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GROUP ACTIONS ON AFFINE CONES

TAKASHI KISHIMOTO, YURI PROKHOROV, AND MIKHAIL ZAIDENBERG

ABSTRACT. We address the following question:

Determine the affine cones over smooth projective varieties which admit an action of a connected algebraic group different from the standard \mathbb{C}^ -action by scalar matrices and its inverse action.*

We show in particular that the affine cones over anticanonically embedded smooth del Pezzo surfaces of degree ≥ 4 possess such an action. A question in [FZ₁] whether this is also true for cubic surfaces, occurs to be out of reach for our methods. Nevertheless, we provide a general geometric criterion that could be helpful also in this case.

INTRODUCTION

All varieties in this paper are defined over \mathbb{C} . By Corollary 1.13 in [FZ₁], an isolated Cohen-Macaulay singularity (X, x) of a normal quasiprojective variety X must be rational provided that X admits an effective action of a connected non-abelian algebraic group, in particular of the additive group \mathbb{C}_+ . On the other hand, consider the affine Fermat cubic in \mathbb{A}^4

$$x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0.$$

It has a rational singularity at the origin. The question was raised [FZ₁, Question 2.22] whether it also admits a non-diagonal action of a connected algebraic group, in particular, a \mathbb{C}_+ -action. So far, we do not know the answer. However, we answer in affirmative a similar question for all del Pezzo surfaces of degree $d \geq 4$.

Theorem 0.1. *Let Y_d be a smooth del Pezzo surface of degree d anticanonically embedded into \mathbb{P}^d , and let $X_d \subseteq \mathbb{A}^{d+1}$ be the affine cone over Y_d . If $4 \leq d \leq 9$ then X_d admits a nontrivial \mathbb{C}_+ -action. Consequently, the automorphism group $\text{Aut}(X_d)$ is infinite dimensional.*

One of our main results (Theorem 3.9) provides a necessary and sufficient condition for the existence of a nontrivial \mathbb{C}_+ -action on an affine cone. As a corollary, for affine cones of dimension 3 we obtain the following geometric criterion.

Theorem 0.2. *Let Y be a smooth projective rational surface with a principal polarization $\varphi_{|H|} : Y \hookrightarrow \mathbb{P}^n$, and let $X \subseteq \mathbb{A}^{n+1}$ be the affine cone over $Y \subseteq \mathbb{P}^n$. Then X admits a nontrivial \mathbb{C}_+ -action if and only if Y contains a cylindrical Zariski open set*

$$U = Y \setminus \text{supp } D \simeq Z \times \mathbb{A}^1,$$

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where Z is an affine curve and $D \in |dH|$ is an effective divisor on \mathbb{P}^n .

Using this criterion, we show in Proposition 3.12 that for every smooth projective rational surface Y , there exists an embedding $Y \hookrightarrow \mathbb{P}^n$ such that the corresponding affine cone possesses an effective action of \mathbb{C}_+ .

A theorem due to Matsumura, Monsky and Andreotti (see [MM], or [GH, §I.4], or Section 1 below) claims that any automorphism of a smooth hypersurface Y in \mathbb{P}^n of degree d , where $d, n \geq 3$ and $(d, n) \neq (4, 3)$, is restriction of a unique projective linear transformation, and $\text{Aut}(Y)$ is a finite group. In Corollary 2.4 we show that the automorphism group $\text{Aut}(X)$ of the affine cone X over a smooth, non-birationally ruled projective variety Y is a linear group, and actually a central extension of \mathbb{C}^* by a finite group. Consequently, among the affine cones over smooth projective surfaces in \mathbb{P}^3 , only those of degree ≤ 3 can admit a nontrivial action of a connected algebraic group, and their automorphism groups can be infinite-dimensional. Actually, the 3-fold affine quadric cone possesses an effective linear action of the additive group \mathbb{C}_+^2 , see Example 3.3 below and [Sh].

In Section 1 we give a short overview of the known results on the automorphism groups. In Section 2 we collect generalities on automorphisms of affine cones. Theorems 0.1 and 0.2 are proven in Section 3. In the final Section 4 we summarize some geometric facts that could be useful (in view of the criterion of Theorem 0.2) in order to answer Question 2.22 in [FZ₁].

1. GROUP ACTIONS ON PROJECTIVE VARIETIES

In this section we recall some well known facts about the automorphism groups of projective or quasiprojective varieties; see e.g., [GH, §I.4], [LZ, §II.3]. For an algebraic variety Y , we let $\text{Aut}(Y)$ denote the group of all biregular automorphisms of Y and $\text{Bir}(Y)$ the group of all birational transformations of Y into itself. For a projective or affine embedding $Y \hookrightarrow \mathbb{P}^n$ ($Y \hookrightarrow \mathbb{A}^n$, respectively) we let $\text{Lin}(Y)$ denote the group of all automorphisms of Y which extend linearly to the ambient space.

1.1. Automorphisms of smooth projective hypersurfaces. In the following theorems we gather some results concerning the groups Lin , Aut , and Bir for projective hypersurfaces; see Matsumura and Monsky [MM], Iskovskikh and Manin [IM], Pukhlikov [Pu1]-[Pu3], Cheltsov [Ch], de Fernex, Ein and Mustăța [DEM].

Theorem 1.1. *Let Y be a smooth hypersurface in \mathbb{P}^n of degree d . Then for all $d, n \geq 3$ except for $(d, n) = (4, 3)$,*

$$\text{Aut}(Y) = \text{Lin}(Y)$$

and this group is finite. It is trivial for a general hypersurface of degree $d \geq 3$.

Theorem 1.2. *For $Y \subseteq \mathbb{P}^n$ as above and for all $d > n \geq 2$ except for $(d, n) = (3, 2)$,*

$$\text{Bir}(Y) = \text{Aut}(Y).$$

This group is finite except in the case $(d, n) = (4, 3)$ of a smooth quartic surface $Y \subseteq \mathbb{P}^3$, where it is discrete, but can be infinite and different from $\text{Lin}(Y)$ which is finite. The group $\text{Bir}(Y)$ of a very general quartic surface $Y \subseteq \mathbb{P}^3$ is trivial.

The case $d \leq n$ is much more complicated, however, there are deep partial results, see e.g., [IM, Pu1].

Let us indicate briefly some ideas used in the proofs. In case $d \neq n + 1$ the proof is easy and exploits the fact that the canonical divisor $K_Y = \mathcal{O}_Y(d - n - 1)$ is $\text{Aut}(Y)$ -stable. In case $d = n + 1$ the equalities $\text{Bir}(Y) = \text{Aut}(Y) = \text{Lin}(Y)$ follow since such hypersurfaces represent Mori minimal models. Indeed a birational map between minimal models is an isomorphism in codimension 1, see e.g., [KM], hence it induces an isomorphism of the corresponding Picard groups. If $n \geq 4$ then $\text{Pic}(Y) \simeq \mathbb{Z}$ by the Lefschetz Hyperplane Section Theorem. Therefore any birational transformation φ of Y acts trivially on $\text{Pic}(Y)$ and so preserves the complete linear system of hyperplane sections $|\mathcal{O}_Y(1)|$. Since Y is linearly normal¹, φ is induced by a projective linear transformation of the ambient projective space \mathbb{P}^n . For the proof of finiteness of the group $\text{Lin}(Y)$ and its triviality for general hypersurfaces, we refer to the classical paper of Matsumura and Monsky [MM].

By virtue of the Noether-Lefschetz Theorem, these arguments apply also to very general smooth surfaces in \mathbb{P}^3 of degree $d \geq 4$. For an arbitrary smooth surface Y in \mathbb{P}^3 , the minimality of Y should be combined with the fact that $\text{Pic}(Y)$ is torsion free. Indeed, any smooth surface in \mathbb{P}^3 of degree $d \geq 4$ represents a minimal model and so is not birationally ruled, hence its birational automorphisms are biregular; see e.g., [Mat, Theorem 1-8-6]. For $d > 4$ the canonical class K_Y defines an equivariant polarization of Y , and $\mathcal{O}_Y(1) \sim \frac{1}{d-4}K_Y$. Since $\text{Pic}(Y)$ is torsion free and $H^0(\mathcal{O}_{\mathbb{P}^3}(1)) \rightarrow H^0(\mathcal{O}_Y(1))$ is a surjection, all automorphisms of Y are linear. This is not true, in general, in the case of a smooth quartic surface in \mathbb{P}^3 . An example of such a surface with infinite automorphism group due to Fano and Severi is discussed in [MM, Theorem 4]. A non-linear biregular involution exists on any smooth quartic in \mathbb{P}^3 containing skew lines, for instance, on the Fermat quartic $x^4 + y^4 + z^4 + u^4 = 0$; see Takahashi [Ta].

For a quadric hypersurface $X \subseteq \mathbb{A}^{n+1}$ of dimension $n \geq 2$, the group $\text{Aut}(X)$ is infinite-dimensional [To, Lemma 1.1], cf. also example 3.3 below. For $n = 2$ this group has an amalgamated product structure [DG]; cf. also [ML].

For a smooth cubic surface $Y \subseteq \mathbb{P}^3$ the group $\text{Aut}(Y) = \text{Lin}(Y)$ is finite, while the Cremona group $\text{Bir}(Y) \simeq \text{Bir}(\mathbb{P}^2)$ is infinite-dimensional. The automorphism groups of such surfaces were listed by Hosoh [Ho₁] who corrected an earlier classification by Segre [Se]; see also Manin [Man] and Dolgachev [Dol₁]. The largest order of such a group is 648; up to isomorphism this bound is achieved only for the Fermat cubic surface, see [Ho₂]. The least common multiple of the orders of all these automorphism groups is $3240 = 2^3 \cdot 3^4 \cdot 5$; see Gorinov [Go].

There is similar characterizations in terms of the automorphism group for the Fermat quartic $x^4 + y^4 + z^4 + u^4 = 0$ and the smooth quartic $x^4 + y^4 + z^4 + u^4 + 12xyzu = 0$ in \mathbb{P}^3 , see Mukai [Mu], Kondo [Ko₁], and Oguiso [Og].

1.2. Automorphisms of smooth projective varieties. According to the well known Matsumura Theorem ² the group $\text{Bir}(Y)$ of a smooth projective variety Y of general type is finite, hence also the group $\text{Aut}(Y)$ is. In particular, this holds if $c_1(Y) < 0$. See e.g., Xiao [Xi₁, Xi₂] for effective bounds of orders of automorphism groups for the general type surfaces. A vast literature is devoted to automorphism groups of K3 and

¹i.e., $H^0(\mathcal{O}_{\mathbb{P}^n}(1)) \rightarrow H^0(\mathcal{O}_Y(1))$ is a surjection.

