$(\alpha, \beta) \cdot (x, y, z) = (x + \alpha z, y + \beta z, z)$$

descends to an action on  $\mathbb{P}^2$  fixing the line  $z = 0$  pointwise. The blowup at two points on this line preserves the action. Likewise one defines an  $\mathbb{A}_+^2$ -action on  $Y$  for  $d = 8$  or  $d = 9$ .

**1.3. Homogeneous and almost homogeneous varieties.** By the Borel-Remmert Theorem [Ak<sub>3</sub>, 3.9] any connected, compact, homogeneous Kähler manifold  $V$  is biholomorphic to the product  $\mathrm{Alb}(V) \times Y$  of the Albanese torus and a (generalized) flag variety  $Y = G/P$  (i.e.,  $Y$  is the quotient of a connected semisimple linear algebraic group by a parabolic subgroup)<sup>3</sup>. It follows that every simply connected homogeneous compact Kähler manifold is a flag variety and the same is true for a rational projective homogeneous variety (for homogeneous compact complex manifolds satisfying both conditions this was established by Goto [Got]). Furthermore, Grauert and Remmert [GR] carried over a result of Chow [Cho] from abstract algebraic to Moishezon varieties. Namely, they proved that a homogeneous Moishezon variety is projective algebraic. Thus, if such a variety is simply connected or rational, it is a flag variety.

Every flag variety  $G/P$  is a projective rational Fano variety (see [Sn]). Every ample line bundle  $L$  on  $G/P$  is very ample (see e.g., [Chev<sub>2</sub>], [Ja], [La, §3.3.2], or [Te, Theorem 7.52]). The complete linear system  $|L|$  defines a  $G$ -equivariant embedding  $Y \hookrightarrow \mathbb{P}^n$  with a projectively normal image [RR, Theorem 1.iii].

For a maximal parabolic subgroup  $P_{\max} \subseteq G$ , the Picard group  $\mathrm{Pic}(G/P_{\max}) \cong \mathbb{Z}$  is generated by the class of a unique Schubert divisorial cycle in  $G/P_{\max}$ , and this class is very ample. In the case of a Grassmannian this class gives the Plücker embedding. For an arbitrary flag variety  $G/P$ , its Picard group  $\mathrm{Pic}(G/P)$  is also generated by the classes of the Schubert divisorial cycles; see e.g., [Chev<sub>2</sub>] or [Po<sub>2</sub>]. The set of maximal parabolic subgroups  $P_{\max}$  of  $G$  which contain  $P$  is finite. Every Schubert divisor class in  $\mathrm{Pic}(G/P)$  is lifted via a surjection  $G/P \rightarrow G/P_{\max}$ , see e.g. [LL] or [Sn]. A linear combination of these divisors is very ample if and only if its coefficients are all positive (see [Br] for the case of a full flag variety; the general case is similar [Te, Theorem 7.52]).

For a description of the automorphism groups of flag varieties see e.g., [Ak<sub>3</sub>, §3.3].

<sup>3</sup>See also [SDS] for a more general result in the projective case.

The following “cone theorem” describes certain *almost homogeneous* complex varieties. It is due to Akhiezer [Ak<sub>1</sub>, Theorem 3] in algebraic context and to Huckleberry and E. Oeljeklaus [HO<sub>1</sub>] in analytic one<sup>4</sup>.

**Theorem 1.7.** *Let  $X$  be an irreducible reduced complex space of dimension  $\geq 2$ . Suppose that a connected complex Lie group acts by biholomorphic transformations on  $X$  with an open orbit  $\Omega \subseteq X$  such that the complement  $E = X \setminus \Omega$  is a proper analytic subset with an isolated point, say,  $0 \in E$ . Then the normalization  $\nu: \tilde{X} \rightarrow X$  is one-to-one and  $\tilde{X}$  is biholomorphic to a projective or an affine cone over a flag variety  $G/P$  of some semisimple linear algebraic group  $G$  under a certain equivariant projective embedding. The isolated point  $0 \in E$  corresponds to the vertex of the cone. In particular, if  $(X, 0)$  is smooth then  $X \simeq \mathbb{A}^n$  or  $X \simeq \mathbb{P}^n$ .*

Thus the variety  $X$  as in the theorem equipped with an appropriate algebraic structure carries a regular almost transitive group action. If the initial group is a complex linear algebraic group, then  $G$  is its maximal semisimple subgroup [Ak<sub>1</sub>]. Given any  $G$ -equivariant projective embedding  $\varphi_{|H|}: Y = G/P \hookrightarrow \mathbb{P}^n$ , where  $\dim Y \geq 1$ , the affine cone  $\text{AffCone}_H(Y)$  over the image admits a regular action transitive off the vertex of a locally direct product  $\tilde{G} \cdot \mathbb{C}^*$ , with  $\mathbb{C}^*$  acting by homotheties, where  $\tilde{G} \rightarrow G$  is a finite group cover.

A similar description exists for the class of quasi-projective  $G$ -varieties  $X$ , where  $G$  is a connected linear algebraic group acting on  $X$  with an open orbit  $\Omega$ , provided that there is an equivariant completion  $\tilde{X}$  of  $X$  with disconnected complement  $\tilde{X} \setminus \Omega$  [Ak<sub>1</sub>, Theorem 2]. See also [Ak<sub>2</sub>] for the case that  $\tilde{X} \setminus \Omega$  is a  $G$ -orbit of codimension 1 in  $\tilde{X}$  (in this case it is connected).

An explicit description of almost homogeneous 2-dimensional affine cones over smooth projective curves is due to Popov [Po<sub>1</sub>] (see also [FZ<sub>2</sub>] for an alternative proof). We recall that a Veronese cone  $V_d$  is the affine cone over a smooth rational normal curve  $\Gamma_d \subset \mathbb{P}^d$  i.e., a linearly non-degenerate<sup>5</sup> smooth curve in  $\mathbb{P}^d$  of degree  $d$ . All such curves in  $\mathbb{P}^d$  are projectively equivalent and rational. For normal 2-dimensional cones, Popov’s Theorem can be stated as follows.

**Theorem 1.8** (V. Popov). *Let  $X$  be the affine cone over a smooth projective curve  $Y$ . If  $X$  is normal and admits an algebraic group action transitive in  $X \setminus \{0\}$ , then  $X$  is a Veronese cone  $V_d$  for some  $d \geq 1$ , and  $Y$  is a rational normal curve  $\Gamma_d$ .*

Popov [Po<sub>1</sub>] actually classified all almost homogeneous cones in dimension 2 with an isolated singularity (not necessarily normal). Every such cone possesses a linear  $\text{SL}(2, \mathbb{C})$ -action transitive off the vertex. The group  $\text{Aut}(X)$  of a Veronese cone is infinite dimensional and so cannot be linearized under an affine embedding; see Section 2.3 below.

## 2. GROUPS ACTING ON AFFINE CONES

**2.1. Linear automorphisms of affine cones.** Let us start with the following result.

<sup>4</sup>The smooth compact case was done first in [Oe]. See also [HO<sub>2</sub>, Ch. 2, §3, Theorem 1] for real groups.

<sup>5</sup>A projective variety  $Y \subseteq \mathbb{P}^n$  is linearly non-degenerate if it is not contained in any hyperplane.



**Proposition 2.1.** *Given two affine cones  $X_i = \text{AffCone}(Y_i) \subseteq \mathbb{A}^{n_i+1}$  over smooth, linearly non-degenerate, projective varieties  $Y_i \subseteq \mathbb{P}^{n_i}$  ( $i = 1, 2$ ) and an isomorphism  $\varphi : X_1 \xrightarrow{\cong} X_2$ , the differential  $d\varphi(0)$  provides a linear isomorphism  $\Phi : \mathbb{A}^{n_1+1} \xrightarrow{\cong} \mathbb{A}^{n_2+1}$  which restricts to an isomorphism  $\Phi|_{X_1} : X_1 \xrightarrow{\cong} X_2$ . In particular  $n_1 = n_2$ , and  $Y_1$  and  $Y_2$  are projectively equivalent.*

*Proof.* By the linear non-degeneracy assumption

$$T_0X_i = \mathbb{A}^{n_i+1}, \quad C_0X_i = X_i, \quad \text{and} \quad \mathbb{P}(C_0X_i) = Y_i, \quad i = 1, 2,$$

where  $T_0X_i$  is the Zariski tangent space to  $X_i$  at the vertex  $0 \in X_i$ , and  $C_0X_i$  is the tangent cone in 0 (see e.g., [CLS, §9.7]). Now the assertion follows since  $d\varphi(0)$  provides an isomorphism of the Zariski tangent spaces and sends the cone  $C_0X_1$  onto the cone  $C_0X_2$  [Da, §7.3]. In fact  $d\varphi(0)$  lifts to an isomorphism of blowups  $\text{Bl}_0(X_1) \xrightarrow{\cong} \text{Bl}_0(X_2)$  preserving the exceptional divisors. These divisors are isomorphic to  $Y_1$  and  $Y_2$ , respectively, and  $d\varphi(0)$  induces a linear isomorphism  $Y_1 \xrightarrow{\cong} Y_2$ .  $\square$

**Remark 2.2.** The isomorphism  $\varphi$  as in Proposition 2.1 does not need to be linear itself. However, this is the case under the additional assumption that  $Y_1$  is not birationally ruled (see Proposition 2.3 below). A birationally ruled projective variety is a variety birationally equivalent to a product  $Z \times \mathbb{P}^1$ . Recall also that a birational map  $\bar{f} : \bar{X}_1 \dashrightarrow \bar{X}_2$  is said to be *isomorphism in codimension one* if there are subsets  $B_i \subseteq \bar{X}_i$  of codimension at least 2 such that

$$\bar{f}|(\bar{X}_1 \setminus B_1) : \bar{X}_1 \setminus B_1 \rightarrow \bar{X}_2 \setminus B_2$$

is an isomorphism.

**Proposition 2.3.** *Consider the affine cones  $X_i = \text{AffCone}(Y_i) \subseteq \mathbb{A}^{n_i+1}$  over projective varieties  $Y_i \subsetneq \mathbb{P}^{n_i}$ ,  $i = 1, 2$ . Suppose that  $Y_1$  and  $Y_2$  are smooth, irreducible, and linearly non-degenerate. If  $Y_1$  is not birationally ruled then every isomorphism  $\varphi : X_1 \xrightarrow{\cong} X_2$  extends to a unique linear isomorphism  $\mathbb{A}^{n_1+1} \xrightarrow{\cong} \mathbb{A}^{n_2+1}$ . In particular  $n_1 = n_2$ , and  $Y_1$  and  $Y_2$  are projectively equivalent.*

The proposition follows immediately from Lemmas 2.7 and 2.8 below. Before passing to the lemmas, let us give two corollaries, which are the main results of this subsection.

**Corollary 2.4.** *Let  $X = \text{AffCone}(Y) \subseteq \mathbb{A}^{n+1}$  be the affine cone over a smooth projective variety  $Y \subseteq \mathbb{P}^n$ . If  $Y$  is not birationally ruled then  $\text{Aut}(X) = \text{Lin}(X)$ . Moreover,  $\text{Aut}(X)$  is a central extension of the group  $\text{Lin}(Y)$  by  $\mathbb{C}^*$ .*

Indeed, the exact sequence

$$0 \rightarrow \mathbb{C}^* \rightarrow \text{GL}(n+1, \mathbb{C}) \rightarrow \text{PGL}(n+1, \mathbb{C}) \rightarrow 0$$

yields the following one:

$$(2) \quad 0 \rightarrow \mathbb{C}^* \longrightarrow \text{Lin}(X) \xrightarrow{\pi} \text{Lin}(Y) \rightarrow 0.$$

**Corollary 2.5.** *Let  $X = \text{AffCone}(Y)$  be the affine cone over a smooth projective 3-fold  $Y$ . Suppose that  $Y$  is rationally connected and non-rational. Then  $\text{Aut}(X) = \text{Lin}(X)$ .*

*Proof.* Indeed if  $Y$  were birationally ruled i.e., birational to a product  $Z \times \mathbb{P}^1$ , then  $Z$  would be rationally connected and so a rational surface. Hence  $Y$  would be rational too, contrary to our assumption. Thus Corollary 2.4 applies and gives the assertion.  $\square$

**Example 2.6.** For instance, if  $Y \subseteq \mathbb{P}^n$  is a non-rational Fano 3-fold and  $X = \text{AffCone}(Y)$ , then  $\text{Aut}(X) = \text{Lin}(X)$ . As an example, one can consider any smooth cubic or quartic 3-fold  $Y \subseteq \mathbb{P}^4$ .

For the proof of the next lemma we refer the reader to [To, Theorem 2.2] or [Co, Proposition 2.7].

**Lemma 2.7.** *Let  $X_i = \bar{X}_i \setminus D_i$ ,  $i = 1, 2$ , where  $\bar{X}_i$  is a projective variety and  $D_i$  an irreducible divisor on  $\bar{X}_i$ . Suppose that  $X_i$  is regular near  $D_i$  for  $i = 1, 2$ . If  $D_1$  is not birationally ruled then any isomorphism  $f : X_1 \rightarrow X_2$  extends to a birational map  $\bar{f} : \bar{X}_1 \dashrightarrow \bar{X}_2$  which is an isomorphism in codimension 1. If in addition the divisors  $D_1$  and  $D_2$  are ample then  $\bar{f} : \bar{X}_1 \xrightarrow{\cong} \bar{X}_2$  is an isomorphism.*

Proposition 2.3 is now a direct consequence of the following lemma.

**Lemma 2.8.** *Consider two projective varieties  $Y_i \subseteq \mathbb{P}^{n_i}$ , where  $n_i \geq 2$ ,  $i = 1, 2$ . Suppose that  $Y_1$  and  $Y_2$  are smooth, irreducible, and linearly non-degenerate. Consider also the affine cones  $X_i = \text{AffCone}(Y_i) \subseteq \mathbb{A}^{n_i+1}$  over  $Y_i$  and the projective cones  $\bar{X}_i \subseteq \mathbb{P}^{n_i+1}$ ,  $i = 1, 2$ . Let  $\varphi : X_1 \xrightarrow{\cong} X_2$  be an isomorphism such that the induced birational map  $\bar{\varphi} : \bar{X}_1 \dashrightarrow \bar{X}_2$  is an isomorphism in codimension 1. Then  $\varphi$  extends to a unique linear isomorphism  $\Phi : \mathbb{A}^{n_1+1} \xrightarrow{\cong} \mathbb{A}^{n_2+1}$ . In particular  $n_1 = n_2$ , and  $Y_1$  and  $Y_2$  ( $\bar{X}_1$  and  $\bar{X}_2$ , respectively) are projectively equivalent.*

*Proof.* We let  $D_i = \bar{X}_i \setminus X_i$  denote the divisor at infinity; it is a scheme-theoretic hyperplane section. Since  $D_1$  and  $D_2$  are ample then (similarly as in Lemma 2.7)  $\varphi$  extends to an isomorphism  $\bar{\varphi} : \bar{X}_1 \rightarrow \bar{X}_2$ , which sends  $0 \in \mathbb{A}^{n_1+1}$  to  $0 \in \mathbb{A}^{n_2+1}$ . Indeed, these points are the only singular points of the projective cones  $\bar{X}_1$  and  $\bar{X}_2$ . Moreover,  $\bar{\varphi}$  sends the generators of the cone<sup>6</sup>  $\bar{X}_1$  into generators of  $\bar{X}_2$ . Indeed, every generator  $l_1$  of  $\bar{X}_1$  meets  $D_1$  transversally in one point. The image  $l_2 = \varphi(l_1) \subseteq \bar{X}_2$  possesses similar properties, hence  $l_2$  is again a projective line through the origin i.e., a generator of the cone  $\bar{X}_2$ .

It follows that the orbits of the  $\mathbb{C}^*$ -action on  $\bar{X}_1$  are sent to the orbits of the  $\mathbb{C}^*$ -action on  $\bar{X}_2$ . Furthermore  $\bar{\varphi}$  is  $\mathbb{C}^*$ -equivariant, hence it induces an isomorphism  $\varphi^* : \mathcal{O}(Y_2) \xrightarrow{\cong} \mathcal{O}(Y_1)$  of the homogeneous coordinate rings. These graded rings are the coordinate rings of the affine cones  $X_1$  and  $X_2$ , respectively, generated by their first graded pieces<sup>7</sup>. The graded isomorphism  $\varphi^*$  restricts to a linear isomorphism, say,  $\Psi : \mathbb{A}^{n_2+1} \xrightarrow{\cong} \mathbb{A}^{n_1+1}$  between these first graded pieces. The dual isomorphism  $\Phi = \Psi^\vee : \mathbb{A}^{n_1+1} \xrightarrow{\cong} \mathbb{A}^{n_2+1}$  provides a desired linear extension of  $\varphi$ . The uniqueness of such an extension follows immediately, since  $Y_1$  and  $Y_2$  are assumed to be linearly non-degenerate.  $\square$

For a projective variety  $Y \subseteq \mathbb{P}^n$  with affine cone  $X = \text{AffCone}(Y)$  it can happen that  $\text{Aut}(Y) \neq \text{Lin}(Y)$ , while  $\text{Aut}(X) = \text{Lin}(X)$ , as in the following examples.

**Examples 2.9.** 1. Let  $A$  be an abelian variety. Consider a projective embedding  $A \hookrightarrow \mathbb{P}^n$  (for instance, a smooth cubic in  $\mathbb{P}^2$ ) with affine cone  $X = \text{AffCone}(A)$ . By Corollary 2.4  $\text{Aut}(X) = \text{Lin}(X)$ . While  $\text{Lin}(A)$  is a finite group (see [GH, §II.6] or

<sup>6</sup>That is the projective lines on  $\bar{X}_1$  passing through the origin.

<sup>7</sup>Consisting of the restrictions to  $X_i$  of linear functions on  $\mathbb{A}^{n_i+1}$ ,  $i = 1, 2$ .