²Which generalizes earlier results by Andreotti for surfaces of general type; see Kobayashi-Ochiai [KO] and Noguchi-Sunada [NS] for further generalizations.

Enriques surfaces. These groups are discrete, and infinite in many cases; finite such groups in the case of $K3$ -surfaces were classified e.g., in [Dol₂, IS, Ko₂, Mas, Ni, Xi₂].

For varieties of non-general type we have the following result due to Kalka-Shiffman-Wong [KSW] and Lin [LZ, Theorem II.3.1.2].

Theorem 1.3. *Let Y be a smooth projective variety of dimension n . Suppose that not all Chern numbers of Y vanish and either $c_1(Y) \leq 0$ or $H^{n,0}(Y) \neq 0$. Then $\text{Aut}(Y)$ is a discrete group.*

The first assumption is fulfilled, for instance, if $e(Y) = c_n(Y) \neq 0$, or $e(\mathcal{O}_Y) \neq 0$, or $c_1^n(Y) \neq 0$, where e stands for the Euler characteristic.

Clearly, it is not fulfilled for an abelian variety. For a projective embedding $A \hookrightarrow \mathbb{P}^N$ of an abelian variety A , the group $\text{Lin}(A)$ is finite, see [GH, §II.6], while the group $\text{Aut}(A) \supseteq A$ is infinite. Conversely, by Théorème Principale I of Blanchard [Bl] for any finite subgroup $G \subseteq \text{Aut}(A)$ there exists a projective embedding $A \hookrightarrow \mathbb{P}^N$ which linearizes G . A general form of Blanchard's Theorem is as follows (cf. [Ak₂, Theorem 3.2.1]).

Theorem 1.4. *Let Y be a smooth projective variety and $G \subseteq \text{Aut}(Y)$ a subgroup which acts finitely on $\text{Pic}(Y)$. Then there is a G -equivariant projective embedding $Y \hookrightarrow \mathbb{P}^N$.*

Indeed, such an embedding exists if and only if $\text{Pic}(Y)$ contains an ample G -invariant divisor class. If G acts finitely on $\text{Pic}(Y)$ then the orbit of any ample class is an ample G -invariant class.

For instance, if $\text{Pic}(Y)$ is discrete and G is connected then G acts trivially on $\text{Pic}(Y)$ and so, there is a G -equivariant projective embedding $Y \hookrightarrow \mathbb{P}^N$.

1.3. Homogeneous and almost homogeneous varieties. By the Borel-Remmert Theorem [Ak₂, 3.9], any connected, compact, homogeneous Kähler manifold V is biholomorphic to the product $Y \times \text{Alb}(V)$ of a flag variety Y and the Albanese torus $\text{Alb}(V)$.³ For a description of the automorphism groups of flag varieties see e.g., [Ak₂, §3.3].

The following ‘‘cone theorem’’ describes certain *almost homogeneous* complex varieties. It is due to Akhiezer [Ak₁] in algebraic context and to Huckleberry and Oeljeklaus [HO] in analytic one; see also [Hu] for some generalizations.

Theorem 1.5. *Let X be an irreducible reduced complex space of dimension ≥ 2 . Suppose that a Lie group G acts by biholomorphic transformations on X with an open orbit $\Omega = G(x) \subseteq X$ such that the complement $E = X \setminus \Omega$ is a proper analytic subset with an isolated point, say, $0 \in E$. Then the normalization $\nu: \tilde{X} \rightarrow X$ is one-to-one and \tilde{X} is biholomorphic to a projective or an affine cone over a projective flag variety G/P . The isolated point $0 \in E$ corresponds to the vertex of the cone. In particular, if $(X, 0)$ is smooth then $X \simeq \mathbb{A}^n$ or $X \simeq \mathbb{P}^n$ (Oeljeklaus 1985).*

An explicit description of almost homogeneous 2-dimensional affine cones over smooth projective curves is due to Popov [Po]; see also [FZ₂]. We recall that a Veronese cone V_d is the affine cone over a smooth rational normal curve $\Gamma_d \subset \mathbb{P}^d$ i.e., a linearly

³Recall that a flag variety is a (rational, projective) homogeneous variety G/P , where G is a semisimple linear algebraic group and $P \subseteq G$ is a parabolic subgroup, i.e. a subgroup of G which contains a Borel subgroup.

non-degenerate⁴ smooth curve in \mathbb{P}^d of degree d ; all such curves in \mathbb{P}^d are rational and projectively equivalent. For normal 2-dimensional cones, Popov's Theorem can be stated as follows.

Theorem 1.6. *Let X be the affine cone over a smooth projective curve Y . If X is normal and admits an algebraic group action transitive in $X \setminus \{0\}$, then X is a Veronese cone V_d for some $d \geq 1$, and Y is a rational normal curve Γ_d .*

Popov [Po] actually classified all almost homogeneous cones with an isolated singularity and not only the normal ones. Every such cone possesses a linear $\mathrm{SL}(2, \mathbb{C})$ -action transitive off the vertex. The group $\mathrm{Aut}(X)$ of a Veronese cone is infinite dimensional and so cannot be linearized under a projective embedding; see Section 2.3 below.

2. GROUP ACTING ON AFFINE CONES

2.1. Linear automorphisms of affine cones. Let us recall the following notions. A birational map $\bar{f} : \bar{X}_1 \dashrightarrow \bar{X}_2$ is said to be an isomorphism in codimension 1 if there are subsets $B_i \subseteq \bar{X}_i$ of codimension at least 2 such that \bar{f} is an isomorphism from $\bar{X}_1 \setminus B_1$ to $\bar{X}_2 \setminus B_2$. A projective variety Y is said to be birationally ruled if Y is birationally equivalent to a product $Z \times \mathbb{P}^1$ for some variety Z . For the proof of the following well known lemma we refer the reader to [To, Theorem 2.2] or [Co, Proposition 2.7].

Lemma 2.1. *Let $X_i = \bar{X}_i \setminus D_i$, $i = 1, 2$, where \bar{X}_i is a projective variety and D_i an irreducible divisor on \bar{X}_i . Suppose that X_i is regular near D_i for $i = 1, 2$. If D_1 is not birationally ruled then any isomorphism $f : X_1 \rightarrow X_2$ extends to a birational map $\bar{f} : \bar{X}_1 \dashrightarrow \bar{X}_2$ which is an isomorphism in codimension 1. If in addition the divisors D_1 and D_2 are ample then $\bar{f} : \bar{X}_1 \xrightarrow{\cong} \bar{X}_2$ is an isomorphism.*

We also need the following fact.

Lemma 2.2. *Consider two projective varieties $Y_i \subseteq \mathbb{P}^{n_i}$, where $n_i \geq 2$, $i = 1, 2$. Suppose that Y_1 and Y_2 are proper, smooth, irreducible, and linearly non-degenerate. Consider also the affine cones $X_i \subseteq \mathbb{A}^{n_i+1}$ and the projective cones $\bar{X}_i \subseteq \mathbb{P}^{n_i+1}$ over Y_i , $i = 1, 2$. Let $\varphi : X_1 \xrightarrow{\cong} X_2$ be an isomorphism such that the induced birational map $\bar{\varphi} : \bar{X}_1 \dashrightarrow \bar{X}_2$ is an isomorphism in codimension 1. Then φ extends to a unique linear isomorphism $\Phi : \mathbb{A}^{n_1+1} \xrightarrow{\cong} \mathbb{A}^{n_2+1}$. In particular $n_1 = n_2$, and Y_1 and Y_2 are projectively equivalent.*

Proof. We let $D_i = \bar{X}_i \setminus X_i$ denote the divisor at infinity; it is a scheme-theoretic hyperplane section. Since D_1 and D_2 are ample then (similarly as in Lemma 2.1) φ extends to an isomorphism $\bar{\varphi} : \bar{X}_1 \rightarrow \bar{X}_2$, which sends $0 \in \mathbb{A}^{n_1+1}$ to $0 \in \mathbb{A}^{n_2+1}$. Indeed, these points are the only singular points of the projective cones \bar{X}_1 and \bar{X}_2 . Moreover, $\bar{\varphi}$ sends the generators of the cone⁵ \bar{X}_1 into generators of \bar{X}_2 . Indeed, every generator l_1 of \bar{X}_1 meets D_1 transversally in one point. The image $l_2 = \varphi(l_1) \subseteq \bar{X}_2$ possesses similar properties, hence l_2 is again a projective line through the origin i.e., a generator of the cone \bar{X}_2 .

It follows that the orbits of the \mathbb{C}^* -action on \bar{X}_1 are sent to the orbits of the \mathbb{C}^* -action on \bar{X}_2 . Furthermore $\bar{\varphi}$ is \mathbb{C}^* -equivariant, hence it induces an isomorphism $\varphi^* :$

⁴A projective variety $Y \subseteq \mathbb{P}^n$ is linearly non-degenerate if it is not contained in any hyperplane.

⁵That is the projective lines on \bar{X}_1 passing through the origin.

$\mathcal{O}(Y_2) \xrightarrow{\cong} \mathcal{O}(Y_1)$ of the homogeneous coordinate rings. These graded rings are the coordinate rings of the affine cones X_1 and X_2 , respectively, generated by their first graded pieces.⁶ The graded isomorphism φ^* restricts to a linear isomorphism, say, $\Psi : \mathbb{A}^{n_2+1} \xrightarrow{\cong} \mathbb{A}^{n_1+1}$ between these first graded pieces. The dual isomorphism $\Phi = \Psi^\vee : \mathbb{A}^{n_1+1} \xrightarrow{\cong} \mathbb{A}^{n_2+1}$ provides a desired linear extension of φ . The uniqueness of such an extension follows immediately from the linear non-degeneracy. \square

Lemmas 2.1 and 2.2 yield the following result.

Proposition 2.3. *Consider the affine cones $X_i \subseteq \mathbb{A}^{n_i+1}$ over projective varieties $Y_i \subseteq \mathbb{P}^{n_i}$, $i = 1, 2$. Suppose that Y_1 and Y_2 are smooth, irreducible, and linearly non-degenerate. If Y_1 is not birationally ruled then every isomorphism $\varphi : X_1 \xrightarrow{\cong} X_2$ extends to a unique linear isomorphism $\mathbb{A}^{n_1+1} \xrightarrow{\cong} \mathbb{A}^{n_2+1}$. In particular $n_1 = n_2$, and Y_1 and Y_2 are projectively equivalent.*

For a projective variety $Y \subseteq \mathbb{P}^n$ and its affine cone $X \subseteq \mathbb{A}^{n+1}$, the exact sequence

$$0 \rightarrow \mathbb{C}^* \rightarrow \mathrm{GL}(n+1, \mathbb{C}) \rightarrow \mathrm{PGL}(n+1, \mathbb{C}) \rightarrow 0$$

yields the following one:

$$0 \rightarrow \mathbb{C}^* \rightarrow \mathrm{Lin}(X) \rightarrow \mathrm{Lin}(Y) \rightarrow 0.$$

This leads to the following corollaries.