Section 1 above), the group  $\text{Aut}(A)$  contains the subgroup of translations and so is infinite. Thus  $\text{Aut}(Y) \neq \text{Lin}(Y)$  (cf. Blanchard's Theorem 1.4).

2. A smooth quartic  $Y \subseteq \mathbb{P}^3$  is a K3-surface and so is not birationally ruled. Hence again  $\text{Aut}(X) = \text{Lin}(X)$ , where  $X = \text{AffCone}(Y) \subseteq \mathbb{A}^4$ . Moreover,  $\text{Lin}(Y)$  is a finite group, while the group  $\text{Aut}(Y)$  can be infinite, see the discussion in §1.1. Clearly, non-linear automorphisms of  $Y$  are not induced by automorphisms of  $X$ .

**2.2. Lifting  $G$ -actions to affine cones.** In this subsection we address the following questions.

- (1) *When a  $G$ -action on  $Y$  is induced by a  $G$ -action on  $X$ ?*
- (2) *When a  $G$ -action on  $Y$  is induced by a  $\tilde{G}$ -action on  $X$ ?*

A related question is:

*Which projective representations can be lifted to linear ones?*

Simple examples show that one needs some restrictions on such a projective representation. In the first example below the group  $G$  is finite, and is connected algebraic in the second.

**Examples 2.10.** 1. The standard representation on  $\mathbb{A}^2$  of the group of quaternions  $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$  induces a faithful representation of  $Q_8$  on any Veronese cone  $V_d \simeq \mathbb{A}^2/\mathbb{Z}_d$  with  $d$  odd (cf. Subsection 2.4 below). The latter representation descends to an effective linear action on  $\mathbb{P}^1$  of the dihedral group

$$D_2 = Q_8/Z(Q_8) \simeq (\mathbb{Z}/2\mathbb{Z})^2.$$

However, this  $D_2$ -action on  $\mathbb{P}^1$  cannot be lifted to a  $D_2$ -action on  $\mathbb{A}^2$  or on any of the Veronese cones  $V_d$  with  $d$  odd. Indeed, otherwise the exact sequence

$$0 \longrightarrow Z(Q_8) \longrightarrow Q_8 \longrightarrow D_2 \longrightarrow 0$$

would split, which is not the case. In other words, the faithful projective representation  $D_2 \rightarrow \text{PGL}_{n+1}(\mathbb{C})$  induced by the Veronese embedding  $\varphi|_{\mathcal{O}_{\mathbb{P}^1}(n)} : \mathbb{P}^1 \rightarrow \mathbb{P}^n$  lifts to a linear representation  $D_2 \rightarrow \text{GL}_{n+1}(\mathbb{C})$  if and only if  $n = 2k > 0$  is even and so  $\mathcal{O}_{\mathbb{P}^1}(n) = -kK_{\mathbb{P}^1}$ .

2. The standard projective representation of  $G = \text{PGL}_2(\mathbb{C})$  on  $\mathbb{P}^1$  induces a linear  $G$ -action on the rational normal curve  $\Gamma_d \subseteq \mathbb{P}^d$ . Suppose that the latter action can be lifted to the Veronese cone  $V_d = \text{AffCone}(\Gamma_d) \subseteq \mathbb{A}^{d+1}$ . This would give an irreducible representation of  $G = \text{PGL}_2(\mathbb{C})$  of dimension  $d + 1$ . However, such a representation does exist only for  $d$  even. Indeed, every irreducible representation of  $\text{PGL}_2(\mathbb{C})$  yields an irreducible representation of  $\text{SL}_2(\mathbb{C})$  trivial on the center, and vice versa.

**Remark 2.11.** Concerning question (2), recall that for any perfect group  $G$  there exists a unique universal central extension (or Schur cover)  $G'$  of  $G$  such that every projective representation of  $G$  is induced by a linear representation of  $G'$  (see [St, §7]). For a finite perfect group  $G$ , the Schur cover  $G'$  is again finite. For a perfect (e.g., semi-simple) connected linear algebraic group  $G$  over  $\mathbb{C}$ , the Schur cover is just the simply connected universal covering group  $G'$  of  $G$ .

Any linear action  $G \rightarrow \text{Lin}(X)$  on the affine cone  $X = \text{AffCone}(Y)$  induces (via the exact sequence (2)) a linear action  $G \rightarrow \text{Lin}(Y)$  on  $Y$ . The latter factorizes through the action on  $Y$  of the quotient group  $G/(G \cap \mathbb{C}^*)$ . Answering question (2) above,

in the following proposition we provide a simple criterion as to when a  $G$ -action on  $Y$  is induced by a  $\tilde{G}$ -action on  $X = \text{AffCone}(Y)$ , where  $\tilde{G}$  is a central extension of  $G$  (which is not a Schur cover). The proof is straightforward. Let us remind that any linear action  $G \rightarrow \text{Lin}(Y)$  of a group  $G$  on a projective variety  $Y \subseteq \mathbb{P}^n$  stabilizes the very ample divisor class  $[\mathcal{O}_Y(1)] \in \text{Pic}(Y)$ .

**Proposition 2.12.** (a) *Let  $Y$  be a smooth projective variety and  $G \rightarrow \text{Aut}(Y)$  be a group action on  $Y$ . If this action stabilizes a very ample divisor class  $|H| \in \text{Pic}(Y)$ , then it extends linearly to the ambient projective space  $\mathbb{P}^n = \mathbb{P}H^0(Y, \mathcal{O}_Y(H))$ .*

(b) *Furthermore, let  $X = \text{AffCone}_H(Y)$  be the affine cone over  $\varphi_{|H|}(Y)$ . Consider the central extension  $\tilde{G} = \pi^{-1}(G) \subseteq \text{Lin}(X)$  of  $G$  by  $\mathbb{C}^*$ , where  $\pi : \text{Lin}(X) \rightarrow \text{Lin}(Y)$  is as in (2). Then the group  $\tilde{G}$  acts linearly on  $X$  inducing the given  $G$ -action on  $Y$ .*

**Corollary 2.13.** *Let  $G$  be a connected linear algebraic group. Then any regular  $G$ -action on a smooth projective variety  $Y \subseteq \mathbb{P}^n$  is induced by a regular  $\tilde{G}$ -action on the affine cone  $\text{AffCone}(Y)$ , where  $\tilde{G} = \pi^{-1}(G) \subseteq \text{Lin}(X)$  is a central extension of  $G$  by  $\mathbb{C}^*$ .*

*Proof.* Since  $G$  is connected,  $G$  acts on  $\text{Pic}_0(Y)$ . The group  $G$  being a rational variety [Chev<sub>1</sub>], every morphism of  $G$  to the abelian variety  $\text{Pic}_0(Y)$  is constant. Hence the  $G$ -action on  $\text{Pic}_0(Y)$  is trivial, and so is the induced action on the Neron-Severi group  $NS(Y) = \text{Pic}(Y)/\text{Pic}_0(Y)$ . Thus  $G$  acts trivially on  $\text{Pic}(Y)$ . By Proposition 2.12(a) the  $G$ -action on  $Y$  extends linearly to  $\mathbb{P}^n$ . Now the result follows.  $\square$

**Remark 2.14.** Instead of referring to [Chev<sub>1</sub>] one can show directly that every morphism  $f : G \rightarrow A$  to an abelian variety  $A$  is constant. Clearly,  $f$  is constant on any abelian subgroup of  $G$  and on its cosets. Hence  $f$  is also constant on any solvable subgroup. In particular, it is constant on  $\text{Rad}(G)$  and on its cosets. Thus  $f$  induces a morphism  $G/\text{Rad}(G) \rightarrow A$ . So we may assume that  $G$  is semisimple. Consider a maximal torus  $\mathbb{T} \subseteq G$  and the collection of its root vectors  $(H_\alpha)_\alpha \subseteq T_e G = \text{lie}(G)$ . The subset  $T_e \mathbb{T} \cup (H_\alpha)_\alpha$  consists of the tangent vectors of algebraic one-parameter subgroups of  $G$  and spans the tangent space  $T_e G$ . Hence the differential  $df(e)$  vanishes. Now the assertion follows. Indeed, applying left shifts one can produce a similar situation in any point  $g$  of  $G$ .

A stronger statement holds for pluri-canonical or pluri-anticanonical embeddings.

**Proposition 2.15.** *Let  $Y$  be a smooth projective variety. Suppose that for some  $m \in \mathbb{Z}$  there is an embedding  $\varphi = \varphi_{|mK_Y|} : Y \hookrightarrow \mathbb{P}^n$ , and let  $X = \text{AffCone}(\varphi(Y))$ . Then*

$$\text{Lin}(X) = \mathbb{C}^* \times \text{Lin}(\varphi(Y)) \simeq \mathbb{C}^* \times \text{Aut}(Y),$$

where  $\mathbb{C}^*$  acts on the cone  $X$  by scalar matrices.

*Proof.* Indeed, the group  $\text{Aut}(Y)$  acts on the linear system  $|mK_Y|$  yielding an isomorphism  $\text{Aut}(Y) \simeq \text{Lin}(\varphi(Y))$ . Moreover,  $\text{Aut}(Y)$  acts on the linear bundle  $\mathcal{O}(mK_Y)$ . Hence it acts linearly on  $H^0(Y, \mathcal{O}(mK_Y))$ . The dual action on  $H^0(Y, \mathcal{O}(mK_Y))^\vee$  preserves the cone  $X$ . This gives an embedding  $\text{Aut}(Y) \hookrightarrow \text{Lin}(X)$  and a splitting of the exact sequence

$$0 \rightarrow \mathbb{C}^* \rightarrow \text{Lin}(X) \rightarrow \text{Lin}(\varphi(Y)) \simeq \text{Aut}(Y) \rightarrow 0.$$

Since the subgroup  $\mathbb{C}^* \subseteq \text{Lin}(X)$  is central, the assertions follow.  $\square$

This proposition can be applied to the anticanonical embeddings of del Pezzo surfaces. In the case where there is a  $\mathbb{C}_+$ -action on  $X$  the group  $\text{Aut}(X)$  is infinite dimensional. For instance, this is so for the cones over del Pezzo surfaces of degree  $d \geq 4$ . For  $d \geq 7$  there exists a linear  $\mathbb{A}_+^2$ -action on  $X$ . While for  $6 \geq d \geq 4$  the group  $\text{Aut}(Y)$  is finite or toric, hence any  $\mathbb{C}_+$ -action on  $X$  is non-linear; cf. Theorem 0.1 in the Introduction and also Theorems 1.5 and 3.19.

**2.3. Groups acting on affine cones.** Similarly as in Proposition 2.1, in the case of a reductive group action a weaker analog of Corollary 2.4 holds without the assumption of birational non-ruledness.

**Lemma 2.16.** *Suppose that a connected reductive group  $G$  acts effectively on the affine cone  $X \subseteq \mathbb{A}^{n+1}$  over a smooth linearly non-degenerate projective variety  $Y \subsetneq \mathbb{P}^n$ . Then there is a faithful representation  $\rho : G \rightarrow \text{GL}(n+1, \mathbb{C})$ , which restricts to an effective linear  $G$ -action on  $X$  inducing a linear action of  $G$  on  $Y$ .*

*Proof.* The vertex  $0 \in X$  is an isolated singular point of  $X$ , hence a fixed point of  $G$ . Since  $G$  is reductive, the induced representation  $\rho$  of  $G$  on the Zariski tangent space  $T_0X$  is faithful (see e.g., [Ak<sub>3</sub>] or [FZ<sub>2</sub>, Lemma 2.7(b)]) and descends to  $Y$  via the projective representation  $\bar{\rho} : G \rightarrow \text{PGL}(n+1, \mathbb{C}) = \text{GL}(n+1, \mathbb{C})/\mathbb{C}^*$ .  $\square$

Let us note that for a non-reductive group action,  $\rho$  as above can be trivial. For instance, this is the case for the  $\mathbb{C}_+$ -action  $t.(x, y) = (x + ty^2, y)$  on  $X = \mathbb{A}^2$ .

The following theorem is complementary to Corollary 2.4; cf. also [HO<sub>1</sub>] for (a).

**Theorem 2.17.** *We let  $X \subseteq \mathbb{A}^n$  ( $n \geq 2$ ) be the affine cone over a smooth projective variety  $Y \subseteq \mathbb{P}^{n-1}$ . Suppose that*

- *The group  $\text{Aut}(Y)$  is finite.*
- *A connected algebraic group  $G$  of dimension  $\geq 2$  acts effectively on  $X$  and contains a 1-dimensional torus  $\mathbb{T} \simeq \mathbb{C}^*$  acting on  $\mathbb{A}^n$  via scalar matrices.*

*Then the following hold.*

- (a)  *$G$  is a solvable group of rank 1.*
- (b) *There exists an  $\mathbb{A}^1$ -fibration  $\theta : X \rightarrow Z$ , where  $Z$  is an affine variety equipped with a good  $\mathbb{C}^*$ -action and  $\theta$  is equivariant with respect to the standard  $\mathbb{C}^*$ -action on  $X$ . Furthermore,  $Z$  is normal if  $X$  is.*
- (c)  *$Y$  is uniruled via a family of rational curves parameterized by  $(Z \setminus \{\theta(0)\})/\mathbb{C}^*$ .*

*Proof.* Consider a Levi decomposition  $G = \text{Rad}_u(G) \rtimes L$ , where  $L \subseteq G$  is a Levi subgroup (i.e., a maximal connected reductive subgroup) containing  $\mathbb{T}$ . By Lemma 2.16 the induced representation  $\rho$  of  $L$  on the Zariski tangent space  $T_0X$  is faithful. Moreover  $\mathbb{T}$  (which acts on  $T_0X$  by scalar matrices) is a central subgroup of  $L$ . Since the group  $\text{Aut}(Y)$  is finite and  $L$  is connected, the induced action of the quotient group  $L/\mathbb{T}$  on  $Y$  is trivial. Thus  $L = \mathbb{T}$  is a maximal torus of  $G = \text{Rad}_u(G) \rtimes \mathbb{T}$ , and so  $G$  is solvable of rank 1.

By our assumption  $\dim_{\mathbb{C}}(G) \geq 2$ . Hence the unipotent radical  $\text{Rad}_u(G)$  is non-trivial and contains a one-parameter subgroup  $U \simeq G_a$ . All orbits of  $U$  are closed in  $X$ , and the one-dimensional orbits are isomorphic to the affine line  $\mathbb{A}^1$ . Therefore  $X$  is affine

uniruled. Its coordinate ring  $A = \mathcal{O}_X$  is graded by the dual lattice  $\mathbb{T}^\vee \simeq \mathbb{Z}$ . This grading is actually positive:

$$A = \bigoplus_{k \geq 0} A_k.$$

The infinitesimal generator  $\partial$  of the induced  $G_a$ -action on  $A$  is a homogeneous locally nilpotent derivation of  $A$  (see e.g., [Re] or [FZ<sub>2</sub>]). The ring of invariants  $B = \ker(\partial) = A^{G_a}$  is a graded subalgebra of  $A$  with  $B_0 = A_0 = \mathbb{C}$ . Therefore the affine variety  $Z = \text{spec}(B)$  is endowed by a  $\mathbb{T}$ -action with a unique attractive fixed point  $0' = \theta(0)$ , where  $\theta : X \rightarrow Z$  is the orbit map of the  $G_a$ -action on  $X$ . Thus  $\theta$  is a  $\mathbb{T}$ -equivariant surjection induced by the inclusion  $B \subseteq A$  of graded rings. If  $A$  is integrally closed in  $\text{Frac}(A)$  then also  $B$  is. Indeed, let  $Z' = \text{spec}(\bar{B})$  be the normalization of  $Z$ , where  $\bar{B}$  is the integral closure of  $B$  in  $\text{Frac}(A)$ . Since  $X$  is normal the morphism  $X \rightarrow Z$  factorizes as  $X \rightarrow Z' \xrightarrow{\nu} Z$ . The locally nilpotent derivation  $\partial$  stabilizes  $\bar{B}$  (see e.g., [Sei], [Vas], or [FZ<sub>1</sub>, Lemma 2.15]) and so the morphisms  $X \rightarrow Z' \xrightarrow{\nu} Z$  are equivariant with respect to the induced  $\mathbb{C}_+$ -actions. The  $\mathbb{C}_+$ -action is trivial on  $Z$ , hence also on  $Z'$  since  $Z' \rightarrow Z$  is finite. Thus  $B \subseteq \bar{B} \subseteq \ker \partial = B$ , so  $B = \bar{B}$  is normal as soon as  $A$  is.

Since a general one-dimensional orbit of  $U \simeq G_a$  in  $X$  does not pass through the vertex  $0 \in X$  and is not contained in an orbit closure of  $T$  (i.e., in a generator of the cone), there is a Zariski open subset, say,  $\Omega$  of  $Y$  covered by the images of these orbits. Taking Zariski closures yields a family of rational curves parameterized by  $(Z \setminus \{0'\})/\mathbb{C}^*$ . Thus  $Y$  is uniruled, as claimed.  $\square$

**2.4. Group actions on 2-dimensional affine cones.** The following corollary is immediate from Proposition 2.1.

**Corollary 2.18.** *Consider two smooth linearly non-degenerate curves  $Y_i \subseteq \mathbb{P}^{n_i}$  ( $i = 1, 2$ ) of degrees  $d_i$ , and let  $X_i = \text{AffCone}(Y_i) \subseteq \mathbb{A}^{n_i+1}$  be the corresponding affine cones. Then  $X_1 \simeq X_2$  if and only if these cones are linearly isomorphic, if and only if  $n_1 = n_2$ ,  $d_1 = d_2$  and  $Y_1$  and  $Y_2$  are projectively equivalent.*

Similarly, from Corollary 2.4 we deduce the following one.

**Corollary 2.19.** *Let  $X = \text{AffCone}(Y) \subseteq \mathbb{A}^{n+1}$  be the affine cone over a smooth, non-rational projective curve  $Y \subseteq \mathbb{P}^n$ . Then  $\text{Aut}(X) = \text{Lin}(X)$ , and this group is a central extension of the finite group  $\text{Lin}(Y)$  by  $\mathbb{C}^*$ .*

**Remarks 2.20.** 1. However,  $\text{Aut}(Y) \neq \text{Lin}(Y)$  for an elliptic curve  $Y \subseteq \mathbb{P}^n$ , see Example 2.9(1). Consider further a smooth rational curve  $Y \subseteq \mathbb{P}^n$  of degree  $d > n$ . Then  $Y$  is neither linearly nor projectively normal. Indeed,  $Y$  is a linear projection of the rational normal curve  $\Gamma_d \subseteq \mathbb{P}^d$ , and  $X = \text{AffCone}(Y)$  is a linear projection of the Veronese cone  $V_d = \text{AffCone}(\Gamma_d)$ . The latter projection gives a normalization of  $X$ . This is not an isomorphism as it diminishes the dimension of the Zariski tangent space at the vertex.

2. The normalizations of the affine cones  $X_1$  and  $X_2$  over two smooth rational curves  $Y_1$  and  $Y_2$ , respectively, are isomorphic if and only if  $\deg(Y_1) = \deg(Y_2)$ . While in general the (non-normal) affine surface  $X = \text{AffCone}(Y)$  admits non-trivial equisingular deformations arising from deformations of the projective embedding  $Y \hookrightarrow \mathbb{P}^n$ . For instance, smooth rational curves  $Y$  of type  $(1, a)$  on a quadric  $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$  vary in

a family of projective dimension  $2a + 1$ . Hence for any  $a \geq 3$  the group  $\mathrm{PSO}(4, \mathbb{C})$  cannot act transitively on this family.

3. Since any group action on an affine cone  $X$  lifts to the normalization, it is enough to restrict to normal cones. For the normal Veronese cone  $X = V_d \subseteq \mathbb{A}^{d+1}$  over a rational normal curve  $Y = \Gamma_d \subseteq \mathbb{P}^d$  we have  $\mathrm{Aut}(X) \neq \mathrm{Lin}(X)$ . Moreover,  $V_d \simeq \mathbb{A}^2/\mathbb{Z}_d$  being a toric surface, for every  $d \geq 1$  the group  $\mathrm{Aut}(V_d)$  is infinite dimensional. In particular this is not an algebraic group. Indeed, the graded coordinate ring  $\mathcal{O}(V_d)$  admits a nonzero locally nilpotent derivation  $\partial$  corresponding to an effective  $\mathbb{C}_+$ -action on  $V_d$  [FZ<sub>2</sub>]. The kernel  $\ker(\partial) \subseteq \mathcal{O}(V_d)$  is isomorphic to the polynomial ring  $\mathbb{C}[t]$ . For any  $p \in \mathbb{C}[t]$ , the derivation  $p \cdot \partial$  is again locally nilpotent. Thus  $\mathbb{C}[t] \cdot \partial$  is the Lie algebra of an infinite dimensional abelian subgroup  $G \subseteq \mathrm{Aut}(V_d)$ .

4. There are actually two independent  $\mathbb{C}_+$ -actions on  $V_d$  with different orbits, and even a continuous family of such actions; see e.g., [FZ<sub>2</sub>]. Danilov and Gizatullin [DG] studied the structure of an amalgamated product on the group  $\mathrm{Aut}(V_d)$ , while Makar-Limanov [ML] provided an explicit description of this group.

5. Similarly, independent  $\mathbb{C}_+$ -actions, and an amalgamated product structure, exist on any normal affine toric surface different from  $\mathbb{A}_*^1 \times \mathbb{A}^1$  or  $\mathbb{A}_*^1 \times \mathbb{A}_*^1$ . Every such surface is of the form  $V_{d,e} = \mathbb{A}^2/\mathbb{Z}_d$  for an appropriate diagonal action

$$\zeta \cdot (x, y) = (\zeta x, \zeta^e y), \quad \text{where } \zeta^d = 1 \quad \text{and} \quad \mathrm{gcd}(e, d) = 1,$$

of the cyclic group  $\mathbb{Z}_d = \langle \zeta \rangle$  on the affine plane  $\mathbb{A}^2$ . Choosing a system of homogeneous generators in the graded coordinate ring  $\mathcal{O}(V_{d,e})$  yields an embedding  $V_{d,e} \hookrightarrow \mathbb{A}^{N+1}$ , which is equivariant with respect to a suitable diagonal  $\mathbb{C}^*$ -action on  $\mathbb{A}^{N+1}$  with positive weights  $w = (w_0, \dots, w_N)$ . In this way  $V_{d,e}$  can be realized as the affine cone over a smooth rational curve in the corresponding weighted projective space  $\mathbb{P}_w^N = (\mathbb{A}^{N+1} \setminus \{0\})/\mathbb{C}^*$ .

6. By Popov's Theorem [Po<sub>1</sub>]<sup>8</sup> the Veronese cones  $V_d = V_{d,1}$  can be characterized as normal affine surfaces on which an algebraic group acts with an open orbit and a fixed point.

Let us construct an explicit example of a non-linear biregular automorphism of a Veronese cone  $V_d$  for every  $d \geq 1$ .

**Example 2.21.** Consider as before the Veronese cone  $V_d \subseteq \mathbb{A}^{d+1}$  over a rational normal curve  $\Gamma_d \subseteq \mathbb{P}^d$ . Let  $\bar{V}_d \subset \mathbb{P}^{d+1}$  be the Zariski closure, and  $\tilde{V}_d$  be the blowup of  $\bar{V}_d$  at the vertex  $0 \in V_d$ . It is well known [DG] that  $\tilde{V}_d \simeq \Sigma_d$ , where  $\Sigma_d$  denotes a Hirzebruch surface with the exceptional section  $S_0$  and a disjoint section at infinity, say,  $S_\infty$  with  $S_0^2 = -d$  and  $S_\infty^2 = d$ . Therefore

$$V_d \simeq (\Sigma_d \setminus S_\infty)/S_0.$$

To exhibit a non-linear automorphism of  $V_d$  is the same as to exhibit an automorphism of  $\Sigma_d \setminus S_\infty$ , which extends to a birational transformation of  $\Sigma_d$  preserving the exceptional section  $S_0$  but not the ruling  $\pi : \Sigma_d \rightarrow \mathbb{P}^1$  (or, equivalently, which blows down the curve  $S_\infty$ ). On the level of dual graphs, such a birational transformation consists e.g., in the following sequence of blowups and blowdowns [FKZ]:

<sup>8</sup>This is actually an earlier version of the Cone Theorem in dimension 2; see Section 1.

$$\begin{array}{ccccccc}
d & & -1 & -1 & A_d & & 0 & A_d & & -1 & -1 & A_d & & d \\
\circ & \dashrightarrow & \circ & \text{---} & \square & \dashrightarrow & \square & \dashrightarrow & \square & \text{---} & \square & \dashrightarrow & \circ & . \\
S_\infty & & S_\infty & & v_{d+1} & & v_{d+1} & & S'_\infty & & v_{d+1} & & S'_\infty & .
\end{array}$$

Here a box marked  $A_d$  represents the linear chain  $[[ -2, \dots, -2 ]]$  of length  $d$ . The centers of blowups on the curves  $S_\infty$  and  $v_{d+1}$  can vary. Anyhow, the section  $S_\infty$  being contracted, the resulting biregular transformation of the Veronese cone  $V_d$  is non-linear.

### 3. GROUP ACTIONS ON 3-DIMENSIONAL AFFINE CONES

The main result of this section is the existence of a  $\mathbb{C}_+$ -actions on the affine cones over every smooth del Pezzo surface of degree  $\geq 4$ . The proof exploits a general geometric criterion for the existence of such an action.

#### 3.1. Existence of $\mathbb{C}_+$ -actions on affine cones: a geometric criterion.

**3.1.** Let  $Y$  be a smooth projective variety, and let  $H \in \text{Div}(Y)$  be an ample polarization of  $Y$ . Consider the total space  $\hat{X}$  of the line bundle  $\mathcal{O}_Y(H)$  with the zero section  $S_0 \subseteq \hat{X}$ . Under the natural identification  $S_0 \simeq Y$ , we have  $\mathcal{O}_{S_0}(S_0) = \mathcal{O}_Y(-H)$ . Hence  $S_0$  is contractible, i.e., there is a birational contraction  $v : \hat{X} \rightarrow X$ , where  $X$  is a normal affine variety and  $v(S_0)$  is a point. In this situation, we call  $X$  a *generalized cone* over  $(Y, H)$ . If  $H$  is very ample, then  $X$  coincides with the normalization of the usual affine  $\text{AffCone}(Y)$  cone over  $Y \hookrightarrow \mathbb{P}^n$ , where the embedding is given by the linear system  $|H|$ . So we write  $X = \text{AffCone}_H(Y)_{\text{norm}}$ . In this section we provide a criterion of existence of a  $\mathbb{C}_+$ -action on a generalized cone.

Let us note that  $X$  can be compactified to the projective cone  $\bar{X}$  over  $Y$  by adding a divisor at infinity  $S_\infty \simeq Y$ . The divisor  $S_\infty$  on  $\bar{X}$  being ample, the variety  $X = \bar{X} \setminus S_\infty$  is affine.

**3.2.** For instance, the affine cone over  $\mathbb{P}^2$  in  $\mathbb{A}^3$  coincides with  $\mathbb{A}^3$  and so admits a transitive action of the additive group  $\mathbb{C}_+^3$ . In the following example we exhibit an effective  $\mathbb{C}_+^2$ -action on the affine cone  $X \subseteq \mathbb{A}^4$  over a smooth quadric  $Y \subseteq \mathbb{P}^3$  (cf. another constructions in [Sh]). The automorphism groups of affine quadrics were studied e.g., in [DG, Doe, To]. Over a general base field, this group is infinite dimensional as soon as the corresponding quadratic form is isotropic [To, Lemma 1.1]. The proof of Lemma 1.1 in [To] provides a nontrivial linear  $\mathbb{C}_+$ -action on any quadric over  $\mathbb{C}$ . In the following example we exhibit an explicit effective  $\mathbb{C}_+^2$ -action on the affine cone over a smooth quadric in  $\mathbb{P}^3$ .

**Example 3.3.** All smooth quadrics in  $\mathbb{P}^3$  are projectively equivalent. Choosing for instance the quadric

$$Y = \{xy = zu\}$$

we can define a linear  $\mathbb{C}_+^2$ -action on  $X = \text{AffCone}(Y) \subseteq \mathbb{A}^4$  by the following pair of commuting locally nilpotent derivations on the ring  $A = \mathcal{O}(X)$ :

$$\partial_1 = u\partial/\partial x + y\partial/\partial z \quad \text{and} \quad \partial_2 = u\partial/\partial y + x\partial/\partial z.$$

See [AS, Sh] for a more thorough treatment on the subject.

Let us introduce the following notion.



**Definition 3.4.** Let  $X$  be an affine variety. For a function  $f \in \mathcal{O}(X)$  we let

$$\mathbb{D}_+(f) = X \setminus \mathbb{V}_+(f), \quad \text{where } \mathbb{V}_+(f) := f^{-1}(0).$$

We say that  $X$  is *cylindrical* if  $X$  contains a dense principal Zariski open subset  $U = \mathbb{D}_+(f)$  isomorphic to the cylinder  $Z \times \mathbb{A}^1$  over an affine variety  $Z$ .

The following proposition generalizes Lemma 1.6 in [FZ<sub>2</sub>].

**Proposition 3.5.** *For an irreducible affine variety  $X$ , the following conditions are equivalent:*

- (i)  $X$  possesses an effective  $\mathbb{C}_+$ -action.
- (ii)  $X$  is cylindrical.

*Proof.* First we suppose that  $X$  possesses an effective  $\mathbb{C}_+$ -action  $\psi$  with the associate locally nilpotent derivation  $\partial \neq 0$ . The filtration

$$(3) \quad 0 \in \ker \partial \subsetneq \ker \partial^{(2)} \subsetneq \ker \partial^{(3)} \dots$$

being strictly increasing, we can find  $g \in \mathcal{O}(X)$  such that  $\partial^{(2)}g = 0$  but  $h := \partial g \neq 0$ . Thus  $\partial h = 0$  and so  $h \in \mathcal{O}(X)$  is  $\psi$ -invariant. Letting  $s = g/h$  and  $U = \mathbb{D}_+(h)$  the function  $s \in \mathcal{O}(U)$  gives a *slice* of  $\partial$  that is,  $\partial(s) = 1$ . Consequently, the restriction of  $s$  to any 1-dimensional orbit  $O$  of  $\psi$  in  $U$  is an affine coordinate on  $O \simeq \mathbb{A}^1$ . By the Slice Theorem ([Fr, Cor. 1.22]),  $\mathcal{O}(U) \simeq \ker(\partial)[s]$  and  $\partial = \partial/\partial s$ . Therefore  $U \simeq Z \times \mathbb{A}^1$ , where  $Z = \text{Spec}(\ker \partial) \simeq s^{-1}(0)$ . This yields (ii).

To show the converse, assume that  $X$  is cylindrical. Let  $U = \mathbb{D}_+(f) \simeq Z \times \mathbb{A}^1$  be a principal cylinder in  $X$  as in Definition 3.4. We consider the natural  $\mathbb{C}_+$ -action  $\phi$  on  $U$  by translations along the second factor. Since  $f|_U$  does not vanish it is constant along any orbit of  $\phi$  and so  $\phi$ -invariant. Letting  $\partial$  denote the locally nilpotent derivation on  $\mathcal{O}(U)$  associated to  $\phi$ , the derivation  $\partial_n := f^n \partial \in \text{Der}(\mathcal{O}(U))$  is again locally nilpotent for any  $n \in \mathbb{N}$ . Let  $a_1, \dots, a_k$  be a system of generators of  $\mathcal{O}(X)$ , and let  $N \in \mathbb{N}$  be sufficiently large so that  $f^N \partial a_i \in \mathcal{O}(X)$  for any  $i = 1, \dots, k$ . Then  $\partial_N(a_i) \in \mathcal{O}(X)$  for any  $i = 1, \dots, k$ , hence  $\partial_N(\mathcal{O}(X)) \subseteq \mathcal{O}(X)$ . Thus the derivation  $\partial_N|_{\mathcal{O}(X)} \in \text{Der}(\mathcal{O}(X))$  is locally nilpotent and so generates an effective  $\mathbb{C}_+$ -action  $\psi$  on  $X$ . Therefore (i) holds.  $\square$

**Remark 3.6.** Clearly the  $\mathbb{C}_+$ -actions  $\phi$  and  $\psi|_U$  as in the proof have the same orbits, and  $\mathbb{V}_+(f) = \{f = 0\}$  consists of fixed points of  $\psi$ .

In the case of affine cones, Theorem 3.9 below gives a more practical criterion. We need the following definition.

**Definition 3.7.** For a projective variety  $Y$  with a (very ample) polarization  $\varphi_{|H|} : Y \hookrightarrow \mathbb{P}^n$ , we call an *H-polar subset* any Zariski open subset of the form  $U = Y \setminus \text{supp } D$ , where  $D \in |dH|$  is an effective divisor on  $\mathbb{P}^n$ .

**3.8.** Recall that an affine ruling on a variety  $U$  is a morphism  $\pi : U \rightarrow Z$  such that every scheme theoretic fiber of  $\pi$  is isomorphic to the affine line  $\mathbb{A}^1$ . By a theorem of Kambayashi and Miyanishi [KaMi] (see also [KaWr, RS, Du]), every affine ruling  $\pi : U \rightarrow Z$  on a normal variety  $U$  over a normal base  $Z$  is a locally trivial  $\mathbb{A}^1$ -bundle.

**Theorem 3.9.** *Let  $Y$  be a smooth projective variety with a very ample polarization  $\varphi_{|H|} : Y \hookrightarrow \mathbb{P}^n$ . Then the following hold.*

- (a) If the affine cone  $X = \text{AffCone}_H(Y)$  admits an effective  $\mathbb{C}_+$ -action, then  $Y$  possesses an  $H$ -polar open subset  $U$ , which is the total space of a line bundle  $U \rightarrow Z$ .
- (b) Conversely, if  $Y$  possesses an  $H$ -polar open subset  $U$  equipped with an affine ruling  $U \xrightarrow{\mathbb{A}^1} Z$  and a section  $Z \rightarrow U$ , where  $Z$  is smooth and  $\text{Pic}(Z) = 0$ , then the affine cone  $X = \text{AffCone}_H(Y)$  admits an effective  $\mathbb{C}_+$ -action.

*Proof.* (a) Let  $\psi'$  be an effective  $\mathbb{C}_+$ -action on  $X$  with associate locally nilpotent derivation  $\partial' \neq 0$ . Using the natural grading of the coordinate ring

$$A = \mathcal{O}(X) = \bigoplus_{i \geq 0} A_i,$$

$\partial'$  can be decomposed into a finite sum of homogeneous derivations  $\partial' = \sum_{i=1}^n \partial'_i$ , where the principal component  $\partial := \partial'_n \neq 0$  is again locally nilpotent. The  $\mathbb{C}_+$ -action  $\psi$  on  $X$  generated by  $\partial$  extends to an effective action of a semi-direct product  $G = \mathbb{C}_+ \rtimes \mathbb{C}^*$  on  $X$ .

The filtration (3) from the proof of Proposition 3.5 consists now of graded subrings. Hence we can find homogeneous elements  $\hat{g}, \hat{h} \in A$  such that  $\partial \hat{g} = \hat{h}$  and  $\partial \hat{h} = 0$ . In the notation of 3.4 we let

$$\hat{U} = \mathbb{D}_+(\hat{h}) \subseteq X \quad \text{and} \quad \hat{Z} = \mathbb{V}_+(\hat{g}) \setminus \mathbb{V}_+(\hat{h}) \subseteq \hat{U}.$$

Likewise in the proof of Proposition 3.5, we obtain a decomposition  $\hat{U} \simeq \hat{Z} \times \mathbb{A}^1$ .