Corollary 2.4. *Let $X \subseteq \mathbb{A}^{n+1}$ be the affine cone over a smooth projective variety $Y \subseteq \mathbb{P}^n$. If Y is not birationally ruled then $\mathrm{Aut}(X) = \mathrm{Lin}(X)$. Moreover, $\mathrm{Aut}(X)$ is a central extension of \mathbb{C}^* by the group $\mathrm{Lin}(Y)$.*

Corollary 2.5. *Let X be the affine cone over a smooth projective 3-fold Y . Suppose that Y is rationally connected and non-rational. Then $\mathrm{Aut}(X) = \mathrm{Lin}(X)$.*

Proof. Indeed if Y were birationally ruled i.e., birational to a product $Z \times \mathbb{P}^1$, then Z would be rationally connected and so a rational surface, hence Y would be rational too, contrary to our assumption. Hence Corollary 2.4 applies and gives the assertion. \square

This corollary can be applied, for instance, to non-rational Fano 3-folds. As examples, one can consider the affine cone $X \subseteq \mathbb{A}^5$ over a smooth cubic or quartic 3-fold in $Y \subseteq \mathbb{P}^4$.

2.2. Lifting G -actions. A natural question arises:

Which actions of an algebraic group G on Y are induced by G -actions on X ?

The next example shows that there should be some restrictions on such an action.

Example 2.6. Take for $Y \subseteq \mathbb{P}^n$ a projective embedding of an abelian variety A (for instance, a smooth cubic in \mathbb{P}^2). Thus Y is non-uniruled, hence by Corollary 2.4 any effective G -action on Y which lifts to a G -action on X is linear. Therefore $G \subseteq \mathrm{Lin}(Y)$ is a finite group, while the group $\mathrm{Aut}(Y)$ contains the subgroup of translations and so is infinite (see [GH, §II.6], or Section 1 above).

The following lemma provides a criterion for a G -action on Y to admit a lift to X .

⁶Consisting of the restrictions to X_i of linear functions on \mathbb{A}^{n_i+1} , $i = 1, 2$.

Lemma 2.7. *Let X be the affine cone over a smooth projective variety Y of degree > 1 . A G -action on Y lifts to a G -action on X if and only if the induced G -action on $\text{Pic } Y$ stabilizes the class $[\mathcal{O}_Y(1)]$.*

Proof. Since $\deg Y > 1$, the vertex $0 \in X$ is an isolated singular point. Given an action of an algebraic group G on X , $0 \in X$ is a fixed point of G . Consider the blowup $\sigma: \tilde{X} \rightarrow X$ of the vertex $0 \in X$ with the exceptional divisor $S_0 \simeq Y$. The induced action of G on \tilde{X} restricts to S_0 and so the given G -action on X descends to a G -action on $Y \simeq S_0$. Furthermore, G acts on the normal bundle $N_{S_0/\tilde{X}} \simeq \mathcal{O}_Y(-1)$. Hence the class $[\mathcal{O}_Y(1)]$ in $\text{Pic } Y$ is G -invariant.

Conversely, suppose that G acts on Y stabilizing the class $[\mathcal{O}_Y(1)] \in \text{Pic } Y$. Then G acts on the total space of $\mathcal{O}_Y \oplus \mathcal{O}_Y(1)$, which is isomorphic to \tilde{X} . Hence the G -action on Y lifts naturally to \tilde{X} and descends to X so that the projection $\pi: X \setminus \{0\} \rightarrow Y$ is G -equivariant. \square

The following weaker analog of Proposition 2.3 does not need the assumption of non-ruledness.

Proposition 2.8. *Consider the affine cones $X_i \subseteq \mathbb{A}^{n_i+1}$ over smooth, linearly non-degenerate, projective varieties $Y_i \subseteq \mathbb{P}^{n_i}$, $i = 1, 2$. Given an isomorphism $\varphi: X_1 \xrightarrow{\simeq} X_2$, the differential $d\varphi(0)$ provides a linear isomorphism $\Phi: \mathbb{A}^{n_1+1} \xrightarrow{\simeq} \mathbb{A}^{n_2+1}$ which restricts to an isomorphism $\Phi|_{X_1}: X_1 \xrightarrow{\simeq} X_2$. In particular $n_1 = n_2$, and Y_1 and Y_2 are projectively equivalent.*

Proof. Due to the assumption of linear non-degeneracy, the Zariski tangent space to X_i at the vertex $0 \in X_i$ is $T_0X_i = \mathbb{A}^{n_i+1}$, the tangent cone is $C_0X_i = X_i$, and the projective tangent cone is Y_i , $i = 1, 2$ (see e.g., [CLS, §9.7]). Now the assertion follows since $d\varphi(0)$ provides an isomorphism of the Zariski tangent spaces and sends the tangent cone C_0X_1 onto the cone C_0X_2 [Da, §7.3]. \square

Similarly as in Proposition 2.8, in case of a reductive group action a weaker analog of Corollary 2.4 holds without the assumption of non-ruledness.

Lemma 2.9. *Suppose that a reductive group G acts effectively on the affine cone $X \subseteq \mathbb{A}^{n+1}$ over a linearly non-degenerate proper projective variety $Y \subsetneq \mathbb{P}^n$. Then there is a faithful representation $\rho: G \rightarrow \text{GL}(n+1, \mathbb{C})$, which restricts to an effective linear G -action on X inducing an action of G on Y .*

Proof. The vertex $0 \in X$ is an isolated singular point of X , hence a fixed point of G . Since G is reductive, the induced representation ρ of G on the Zariski tangent space T_0X is faithful (see e.g., [Ak₂, FZ₂, Lemma 2.7(b)]) and descends to Y via the projective representation $\bar{\rho}: G \rightarrow \text{PGL}(n+1, \mathbb{C}) = \text{GL}(n+1, \mathbb{C})/\mathbb{C}^*$. \square

We note that for a non-reductive group action, ρ as above can be trivial. This is the case for instance for the \mathbb{C}_+ -action $t.(x, y) = (x + ty^2, y)$ on $X = \mathbb{A}^2$.

The following theorem is complementary to Corollary 2.4.

⁷We note that $d\varphi(0)$ lifts to an isomorphism of blowups $\text{Bl}_0(X_1) \xrightarrow{\simeq} \text{Bl}_0(X_2)$ preserving the exceptional divisors. These divisors are isomorphic to Y_1 and Y_2 , respectively, and $d\varphi(0)$ induces a linear isomorphism $Y_1 \xrightarrow{\simeq} Y_2$.

Theorem 2.10. *We let $X \subseteq \mathbb{A}^n$ ($n \geq 2$) be the affine cone over a smooth projective variety $Y \subseteq \mathbb{P}^{n-1}$. Suppose that*

- *The group $\text{Aut}(Y)$ is finite.*
- *A connected algebraic group G of dimension ≥ 2 acts effectively on X and contains a 1-dimensional torus \mathbf{T}_1 acting on \mathbb{A}^n by scalar matrices.*

Then the following hold.

- (a) *G is a solvable group of rank 1.*
- (b) *There exists an \mathbb{A}^1 -fibration $\theta : X \rightarrow Z$, where Z is an affine variety equipped with a good \mathbb{C}^* -action and θ is equivariant with respect to the standard \mathbb{C}^* -action on X . Furthermore, Z is normal if X is.*
- (c) *Y is uniruled via a family of rational curves parameterized by $(Z \setminus \{\theta(0)\})/\mathbb{C}^*$.*

Proof. Consider a Levi decomposition $G = \text{Rad}_u(G) \rtimes H$, where $H \subseteq G$ is a maximal reductive subgroup (a Levi subgroup) containing \mathbf{T}_1 . By Lemma 2.9 the induced representation ρ of H on the Zariski tangent space T_0X is faithful. Moreover \mathbf{T}_1 acting on T_0X by scalar matrices is a central subgroup of $\text{Aut}(X)$. Since the group $\text{Aut}(Y)$ is finite and H is connected, the induced action of the quotient group H/\mathbf{T}_1 on Y is trivial. Hence $H = \mathbf{T}_1$ is a maximal torus of $G = \text{Rad}_u(G) \rtimes \mathbf{T}_1$, and so G is solvable of rank 1.

By our assumption $\dim(G) \geq 2$, hence the unipotent radical $\text{Rad}_u(G)$ is non-trivial and contains a one-parameter subgroup $U \simeq G_a$. All orbits of U are closed in X , and the one-dimensional orbits are isomorphic to the affine line \mathbb{A}^1 . Hence X is affine uniruled. Consider the naturally graded coordinate ring

$$\mathcal{O}_X =: A = \bigoplus_{k \geq 0} A_k.$$

The infinitesimal generator ∂ of the induced G_a -action on A is a homogeneous locally nilpotent derivation of A ; see [Re], [FZ₂]. The ring of invariants $B = \ker(\partial) = A^{G_a}$ is a graded subalgebra of A with $B_0 = A_0 = \mathbb{C}$. Therefore the affine variety $Z = \text{spec}(B)$ is endowed with a good \mathbb{C}^* -action with the attractive fixed point $0' = \theta(0)$. The orbit map $X \rightarrow Z$ of the G_a -action on X is a \mathbb{C}^* -equivariant surjection induced by the inclusion $B \subseteq A$ of graded rings. If A is integrally closed in $\text{Frac}(A)$ then also B is. Indeed, let $Z' = \text{spec } \bar{B}$ be the normalization of Z , where \bar{B} is the integral closure of B in $\text{Frac}(A)$. Since X is normal the morphism $X \rightarrow Z$ factorizes as $X \rightarrow Z' \xrightarrow{\nu} Z$. The locally nilpotent derivation ∂ stabilizes \bar{B} (see e.g., [Sei], [Vas], or [FZ₁, Lemma 2.15]) and so the morphisms $X \rightarrow Z' \xrightarrow{\nu} Z$ are equivariant with respect to the induced \mathbb{C}_+ -actions. The \mathbb{C}_+ -action is trivial on Z , hence also on Z' since $Z' \rightarrow Z$ is finite. Thus $B \subseteq \bar{B} \subseteq \ker \partial = B$, so $B = \bar{B}$ is normal as soon as A is.