Furthermore,  $G$  acts on  $\hat{U} \simeq \hat{Z} \times \mathbb{A}^1$  respecting the product structure. More precisely,  $\mathbb{C}_+$  acts by shifts on the second factor i.e., along the fibers of the morphism  $\hat{\pi} : \hat{U} \rightarrow \hat{Z}$ . Since  $\hat{g}, \hat{h}$ , and  $\partial$  are homogeneous,  $\mathbb{C}^*$  acts on  $\hat{U}$  stabilizing  $\hat{Z}$  and sending the fibers of  $\hat{\pi}$  into fibers. The factorization by the  $\mathbb{C}^*$ -action on  $\hat{U}$  yields a Zariski open subset  $U = \hat{U}/\mathbb{C}^* \subseteq Y$  and a divisor  $Z = \hat{Z}/\mathbb{C}^*$  on  $U$  so that

$$(4) \quad \hat{U} = \text{AffCone}(U) \setminus \{0\} \quad \text{and} \quad \hat{Z} = \text{AffCone}(Z) \setminus \{0\}.$$

The map  $\hat{\pi}$  defines an affine ruling  $\pi : U \xrightarrow{\mathbb{A}^1} Z$  with a section  $Z \hookrightarrow U$ . Each fiber of  $\pi$  is the quotient of a  $G$ -orbit in  $U$  by the  $\mathbb{C}^*$ -action. Since  $Y$  and  $Z$  are smooth, by the Kambayashi-Miyayashi Theorem cited in 3.8,  $\pi$  is an  $\mathbb{A}^1$ -bundle and, moreover, a vector bundle since it possesses a section.

Finally, since  $D := \hat{h}^*(0) \in |dH|$ , where  $d = \deg(\hat{h})$ , the open set  $U \subseteq Y$  is  $H$ -polar (see Definition 3.7). This shows (a).

To show (b), suppose that  $Y$  possesses an  $H$ -polar open subset  $U$  with an affine ruling  $\pi : U \xrightarrow{\mathbb{A}^1} Z$  and a section  $Z \rightarrow U$ , where  $Z$  is smooth and  $\text{Pic}(Z) = 0$ . Since both  $U$  and  $Z$  are smooth,  $\pi$  is locally trivial by the Kambayashi-Miyayashi Theorem. Since  $\pi$  has a section  $Z$ ,  $\pi : U \rightarrow Z$  is a line bundle and  $Z$  is the zero section. This bundle is trivial since  $\text{Pic}(Z) = 0$ . Thus  $U \simeq Z \times \mathbb{A}^1$ . In particular  $\text{Pic}(U) = 0$ .

Let further  $\sigma : \tilde{X} \rightarrow X$  be the blowup of the vertex  $0 \in X$ . The induced morphism  $\rho : \tilde{X} \rightarrow Y$  has a natural structure of a line bundle with the exceptional divisor  $E = \sigma^{-1}(0)$  as the zero section. Since  $\text{Pic}(U) = 0$ , the restriction  $\rho|_{\tilde{U}} : \tilde{U} \rightarrow U$  to  $\tilde{U} := \rho^{-1}(U) \subseteq \tilde{X}$  yields a trivial line bundle. Hence

$$\tilde{U} \simeq U \times \mathbb{A}^1 \simeq Z \times \mathbb{A}^1 \times \mathbb{A}^1 \quad \text{and, similarly,} \quad \tilde{Z} := \rho^{-1}(Z) \simeq Z \times \mathbb{A}^1.$$

Under this isomorphism  $E \cap \tilde{U}$  is sent to  $U \times \{0\} \simeq Z \times \mathbb{A}^1 \times \{0\}$  and  $E \cap \tilde{Z}$  to  $Z \times \{0\}$ . For  $\hat{U} = \tilde{U} \setminus E$  and  $\hat{Z} = \tilde{Z} \setminus E$  as in (4) we obtain

$$(5) \quad \hat{U} \simeq Z \times \mathbb{A}^1 \times \mathbb{C}^* \quad \text{and} \quad \hat{Z} \simeq Z \times \{0\} \times \mathbb{C}^*.$$

Thus  $\hat{U} \simeq \hat{Z} \times \mathbb{A}^1$  is a cylinder in  $X$ . Since  $U \subseteq Y$  is  $H$ -polar,  $\hat{U} \subseteq X$  is a principal Zariski open subset and so  $X$  is cylindrical. Now (b) follows by Proposition 3.5.  $\square$

**Remarks 3.10.** 1. This theorem, with the same proof, holds also for generalized cones (see 3.1). In particular, we may assume that  $H$  is just an ample divisor.

2. It is easily seen that if a cone  $X = \text{AffCone}_H(Y)$  admits an effective  $\mathbb{C}_+$ -action, then also the cone  $X_k = \text{AffCone}_{kH}(Y)$  admits such an action for any  $k \geq 1$ . Moreover, this cone  $X_k$  is normal for  $k \gg 1$ , see [Ha, Ch. II, Ex5.14].

**Remark 3.11.** The construction of a  $\mathbb{C}_+$ -action on  $X$  as in the proof of (b) can be made more explicite. The product  $G = \mathbb{C}_+ \times \mathbb{C}^*$  acts on  $\hat{U}$  preserving the product structure in (5):

$$G \ni (a, \lambda) : \hat{U} \rightarrow \hat{U}, \quad (z, x, y) \mapsto (z, x + a, \lambda y).$$

The generators  $\partial/\partial x$  and  $y\partial/\partial y$  of the  $\mathbb{C}_+$ - and  $\mathbb{C}^*$ -actions commute. Letting  $D = X \setminus \hat{U}$ , there is a regular function  $f \in \mathcal{O}(X)$  such that  $\text{div}(f) = nD$ . Moreover, we can choose  $f$  of the form  $f = y^k g(z)$ , where  $g \neq 0$ . For  $N \gg 1$  the  $\mathbb{C}_+$ -action generated by  $\partial = f^N \partial/\partial x$  extends to the cone  $X$ , see the proof of Proposition 3.5. With this new  $\mathbb{C}_+$ -action, a semidirect product  $\mathbb{C}_+ \times \mathbb{C}^*$  acts effectively on  $X$ . However, the factors do not commute any more.

Theorem 3.9 yields the following criterion of existence of a  $\mathbb{C}_+$ -action on certain 3-dimensional affine cones.

**Corollary 3.12.** *Let  $Y$  be a rational smooth projective surface with a polarization  $\varphi_{|H|} : Y \hookrightarrow \mathbb{P}^n$ , and let  $X = \text{AffCone}_H(Y) \subseteq \mathbb{A}^{n+1}$  be the affine cone over  $Y$ . Then  $X$  admits a nontrivial  $\mathbb{C}_+$ -action if and only if  $Y$  possesses an  $H$ -polar cylinder  $U \simeq Z \times \mathbb{A}^1$ , where  $Z$  is a smooth affine curve.*

*Proof.* Since  $Y$  is smooth and rational,  $Z$  as in Theorem 3.9 is a non-complete smooth rational curve. Thus  $\text{Pic}(Z) = 0$ . Hence the affine rulings from Theorem 3.9 and its proof are actually direct products. Our assertion can be easily deduced now from Theorem 3.9.  $\square$

Using this criterion, we show next that for an arbitrary smooth rational surface  $Y$ , some affine cone over  $Y$  admits a nontrivial  $\mathbb{C}_+$ -action.

**Proposition 3.13.** *Let  $Y$  be a rational smooth projective surface. Then there is an embedding  $\varphi : Y \hookrightarrow \mathbb{P}^n$  such that the affine cone  $X = \text{AffCone}(\varphi(Y))$  is normal and admits an effective  $\mathbb{C}_+$ -action.*

*Proof.* Any point  $Q \in Y$  possesses an affine neighborhood  $U \simeq \mathbb{A}^2$ . An argument from [Fu, (2.5)] shows that  $Y \setminus U$  supports an ample divisor. Indeed,  $\text{Pic}(Y)$  is a free abelian group generated by the components  $\Delta_i$  of the divisor  $Y \setminus U$ . Hence  $\Delta_j^2 > 0$  for some  $j$ .

Choose a nef and big effective divisor  $D = \sum \delta_i \Delta_i$  such that  $D \cdot \Delta_i > 0$  whenever  $\delta_i > 0$ , with a maximal possible value of  $\lambda(D) := \text{card}\{i \mid \delta_i > 0\}$ . Assume on the contrary that  $\text{supp}(D) \neq \text{supp}(\sum \Delta_i)$ , i.e.,  $\delta_i = 0$  for some  $i$ . Since  $\text{supp}(\sum \Delta_i)$  is

connected, there is a component  $\Delta_k \not\subseteq \text{supp}(D)$  with  $D \cdot \Delta_k > 0$ . Then for  $t \gg 0$  the divisor  $tD + \Delta_k$  is again nef and big. This contradicts our maximality assumption for  $\lambda(D)$ . Therefore  $\text{supp}(D) = \text{supp}(\sum \Delta_i)$  is ample. So for  $m \gg 1$  the linear system  $|mD|$  gives an embedding  $Y \hookrightarrow \mathbb{P}^n$  with a projectively normal image, see Exercise 5.14 in [Ha, Ch. II]. Since  $Y$  admits an  $|mD|$ -polar cylinder,  $X$  is normal and cylindrical. By Corollary 3.12,  $X$  admits an effective  $\mathbb{C}_+$ -action, as required.  $\square$

The following question arises.

**3.14. Question.** *Does there exist a polarized smooth rational surface  $(Y, H)$  without any  $H$ -polar cylinder?*

**Remark 3.15.** If  $U$  is an  $H$ -polar cylinder on  $Y$  then it is also  $kH$ -polar for any  $k \in \mathbb{N}$ , and vice versa. Thus the existence of an  $H$ -polar cylinder depends only on the ray of  $H$  in the ample cone of  $Y$ . Moreover, since the irreducible components of the divisor  $D = Y \setminus U$  span the Picard group  $\text{Pic}(Y)$  and the ample cone is open, the property of a cylinder  $U$  to be  $H$ -polar is stable under small perturbation of  $H$ .

For any smooth rational projective surface, the  $H$ -polar cylinder from Proposition 3.13 can be chosen to be isomorphic to the affine plane. Let us provide similar examples in higher dimensions.

**Example 3.16.** Consider a flag variety  $G/P$  with an ample polarization  $H$  (see §1.3). By Corollary 2.13 the  $G$ -action on  $G/P$  lifts to a  $\tilde{G}$ -action on the cone  $\text{AffCone}_H(G/P)$ , where  $\tilde{G}$  is the universal cover of  $G$ . The actions of one-parameter unipotent subgroups of  $\tilde{G}$  yield effective  $\mathbb{C}_+$ -actions on the cone. Actually  $G/P$  contains an  $H$ -polar open cylinder  $U$  isomorphic to an affine space  $\mathbb{A}^n$  (cf. Theorem 3.9(a)).

Indeed, let  $B_+ \subseteq P$  be a Borel subgroup of  $G$ , and let  $B_-$  be the opposite Borel subgroup so that  $B_+ \cap B_-$  is a Cartan subgroup. Then  $B_- \cdot P$  is open in  $G$  and so the  $B_-$ -orbit  $U$  of  $e \cdot P$  is open in  $G/P$ . Thus  $U$  is a big Schubert cell. Since  $U$  is also an orbit of the maximal unipotent subgroup  $B_u \subseteq B_-$ , it is isomorphic to  $\mathbb{A}^n$ . In particular,  $U$  is a cylinder in  $Y$ . Letting  $D = Y \setminus U = \bigcup_i D_i$ , the Schubert divisors  $D_i$  form a basis in  $\text{Pic}(G/P)$ . In this basis  $H = \sum_i \alpha_i D_i$ , where  $\alpha_i > 0$  for all  $i$  since the divisor  $H$  is ample, see [Sn] or [Te, Theorem 7.53]. Hence  $U$  is an  $H$ -polar cylinder in  $G/P$ .

We note that the action of  $\tilde{G}$  on the affine cone  $\tilde{X} := \text{AffCone}_H(G/P)$  is transitive off the vertex  $0 \in \tilde{X}$ . Indeed, we may suppose that  $\tilde{X} = \tilde{G}/\tilde{P}$ , where  $\tilde{G}$  is semisimple, simply connected, and  $\tilde{P} \subseteq \tilde{G}$  is parabolic. Since  $\tilde{X}$  is affine, the stabiliser  $\text{Stab}_{\tilde{G}}(x)$  of a point  $x \in \tilde{X} \setminus \{0\}$  cannot contain a parabolic subgroup. Hence the stabilizer  $\text{Stab}_{\tilde{G}}([x])$  (conjugate to  $\tilde{P}$ ) acts non-trivially on the generator of the cone through  $x$ . By Corollary 1.5 in [Po<sub>4</sub>] the Makar-Limanov invariant  $ML(\tilde{X})$  is trivial (cf. Theorem 3.26; see Section 3.3 below for the definition of the Makar-Limanov invariant).

**Remark 3.17.** The existence of a cylinder in a projective variety isomorphic to an affine space is rather exceptional. For instance, none of smooth rational cubic 4-folds in  $\mathbb{P}^5$ , and none of smooth 3-fold intersections of a pair of quadrics in  $\mathbb{P}^5$  contains a Zariski open set isomorphic to an affine space, see [PS] and [Pr<sub>2</sub>]. At the same time, every smooth intersection  $Y$  of a pair of quadrics in  $\mathbb{P}^5$  contains a  $(-K_Y)$ -polar cylinder, see Proposition 5.1 below and its proof.

**3.2.  $\mathbb{C}_+$ -actions on affine cones over del Pezzo surfaces.** Let us explain the reason why are we interested in the affine cones over del Pezzo surfaces.

**3.18.** A normal variety  $X$  is  $\mathbb{Q}$ -Gorenstein if some multiple  $nK_X$  of the canonical Weil divisor  $K_X$  is Cartier. This notion is important in the Mori minimal model program (MMP). It is easily seen that the generalized cone  $X = \text{AffCone}_H(Y)$  over a smooth polarized variety  $(Y, H)$  is  $\mathbb{Q}$ -Gorenstein if and only if  $aH \sim -bK_Y$  for some  $a \in \mathbb{N}$ ,  $b \in \mathbb{Z}$  ([Kol<sub>1</sub>, Example 3.8]). (If, moreover,  $H \sim -K_Y$ , then  $X$  is Gorenstein and has at most canonical singularity at the origin.)

On the other hand, if  $\text{Aut}(X) \neq \text{Lin}(X)$  then by Corollary 2.4  $Y$  is birationally ruled, hence the Kodaira dimension of  $Y$  is negative, see [Kol<sub>2</sub>]. Thus  $b > 0$ , i.e.,  $-K_Y$  is ample. Consequently,  $Y$  is a Fano variety.

Therefore, if the affine cone  $X$  over  $(Y, H)$  is  $\mathbb{Q}$ -Gorenstein and admits an effective non-linear  $\mathbb{C}_+$ -action, then  $Y$  is a Fano variety and  $H \in \mathbb{Q}_{>0}[-K_Y]$ . In particular, if  $\dim(Y) = 2$  then  $Y$  is a del Pezzo surface with its pluri-anticanonical embedding.<sup>9</sup>

From now on we assume that  $Y$  is a del Pezzo surface of degree  $d \geq 3$  and  $H = -K_Y$  is the anti-canonical polarization. Thus the linear system  $|-K_Y|$  is very ample and provides an embedding  $Y \hookrightarrow \mathbb{P}^d$  onto a projectively normal smooth surface of degree  $d$ , see e.g., [Dol<sub>1</sub>]. The affine cone  $X = \text{AffCone}_{-K_Y}(Y)$  has a normal, canonical, Gorenstein (hence also Cohen-Macaulay) singularity at the vertex.

The following theorem is the main result of this subsection (see Theorem 0.1 in the Introduction).

**Theorem 3.19.** *Let  $Y_d$  be a smooth del Pezzo surface of degree  $d$  anticanonically embedded into  $\mathbb{P}^d$ , where  $4 \leq d \leq 9$ , and let  $X_d \subseteq \mathbb{A}^{d+1}$  be the affine cone over  $Y_d$ . Then  $X_d$  admits a nontrivial  $\mathbb{C}_+$ -action.*

*Proof.* Consider a pencil  $\mathcal{L}_{\mathbb{P}^2} = \langle C_1, C_2 \rangle$  on  $\mathbb{P}^2$  generated by a smooth conic  $C_1$  and a double line  $C_2 = 2l$ , where  $l$  is tangent to  $C_1$  at a point  $P_0 \in C_1$ . Then  $L \setminus \{P_0\} \simeq \mathbb{A}^1$ , where  $L$  is a general member of  $\mathcal{L}_{\mathbb{P}^2}$ . Moreover,  $U = \mathbb{P}^2 \setminus (C_1 \cup C_2)$  is a cylinder over  $\mathbb{A}_*^1$ . Blowing up at  $9 - d$  distinct points  $Q_i$  on  $C_1 \setminus \{P_0\}$ , where  $9 \geq d \geq 4$ , we obtain a del Pezzo surface  $Y$  of degree  $d$  with a contraction  $\sigma : Y \rightarrow \mathbb{P}^2$ , and any such surface can be obtained in this way, except for  $\mathbb{P}^1 \times \mathbb{P}^1$ .

The cylinder  $U' = \sigma^{-1}(U) \simeq U$  is  $(-K_Y)$ -polar (see Definition 3.7). Indeed, let  $E_i = \sigma^{-1}(Q_i)$ ,  $i = 1, \dots, 9 - d$ . For any  $1 \gg \varepsilon > 0$  we have

$$-K_{\mathbb{P}^2} \equiv (1 + \varepsilon)C_1 + (1 - 2\varepsilon)l.$$

Hence,

$$-K_Y = \sigma^*(-K_{\mathbb{P}^2}) - \sum_{i=1}^{9-d} E_i \equiv (1 + \varepsilon)\Delta_1 + (1 - 2\varepsilon)\Delta_2 + \varepsilon \sum_{i=1}^{9-d} E_i,$$

where  $\Delta_1$  and  $\Delta_2$  are the proper transforms in  $Y$  of  $C_1$  and  $l$ , respectively. Thus  $U'$  is a  $(-K_Y)$ -polar cylinder on  $Y$ .