Since none of the one-dimensional orbits of $U \simeq G_a$ in X passes through the vertex $0 \in X$ or is contained in an orbit closure of T_1 (i.e., in a generator of the cone), there is a Zariski open subset, say, Ω of Y covered by their images. Taking Zariski closures yields a family of rational curves parameterized by $(Z \setminus \{0'\})/\mathbb{C}^*$. Thus Y is uniruled. \square

2.3. On groups acting on 2-dimensional affine cones. The following corollary is immediate from Proposition 2.8.

Corollary 2.11. *Let Y_i be a smooth linearly non-degenerate curve in \mathbb{P}^{n_i} of degree d_i , and $X_i \subseteq \mathbb{A}^{n_i+1}$ be the affine cone over Y_i , $i = 1, 2$. Then the cones X_1 and X_2 are*

isomorphic if and only if they are linearly isomorphic, if and only if $n_1 = n_2$, $d_1 = d_2$ and Y_1 and Y_2 are projectively equivalent.

Similarly, from Corollary 2.4 we deduce the following one.

Corollary 2.12. *Let $X \subseteq \mathbb{A}^{n+1}$ be the affine cone over a smooth projective curve $Y \subseteq \mathbb{P}^n$. If Y is non-rational then $\text{Aut}(X) = \text{Lin}(X)$, and this group is a central extension of \mathbb{C}^* by the finite group $\text{Lin}(Y)$.*

2.13. However, $\text{Aut}(Y) \neq \text{Lin}(Y)$ for an elliptic curve Y . Consider further a smooth rational curve $Y \subseteq \mathbb{P}^n$ of degree d . If $d > n$ then Y is not linearly normal, and the affine surface X is not normal. Indeed, Y is a linear projection of the rational normal curve $\Gamma_d \subset \mathbb{P}^d$, and X is a linear projection of the affine cone V_d over Γ_d (a Veronese cone). The latter projection gives a normalization of X ; this is not an isomorphism as it diminishes the dimension of the Zariski tangent space at the vertex.

The normalizations of two such cones X_1 over Y_1 and X_2 over Y_2 are isomorphic if and only if $\deg Y_1 = \deg Y_2$. While in general the affine surface X admits non-trivial equisingular deformations arising from deformations of the projective embedding $Y \hookrightarrow \mathbb{P}^n$. For instance, smooth rational curves Y of type $(1, a)$ on a quadric $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ vary in a family of projective dimension $2a + 1$. Hence for any $a \geq 3$ the group $\text{PSO}(4, \mathbb{C})$ cannot act transitively on this family.

2.14. Since any group action on an affine cone X lifts to the normalization of X , it is enough to restrict to normal cones. For a normal affine cone $X \subseteq \mathbb{A}^{d+1}$ over a rational curve $Y \subseteq \mathbb{P}^d$ we have $\text{Aut}(X) \neq \text{Lin}(X)$. Indeed, by 2.13 $Y = \Gamma_d$ is a rational normal curve and $X = V_d$ the Veronese cone over Γ_d , which is a toric surface: $V_d \simeq \mathbb{A}^2/\mathbb{Z}_d$. We claim that for any $d \geq 1$ the group $\text{Aut}(V_d)$ is infinite dimensional and so it is not an algebraic group. Indeed, the graded coordinate ring $\mathcal{O}(V_d)$ admits a nonzero locally nilpotent derivation ∂ corresponding to an effective \mathbb{C}_+ -action on V_d [FZ₂]. The kernel $\ker(\partial) \subseteq \mathcal{O}(V_d)$ is isomorphic to the polynomial ring $\mathbb{C}[t]$. For any $p \in \mathbb{C}[t]$, the derivation $p \cdot \partial$ is again locally nilpotent. Thus $\mathbb{C}[t] \cdot \partial$ is the Lie algebra of an infinite dimensional abelian subgroup $G \subseteq \text{Aut}(V_d)$.

Remark 2.15. In fact there are two independent \mathbb{C}_+ -actions on V_d with different orbits, and even a continuous family of such actions; see e.g., [FZ₂]. Danilov and Gizatullin [DG] studied the structure of an amalgamated product on the group $\text{Aut}(V_d)$, while Makar-Limanov [ML] provided an explicit description of this group. Similarly, independent \mathbb{C}_+ -actions, and an amalgamated product structure, exist on any normal affine toric surface different from $\mathbb{A}_*^1 \times \mathbb{A}^1$ or $\mathbb{A}_*^1 \times \mathbb{A}_*^1$. Every such surface is of the form $V_{d,e} = \mathbb{A}^2/\mathbb{Z}_d$ for an appropriate diagonal action

$$\zeta.(x, y) = (\zeta x, \zeta^e y), \quad \text{where } \zeta^d = 1 \quad \text{and} \quad \gcd(e, d) = 1,$$

of the cyclic group $\mathbb{Z}_d = \langle \zeta \rangle$ on the affine plane \mathbb{A}^2 . We have $V_d = V_{d,1}$. Choosing a system of homogeneous generators in the graded coordinate ring $\mathcal{O}(V_{d,e})$ yields an equivariant embedding $V_{d,e} \hookrightarrow \mathbb{A}^{N+1}$ into an affine space equipped by a suitable diagonal \mathbb{C}^* -action with positive weights $w = (w_0, \dots, w_N)$. In this way $V_{d,e}$ can be realized as the affine cone over a smooth rational curve in the corresponding weighted projective space $\mathbb{P}_w^N = (\mathbb{A}^{N+1} \setminus \{0\})/\mathbb{C}^*$.

By Popov's Theorem [Po]⁸ the Veronese cones $V_d = V_{d,1}$ can be characterized as the almost homogeneous normal affine surfaces on which an algebraic group acts with a "big" orbit i.e., an open orbit admitting a zero-dimensional component in the complement.

In the next example we give an explicit construction of a non-linear biregular automorphism of a Veronese cone V_d for $d \geq 4$.

Example 2.16. Consider as before the Veronese cone $V_d \subseteq \mathbb{A}^{d+1}$ over a rational normal curve $\Gamma_d \subseteq \mathbb{P}^d$. Let $\bar{V}_d \subset \mathbb{P}^{d+1}$ be the Zariski closure, and \tilde{V}_d be the blowup of \bar{V}_d at the vertex $0 \in V_d$. It is well known [DG] that $\tilde{V}_d \simeq \Sigma_d$, where Σ_d denotes a Hirzebruch surface with the exceptional section S_0 and a disjoint section at infinity, say, S_∞ with $S_0^2 = -d$ and $S_\infty^2 = d$, so that

$$V_d \simeq (\Sigma_d \setminus S_\infty)/S_0.$$

To exhibit a non-linear automorphism of V_d is the same as to exhibit an automorphism of $\Sigma_d \setminus S_\infty$, which extends to a birational transformation of Σ_d preserving the exceptional section S_0 but not the ruling $\pi : \Sigma_d \rightarrow \mathbb{P}^1$ (or, equivalently, which blows down the curve S_∞). On the level of dual graphs, such a birational transformation consists e.g., in the following sequence of blowups and blowdowns [FKZ]:

$$\begin{array}{ccccccc} d & & -1 & -1 & A_d & & 0 & A_d & & -1 & -1 & A_d & & d \\ \circ & \dashrightarrow & \circ & \text{---} & \square & \dashrightarrow & \circ & \square & \dashrightarrow & \circ & \text{---} & \square & \dashrightarrow & \circ \\ S_\infty & & S_\infty & & v_{d+1} & & v_{d+1} & & S'_\infty & & v_{d+1} & & S'_\infty \end{array}.$$

Here a box marked A_d represents the linear chain $[[-2, \dots, -2]]$ of length d . The center of blowup on the curve v_{d+1} can vary, and each time the section S_∞ is contracted. Hence the resulting biregular transformation of the Veronese cone V_d is non-linear.

3. GROUP ACTIONS ON 3-DIMENSIONAL AFFINE CONES

3.1. Let Y be a smooth projective variety, and H be a polarization i.e., an ample divisor on Y . Consider the total space \hat{X} of the linear bundle $\mathcal{O}_Y(H)$ with the zero section $S_0 \subset \hat{X}$. Under the natural identification $S_0 \simeq Y$, we have $\mathcal{O}_{S_0}(S_0) = \mathcal{O}_Y(-H)$. Hence S_0 is contractible, i.e., there is a birational contraction $v : \hat{X} \rightarrow X$, where X is a normal affine variety and $v(S_0)$ is a point. In this situation, X is called a *generalized cone* over (Y, H) . If H is very ample, then X coincides with the normalization of the usual cone over $Y \subset \mathbb{P}^{n-1}$, where the embedding is given by the linear system $|H|$. We are interested in group actions on (generalized) cones.

3.2. For instance, the affine cone over \mathbb{P}^2 in \mathbb{A}^3 coincides with \mathbb{A}^3 and so admits a transitive action of the additive group \mathbb{C}_+^3 . In the following example we exhibit an effective \mathbb{C}_+^2 -action on the affine cone $X \subseteq \mathbb{A}^4$ over a smooth quadric $Y \subseteq \mathbb{P}^3$ (cf. another constructions in [Sh]). The automorphism groups of affine quadrics were studied e.g., in [DG, Doe, To]. Over a general base field, this group is infinite dimensional as soon as the corresponding quadratic form is isotropic [To, Lemma 1.1]. This also holds over any algebraically closed field. The proof of Lemma 1.1 in [To] provides a nontrivial

⁸This is actually an earlier version of the Cone Theorem in dimension 2; see Section 1.

linear \mathbb{C}_+ -action on a quadric. In the following example we exhibit an explicit effective \mathbb{C}_+^2 -action on the affine cone $X \subseteq \mathbb{A}^4$ over a smooth quadric $Y \subseteq \mathbb{P}^3$.

Example 3.3. All smooth quadrics in \mathbb{P}^3 are projectively equivalent. So we can chose the quadric

$$xy = zu$$

as a model for $Y \subseteq \mathbb{P}^3$. A linear \mathbb{C}_+^2 -action on X can be defined using the following pair of commuting locally nilpotent derivations on the ring $A = \mathcal{O}(X)$, which are not proportional over $\text{Frac}(A)$:

$$\partial_1 = u\partial/\partial x + y\partial/\partial z \quad \text{and} \quad \partial_2 = u\partial/\partial y + x\partial/\partial z.$$

To construct \mathbb{C}_+ -actions on the affine cones over other smooth del Pezzo surfaces needs some preparation. Let us introduce the following notion.

Definition 3.4. Let X be an affine variety. For a function $f \in \mathcal{O}(X)$ we let $\mathbb{D}_+(f) := X \setminus \mathbb{V}_+(f)$, where $\mathbb{V}_+(f) := f^{-1}(0)$. We say that X is *cylindrical* if X contains a dense principal Zariski open subset $U = \mathbb{D}_+(f)$ isomorphic to the cylinder $Z \times \mathbb{A}^1$ over an affine variety Z .

The following proposition generalizes Lemma 1.6 in [FZ₂].

Proposition 3.5. *For an irreducible affine variety X , the following conditions are equivalent:*

- (i) X possesses an effective \mathbb{C}_+ -action.
- (ii) X is cylindrical.

Proof. First we suppose that X possesses an effective \mathbb{C}_+ -action ψ with the associate locally nilpotent derivation $\partial \neq 0$. The filtration

$$(1) \quad 0 \in \ker \partial \subsetneq \ker \partial^{(2)} \subsetneq \ker \partial^{(3)} \dots$$

being strictly increasing, we can find $g \in \mathcal{O}(X)$ such that $\partial^{(2)}g = 0$ but $h := \partial g \neq 0$. Thus $\partial h = 0$ and so $h \in \mathcal{O}(X)$ is ψ -invariant. Letting $\tilde{g} = g/h$ and $U = \mathbb{D}_+(h)$ the function $\tilde{g} \in \mathcal{O}(U)$ gives a *slice* of ∂ that is, $\partial(\tilde{g}) = 1$. Consequently, the restriction of \tilde{g} to any 1-dimensional orbit O of ψ in U gives a coordinate on $O \cong \mathbb{A}^1$. Moreover, for the restriction to U of the sequence (1) we obtain

$$\ker(\partial^{(k+1)}|_{\mathcal{O}(U)}) = \tilde{g} \ker(\partial^{(k)}|_{\mathcal{O}(U)}) = \tilde{g}^k \ker(\partial|_{\mathcal{O}(U)}).$$

Hence, $\mathcal{O}(U) \simeq \ker(\partial)[\tilde{g}]$ and so $U \simeq Z \times \mathbb{A}^1$, where $Z = \text{Spec}(\ker \partial)$ ⁹. This yields (i).

To show the converse, we assume that X is cylindrical. Let $U = \mathbb{D}_+(f) \simeq Z \times \mathbb{A}^1$ be a principal cylinder in X as in Definition 3.4. We consider the natural \mathbb{C}_+ -action ϕ on U by translations along the second factor. Since $f|_U$ does not vanish it is constant along any orbit of ϕ and so ϕ -invariant. Letting ∂ denote the locally nilpotent derivation on $\mathcal{O}(U)$ associated to ϕ , the derivation $\partial_n := f^n \partial \in \text{Der}(\mathcal{O}(U))$ is again locally nilpotent for any $n \in \mathbb{N}$. Let a_1, \dots, a_k be a system of generators of $\mathcal{O}(X)$, and let $N \in \mathbb{N}$ be sufficiently large so that $f^N \partial a_i \in \mathcal{O}(X)$ for any $i = 1, \dots, k$. Then $\partial_N(a_i) \in \mathcal{O}(X)$ for any $i = 1, \dots, k$, hence $\partial_N(\mathcal{O}(X)) \subseteq \mathcal{O}(X)$. Thus the derivation $\partial_N|_{\mathcal{O}(X)} \in \text{Der}(\mathcal{O}(X))$ is locally nilpotent and so generates an effective \mathbb{C}_+ -action ψ on X . This proves (ii). \square

⁹In fact $Z \simeq \tilde{g}^{-1}(0)$.

Remark 3.6. Clearly the \mathbb{C}_+ -actions ϕ and $\psi|_U$ as in the proof have the same orbits, and $\mathbb{V}_+(f) = \{f = 0\}$ consists of fixed points of ψ .

Theorem 3.9 below provides a more practical criterion, which applies to affine cones. We need the following definition.

Definition 3.7. For a projective variety Y with a (very ample) polarization $\varphi_{|H|} : Y \hookrightarrow \mathbb{P}^n$, we call an H -polar subset any Zariski open subset of the form $U = Y \setminus \text{supp } D$, where $D \in |dH|$ is an effective divisor on \mathbb{P}^n .

3.8. Recall that an affine ruling on a variety U is a morphism $\pi : U \rightarrow Z$ such that every scheme theoretic fiber of π is isomorphic to the affine line \mathbb{A}^1 . By a theorem of Kambayashi and Miyanishi [KaMi] (see also [KaWr, RS, Du]), every affine ruling $\pi : U \rightarrow Z$ on a normal variety U over a normal base Z is a locally trivial \mathbb{A}^1 -bundle.

Theorem 3.9. *Let Y be a smooth projective variety with a polarization $\varphi_{|H|} : Y \hookrightarrow \mathbb{P}^n$, and let $X = \text{AffCone}_H Y \subseteq \mathbb{A}^{n+1}$ denote the associated affine cone over Y .*

- (a) *If X admits an effective \mathbb{C}_+ -action then Y possesses an H -polar open subset U which is a total space of a line bundle $U \rightarrow Z$.*
- (b) *Conversely, if Y possesses an H -polar open subset U equipped with an affine ruling $U \xrightarrow{\mathbb{A}^1} Z$ and a section $Z \rightarrow U$, where Z is smooth and $\text{Pic } Z = 0$, then X admits an effective \mathbb{C}_+ -action.*

Proof. (a) Let ψ' be an effective \mathbb{C}_+ -action on X with associate locally nilpotent derivation $\partial' \neq 0$. Using the natural grading of the coordinate ring

$$A = \mathcal{O}(X) = \bigoplus_{i \geq 0} A_i,$$

∂' can be decomposed into a finite sum of homogeneous derivations $\partial' = \sum_{i=1}^n \partial'_i$, where the principal component $\partial := \partial'_n \neq 0$ is again locally nilpotent. The effective \mathbb{C}_+ -action ψ on X generated by ∂ extends to an effective action of a semi-direct product $G := \mathbb{C}_+ \rtimes \mathbb{C}^*$ on X .

The filtration (1) from the proof of Proposition 3.5 consists now of graded subrings. Hence we can find homogeneous elements $\hat{g}, \hat{h} \in A$ such that $\partial \hat{g} = \hat{h}$ and $\partial \hat{h} = 0$. Letting

$$\hat{U} = \mathbb{D}_+(\hat{h}) \subseteq X \quad \text{and} \quad \hat{Z} = \mathbb{V}_+(\hat{g}) \setminus \mathbb{V}_+(\hat{h}) \subseteq \hat{U},$$

we obtain a decomposition $\hat{U} \simeq \hat{Z} \times \mathbb{A}^1$, similarly as in the proof of Proposition 3.5.

Furthermore, G acts on $\hat{U} \simeq \hat{Z} \times \mathbb{A}^1$ respecting the product structure. More precisely, \mathbb{C}_+ acts by shifts along the fibers of the second projection $\hat{\pi} : \hat{U} \rightarrow \hat{Z}$. Since \hat{g}, \hat{h} , and ∂ are homogeneous, \mathbb{C}^* acts on \hat{U} stabilizing \hat{Z} and sending the fibers of $\hat{\pi}$ into fibers. The factorization by the \mathbb{C}^* -action on \hat{U} yields a Zariski open subset $U = \hat{U}/\mathbb{C}^* \subseteq Y$ and a divisor $Z = \hat{Z}/\mathbb{C}^*$ on U so that $\hat{U} = \text{AffCone}(U)$ and $\hat{Z} = \text{AffCone}(Z)$. The map $\hat{\pi}$ factorizes to an affine ruling $\pi : U \xrightarrow{\mathbb{A}^1} Z$ with a section $Z \hookrightarrow U$. Each fiber of π is the quotient of a G -orbit in U by the \mathbb{C}^* -action. Since Y and Z are smooth, by the Kambayashi-Miyanishi Theorem cited in 3.8, π is an \mathbb{A}^1 -bundle and, moreover, a vector bundle since it possesses a section.

Finally, since $D := \hat{h}^*(0) \in |dH|$, where $d = \deg(\hat{h})$, the open set $U \subseteq Y$ is H -polar (see Definition 3.7). This shows (a).

To show (b), suppose that Y possesses an H -polar open subset U with an affine ruling $\pi : U \xrightarrow{\mathbb{A}^1} Z$ and a section $Z \rightarrow U$, where Z is smooth and $\text{Pic } Z = 0$. Since both U and Z are smooth, by the Kambayashi-Miyayashi Theorem, π is locally trivial. Since π has a section Z , $\pi : U \rightarrow Z$ is a line bundle and Z is the zero section. It is trivial since $\text{Pic } Z = 0$. Thus $U \simeq Z \times \mathbb{A}^1$ and so $\text{Pic } U = 0$.

Let further $\sigma : \tilde{X} \rightarrow X$ be the blowup at the origin. The induced surjection $\rho : \tilde{X} \rightarrow Y$ has a natural structure of a line bundle with the exceptional divisor $E = \sigma^{-1}(0)$ as the zero section. Since $\text{Pic } U = 0$, the restriction $\rho|_{\tilde{U}} : \tilde{U} \rightarrow U$, where $\tilde{U} := \rho^{-1}(U) \subseteq \tilde{X}$, is a trivial line bundle. Hence $\tilde{U} \simeq U \times \mathbb{A}^1 \simeq Z \times \mathbb{A}^1 \times \mathbb{A}^1$ and, similarly, $\tilde{Z} := \rho^{-1}(Z) \simeq Z \times \mathbb{A}^1$. Under this isomorphism $E \cap \tilde{U}$ is sent to $U \times \{0\} \simeq Z \times \mathbb{A}^1 \times \{0\}$ and $E \cap \tilde{Z}$ to $Z \times \{0\}$. Therefore the contraction of E preserves the product structure and so $\hat{U} \simeq \hat{Z} \times \mathbb{A}^1$ is a cylinder in X , where as before $\hat{U} = \text{AffCone}(U)$ and $\hat{Z} = \text{AffCone}(Z)$. Since $U \subseteq Y$ is H -polar, $\hat{U} \subseteq X$ is principal and so X is cylindrical. Now (b) follows by Proposition 3.5. \square

Remark 3.10. This theorem, with the same proof, holds also for generalized cones (see 3.1) i.e., we may assume that H is just an ample divisor.

Theorem 3.9 yields the following criterion of existence of a \mathbb{C}_+ -action on certain 3-dimensional affine cones.