In the remaining case where  $Y = \mathbb{P}^1 \times \mathbb{P}^1$ , the natural embedding  $\mathbb{A}^2 = \mathbb{A}^1 \times \mathbb{A}^1$  into  $Y$  yields a  $(-K_Y)$ -polar cylinder on  $Y$ . Applying now Corollary 3.12 ends the proof.  $\square$

<sup>9</sup>Cf. Remark 2.9(3).

The proof exploits a  $(-K_{\mathbb{P}^2})$ -polar cylinder on  $\mathbb{P}^2$  made of a pencil of conics with a common tangent line. Based on the same idea, we give below some alternative constructions of polar cylinders on anticanonically polarized del Pezzo surfaces of degrees  $\geq 4$ . Due to Corollary 3.12, this leads to new  $\mathbb{C}_+$ -actions on the cones over del Pezzo surfaces of degree  $\geq 4$  under their anticanonical embeddings. These examples will be useful in the sequel.

**Example 3.20.** Consider a pencil of rational curves on  $\mathbb{P}^2$  with a unique base point  $P$ . (Similarly, one can find such a pencil on the quadric  $\mathbb{P}^1 \times \mathbb{P}^1$ .) Then the complement of the union of its degenerate members (or of a general one, if all members are non-degenerate) is a  $(-K_{\mathbb{P}^2})$ -polar cylinder on  $\mathbb{P}^2$ . In [MiSu] an example was proposed of such a pencil of quintic curves. Moreover, there is a smooth conic  $C_1$  and a rational unicuspidal quintic  $C_2$  from the pencil as in [MiSu] that meet in one point, the cuspidal point of the quintic.

These two curves generate a pencil  $\mathcal{L}_{\mathbb{P}^2} = \langle 5C_1, 2C_2 \rangle$  of rational curves of degree 10 with a unique base point such that  $\mathbb{P}^2 \setminus (C_1 \cup C_2)$  is a cylinder. Similarly as in the proof above, every del Pezzo surface  $Y$  of degree  $d \geq 4$  can be obtained, along with a  $(-K_Y)$ -polar cylinder, by blowing up a certain set of  $9 - d$  points on  $C_1$ . Indeed, we can write

$$-K_{\mathbb{P}^2} \equiv \left(\frac{3}{2} - \varepsilon\right)C_1 + \frac{2}{5}\varepsilon C_2$$

with an appropriate  $\varepsilon > 0$ , and then proceed in the same fashion as in the proof.

**Example 3.21.** Picking up four points  $P_1, \dots, P_4$  in  $\mathbb{P}^2$  in general position, we consider the pencil of lines  $\mathcal{L}_{\mathbb{P}^2}$  on  $\mathbb{P}^2$  generated by  $l_1 = (P_1P_2)$  and  $l_2 = (P_3P_4)$ . The blowup  $\sigma : Y \rightarrow \mathbb{P}^2$  of these points yields a del Pezzo surface  $Y$  of degree 5. We have

$$-K_{\mathbb{P}^2} \equiv \frac{3}{2}l_1 + \frac{3}{2}l_2 \quad \text{and so} \quad -K_Y \equiv \frac{3}{2}l'_1 + \frac{3}{2}l'_2 + \frac{1}{2} \sum_{i=1}^4 E_i,$$

where  $l'_i$  is the proper transform of  $l_i$ ,  $i = 1, 2$ , and  $E_i$  is the exceptional  $(-1)$ -curve over  $P_i$ ,  $i = 1, \dots, 4$ . Then  $L^{(1)} = l'_1 + E_1 + E_2$  and  $L^{(2)} = l'_2 + E_3 + E_4$  are the only degenerate fibers of the pencil  $\mathcal{L} = \sigma_*^{-1}(\mathcal{L}_{\mathbb{P}^2})$  on  $Y$ . Since

$$D := \frac{1}{2}(L^{(1)} + L^{(2)}) + (l'_1 + l'_2) \equiv -K_Y,$$

the open set

$$Y \setminus \text{supp}(D) = \mathbb{P}^2 \setminus (l_1 \cup l_2) \simeq \mathbb{A}_*^1 \times \mathbb{A}^1$$

is a  $(-K_Y)$ -polar cylinder on  $Y$ . A similar construction can be applied to any del Pezzo surface of degree  $d \geq 5$ .

**Example 3.22.** Consider the pencil  $\mathcal{L}_{\mathbb{P}^2}$  of unicuspidal rational curves  $\alpha yz^{n-1} + \beta x^n = 0$  in  $\mathbb{P}^2$ , where  $n \geq 1$ . Blowing up  $k \leq 4$  points in  $\mathbb{P}^2$ , at most two on each of the lines  $x = 0$  and  $y = 0$  off their common point  $(0 : 0 : 1)$  we obtain examples of  $(-K_Y)$ -polar cylinders on an arbitrary del Pezzo surface of degree  $d \geq 5$ . For  $n = 1$  and  $k = 4$  this gives again the cylinder from Example 3.21.

**Remark 3.23.** The idea to start with a  $(-K_{\mathbb{P}^2})$ -polar cylinder on  $\mathbb{P}^2$  cannot be carried out any more in case of a smooth cubic surface  $Y \subseteq \mathbb{P}^3$ . Indeed, suppose we are given a cylinder  $\mathbb{P}^2 \setminus C \simeq Z \times \mathbb{A}^1$ , where  $C$  is a reduced plane curve of degree  $d$ , not necessarily smooth or irreducible, and let  $\mathcal{L}_{\mathbb{P}^2}$  be the corresponding pencil. Then  $\text{Bs}(\mathcal{L}_{\mathbb{P}^2})$  consists

of one point  $P_0 \in C$ , and  $C \setminus \{P_0\}$  is a disjoint union of components isomorphic to  $\mathbb{A}^1$ . Performing a blowup  $\sigma : Y \rightarrow \mathbb{P}^2$  of  $m$  points  $P_i \in C \setminus \{P_0\}$  with exceptional curves  $E_j = \sigma^{-1}(p_j)$ ,  $i = 1, \dots, m$ , from the equalities

$$3\sigma^*(C) \sim d\sigma^*(-K_{\mathbb{P}^2}) = -dK_Y + d \sum_{j=1}^m E_j \quad \text{and} \quad 3\sigma^*(C) = 3C' + 3 \sum_{j=1}^m E_j,$$

where  $C'$  is the proper transform of  $C$  on  $Y$ , we obtain

$$-dK_Y \sim 3C' + (3-d) \sum_{j=1}^m E_j =: D.$$

Here  $D$  is an effective divisor with  $\text{supp}(D) = C' + \sum_{j=1}^m E_j$  if and only if  $d \leq 2$  i.e.,  $C$  is a line or a conic. Since the centers of blowup  $P_i$ ,  $i = 1, \dots, m$ , are situated on  $C$  and  $Y$  must be del Pezzo, we have  $m \leq 5$  and so  $\text{deg}(Y) \geq 4$ .

In the next example, starting with a pencil on  $\mathbb{P}^2$  with five base points, we construct a  $(-K_Y)$ -polar cylinder of different type on arbitrary del Pezzo surface  $Y$  of degree  $d = 5$ , by resolving all the base points but one.

**Example 3.24.** Consider the following pencil  $\mathcal{L}_{\mathbb{P}^2}$  of rational plane sextics:

$$\alpha(y^2z - x^3)^2 + \beta(y^2 - xz)(y^4 - x^4) = 0.$$

The base locus of  $\mathcal{L}_{\mathbb{P}^2}$  consists of the points  $P_0, \dots, P_4$ , where  $P_0 = (0 : 0 : 1)$  and  $\{P_1, \dots, P_4\} = (x^2 = z^2, \quad xz = y^2)$ . Furthermore,  $\mathcal{L}_{\mathbb{P}^2}$  has no fixed component and so its general member  $L$  is irreducible. Since  $\text{mult}_{P_0}(L) = 4$  and  $\text{mult}_{P_i}(L) = 2$  for  $i = 1, \dots, 4$ , the curve  $L$  is rational. Any singular point  $P_i$  of  $L$  is resolved by one blowup, and the singularity of  $L$  at  $P_0$  is cuspidal. No three of the points  $P_1, \dots, P_4$  are collinear. Therefore the blowup  $\sigma : Y \rightarrow \mathbb{P}^2$  of the latter points yields a del Pezzo surface  $Y$  of degree 5, and any such surface arises in this way. Let  $\mathcal{L}$  be the proper transform of  $\mathcal{L}_{\mathbb{P}^2}$  on  $Y$ , and let  $P = \sigma^{-1}(P_0)$ . Then  $\mathcal{L}$  is a pencil of rational curves with a cuspidal singularity at the unique base point  $P$ , smooth and disjoint outside  $P$ .

There are exactly two degenerate members of  $\mathcal{L}_{\mathbb{P}^2}$ , namely the double cuspidal cubic  $C' = 2(y^2z = x^3)$  and the union  $C''$  of the conic  $(y^2 = xz)$  and the four lines  $(y^4 = x^4)$ . We have  $-K_{\mathbb{P}^2} \equiv D_{\mathbb{P}^2} := \frac{1}{4}C' + \frac{1}{4}C''$ . Let  $D$  be the proper transform of  $D_{\mathbb{P}^2}$  on  $Y$ . Since  $\text{mult}_{P_i}(D_{\mathbb{P}^2}) = 1$  for  $i = 1, \dots, 4$ , we have  $K_Y + D = \sigma^*(K_{\mathbb{P}^2} + D_{\mathbb{P}^2}) \equiv 0$ . Hence  $U = Y \setminus (C' \cup C'')$  is a  $(-K_Y)$ -polar cylinder on  $Y$  (see also Example 4.17 below).

### 3.3. The Makar-Limanov invariant on affine cones over del Pezzo surfaces.

**3.25.** For an algebra  $A$  over a field  $k$ , its Makar-Limanov invariant  $\text{ML}(A)$  is defined as the intersection of the kernels of all locally nilpotent derivations on  $A$ . It is trivial if  $\text{ML}(A) = k$ . Following [MiMa] we say that  $A$  is of class  $\text{ML}_i$  if the quotient field  $\text{Frac}(\text{ML}(A))$  has transcendence degree  $i$ . If  $\text{ML}(A)$  is finitely generated then  $i = \dim(Z)$ , where  $Z = \text{spec ML}(A)$ . Thus  $A \in \text{ML}_0$  whenever  $A$  has trivial Makar-Limanov invariant. For instance,  $\mathbb{A}^3 \in \text{ML}_0$  (regarded as the affine cone over  $\mathbb{P}^2$ ).

For  $A$  graded there are graded versions  $\text{ML}^{(h)}(A)$  and  $\text{ML}_i^{(h)}$  of  $\text{ML}(A)$  and  $\text{ML}_i$ , respectively [FZ<sub>2</sub>], where one restricts to homogeneous locally nilpotent derivations. Clearly,  $\text{ML}(A) \subseteq \text{ML}^{(h)}(A)$ . Hence the usual ML invariant is trivial if the homogeneous is.

**Theorem 3.26.** *Let  $X$  be the affine cone over a smooth, anticanonically embedded del Pezzo surface  $Y \subseteq \mathbb{P}^d$  of degree  $d \geq 4$ . Then the homogeneous Makar-Limanov invariant  $\text{ML}^{(h)}(X)$  is trivial i.e.,  $X \in \text{ML}_0^{(h)}$ .*

*Proof.* By Theorem 1.5 and Proposition 2.15, for  $d \geq 6$  the surface  $Y$  and the cone  $X = \text{AffCone}_{-K_Y}(Y)$  are toric. Since  $X$  is not isomorphic to a product  $X' \times \mathbb{A}_*^1$ , by Lemma 4.5 in [Li] the homogeneous Makar-Limanov invariant of  $X$  is trivial.

It remains to show that  $X \in \text{ML}_0^{(h)}$  for  $d = 4, 5$ . Note that for an arbitrary graded algebra  $A = \bigoplus_i A_i$ , the graded subalgebra  $\text{ML}^{(h)}(A)$  is non-trivial if and only if there exists a non-constant homogeneous element  $h \in A_n \cap \text{ML}^{(h)}(A)$  (so  $h$  is annihilated by all homogeneous locally nilpotent derivations on  $A$ ). In the case of an affine cone  $X$ , the degree  $n = \deg(h)$  is positive. Hence  $\Gamma = \mathbb{V}(h) \in |n(-K_Y)|$  is an effective ample divisor on  $Y$ .

Let as before  $Y \subseteq \mathbb{P}^d$  be a del Pezzo surface of degree  $d \geq 4$ . Suppose on the contrary that  $\text{ML}^{(h)}(X) \neq \mathbb{C}$ , and let  $h \in A_n \cap \text{ML}^{(h)}(A)$  be nonconstant. Then the affine cone over the curve  $\text{supp}(\Gamma)$  is a divisor on  $X$  stable under any  $\mathbb{C}_+$ -action defined by a homogeneous locally nilpotent derivation on  $\mathcal{O}(X)$ . Hence for every  $(-K_Y)$ -polar cylinder  $U$  on  $Y$ , the curve  $\text{supp}(\Gamma)$  consists of components of the members of the linear pencil  $\mathcal{L}$  on  $Y$  associated with  $U$ . In particular, for all  $(-K_Y)$ -polar cylinders on  $Y$  there must be a common component of the associated linear pencils  $\mathcal{L}$ .

For  $d = 5$ , we let  $\sigma : Y \rightarrow \mathbb{P}^2$  be the blowup of four points  $P_1, \dots, P_4$  in  $\mathbb{P}^2$  with exceptional curves  $E_i = \sigma^{-1}(P_i)$ . There are exactly ten lines on  $Y$ . Besides  $E_1, \dots, E_4$  these are the proper transforms  $l_{ij}$  of the lines  $(P_i P_j)$  on  $\mathbb{P}^2$ , where  $1 \leq i < j \leq 4$ . For every pair of lines  $(l_{ij}, l_{i'j'})$  with distinct indices  $i, j, i', j'$ , the curves

$$L_1 := l_{ij} + E_i + E_j \quad \text{and} \quad L_2 := l_{i'j'} + E_{i'} + E_{j'}$$

are the only degenerate members of a cylindrical linear pencil on  $Y$  (cf. Example 3.21). The 3 such pencils have no common component except for the lines  $E_1, \dots, E_4$ .

Let us replace the lines  $E_1, \dots, E_4$  on  $Y$  by some other four disjoint lines, e.g. by  $l_{12}, l_{13}, l_{23}, E_4$ . We consider also the three associated cylindrical pencils on  $Y$  e.g., that with degenerate members

$$L'_1 := E_2 + l_{12} + l_{23} \quad \text{and} \quad L'_2 := E_4 + l_{13} + l_{24}.$$

These pencils have no common component except for the lines  $l_{12}, l_{13}, l_{23}, E_4$ . The line  $E_4$  is the only common component of all six above pencils. With yet further choice of a pencil, we can eliminate also this latter line. Thus the homogeneous Makar-Limanov invariant of  $Y$  is trivial, as stated.

Let further  $d = 4$ , and let  $\sigma_0 : Y \rightarrow \mathbb{P}^2$  be the blowup of five points  $P_1, \dots, P_5$  in general position in  $\mathbb{P}^2$ , with exceptional curves  $E_i = \sigma_0^{-1}(P_i)$ ,  $i = 1, \dots, 5$ . We let  $C$  denote the unique smooth conic through the points  $P_i$ . Given a point  $Q \in C$  different from the  $P_i$ , similarly as in the proof of Theorem 3.19 we consider the pencil of conics on  $\mathbb{P}^2$  generated by  $C$  and  $2l_Q$ , where  $l_Q$  is the tangent line to  $C$  at  $Q$ . Two different such pencils on  $\mathbb{P}^2$  have no common member except for the conic  $C$  itself. Thus  $C'$  and the lines  $E_i$ ,  $i = 1, \dots, 5$ , are the only common components of the induced cylindrical pencils on the del Pezzo surface  $Y$  of degree 4.

Consider next the contraction  $\sigma_1 : Y \rightarrow \mathbb{P}^2$  of the five disjoint lines  $C', l_{12}, l_{13}, l_{14}, l_{15}$  on  $Y$ . Then  $\sigma_1(E_1)$  is a conic in  $\mathbb{P}^2$ , which plays now the role of  $C$ . Once again, the



only common components of the induced cylindrical pencils on  $Y$  are  $E_1$  and the five disjoint lines above meeting  $E_1$ .

Likewise, for the six different contractions  $\sigma_i : Y \rightarrow \mathbb{P}^2$ ,  $i = 0, \dots, 5$ , the only common component of the induced cylindrical pencils on  $Y$  is  $C'$ . However, the ample divisor  $\Gamma$  as at the beginning of the proof cannot be supported by  $C'$ . This contradiction finishes the proof.  $\square$

**Problem.** *Describe all affine cones whose homogeneous Makar-Limanov invariant is trivial.*

#### 4. ON EXISTENCE OF $\mathbb{C}_+$ -ACTIONS ON CONES OVER CUBIC SURFACES

In this section we analyze in detail the case of a smooth cubic surface  $Y \subseteq \mathbb{P}^3$ . We do not know whether the affine cone  $X = \text{AffCone}(Y)$  carries a  $\mathbb{C}_+$ -action. However, we obtain in Proposition 4.21 and Theorem 4.23 below a detailed information on an eventual anticanonical polar cylinder in  $Y$ . This makes the criterion 3.12 of the existence of a  $\mathbb{C}_+$ -action much more concrete in our particular case. We adopt the following convention.