Corollary 3.11. *Let Y be a smooth projective rational surface with a polarization $\varphi_{|H|} : Y \hookrightarrow \mathbb{P}^n$, and let $X = \text{AffCone}_H Y \subseteq \mathbb{A}^{n+1}$ be the affine cone over Y . Then X admits a nontrivial \mathbb{C}_+ -action if and only if Y possesses an H -polar cylinder $U \simeq Z \times \mathbb{A}^1$, where Z is a smooth rational affine curve.*

Proof. Since Y is rational and smooth, Z as in Theorem 3.9 is also rational and smooth. Moreover, being non-complete it has $\text{Pic } Z = 0$. Hence the affine rulings as in Theorem 3.9 and its proof are actually direct products. Now it is easy to deduce the assertion from Theorem 3.9. \square

Given an arbitrary rational surface Y , using this criterion we can construct an affine cone over Y which possesses a nontrivial \mathbb{C}_+ -action.

Proposition 3.12. *Let Y be any smooth projective rational surface. Then there is an embedding $Y \hookrightarrow \mathbb{P}^n$ such that the affine cone $X \subset \mathbb{A}^{n+1}$ over Y is cylindrical and hence possesses an effective \mathbb{C}_+ -action.*

Proof. It is well known that for any point $Q \in Y$ there is an affine neighborhood U of Q in Y such that $U \simeq \mathbb{A}^2$. To the open acyclic surface \mathbb{A}^2 with a smooth completion Y we can apply an argument in [Fu, (2.5)]. For the completeness purposes, we repeat it here. Since $\text{Pic } Y$ is a free abelian group generated by the components Δ_i of the divisor $Y \setminus U$, there is a component Δ_i with $\Delta_i^2 > 0$.

There exists also an effective divisor $D = \sum \delta_i \Delta_i$ such that D is nef and big, $D \cdot \Delta_i > 0$ whenever $\delta_i > 0$, and such that the number $\lambda(D) := \text{card}\{i \mid \delta_i > 0\}$ is maximal. Assume on the contrary that $\text{supp } D \neq \text{supp } \sum \Delta_i$, i.e., $\delta_i = 0$ for some i . Since $\text{supp } \sum \Delta_i$ is connected, there is a component $\Delta_k \not\subseteq \text{supp } D$ such that $D \cdot \Delta_k > 0$. Then for $t \gg 0$, the divisor $tD + \Delta_k$ is again nef and big. This contradicts our maximality assumption for $\lambda(D)$. Therefore, $\text{supp } D = \text{supp } \sum \Delta_i$. In this case, D

must be ample and we can take an embedding $Y \hookrightarrow \mathbb{P}^n$ given by the linear system $|mD|$ for $m \gg 1$. \square

3.13. A normal variety X is \mathbb{Q} -Gorenstein if some multiple nK_X of the canonical Weil divisor K_X is Cartier. This notion is important in the minimal model program. It is easy to see that the generalized cone X over a smooth polarized variety (Y, H) is \mathbb{Q} -Gorenstein if and only if $aH \sim -bK_Y$ for some $a \in \mathbb{N}$, $b \in \mathbb{Z}$ ([Kol₁, Example 3.8]).

On the other hand, it is plausible that $\text{Aut}(X) \neq \text{Lin}(X)$ only if Y is birationally ruled. This implies that the Kodaira dimension of Y is negative, see [Kol₂]. Hence $b > 0$, i.e., $-K_Y$ is ample and so Y is a Fano variety.

If, moreover, $H \sim -K_Y$, then X is Gorenstein and so has at most canonical singularity at the origin. We concentrate on the case where Y is a surface. Two-dimensional Fano varieties are del Pezzo surfaces.

Thus from now on we assume that Y is a del Pezzo surface and our polarization H is the anti-canonical one: $H = -K_Y$. Then for $d = K_Y^2 \geq 3$ the linear system $|-K_Y|$ is very ample and determines embedding $Y \hookrightarrow \mathbb{P}^d$ whose image is a projectively normal smooth surface of degree d . Hence in this situation, the corresponding affine cone is normal.

The following theorem is the main result of this section.

Theorem 3.14. *Let Y_d be a smooth del Pezzo surface of degree d anticanonically embedded into \mathbb{P}^d , where $4 \leq d \leq 9$, and let $X_d \subseteq \mathbb{A}^{d+1}$ be the affine cone over Y_d . Then X_d admits a nontrivial \mathbb{C}_+ -action.*

Proof. Consider a pencil $\mathcal{L}_{\mathbb{P}^2} = \langle C_1, C_2 \rangle$ on \mathbb{P}^2 generated by a smooth conic C_1 and a double line $C_2 = 2l$, where l is tangent to C_1 at a point $P_0 \in C_1$. Then $L \setminus \{P_0\} \cong \mathbb{A}^1$, where L is a general member of $\mathcal{L}_{\mathbb{P}^2}$. Moreover, $U = \mathbb{P}^2 \setminus (C_1 \cup C_2)$ is a cylinder over \mathbb{A}_*^1 . Blowing up at $9 - d$ distinct points Q_i on $C_1 \setminus \{P_0\}$, where $9 \geq d \geq 4$, we obtain a del Pezzo surface Y of degree d with a contraction $\sigma : Y \rightarrow \mathbb{P}^2$, and any such surface can be obtained in this way, except for $\mathbb{P}^1 \times \mathbb{P}^1$.

The cylinder $U' = \sigma^{-1}(U) \simeq U$ is $-K_Y$ -polar (see Definition 3.7). Indeed, let $E_i = \sigma^{-1}(Q_i)$, $i = 1, \dots, 9 - d$. For any $1 \gg \varepsilon > 0$ we have

$$-K_{\mathbb{P}^2} \equiv (1 + \varepsilon)C_1 + (1 - 2\varepsilon)l.$$

Hence,

$$-K_Y = \sigma^*(-K_{\mathbb{P}^2}) - \sum_{i=1}^{9-d} E_i \equiv (1 + \varepsilon)\Delta_1 + (1 - 2\varepsilon)\Delta_2 + \varepsilon \sum_{i=1}^{9-d} E_i,$$

where Δ_1 and Δ_2 are the proper transforms in Y of C_1 and l , respectively. Thus U' is a $-K_Y$ -polar cylinder on Y .

In the remaining case where $Y = \mathbb{P}^1 \times \mathbb{P}^1$, the natural embedding $\mathbb{A}^2 = \mathbb{A}^1 \times \mathbb{A}^1$ into Y yields a $-K_Y$ -polar cylinder on Y . Applying now Corollary 3.11 ends the proof. \square

Remark 3.15. This proof cannot be applied to a smooth cubic surface $Y \subseteq \mathbb{P}^3$. Indeed, let us start with an effective \mathbb{C}_+ -action ϕ on $\mathbb{P}^2 \setminus C$, where C is a reduced but not necessarily smooth or irreducible plane curve of degree $d > 0$. Let $\mathcal{L}_{\mathbb{P}^2}$ be the pencil of orbit closures of ϕ . It is easily seen that $\text{Bs}(\mathcal{L}_{\mathbb{P}^2})$ consists of at most one point $p \in C$. Furthermore, $C \setminus \{p\}$ is smooth and all its irreducible components are

isomorphic to \mathbb{A}^1 . Let $\sigma_i : Y \rightarrow \mathbb{P}^2$ be the blowup of $m \geq 1$ smooth points $p_i \in C$. From the equalities

$$3C^* \sim d(-K_{\mathbb{P}^2})^* = -dK_Y + d \sum_{j=1}^m E_j \quad \text{and} \quad 3C^* = 3C' + 3 \sum_{j=1}^m E_j,$$

where $E_j = \sigma_i^{-1}(p_j)$ are the exceptional curves on Y , we obtain

$$-dK_Y \sim 3C' + (3-d) \sum_{j=1}^m E_j =: D.$$

Therefore the divisor D is effective and $D_{\text{red}} = C' + \sum_{j=1}^m E_j$ if and only if $d \leq 2$ i.e., C is a line or a conic. Since all the centers p_j of blowup are situated on C and Y is del Pezzo, we must have $m \leq 5$.

In the examples that follow we give alternative constructions of polar cylinders on some anticanonically polarized del Pezzo surfaces. Due to Corollary 3.11, these lead to new \mathbb{C}_+ -actions on the cones over del Pezzo surfaces of degree ≥ 4 under their anticanonical embeddings.

Example 3.16. Picking up four points P_1, \dots, P_4 in \mathbb{P}^2 in general position, we consider the pencil of lines $\mathcal{L}_{\mathbb{P}^2}$ on \mathbb{P}^2 generated by $l_1 = (P_1P_2)$ and $l_2 = (P_3P_4)$. Letting $\sigma : Y \rightarrow \mathbb{P}^2$ be the blowup of these points gives a del Pezzo surface Y of degree 5. We have

$$-K_{\mathbb{P}^2} \equiv \frac{3}{2}l_1 + \frac{3}{2}l_2 \quad \text{and so} \quad -K_Y \equiv \frac{3}{2}l'_1 + \frac{3}{2}l'_2 + \frac{1}{2} \sum_{i=1}^4 E_i,$$

where l'_i is the proper transform of l_i , $i = 1, 2$, and E_i is the exceptional (-1) -curve over P_i , $i = 1, \dots, 4$. Then $L^{(1)} = l'_1 + E_1 + E_2$ and $L^{(2)} = l'_2 + E_3 + E_4$ are the only degenerate fibers of the pencil $\mathcal{L} = \sigma_*^{-1}(\mathcal{L}_{\mathbb{P}^2})$ on Y . Since

$$D := \frac{1}{2}(L^{(1)} + L^{(2)}) + (l'_1 + l'_2) \equiv -K_Y,$$

the open set

$$Y \setminus \text{supp } D = \mathbb{P}^2 \setminus (l_1 \cup l_2) \simeq \mathbb{A}_*^1 \times \mathbb{A}^1$$

is a $-K_Y$ -polar cylinder on Y . Thus by Corollary 3.11 there exists an effective \mathbb{C}_+ -action on the affine cone X of Y . The same construction holds for any del Pezzo surface of degree $d \geq 5$.

Example 3.17. Let $\mathcal{L}_{\mathbb{P}^2}$ be a pencil of rational curves on \mathbb{P}^2 with a unique base point P (see e.g., [MiSu] for an example of degree 5). Then the complement of the union of its degenerate members (or of a general one, if all members are non-degenerate) is a $-K_{\mathbb{P}^2}$ -polar cylinder on \mathbb{P}^2 . Similarly, one can find such a cylinder on the quadric $\mathbb{P}^1 \times \mathbb{P}^1$.

Example 3.18. Take the pencil $\mathcal{L}_{\mathbb{P}^2}$ of unicuspidal rational curves $\alpha yz^{n-1} + \beta x^n = 0$ in \mathbb{P}^2 and blow up four points, two of them on the line $x = 0$ and another two on $y = 0$, off the intersection point. Then for $n > 1$ we obtain another example of a $-K_Y$ -polar cylinder on the del Pezzo surface Y of degree 5, while for $n = 1$ this gives again the cylinder from Example 3.16.

Example 3.19. In [MiSu] an example was constructed of a conic C_1 and a rational unicuspidal quintic C_2 on \mathbb{P}^2 which meet in one point, the cuspidal point of the quintic. These curves generate a pencil $\mathcal{L}_{\mathbb{P}^2} = \langle 5C_1, 2C_2 \rangle$ of rational curves of degree 10 with a unique base point such that $\mathbb{P}^2 \setminus (C_1 \cup C_2)$ is a cylinder. Every del Pezzo surface Y of degree 4 can be obtained, along with a $-K_Y$ -polar cylinder, by blowing up five suitable points on C_1 in general position. Indeed, we can write a decomposition

$$-K_{\mathbb{P}^2} \equiv \left(\frac{3}{2} - \varepsilon\right)C_1 + \frac{2}{5}\varepsilon C_2$$

with an appropriate small $\varepsilon > 0$, and then proceed in the same fashion as before.

See also Example 4.21 in the next section.

3.20. For an algebra A over a field k , its Makar-Limanov invariant $\text{ML}(A)$ is by definition the intersection of the kernels of all locally nilpotent derivations on A . It is trivial if $\text{ML}(A) = k$. One says that A is of class ML_i if the transcendence degree of the quotient field $\text{Frac}(\text{ML}(A))$ is equal to i . Thus $A \in \text{ML}_0$ whenever A has trivial Makar-Limanov invariant. Assuming that A and $\text{ML}(A)$ are both finitely generated, and letting $X = \text{spec } A$ and $Z = \text{spec } \text{ML}(A)$, we have $0 \leq i = \dim Z \leq \dim X$. For instance, for $k = \mathbb{C}$, $\mathbb{A}^3 \in \text{ML}_0$ (regarded as the affine cone over \mathbb{P}^2).

If A is graded then, restricting to homogeneous locally nilpotent derivations, we obtain graded versions $\text{ML}^{(h)}(A)$ and $\text{ML}_i^{(h)}$ of these notions, as considered e.g., in [MiMa] or in [FZ₂].

3.21. It is easily seen that every del Pezzo surface Y of degree ≥ 6 is toric. Moreover, the affine cone X over Y with its anticanonical polarization is toric as well. Since X is not isomorphic to a product $X' \times \mathbb{A}_*^1$, the coordinate ring $\mathcal{O}(X)$ has trivial Makar-Limanov invariant by Lemma 4.5 in [Li].

If Y has degree $d \geq 4$, then $X \in \text{ML}_i^{(h)}$ for some $i \leq 2$ by Theorem 3.14. The equality holds if and only if all homogeneous locally nilpotent derivations on $\mathcal{O}(X)$ are proportional and so define the same \mathbb{A}^1 -fibration on X . This fibration is stable under the standard \mathbb{C}^* -action on X and yields a ‘cylindrical’ pencil \mathcal{L} on Y as before. Along with the construction from the proofs of Proposition 3.5 and Theorem 3.14, any two $-K_Y$ -polar cylinders in Y which correspond to different pencils, say, $\mathcal{L}_1, \mathcal{L}_2$, give rise to non-proportional homogeneous locally nilpotent derivations on $\mathcal{O}(X)$. According to this argument, by Examples 3.16-3.19, 4.21, and the proof of Theorem 3.14, we arrive to the following conclusion.

Corollary 3.22. *Let X be the affine cone over a smooth, anticanonically embedded del Pezzo surface Y . Then $X \in \text{ML}_0^{(h)}$ if $\deg(Y) \geq 6$ and $X \in \text{ML}_0^{(h)} \cup \text{ML}_1^{(h)}$ if $\deg(Y) \geq 4$.*

Question. *Describe all affine cones with trivial (homogeneous) Makar-Limanov invariant.*¹⁰

4. \mathbb{C}_+ -ACTIONS ON CONES OVER CUBIC SURFACES

In this section we analyse in detail the case of a smooth cubic surface $Y \subseteq \mathbb{P}^3$. Assuming that such a surface does admit an anticanonical polar cylinder, we obtain

¹⁰Cf. Liendo [Li].

an information on its structure summarized in Proposition 4.20 below. Thus we adopt the following convention.

4.1. Convention. Let Y be a smooth cubic surface in \mathbb{P}^3 . We assume below that the affine cone $X = \text{AffCone } Y \subseteq \mathbb{A}^4$ admits an effective \mathbb{C}_+ -action. By Corollary 3.11 then Y possesses an H -polar cylinder $U \simeq \mathbb{A}^1 \times Z$, where Z is a smooth affine rational curve and $H = -K_Y$. In other words, $Y \setminus U = \text{supp } D$, where

$$(2) \quad D = \sum_{i=1}^n \delta_i \Delta_i \equiv -K_Y, \quad \delta_i \in \mathbb{Q}_{>0} \quad \forall i,$$

is an effective \mathbb{Q} -divisor on Y .

Let \mathcal{L} be the pencil on Y with general member $L_z = \overline{\text{pr}_2^{-1}(z)}$ for $z \in Z$. It is easily seen that \mathcal{L} has at least one degenerate member. We may suppose in what follows that $\text{supp } D$ does not contain a non-degenerate member of \mathcal{L} . Indeed otherwise, up to numerical equivalence, one can replace such a member by a degenerate one. Under these assumptions, the following hold.

Lemma 4.2. *The support of D is connected and simply connected, and contains at least 7 irreducible components.*

Proof. The projection $\text{pr}_2 : U \rightarrow Z$ extends to a rational map $Y \dashrightarrow \mathbb{P}^1$. The latter defines a pencil, say, \mathcal{L} on Y . A general member L of \mathcal{L} is a rational curve smooth off a unique point $P = L \cap \text{supp } D$, and $L \setminus \text{supp } D = L \cap U \cong \mathbb{A}^1$. Thus L is unibranch at P . The only base point of \mathcal{L} (if exists) is contained in $\text{supp } D$.

Since D is ample, by the Lefschetz Hyperplane Section Theorem $\text{supp } D$ is connected. Resolving, if necessary, the base point of \mathcal{L} by a modification $p : W \rightarrow Y$ yields a rational surface W with a pencil of rational curves $\mathcal{L}_W = p_*^{-1}\mathcal{L}$ that are fibers of $q = q_{|\mathcal{L}_W|} : W \rightarrow \mathbb{P}^1$. By Zariski's Main Theorem, the total transform $p^{-1}(\text{supp } D)$ is still connected, and is a union of a (-1) -section, say, S of \mathcal{L}_W and of some number of rational trees contained in fibers of \mathcal{L}_W . Hence $p^{-1}(\text{supp } D)$ is also a tree of rational curves. Thus $p^{-1}(\text{supp } D)$ is simply connected. The exceptional divisor E of p being a subtree of $p^{-1}(\text{supp } D)$, the contraction of E does not affect the simply-connectedness. This proves the first assertion.

We have $\text{Pic } U = \text{Pic } \mathbb{A}^1 \times \text{Pic } Z = 0$ since Z is a smooth rational affine curve. Hence the exact sequence

$$(3) \quad \sum_{i=1}^n \mathbb{Z}\Delta_i =: G \rightarrow \text{Pic } Y \rightarrow \text{Pic } U = 0 \rightarrow 0$$

gives rise to a surjection $G \rightarrow \text{Pic } Y \simeq \mathbb{Z}^7$, where G is the free abelian group generated by the components Δ_i of D . Therefore, $\text{rk}(G) \geq 7$, which proves the second assertion. \square

Lemma 4.3. *The pencil \mathcal{L} has a (unique) base point, say, P , and $\text{deg } \mathcal{L} \geq 3$.*

Proof. Assume on the contrary that $\text{Bs}(\mathcal{L}) = \emptyset$ and so \mathcal{L} is a pencil of conics on Y with a section, say, $S = \Delta_0$. This pencil defines a morphism $\varphi_{|C|} : Y \rightarrow \mathbb{P}^1$ with exactly 5 degenerate fibers, which extends the projection pr_2 of the cylinder. Each of the degenerate fibers consists of two (-1) -curves, which are lines on Y intersecting transversally at one point. At most one of these two lines, say, l_i meets the cylinder

U , while the other one, say, Δ_i is a components of D . Since D is connected we may suppose that $\Delta_i \cdot S = 1$ and $l_i \cdot S = 0$, $i = 1, \dots, 5$. By the Adjunction Formula we get

$$1 = -K_Y \cdot l_i = D \cdot l_i = \delta_i - x,$$

where $x = 0$ if $l_i \neq \Delta_j \forall j$ and $x = \delta_j > 0$ otherwise. Hence $\delta_i = 1 + x \geq 1$. Similarly, for a general fiber L of \mathcal{L} we get

$$2 = -K_Y \cdot L = D \cdot L = \delta_1 S \cdot L = \delta_1,$$

and so $\delta_1 = 2$.

On the other hand, again by the Adjunction Formula

$$-K_Y \cdot S = 2 + S^2 = D \cdot S = 2S^2 + \sum_{i=1}^5 \delta_i.$$

Therefore,

$$2 = S^2 + \sum_{i=1}^5 \delta_i = S^2 + \sum_{i=1}^5 \delta_i \geq -1 + 5 = 4,$$

a contradiction. The proof of the inequality $-K_Y \cdot \mathcal{L} \geq 3$ is easy and left to the reader. \square

Remarks 4.4. 1. In Lemma 4.8 below we show that actually $n = 8$, and so the lower bound for the degree of \mathcal{L} is in fact essentially higher.

2. The assertion of the lemma holds also for any del Pezzo surface of degree 4 or 5, with a similar proof. However, it fails for degree 6. Indeed, pick 3 points P_0, P_1, P_2 in general position in \mathbb{P}^2 , and consider the pencil generated by the lines $l_i = (P_0 P_i)$, $i = 1, 2$. Blowing up these points we get a del Pezzo surface Y of degree 6 with a base point free pencil. It is easily seen that the complement in Y of the total transform of $l_1 \cup l_2$ is a $-K_Y$ -principal cylinder.