**4.1. Convention.** Let  $Y$  be a smooth cubic surface in  $\mathbb{P}^3$ . Suppose that the affine cone  $X = \text{AffCone}(Y) \subseteq \mathbb{A}^4$  admits an effective  $\mathbb{C}_+$ -action. Then by Corollary 3.12  $Y$  possesses a  $(-K_Y)$ -polar cylinder  $U \simeq \mathbb{A}^1 \times Z$ , where  $Z$  is an affine smooth rational curve. In other words  $Y \setminus U = \text{supp}(D)$ , where

$$(6) \quad D = \sum_{i=1}^n \delta_i \Delta_i \equiv -K_Y$$

is an effective  $\mathbb{Q}$ -divisor on  $Y$  with  $\delta_i \in \mathbb{Q}_{>0} \forall i = 1, \dots, n$ .

**4.1. The linear pencil on a cubic surface compatible with a cylinder.** Let  $\mathcal{L}$  be the pencil on  $Y$  with general member  $L_z = \overline{\text{pr}_2^{-1}(z)}$  for  $z \in Z$ . It is easily seen that  $\mathcal{L}$  has at least one degenerate member. In what follows we suppose that  $\text{supp } D$  does not contain a non-degenerate member of  $\mathcal{L}$  (otherwise, up to numerical equivalence, we replace such a member by a degenerate one). Under these assumptions, the following hold.

**Lemma 4.2.** *The support of  $D$  is connected and simply connected, and contains at least 7 irreducible components.*

*Proof.* The projection  $\text{pr}_2 : U \rightarrow Z$  extends to a rational map  $Y \dashrightarrow \mathbb{P}^1$  defined by the pencil  $\mathcal{L}$  as above. A general member  $L$  of  $\mathcal{L}$  is a rational curve smooth off a unique point  $P$ , where  $\{P\} = L \cap \text{supp}(D)$ , and  $L \setminus \{P\} = L \cap U \simeq \mathbb{A}^1$ . Thus  $L$  is unibranch at  $P$ . The only base point of  $\mathcal{L}$  (if exists) is contained in  $\text{supp}(D)$ .

Since  $D$  is ample, by the Lefschetz Hyperplane Section Theorem  $\text{supp}(D)$  is connected. Resolving, if necessary, the base point of  $\mathcal{L}$  by a modification  $p : W \rightarrow Y$  yields a rational surface  $W$  with a pencil of rational curves  $\mathcal{L}_W = p_*^{-1}\mathcal{L}$  that are fibers of  $q = q|_{\mathcal{L}_W} : W \rightarrow \mathbb{P}^1$ . By Zariski's Main Theorem, the total transform  $p^{-1}(\text{supp } D)$  is still connected, and is a union of a  $(-1)$ -section, say,  $S$  of  $\mathcal{L}_W$  and of some number of rational trees contained in fibers of  $\mathcal{L}_W$ . Hence  $p^{-1}(\text{supp } D)$  is also a tree of rational curves i.e., is connected and simply connected. The exceptional divisor  $E$  of  $p$  being a

subtree of  $p^{-1}(\text{supp } D)$ , the contraction of  $E$  does not affect the simply-connectedness. This proves the first assertion.

Since  $Z$  is a rational smooth affine curve, we have  $\text{Pic}(U) = \text{Pic}(\mathbb{A}^1) \times \text{Pic}(Z) = 0$ . By virtue of the exact sequence

$$(7) \quad G := \sum_{i=1}^n \mathbb{Z}\Delta_i \rightarrow \text{Pic}(Y) \rightarrow \text{Pic}(U) = 0$$

the free abelian group  $G$  generated by the components  $\Delta_i$  of  $D$  surjects onto  $\text{Pic}(Y) \simeq \mathbb{Z}^7$ . Therefore  $\text{rk}(G) \geq 7$ , which proves the second assertion.  $\square$

**Lemma 4.3.** *The pencil  $\mathcal{L}$  has a unique base point, say,  $P$ , and  $\deg(\mathcal{L}) \geq 3$ .*

*Proof.* If on the contrary  $\text{Bs}(\mathcal{L}) = \emptyset$ , then the pencil of conics  $\mathcal{L}$  on  $Y$  with a section, say,  $S = \Delta_0$  defines a morphism  $\varphi_{|C|} : Y \rightarrow \mathbb{P}^1$  (extending the projection  $\text{pr}_2$  of the cylinder) with exactly five degenerate fibers  $L_1, \dots, L_5$ . Each degenerate fiber consists of two lines on  $Y$  intersecting transversally at one point. At most one of these two lines, say,  $l_i$  meets the cylinder  $U$ , while the other one, say,  $\Delta_i$  is a components of  $D$ . Since  $D$  is connected we have  $\Delta_i \cdot S = 1$  and  $l_i \cdot S = 0$ ,  $i = 1, \dots, 5$ . By the Adjunction Formula we get

$$1 = (-K_Y) \cdot l_i = D \cdot l_i = \delta_i - x,$$

where  $x = 0$  if  $l_i \neq \Delta_j \forall j$  and  $x = \delta_j > 0$  otherwise. Hence  $\delta_i = 1 + x \geq 1$ . Similarly, for a general fiber  $L$  of  $\mathcal{L}$ ,

$$2 = (-K_Y) \cdot L = D \cdot L = \delta_0 S \cdot L = \delta_0$$

and so  $\delta_0 = 2$ .

On the other hand, by the Adjunction Formula

$$-K_Y \cdot S = 2 + S^2 = D \cdot S = 2S^2 + \sum_{i=1}^5 \delta_i.$$

Therefore,

$$2 = S^2 + \sum_{i=1}^5 \delta_i = S^2 + \sum_{i=1}^5 \delta_i \geq -1 + 5 = 4,$$

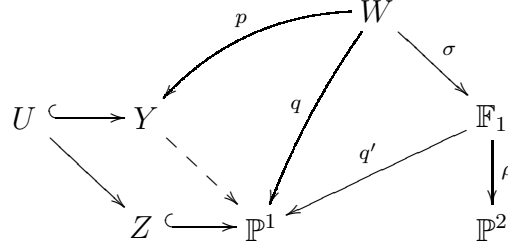
a contradiction. The inequality  $(-K_Y) \cdot \mathcal{L} \geq 3$  is now immediate.  $\square$

**Remarks 4.4.** 1. Actually the degree of  $\mathcal{L}$  must be essentially higher, since by Lemma 4.9 below  $D$  has 8 irreducible components.

2. The assertion of the lemma holds also for any del Pezzo surface of degree 4 or 5, with a similar proof. However, it fails for degree 6. Indeed, pick 3 points  $P_0, P_1, P_2$  in general position in  $\mathbb{P}^2$ , and consider the pencil generated by the lines  $l_i = (P_0P_i)$ ,  $i = 1, 2$ . Blowing up these points we get a del Pezzo surface  $Y$  of degree 6 with a base point free pencil. Then the complement in  $Y$  of the total transform of  $l_1 \cup l_2$  is a  $(-K_Y)$ -polar cylinder.

**4.5.** In the sequel we frequently use the following commutative diagram:

(8)



where  $p : W \rightarrow Y$  is the minimal resolution of the base point  $P$  of  $\mathcal{L}$ ,  $q : W \rightarrow \mathbb{P}^1$  is the induced pencil,  $\sigma : W \rightarrow \mathbb{F}_1$  is composed of the contraction of all components of degenerate fibers of  $q$  except those which meet the exceptional  $(-1)$ -section  $S_W$  of  $q$ , and  $\rho : \mathbb{F}_1 \rightarrow \mathbb{P}^2$  is the contraction of this exceptional section.

**Lemma 4.6.**  $\delta_i < 1$  in (6) for all  $i = 1, \dots, n$ .

*Proof.* It is enough to show that  $\delta_1 < 1$ . By symmetry then  $\delta_i < 1 \forall i$ . Since the anticanonical divisor of  $Y$  is ample, we have  $(-K_Y) \cdot \Delta_i > 0 \forall i = 1, \dots, n$ . We distinguish between the following 3 cases :

- (i)  $(-K_Y) \cdot \Delta_1 \geq 3$ ,
- (ii)  $(-K_Y) \cdot \Delta_1 = 2$ , and
- (iii)  $(-K_Y) \cdot \Delta_1 = 1$ .

In case (i) suppose on the contrary that  $\delta_1 \geq 1$ . Since  $n \geq 7$  by Lemma 4.2 and the divisor  $-K_Y \equiv D$  is ample, we obtain

$$3 = (-K_Y) \cdot D = \sum_{i=1}^n \delta_i (-K_Y) \cdot \Delta_i > \delta_1 (-K_Y) \cdot \Delta_1 \geq 3,$$

which gives a contradiction.

In case (ii)  $\Delta_1 \subseteq Y$  is a conic,  $\Delta_1^2 = 0$ , and  $-K_Y \equiv \Delta_1 + E$ , where  $E$  is the residual line cut out on  $Y$  by a plane in  $\mathbb{P}^3$  through  $\Delta_1$ .

Let  $E = \Delta_i$  for some  $i > 1$ ; we may assume that  $i = 2$ . Then  $-K_Y \equiv \Delta_1 + \Delta_2$ ,  $\Delta_1^2 = 0$  and  $\Delta_1 \cdot \Delta_2 = 2$ . Thus

$$(9) \quad 2 = (-K_Y) \cdot \Delta_1 = D \cdot \Delta_1 \geq \delta_2 \Delta_1 \cdot \Delta_2 = 2\delta_2,$$

so  $\delta_2 \leq 1$ . Furthermore,

$$1 = \Delta_2 \cdot D = \sum_{i=1}^n \delta_i \Delta_2 \cdot \Delta_i \geq 2\delta_1 - \delta_2.$$

Hence  $\delta_1 \leq \frac{1}{2}(\delta_2 + 1) \leq 1$ .

If  $\delta_1 = 1$  then also  $\delta_2 = 1$ . Since  $\Delta_1 + \Delta_2 \equiv -K_Y \equiv D$  we get  $D = \Delta_1 + \Delta_2$ . Thus  $n = 2$ , which contradicts Lemma 4.2. Therefore in this case  $\delta_1 < 1$ , as stated.

If further  $E \neq \Delta_i \forall i$  then

$$(10) \quad 1 = (-K_Y) \cdot E = D \cdot E \geq \delta_1 \Delta_1 \cdot E = 2\delta_1,$$

hence  $\delta_1 \leq 1/2$ . Thus anyway,  $\delta_1 < 1$  in case (ii).

In case (iii)  $\Delta_1$  is a line on  $Y$ . Let  $C$  be a residual conic of  $\Delta_1$ , so that  $\Delta_1 + C \equiv -K_Y$  is a hyperplane section. We have as before

$$2 = (-K_Y) \cdot C = D \cdot C \geq \delta_1 \Delta_1 \cdot C = 2\delta_1,$$

hence  $\delta_1 \leq 1$ .

If  $\delta_1 = 1$  then  $D \cdot C = \Delta_1 \cdot C = 2$  and so  $(D - \Delta_1) \cdot C = 0$ . Therefore the divisor  $\text{supp}(D - \Delta_1)$  is supported on the members of the pencil of conics  $|C|$  on  $Y$ . The curve  $\Delta_1$  meets each fiber twice, and so the morphism  $\varphi|_{C|}$  restricted to  $\Delta_1$  has 2 branch points.

By Lemma 4.2 the curve  $\text{supp } D$  is simply connected, hence it cannot contain the whole fiber of  $\varphi|_{C|}$  which meets the component  $\Delta_1$  of  $D$  at two distinct points. We claim however that if a degenerate fiber  $l_1 + l_2$  of the pencil  $|C|$  contains a component, say,  $\Delta_i = l_1$  of  $D$ , then its second component  $l_2 = \Delta_j$  is also contained in  $\text{supp } D$  and, moreover,  $\delta_i = \delta_j$ . Indeed, since  $\delta_1 = 1$  and  $\Delta_1$  is a line on  $Y$  we have

$$(11) \quad 1 = \Delta_i \cdot D = \Delta_i \cdot \Delta_1 + \Delta_i \cdot (D - \Delta_1) = 1 - \delta_i + \sum_{k \neq 1, i} \delta_k \Delta_i \cdot \Delta_k.$$

The only component of the latter sum that meets  $\Delta_i$  can be the line  $l_2$ . Hence  $l_2 = \Delta_j$  for some  $j \neq 1, i$ . Now (11) shows that  $\delta_i = \delta_j$ , as claimed.

Furthermore, since  $\Delta_i \cup \Delta_j$  meets  $\Delta_1$  twice and  $\text{supp } D$  is a tree, the line  $\Delta_1$  passes through the intersection point  $\Delta_i \cap \Delta_j$ . On the other hand,  $\Delta_1$  is tangent to exactly two members of the pencil  $|C|$ , which are either smooth or consist of two lines  $\Delta_i$  and  $\Delta_j$  meeting  $\Delta_1$  at their common point (an Eckardt point of  $Y$ ). By the simply connectedness of  $\text{supp } D$ , none of the other components of members of  $|C|$  can be contained in  $\text{supp } D$ . Hence  $\text{supp } D$  can contain at most 5 components, namely,  $\Delta_1$  and the components of two degenerate members tangent to  $\Delta_1$ . However, this contradicts Lemma 4.2, since by this lemma  $\text{supp } D$  consists of at least 7 components. Now the proof is completed.  $\square$

**Lemma 4.7.** *Every component of the degenerate members of the pencil  $\mathcal{L}$  on  $Y$  passes through the base point  $P$  of  $\mathcal{L}$ .*

*Proof.* Assume on the contrary that there is a component  $C_0$  of a degenerate member  $L^{(0)}$  of  $\mathcal{L}$  such that  $P \notin C_0$ . By Zariski's Lemma  $C_0^2 < 0$ . Hence  $C_0$  is a  $(-1)$ -curve on  $Y$  and so  $D \cdot C_0 = (-K_Y) \cdot C_0 = 1$ . By an easy argument (cf. the proof of Lemma 4.2) the curve  $\text{supp } L^{(0)}$  is connected and simply connected (this is a tree of rational curves outside  $P$ ). If there is another component  $C_1$  of  $L^{(0)}$  which meets  $C_0$  and does not pass through  $P$ , then  $L^{(0)}$  contains the configuration of two crossing lines  $C_0 + C_1$  which do not pass through  $P$ . Then  $\mathcal{L}$  must be the linear system of conics  $|C_0 + C_1|$ . By Lemma 4.3 this leads to a contradiction. Hence  $C_0$  cannot separate  $\text{supp } L^{(0)}$  and so  $C_0$  meets the complement  $\text{supp}(L^{(0)} - C_0)$  at one point transversally.

It follows that  $\Delta_i \cdot C_0 = 1$  for a unique index  $i$ . If  $C_0$  is not a component of  $D$ , then  $1 = D \cdot C_0 = \delta_i$ , which contradicts Lemma 4.6. Similarly, if  $C_0 = \Delta_j$ , then  $1 = D \cdot C_0 = \delta_i - \delta_j$ . Hence  $\delta_i = 1 + \delta_j > 1$ . Once again, this contradicts Lemma 4.6.  $\square$

**Lemma 4.8.** *The pencil  $\mathcal{L}$  has at most two degenerate members.*

*Proof.* Recall (see 4.5) that  $p : W \rightarrow Y$  stands for the minimal resolution of the base locus of  $\mathcal{L}$  and  $q : W \rightarrow \mathbb{P}^1$  for the fibration given by  $p_*^{-1}\mathcal{L}$ . Write  $p$  as a composition of blowups of points over  $P$ :

$$(12) \quad p : W \xrightarrow{p_1} W_1 \xrightarrow{p_2} \dots \xrightarrow{p_N} W_N = Y,$$

where the exceptional divisor  $S_W$  of  $p_1$  is a  $q$ -horizontal  $(-1)$ -curve on  $W$ . A general fiber  $L$  of  $q$  is a smooth rational curve meeting  $S_W$  at one point. Indeed,  $L \setminus S_W \simeq p(L) \setminus P \simeq \mathbb{A}^1$ . Therefore  $S_W$  is a section of  $q$ .

Let  $C_1, \dots, C_m$  be the components of degenerate fibers  $F_1, \dots, F_m$  of  $q$  meeting  $S_W$ . We claim that all the curves  $C_i$  are  $p$ -exceptional. Indeed, otherwise for some  $i$ , the image  $p(\overline{(F_i \setminus C_i)})$  would be a component of a degenerate member of  $\mathcal{L}$  which does not pass through  $P$ . The latter contradicts Lemma 4.7.

Note that, on each step, the exceptional divisor of  $p_k \circ \dots \circ p_N$  is an SNC tree of rational curves. On the other hand, all the curves  $p_1(C_i)$  on  $W_1$  pass through the point  $p_1(S_W)$ . Therefore  $m \leq 2$ .  $\square$

**Lemma 4.9.** *The pencil  $\mathcal{L}$  has exactly two degenerate members, say,  $L^{(1)}$  and  $L^{(2)}$ . Furthermore,  $\text{supp } D = \text{supp}(L^{(1)} + L^{(2)})$  consists of 8 irreducible components i.e.,  $n = 8$  in (6).*

*Proof.* Assume that the only degenerate member of  $\mathcal{L}$  is  $L^{(1)}$ . In this case,  $\text{supp } D \subseteq \text{supp } L^{(1)}$ ,  $Z \supseteq \mathbb{A}^1$ , and  $U \supseteq \mathbb{A}^2$ .

If  $Z \simeq \mathbb{A}^1$  then  $\text{Pic}(U) = 0$  and  $H^0(U, \mathcal{O}_U) = \mathbb{C}$ , hence  $\text{Pic}(Y) \simeq \sum_{i=1}^n \mathbb{Z} \cdot \Delta_i$ . Thus in this case  $n = 7$  and  $-K_Y = \sum_{i=1}^7 m_i \Delta_i$  for some  $m_i \in \mathbb{Z}$ . On the other hand,  $-K_Y \equiv D = \sum_{i=1}^7 \delta_i \Delta_i$ , where  $0 < \delta_i < 1 \forall i$  according to Lemma 4.6. Since the decomposition of  $-K_Y$  in  $\text{Pic}(Y) \otimes \mathbb{Q}$  is unique, this yields a contradiction.

If further  $Z \simeq \mathbb{P}^1$  then  $\text{Pic}(U) \simeq \mathbb{Z}$  and  $H^0(U, \mathcal{O}_U) = \mathbb{C}$ . From the exact sequence (7), where  $\text{Pic}(U) = 0$  is replaced by  $\text{Pic}(U) \simeq \mathbb{Z}$ , we obtain  $n = 6$ . By Lemma 4.2, this leads again to a contradiction. Therefore  $\mathcal{L}$  has indeed two degenerate members.

As for the second assertion, assuming on the contrary that  $\text{supp } D \neq \text{supp}(L^{(1)} + L^{(2)})$  we would have  $Z \supseteq \mathbb{A}^1$ . Now the same argument as before yields a contradiction. Since the Picard group  $\text{Pic}(Y) \cong \mathbb{Z}^7$  is generated by the irreducible components of  $L^{(1)} + L^{(2)}$  and  $L^{(1)} \equiv L^{(2)}$  is the only relation between these components, we obtain that  $n = 8$ .  $\square$

**Corollary 4.10.** *The pencil  $\mathcal{L}$  is ample. Furthermore, for every irreducible component  $C$  of a member of the pencil  $\mathcal{L}$  on  $Y$  we have  $C \setminus \{P\} \simeq \mathbb{A}^1$ , and two such components have just the point  $P$  in common.*

*Proof.* The first assertion follows immediately from Lemma 4.7 by the Nakai-Moishezon criterion. As for the second one, it follows from the well known fact that on an affine surface  $V = Y \setminus L$ , where  $L$  is a general fiber of  $\mathcal{L}$ , every degenerate fiber of the  $\mathbb{A}^1$ -fibration  $\varphi_{|\mathcal{L}|} : V \rightarrow \mathbb{A}^1$  is a disjoint union of affine lines, see e.g., [Mi], [Za].  $\square$

**Lemma 4.11.** *The pair  $(Y, D)$  is not log canonical at  $P$ .*

*Proof.* Let  $D_W$  denote the crepant pull-back of  $D$  on  $W$  as in (8) i.e., a  $\mathbb{Q}$ -divisor on  $W$  such that

$$K_W + D_W = p^*(K_Y + D) \quad \text{and} \quad p_* D_W = D.$$

The exceptional  $(-1)$ -section  $S$  on  $W$  is the only  $q$ -horizontal component of  $D_W$ . For a general fiber  $L$  of  $q$  we have  $2 = (-K_W) \cdot L = D_W \cdot L$ . Therefore, the discrepancy  $a(S, D)$  (i.e., the coefficient of  $S$  in  $D_W$  with the opposite sign) equals  $-2$ . This proves our assertion.  $\square$

**Corollary 4.12.**  $\text{mult}_P(D) > 1$ .

*Proof.* If  $\text{mult}_P(D) \leq 1$ , then the pair  $(Y, D)$  is canonical at  $P$ , because  $P \in Y$  is a smooth point. In particular, it is log canonical. This contradicts Lemma 4.11.  $\square$

**Lemma 4.13.** *Any line  $l$  on  $Y$  through  $P$  is contained in  $\text{supp } D$ .*

*Proof.* Assuming the contrary we obtain  $1 = (-K_Y) \cdot l = D \cdot l \geq \text{mult}_P(D) > 1$ , a contradiction.  $\square$

**Lemma 4.14.**  *$P$  cannot be an Eckardt point on  $Y$ .*

*Proof.* Suppose the contrary. Then by Lemma 4.13, up to a permutation we may assume that  $\Delta_1, \Delta_2, \Delta_3$  are lines through  $P$ , where  $\delta_1 \leq \delta_2 \leq \delta_3$ . Since  $D \cdot \Delta_1 = (-K_Y) \cdot \Delta_1 = 1$ , for an effective  $\mathbb{Q}$ -divisor on  $Y$

$$(13) \quad D' := \frac{1}{1 - \delta_1}(D - \delta_1(\Delta_1 + \Delta_2 + \Delta_3))$$

we obtain  $D' \cdot \Delta_1 = 1$  and by (13)

$$D \equiv -K_Y \equiv \Delta_1 + \Delta_2 + \Delta_3 \equiv D'.$$

Now the proof of Lemma 4.11 works equally for  $D'$ . Hence the pair  $(Y, D')$  is not canonical at  $P$  and so  $\text{mult}_P(D') > 1$ . This contradicts the inequality  $\text{mult}_P(D') \leq D' \cdot \Delta_1 = 1$ .  $\square$

**Lemma 4.15.** *The fibration  $q : W \rightarrow \mathbb{P}^1$  has exactly two degenerate fibers. The general member  $L$  of  $\mathcal{L}$  is singular at  $P$ .*

*Proof.* The first assertion follows from Lemma 4.9. Let us show the second one. Assuming the contrary, for a smooth rational curve  $L$  on  $Y$  we have by adjunction  $(K_Y + L) \cdot L = -2$ . The Mori cone of  $Y$  being spanned by the  $(-1)$ -curves  $E_1, \dots, E_{27}$ , there is a decomposition

$$L \equiv \sum_{i=1}^{27} \alpha_i E_i,$$

where  $\alpha_i \in \mathbb{Q}_{\geq 0}$ . Hence  $(K_Y + L) \cdot E_i < 0$  for some  $i$ . Thus  $L \cdot E_i < (-K_Y) \cdot E_i = 1$ , and so  $L$  cannot be ample. This contradicts Corollary 4.10.  $\square$

**Remark 4.16.** The minimal resolution  $p : W \rightarrow Y$  of the base point  $P$  of  $\mathcal{L}$  dominates the embedded minimal resolution of the cusp at  $P$  of a general member  $L$  of  $\mathcal{L}$ . The exceptional divisor  $E = p^{-1}(P) \subseteq W$  is a rational comb with the number of teeth equal the length of the Puiseux sequence of the cusp. The only  $(-1)$ -curve  $S_W$  in  $E$  is sitting on the handle of the comb. Hence  $E = E^{(1)} + S_W + E^{(2)}$ , where  $E^{(k)} \subseteq p^{-1}(L^{(k)})$ ,  $k = 1, 2$ , and exactly one of the  $E^{(k)}$  is a negatively definite linear chain of rational curves.

The degenerate members  $L_W^{(1)}, L_W^{(2)}$  of the induced linear system  $\mathcal{L}_W$  on  $W$  have the following structure:  $L_W^{(k)}$  consists of  $E^{(k)}$  and the proper transforms  $\Delta'_i$  of the

components  $\Delta_i \subseteq L^{(k)}$ , called *feathers*. The feathers are disjoint in  $W$ , and each of them meets  $E^{(k)}$  at one point transversally. Let us illustrate this for the surface  $Y$  from Example 3.24, with the pullback  $\mathcal{L}_W$  of the pencil  $\mathcal{L}$  on  $Y$ .

**Example 4.17.** Up to interchanging the degenerate fibers  $L_W^{(1)}$  and  $L_W^{(2)}$  of  $\mathcal{L}_W$ , their structure is as follows. The fiber  $L_W^{(1)}$  contains a component  $E_1$  with self-intersection  $-5$  joint with the  $(-1)$ -section  $S_W$ , and also the  $(-1)$ -feathers  $\Delta'_1, \dots, \Delta'_5$  meeting  $E_1$ . The second fiber  $L_W^{(2)}$  contains the only feather  $\Delta'_0$  of multiplicity 2. The weighted dual graph of the configuration  $L_W^{(1)} + S_W + L_W^{(2)}$  is as follows:

$$(14) \quad \begin{array}{ccccccccc} & & (-1)_5 & & (\Delta'_i)_{i=1,\dots,5} & & -1 & & \Delta'_0 & & \\ & & \square & & & & & & \circ & & \\ & & | & & & & & & | & & \\ & & -5 & & -1 & & -2 & & -2 & & -2 \\ & & \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ \\ & & E_1 & & S_W & & E_2 & & E_3 & & E_4 \end{array}$$

where the box denotes a disjoint union of five  $(-1)$ -feathers  $\Delta'_1, \dots, \Delta'_5$  joint with  $E_1$ . The exceptional divisor of  $p : W \rightarrow Y$  is  $E = E_1 + \dots + E_4$ . While  $\Delta_i = p(\Delta'_i)$  ( $i = 0, \dots, 5$ ) are the components of the degenerate fibers  $L^{(j)} = p(L_W^{(j)})$  ( $j = 1, 2$ ) of the pencil  $\mathcal{L}$  on  $Y$ . More precisely,  $\Delta_0$  is the proper transform in  $Y$  of the cuspidal cubic  $C'$ ,  $\Delta_1$  is that of the conic  $C''$ , while  $\Delta_2, \dots, \Delta_5 \subseteq Y$  are the proper transforms of the four lines ( $y^4 = x^4$ ) in  $\mathbb{P}^2$ .

**4.18.** Let  $Q = (\rho \circ \sigma)(S_W) \in \mathbb{P}^2$ , where as in 4.5  $S_W$  stands for a  $(-1)$ -section of  $q$  contained in  $\text{Exc}(p)$ . Then  $\mathcal{L}_{\mathbb{P}^2} := \rho_*(\sigma_*\mathcal{L}_W)$  is the linear pencil of lines through  $Q$  on  $\mathbb{P}^2$ . We let  $\varphi := \rho \circ \sigma \circ p^{-1} : Y \dashrightarrow \mathbb{P}^2$ , and we let  $\mathcal{H}$  denote the proper transform of  $\mathcal{H}_{\mathbb{P}^2} := |\mathcal{O}_{\mathbb{P}^2}(1)|$  on  $Y$  via  $\varphi$ . With this notation, the following holds.

**Lemma 4.19.**  $\mathcal{L} \subseteq \mathcal{H}$ .

*Proof.* We have

$$\mathcal{H}_W := (\rho \circ \sigma)_*^{-1} \mathcal{H}_{\mathbb{P}^2} = (\rho \circ \sigma)^* \mathcal{H}_{\mathbb{P}^2} = \sigma^*(\rho^* \mathcal{H}_{\mathbb{P}^2}) \supseteq \sigma^*(\rho^* \mathcal{L}_{\mathbb{P}^2}).$$

Indeed,  $\mathcal{H}_{\mathbb{P}^2}$  is base point free. It is clear that  $\rho^* \mathcal{L}_{\mathbb{P}^2} = \mathcal{L}_{\mathbb{F}_1} + S$ , where  $S$  is the exceptional curve of  $\rho$ . Note that the centers of subsequent blowups in  $\sigma$  (including infinitesimally near centers) lie neither on the proper transform of  $S$  nor on that of general members of  $\mathcal{L}_{\mathbb{P}^2}$ . Hence,

$$\sigma^*(\rho^* \mathcal{L}_{\mathbb{P}^2}) = \sigma^*(\mathcal{L}_{\mathbb{F}_1} + S) = \mathcal{L}_W + S_W.$$

Since  $S_W$  is  $p$ -exceptional, applying  $p_*$  yields the assertion.  $\square$

**Corollary 4.20.**  $\mathcal{L}$  cannot be contained in  $|-mK_Y|$  for any  $m \in \mathbb{N}$ .

*Proof.* Assume to the contrary that  $\mathcal{L} \subseteq |-mK_Y|$  for some  $m \in \mathbb{N}$ . Then the mobile linear system  $\mathcal{H}$ , which defines the birational map  $\varphi : Y \dashrightarrow \mathbb{P}^2$  (see 4.18), is also contained in  $|-mK_Y|$ . This contradicts the Segre-Manin theorem as stated in [KSC, Theorem 2.13].  $\square$

The results of this subsection can be summarized as follows.

**Proposition 4.21.** *Let  $Y \subseteq \mathbb{P}^3$  be a smooth cubic surface, and  $X \subseteq \mathbb{A}^4$  be the affine cone over  $Y$ . Then  $X$  admits an effective  $\mathbb{C}_+$ -action if and only if  $Y$  admits a linear pencil  $\mathcal{L}$  with the following properties:*

- (1) *The base locus  $\text{Bs}(\mathcal{L})$  consists of a single point, say,  $P$ , which is not an Eckardt point on  $Y$ .*
- (2) *A general member  $L$  of  $\mathcal{L}$  is singular at  $P$ , and  $L \setminus \{P\} \simeq \mathbb{A}^1$ .*
- (3)  *$\mathcal{L}$  has exactly two degenerate members, say  $L^{(1)}$  and  $L^{(2)}$ , where the curve  $L^{(1)} \cup L^{(2)}$  consist of 8 irreducible components  $\Delta_1, \dots, \Delta_8$ .*
- (4) *All curves  $\Delta_i$ ,  $i = 1, \dots, 8$ , pass through  $P$  and are pairwise disjoint off  $P$ . Furthermore,  $\Delta_i \setminus \{P\} \simeq \mathbb{A}^1 \forall i$ .*
- (5) *Every line on  $Y$  passing through  $P$  is one of the  $\Delta_i$ .*
- (6)  *$-K_Y \equiv D := \sum_{i=1}^8 \delta_i \Delta_i$ , where  $\delta_i \in \mathbb{Q}$  and  $0 < \delta_i < 1 \forall i$ .*
- (7) *The pair  $(Y, D)$  is not log canonical at  $P$ .*
- (8) *For every  $m > 0$ ,  $\mathcal{L}$  is not contained in  $|-mK_Y|$ .*

We do not know so far any example of a cubic surface with such a pencil  $\mathcal{L}$ . For the pencils on del Pezzo surfaces from Examples 3.20-3.24, not all of the properties (1)-(8) are fulfilled. For instance, the pencil of Example 3.24 satisfies (1)-(7), however (8) fails, since  $\mathcal{L} \sim -2K_Y$ .

## 4.2. The inverse nef value.

**4.22.** The nef value plays an important role in the adjunction-theoretic classification of polarized varieties. For a projective variety  $Y$  polarized by a nef divisor  $H$  we define the inverse nef value  $t_0 = t_0(Y, H)$  to be the supremum of  $t$  such that the divisor  $H + tK_Y$  is nef i.e.,

$$(15) \quad H \cdot C \geq t(-K_Y) \cdot C = t \deg(C)$$

for every curve  $C$  on  $Y$ . By the Kawamata rationality theorem [Mat, Thm. 7.1.1],  $t_0$  is achieved and is rational. By the Kawamata-Shokurov base-point-free theorem [Mat, Thm. 6.2.1], the divisor  $H + t_0K_Y$  is semiample i.e., the complete linear system  $|m(H + t_0K_Y)|$  has no base point for all  $m \gg 0$  and defines a surjective morphism  $\varphi: Y \rightarrow Y'$  with connected fibers onto a normal projective variety  $Y'$ . In particular,  $\kappa(H + t_0K_Y) \geq 0$ , where  $\kappa$  stands for the Iitaka-Kodaira dimension.

For a smooth cubic surface  $Y$  in  $\mathbb{P}^3$  satisfying Convention 4.1, we let  $\mathcal{H} = \varphi_*^{-1}(\mathcal{O}_{\mathbb{P}^2}(1))$  be the mobile linear system on  $Y$  constructed in 4.18. In this setting the inverse nef value  $t_0 = t_0(Y, \mathcal{H})$  is a positive integer (indeed, for  $t = t_0$  the equality in (15) is achieved on a  $(-1)$ -curve). Moreover,  $\kappa(\mathcal{H} + t_0K_Y) = 0$  if and only if  $\mathcal{H} + t_0K_Y \equiv 0$ . However, in the latter case by Corollary 4.19 a  $(-K_Y)$ -polar cylinder on  $Y$  cannot exist.

By virtue of Theorem 4.23 below, the same conclusion holds in the case where  $\kappa(\mathcal{H} + t_0K_Y) = 1$ . In the latter case the linear system  $|m(\mathcal{H} + t_0K_Y)|$  defines for  $m \gg 1$  a conic bundle  $Y \rightarrow \mathbb{P}^1$ . Indeed, the image curve is rational since  $Y$  is, and an irreducible general fiber  $F$  with  $F^2 = 0$  and  $-K_F = -K_Y|_F$  ample is a smooth conic. Actually  $|\mathcal{H} + t_0K_Y|$  defines already a conic bundle. For assuming that  $\mathcal{H} + t_0K_Y \equiv \beta F$ , where  $\beta \in \mathbb{Q}$ , and taking intersection with a line  $l$  on  $Y$  such that  $F \cdot l = 1$ , we obtain  $\beta \in \mathbb{N}$ .



**Theorem 4.23.** *Let  $\chi : Y \dashrightarrow \mathbb{P}^2$  be a birational map and  $\mathcal{H} = \chi_*^{-1}(|\mathcal{O}_{\mathbb{P}^2}(1)|)$  be the proper transform on  $Y$  of the complete linear system of lines on  $\mathbb{P}^2$ . Then there is no conic  $F$  on  $Y$  such that*

$$(16) \quad \mathcal{H} \sim -aK_Y + bF \quad \text{for some } a \in \mathbb{N} \text{ and } b \in \mathbb{Z}.$$

*Proof.* We use the methods developed in [Is<sub>1</sub>, Is<sub>2</sub>]. Consider a resolution of indeterminacies of  $\chi$ :

$$\begin{array}{ccc} & \tilde{Y} & \\ p \swarrow & & \searrow q \\ Y & \overset{\chi}{\dashrightarrow} & \mathbb{P}^2 \end{array}$$

Decomposing  $p$  into a sequence of blowups with exceptional curves  $E_1, \dots, E_n$ , the linear system  $\tilde{\mathcal{H}} = q^*(|\mathcal{O}_{\mathbb{P}^2}(1)|)$  on  $\tilde{Y}$  and the line bundle  $K_{\tilde{Y}}$  can be written in  $\text{Pic}(\tilde{Y})$  as

$$(17) \quad \tilde{\mathcal{H}} = p^*(\mathcal{H}) - \sum_{i=1}^n m_i E_i^* \quad \text{and} \quad K_{\tilde{Y}} = p^*(K_Y) + \sum_{i=1}^n E_i^*.$$

Computing the intersection numbers  $\tilde{\mathcal{H}}^2$  and  $\tilde{\mathcal{H}} \cdot K_{\tilde{Y}}$ , by (17) we obtain

$$(18) \quad 1 = \tilde{\mathcal{H}}^2 - \sum_{i=1}^n m_i^2 \quad \text{and} \quad -3 = K_{\tilde{Y}} \cdot \tilde{\mathcal{H}} + \sum_{i=1}^n m_i.$$

Suppose on the contrary that (16) holds for some conic  $F$  on  $Y$ . We choose the minimal possible value of  $a > 0$ . Since  $F^2 = 0$  on  $Y$ , from (16) and (18) we deduce

$$(19) \quad \sum_{i=1}^n m_i^2 = 3a^2 + 4ab - 1 \quad \text{and} \quad \sum_{i=1}^n m_i = 3a + 2b - 3.$$

In the rest of the proof we use the following Claims 1-4.

*Claim 1.* *There is a birational transformation  $Y \dashrightarrow Y'$ , where  $Y'$  is again a smooth cubic surface in  $\mathbb{P}^3$ , such that for the proper transforms  $\mathcal{H}'$  of  $\mathcal{H}$  and  $F'$  of  $F$  on  $Y'$  we have*

$$\mathcal{H}' \sim -aK_{Y'} + b'F'$$

*with the same  $a$  as in (16), and additionally with*

$$m'_i := \max_i \{m'_i\} \leq a \quad \forall i,$$

*where the integers  $m'_i$  have the same meaning on  $Y'$  as the  $m_i$  have on  $Y$ .*

*Proof of Claim 1.* Suppose that  $m_i > a$  for some value of  $i$ . Consider the conic bundle  $\varphi = \varphi|_{F'} : Y \rightarrow \mathbb{P}^1$ . Let us perform an elementary transformation at the point  $P = p(E_i) \in Y$ . First we apply the blowup  $\sigma : \hat{Y} \rightarrow Y$  of  $P$  with exceptional divisor  $E$ . Assuming that  $m_i = \text{mult}_P(\mathcal{H})$ , on the new surface  $\hat{Y}$  we have

$$(20) \quad \hat{\mathcal{H}} := \sigma_*^{-1}(\mathcal{H}) = \sigma^*(\mathcal{H}) - m_i E \quad \text{and} \quad K_{\hat{Y}} = \sigma^*(K_Y) + E.$$

Modulo linear equivalence we may choose the conic  $F$  passing through the point  $P$ . Then  $F$  is irreducible. Indeed, otherwise  $F = F_1 + F_2$ , where  $F_1, F_2$  are two lines on  $Y$  and  $F_1$  passes through  $P$ . So

$$a < m_i \leq (F_1 \cdot \mathcal{H})_P \leq F_1 \cdot \mathcal{H} = F_1 \cdot (-aK_Y + b(F_1 + F_2)) = a,$$

which is impossible. Thus the proper transform  $\widehat{F} \sim \sigma^*(F) - E$  of  $F$  on  $\widehat{Y}$  is a  $(-1)$ -curve.

The contraction  $\sigma' : \widehat{Y} \rightarrow Y'$  of  $\widehat{F}$  to a point  $P' \in Y'$  yields a smooth conic  $F' := \sigma'_*(E)$  passing through  $P'$  on the resulting cubic surface  $Y'$ , such that

$$\mathcal{H}' := \sigma'_*(\widehat{\mathcal{H}}) \sim -aK_{Y'} + b'F' \quad \text{for some } b' \in \mathbb{Z}.$$

Using (16) and (20), on  $Y'$  we obtain

$$\text{mult}_{P'}(\mathcal{H}') = \widehat{\mathcal{H}} \cdot \widehat{F} = \sigma^*(\mathcal{H}) \cdot \widehat{F} - m_i E \cdot \widehat{F} = \mathcal{H} \cdot F - m_i = -aK_Y \cdot F - m_i = 2a - m_i < a.$$

Iterating this procedure we achieve finally that  $m'_i \leq a$  for all values of  $i$ , as required.  $\square$

So we assume in the sequel that

$$(21) \quad m = \max_i \{m_i\} \leq a \quad \forall i.$$

*Claim 2.* Under the assumption (21) we have  $b < 0$ .

*Proof of Claim 2.* From (19) and (21) we obtain

$$(22) \quad 3a^2 + 4ab = 1 + \sum_{i=1}^n m_i^2 \leq 1 + m \sum_{i=1}^n m_i \leq 1 + m(3a + 2b - 3) \leq 1 + a(3a + 2b - 3).$$

It follows by (22) that  $2ab \leq 1 - 3a$ . Since  $a \geq 1$  then  $b \leq \frac{1}{2a} - \frac{3}{2} \leq -1$ .  $\square$

*Claim 3 (the Noether-Fano Inequality).* For  $m$  as in (21) we have

$$a \geq m > a + b.$$

*Proof of Claim 3.* The first inequality follows by (21). To show the second, suppose on the contrary that

$$(23) \quad m \leq a + b.$$

From (22) and (23) we obtain

$$(24) \quad 3a^2 + 4ab \leq 1 + m(3a + 2b - 3) \leq 1 + (a + b)(3a + 2b - 3).$$

Thus by (23) and (24)

$$(25) \quad 3 \leq 3m \leq 3(a + b) \leq 1 + b(a + 2b).$$

We claim that  $a + 2b \geq 0$ . Indeed, let  $C$  be the residual line of the conic  $F$  on  $Y$  so that  $F + C \sim -K_Y$ . Then by (16),

$$0 \leq \mathcal{H} \cdot C = (-aK_Y + bF) \cdot C = a + 2b.$$

Now (25) leads to a contradiction, since  $b < 0$  by Claim 2.  $\square$

*Claim 4.* Consider the morphism  $\varphi : Y \rightarrow \mathbb{P}^1$  defined by the pencil of conics  $|F|$  on  $Y$ . Let  $m = m_i$ , and let  $Q = p(E_i) \in Y$ . If a line  $l$  on  $Y$  passes through  $Q$ , then  $l$  is a component of the fiber of  $\varphi$  through  $Q$ .

*Proof of Claim 4.* We have

$$(26) \quad (\mathcal{H} \cdot l)_Q \geq \text{mult}_Q(\mathcal{H}) = m > a + b.$$

On the other hand,

$$(27) \quad (\mathcal{H} \cdot l)_Q \leq \mathcal{H} \cdot l = (-aK_Y + bF) \cdot l = a + bF \cdot l.$$

By (26) and (27),  $b < bF \cdot l$ , where  $b < 0$  by Claim 2. Therefore  $F \cdot l = 0$ , and the claim follows.  $\square$

Let again  $C$  be the residual  $(-1)$ -curve of the conic  $F$  on  $Y$  so that  $-K_Y \sim F + C$ . We let  $V$  denote a del Pezzo surface of degree 4 obtained by the contraction  $\pi : Y \rightarrow V$  of  $C$ , and  $\mathcal{H}_V, F_V$ , etc. denote the images on  $V$  of  $\mathcal{H}, F$ , etc. Due to (16),

$$(28) \quad -K_V \sim F_V \quad \text{and} \quad \mathcal{H}_V \sim -aK_V + bF_V \sim -(a+b)K_V.$$

By Claim 4 there is no line on  $V$  through the point  $Q_V := \pi(Q)$ . The blowup  $\sigma : Y' \rightarrow V$  at  $Q_V$  yields yet another cubic surface  $Y'$  with the exceptional  $(-1)$ -curve  $E = \sigma^{-1}(Q_V)$ . For the proper transform  $\mathcal{H}' = \sigma_*^{-1}(\mathcal{H}_V)$  on  $Y'$  we obtain by (28):

$$(29) \quad \mathcal{H}' \sim \sigma^*(\mathcal{H}_V) - mE \sim (a+b)\sigma^*(-K_V) - mE \sim (a+b)(-K_{Y'}) + (a+b-m)E.$$

The linear system

$$|F'| = |-K_{Y'} - E|$$

defines a conic bundle on  $Y'$ . Plugging  $E \sim -K_{Y'} - F'$  into (29) we deduce:

$$(30) \quad \mathcal{H}' \sim (2a + 2b - m)(-K_{Y'}) - (a + b - m)F'.$$

Using Claims 2 and 3,

$$(31) \quad 2a + 2b - m = (a + b) + (a + b - m) < a.$$

By virtue of (30) the latter inequality contradicts the minimality of  $a$ . Now the proof of Theorem 4.23 is completed.  $\square$

**Corollary 4.24.** *Under Convention 4.1 the divisor  $\mathcal{H} + t_0K_Y$  in 4.22 is big i.e.,*

$$\kappa(\mathcal{H} + t_0K_Y) = 2.$$

## 5. CONES OVER SOME RATIONAL FANO THREEFOLDS

In this section we provide examples of two families of rational Fano 3-folds such that the affine cones over their anti-canonical embeddings admit nontrivial  $\mathbb{C}_+$ -actions.

**Proposition 5.1.** *Consider a smooth intersection  $Y = Y_{2,2}$  of two quadric hypersurfaces in  $\mathbb{P}^5$ . Then the affine cone  $X$  over  $Y$  admits an effective  $\mathbb{C}_+$ -action.*

*Proof.* According to the criterion of Theorem 3.9, it is enough to construct a  $(-K_Y)$ -polar open cylinder on  $Y$ . Fix a line  $l \subseteq Y$ . Consider the diagram

$$\begin{array}{ccc} & \tilde{Y} & \\ \sigma \swarrow & & \searrow \varphi \\ Y & \overset{\psi}{\dashrightarrow} & \mathbb{P}^3 \end{array}$$

where  $\psi$  is the projection with center  $l$ ,  $\sigma$  is the blowup of  $l$ , and  $\varphi$  is the blowdown of the divisor  $D$  which is swept out by lines meeting  $l$  (i.e.,  $\sigma(D)$  is the union of lines meeting  $l$ ; see [GH, Ch. 6]). It is easily seen that  $\Gamma = \varphi(D)$  is a smooth quintic curve

in  $\mathbb{P}^3$  of genus 2. The image  $Q = \varphi(E)$  of the exceptional divisor  $E$  of  $\sigma$  is a quadric in  $\mathbb{P}^3$ . For a line  $l \subseteq Y$ , the following alternative holds: either

- (1)  $N_{l/X} = \mathcal{O}_l \oplus \mathcal{O}_l$  and  $Q$  is smooth,

or

- (2)  $N_{l/X} = \mathcal{O}_l(1) \oplus \mathcal{O}_l(-1)$  and  $Q$  is singular.

Anyhow,

$$Y \setminus \sigma(D) \simeq \mathbb{P}^3 \setminus Q.$$

Suppose that  $Q$  is singular; then  $Q$  is a quadratic cone. Let  $\Pi$  be a plane in  $\mathbb{P}^3$  passing through the vertex  $P$  of  $Q$ . We claim that  $\mathbb{P}^3 \setminus (Q \cup \Pi)$  is a principal cylinder. Indeed, consider the projection  $\pi_P$  with center  $P$  and its resolution:

$$\begin{array}{ccc} & \tilde{\mathbb{P}}^3 & \\ \sigma' \swarrow & & \searrow \varphi' \\ \mathbb{P}^3 & \overset{\pi_P}{\dashrightarrow} & \mathbb{P}^2 \end{array}$$

Let  $E' \subseteq \tilde{\mathbb{P}}^3$  be the exceptional divisor of  $\sigma'$ , and let  $Q' \subseteq \tilde{\mathbb{P}}^3$  be the proper transform of  $Q$ . Then  $C = \varphi'(Q') \subseteq \mathbb{P}^2$  is a conic, and  $E'$  is a section of the  $\mathbb{P}^1$ -bundle  $\tilde{\mathbb{P}}^3 \rightarrow \mathbb{P}^2$ . Furthermore,

$$\tilde{\mathbb{P}}^3 \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1)).$$

Letting  $\Pi' \subseteq \tilde{\mathbb{P}}^3$  be the proper transform of  $\Pi$ , the image  $\varphi'(\Pi') = H \subseteq \mathbb{P}^2$  is a line. We have

$$\mathbb{P}^3 \setminus (Q \cup \Pi) \simeq \tilde{\mathbb{P}}^3 \setminus (Q' \cup \Pi' \cup E')$$

is an  $\mathbb{A}^1$ -bundle over  $\mathbb{P}^2 \setminus (C \cup H)$ . Since  $\text{Pic}(\mathbb{P}^2 \setminus (C \cup H)) = 0$  we obtain

$$\mathbb{P}^3 \setminus (Q \cup \Pi) \simeq \mathbb{A}^1 \times (\mathbb{P}^2 \setminus (C \cup H)),$$

as required. □

Let us exhibit yet another family of Fano threefolds with Picard rank 1. Their moduli space is 6-dimensional. Every member  $Y$  of this family admits a  $(-K_Y)$ -polar cylinder, whereas the subfamily of completions of  $\mathbb{A}^3$  is only 4-dimensional [Fur], [Pr<sub>1</sub>].

**Proposition 5.2.** *Let  $Y = Y_{22} \subseteq \mathbb{P}^{13}$  be a Fano variety of genus 12 with  $\text{Pic}(Y) = \mathbb{Z} \cdot [-K_Y]$ , anticanonically embedded into  $\mathbb{P}^{13}$ . Then the affine cone  $X$  over  $Y$  admits an effective  $\mathbb{C}_+$ -action.*

*Proof.* Again, it is enough to construct a  $(-K_Y)$ -polar open cylinder on  $Y$ . Then the result follows by applying Theorem 3.9. Picking a line  $l_1 \subseteq Y$  we consider the commutative diagram

$$\begin{array}{ccc} \tilde{Y} & \overset{\chi}{\dashrightarrow} & \tilde{Y}^+ \\ \sigma_1 \downarrow & & \downarrow \varphi_1 \\ \mathbb{P}^{13} \supseteq Y = Y_{22} & \overset{\psi_1}{\dashrightarrow} & Y_5 \subseteq \mathbb{P}^6 \end{array}$$

where  $\sigma_1$  is the blowup of  $l_1$ ,  $\psi_1$  is the double projection with center  $l_1$ <sup>10</sup> onto a Fano threefold  $Y_5$  of degree 5 and of Fano index 2, anticanonically embedded into  $\mathbb{P}^6$ ,  $\varphi_1$  is the blowup of a smooth rational curve  $\Gamma \subseteq Y_5$  of degree 5, and  $\chi$  is a flop; see [IP, §4.3]. We have

$$Y \setminus H_1 \simeq Y_5 \setminus H_2,$$

where  $H_1 \subseteq Y$  is a hyperplane section with  $\text{mult}_{l_1}(H_1) = 3$ , and  $H_2 \subseteq Y_5$  is a hyperplane section passing through  $\Gamma$ . Thus it suffices to show that  $Y_5 \setminus H_2$  contains an  $H_2$ -polar cylinder.

Let further  $l_2 \subseteq Y_5$  be a line. Recall that the family of all lines on  $Y_5$  is parameterized by  $\mathbb{P}^2$ , and either  $N_{l_2/Y_5} \simeq \mathcal{O}_{l_2} \oplus \mathcal{O}_{l_2}$ , or  $N_{l_2/Y_5} \simeq \mathcal{O}_{l_2}(1) \oplus \mathcal{O}_{l_2}(-1)$ . The lines of second type are parameterized by a smooth conic on  $\mathbb{P}^2$ ; see [FN]. There exists a line  $l_2$  on  $Y_5$  of second type contained in  $H_2$ . Consider the projection  $\psi_2$  with center  $l_2$  and its resolution:

$$\begin{array}{ccc} & \tilde{Y}_5 & \\ \sigma_2 \swarrow & & \searrow \varphi_2 \\ \mathbb{P}^6 \supseteq Y_5 & \overset{\psi_2}{\dashrightarrow} & Q \subseteq \mathbb{P}^4 \end{array}$$

where  $\sigma_2$  is the blowup of  $l_2$ , and  $Q \subseteq \mathbb{P}^4$  is a smooth quadric. We have

$$Y_5 \setminus H_2 \simeq Q \setminus H_3,$$

where  $H_2' \subseteq Y_5$  and  $H_3 \subseteq Q$  are hyperplane sections such that  $\text{mult}_{l_2}(H_2') = 2$ , and  $H_2'$  is swept out by lines meeting  $l_2$ . Since  $\psi_2$  is a projection,

$$Y_5 \setminus (H_2 \cup H_2') \simeq Q \setminus (H_3 \cup H_3'),$$

where  $H_3' \subseteq Q$  is another hyperplane section (possibly  $H_3' = H_3$ ). It remains to show that the complement  $Q \setminus (H_3 \cup H_3')$  contains an  $H_3$ -polar cylinder.

We may assume that  $H_3' \neq H_3$ . The projection  $\pi_P : \mathbb{P}^4 \dashrightarrow \mathbb{P}^3$  with center at a general point  $P \in H_3 \cap H_3'$  yields an isomorphism

$$Q \setminus (H_3 \cup H_3') \simeq \mathbb{P}^3 \setminus (\Pi_1 \cup \Pi_2 \cup \Pi_3),$$

where  $\Pi_1, \Pi_2, \Pi_3$  are three planes in  $\mathbb{P}^3$ . So the existence of an  $H_3$ -polar cylinder on  $Q \setminus (H_3 \cup H_3')$  is equivalent to the existence of a  $\Pi_1$ -polar cylinder on  $\mathbb{P}^3 \setminus (\Pi_1 \cup \Pi_2 \cup \Pi_3)$ . Now the assertion easily follows.  $\square$

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<sup>10</sup>That is,  $\psi_1 : Y \dashrightarrow \mathbb{P}^6$  is defined by the linear system  $|\sigma_1^* \mathcal{O}_Y(1) - 2\tilde{E}|$ , where  $\tilde{E} \subseteq \tilde{Y}$  is the exceptional divisor of  $\sigma_1$ .

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