Remark 4.5. The following commutative diagram will be used in the sequel:

(4)

$$\begin{array}{ccccc}
 & & & W & \\
 & & & \swarrow p & \searrow \sigma \\
 U & \hookrightarrow & Y & & \mathbb{F}_1 \\
 & \searrow & \dashrightarrow & \downarrow q & \downarrow \rho \\
 & & Z & \hookrightarrow & \mathbb{P}^1 \\
 & & & & \downarrow \\
 & & & & \mathbb{P}^2
 \end{array}$$

where $p : W \rightarrow Y$ is the minimal resolution of the base point P of \mathcal{L} , $q : W \rightarrow \mathbb{P}^1$ is the induced pencil, $\sigma : W \rightarrow \mathbb{F}_1$ is composed of the contraction of all components in every degenerate fiber of q except the only component which meets the exceptional (-1) -section S_W of q , and $\rho : \mathbb{F}_1 \rightarrow \mathbb{P}^2$ is the contraction of this exceptional section.

Lemma 4.6. $\delta_i < 1$ in (2) for all $i = 1, \dots, n$.

Proof. It is enough to show that $\delta_1 < 1$. By symmetry then $\delta_i < 1 \forall i$. Since the anti-canonical divisor $-K_Y$ is ample, we have $-K_Y \cdot \Delta_i > 0 \forall i = 1, \dots, n$. We distinguish between the following 3 cases :

- (i) $-K_Y \cdot \Delta_1 \geq 3$,
- (ii) $-K_Y \cdot \Delta_1 = 2$, and
- (iii) $-K_Y \cdot \Delta_1 = 1$.

In case (i) suppose on the contrary that $\delta_1 \geq 1$. Since $-K_Y \equiv D$ is ample and by Lemma 4.2 $n \geq 7$ we have

$$3 = -K_Y \cdot D = \sum_{i=1}^n \delta_i (-K_Y) \cdot \Delta_i > \delta_1 (-K_Y) \cdot \Delta_1 \geq 3,$$

which gives a contradiction.

In case (ii) $\Delta_1 \subseteq Y$ is a plane conic with $\Delta_1^2 = 0$. There is also a residual line, say, E on Y such that $\Delta_1 + E \equiv -K_Y$ is cut out on Y by a plane through E in \mathbb{P}^3 .

Let $E = \Delta_i$ for some $i > 1$; we may assume that $i = 2$. Then $-K_Y \equiv \Delta_1 + \Delta_2$, $\Delta_1^2 = 0$ and $\Delta_1 \cdot \Delta_2 = 2$. Thus we obtain

$$(5) \quad 2 = -K_Y \cdot \Delta_1 = D \cdot \Delta_1 \geq \delta_2 \Delta_1 \cdot \Delta_2 = 2\delta_2,$$

so $\delta_2 \leq 1$. Furthermore,

$$1 = \Delta_2 \cdot D = \sum_{i=1}^n \delta_i \Delta_2 \cdot \Delta_i \geq 2\delta_1 - \delta_2.$$

Hence $\delta_1 \leq \frac{1}{2}(\delta_2 + 1) \leq 1$.

If $\delta_1 = 1$ then also $\delta_2 = 1$. Since $\Delta_1 + \Delta_2 \equiv -K_Y \equiv D$ we get $D = \Delta_1 + \Delta_2$. Thus $n = 2$, which contradicts Lemma 4.2. Therefore in this case $\delta_1 < 1$, as stated.

If further $E \neq \Delta_i \forall i$ then

$$(6) \quad 1 = -K_Y \cdot E = D \cdot E \geq \delta_1 \Delta_1 \cdot E = 2\delta_1,$$

hence $\delta_1 \leq 1/2$. This proves our inequality in case (ii).

In case (iii) Δ_1 is a line on Y . Let C be a residual conic of Δ_1 , so that $\Delta_1 + C \equiv -K_Y$ is a hyperplane section. We have as before

$$2 = -K_Y \cdot C = D \cdot C \geq 2\delta_1 \Delta_1 \cdot C = 2\delta_1,$$

hence $\delta_1 \leq 1$.

If $\delta_1 = 1$ then $D \cdot C = \Delta_1 \cdot C = 2$ and so $(D - \Delta_1) \cdot C = 0$. Therefore the divisor $\text{supp}(D - \Delta_1)$ is supported on the fibers of the pencil of conics $|C|$ on Y . The curve Δ_1 meets each fiber twice, and so the morphism $\varphi_{|C|}$ restricted to Δ_1 has 2 branch points.

By Lemma 4.2 the curve $\text{supp } D$ is simply connected, hence it cannot contain the whole fiber of $\varphi_{|C|}$ which meets the component Δ_1 of D at two distinct points. We claim however that if a degenerate fiber $l_1 + l_2$ contains a component, say, $\Delta_i = l_1$ of D , then its second component $l_2 = \Delta_j$ is also contained in $\text{supp } D$ and, moreover, $\delta_i = \delta_j$. Indeed, since $\delta_1 = 1$ and Δ_1 is a line on Y we have

$$(7) \quad 1 = \Delta_i \cdot D = \Delta_i \cdot \Delta_1 + \Delta_i \cdot (D - \Delta_1) = 1 - \delta_i + \sum_{k \neq 1, i} \delta_k \Delta_i \cdot \Delta_k.$$

The only component of the latter sum that meets Δ_i can be the line l_2 . Hence $l_2 = \Delta_j$ for some $j \neq 1, i$. Now (7) shows that $\delta_i = \delta_j$, as claimed.

Furthermore, there are exactly two fibers of $\varphi_{|C|}$ that are tangent to Δ_1 . Such a fiber is either non-degenerate or consists of two lines, say, Δ_2 and Δ_3 , which meet Δ_1 at their common point (this is an Eckardt point of Y). By the above arguments, none

of the other fiber components can be contained in D . Hence D can have at most 5 components, namely, Δ_1 and the components of two degenerate fibers tangent to Δ_1 . However, this contradicts Lemma 4.2, since by this Lemma 4.2, $\text{supp } D$ consists of at least 7 components. Now the proof is completed. \square

Lemma 4.7. *The pencil \mathcal{L} has at most two degenerate members.*

Proof. Recall (see Remark 4.5) that $p : W \rightarrow Y$ stands for the minimal resolution of the base locus of \mathcal{L} and $q : W \rightarrow \mathbb{P}^1$ for the fibration given by $p_*^{-1}\mathcal{L}$. Write p as a composition of blowups of points over P :

$$(8) \quad p : W \xrightarrow{p_1} W_1 \xrightarrow{p_2} \dots \xrightarrow{p_N} W_N = Y,$$

where the exceptional divisor S_W of p_1 is a q -horizontal (-1) -curve. A general fiber L of q is a smooth rational curve meeting S_W at one point. Indeed, $L \setminus S_W \simeq p(L) \setminus P \simeq \mathbb{A}^1$. Therefore S_W is a section of q .

Let C_1, \dots, C_m be the components of degenerate fibers F_1, \dots, F_m of q meeting S_W . We claim that all the C_i are p -exceptional. Indeed, otherwise for some i , the image $p(\overline{(F_i \setminus C_i)})$ would be a component of a degenerate fiber of \mathcal{L} which does not pass through P . The latter contradicts Lemma 4.9.

Note that, on each step, the exceptional divisor of $p_k \circ \dots \circ p_N$ is a tree of rational curves. On the other hand, all the $p(C_i)$ pass through the point $p_1(S_W)$. Therefore, $m \leq 2$. \square

Lemma 4.8. *The pencil \mathcal{L} has exactly two degenerate members, say, $L^{(1)}$ and $L^{(2)}$. Furthermore, $\text{supp } D = \text{supp}(L^{(1)} + L^{(2)})$ has exactly 8 irreducible components i.e., $n = 8$ in (2).*

Proof. Assume that the only degenerate member of \mathcal{L} is $L^{(1)}$. In this case, $\text{supp } D \subseteq \text{supp } L^{(1)}$, $Z \supseteq \mathbb{A}^1$, and $U \supseteq \mathbb{A}^2$.

If $Z \cong \mathbb{A}^1$ then $\text{Pic } U = 0$ and $H^0(U, \mathcal{O}_U) = \mathbb{C}$, hence $\text{Pic } Y \simeq \sum_{i=1}^n \mathbb{Z} \cdot \Delta_i$. Thus in this case $n = 7$ and $-K_Y = \sum_{i=1}^7 m_i \Delta_i$ for some $m_i \in \mathbb{Z}$. On the other hand, $-K_Y \equiv D = \sum_{i=1}^7 \delta_i \Delta_i$. However, according to Lemma 4.6, $0 < \delta_i < 1 \forall i$, which yields a contradiction.

If further $Z \cong \mathbb{P}^1$ then $\text{Pic } U \cong \mathbb{Z}$ and $H^0(U, \mathcal{O}_U) = \mathbb{C}$. From the exact sequence (3), where $\text{Pic } U = 0$ is replaced by $\text{Pic } U \cong \mathbb{Z}$, we obtain $n = 6$. However, this again leads to a contradiction by Lemma 4.2.

As for the second assertion, assuming on the contrary that $\text{supp } D \neq \text{supp}(L^{(1)} + L^{(2)})$ we will have $Z \supseteq \mathbb{A}^1$. Now the same argument as before yields a contradiction. The last assertion now follows easily since the Picard group $\text{Pic } Y \cong \mathbb{Z}^7$ is generated by the irreducible components of $L^{(1)} + L^{(2)}$, and $L^{(1)} \equiv L^{(2)}$ is the only relation between these components. \square

Lemma 4.9. *Every component of the degenerate fibers $L^{(1)}$ and $L^{(2)}$ of the pencil \mathcal{L} on Y passes through the base point P of \mathcal{L} .*

Proof. Assume on the contrary that there is a component C_0 of a degenerate member $L^{(0)}$ of \mathcal{L} such that $P \notin C_0$. By Zariski's Lemma $C_0^2 < 0$. Hence C_0 is a (-1) -curve on Y and so $D \cdot C_0 = -K_Y \cdot C_0 = 1$. By an easy argument (cf. the proof of Lemma 4.2) the curve $\text{supp } L^{(0)}$ is connected and simply connected. Hence C_0 cannot separate this fiber and so C_0 meets the complement $\text{supp}(L^{(0)} - C_0)$ at one point transversally.

Indeed, outside P the fiber $L^{(0)}$ represents a tree of rational curves. If there is another component C_1 of $L^{(0)}$ which meets C_0 and does not pass through P , then $L^{(0)}$ contains the configuration of two crossing lines $C_0 + C_1$ which do not pass through P . Then \mathcal{L} must consist of the members of the linear system of conics $|C_0 + C_1|$. By Lemma 4.3 this leads to a contradiction.

Thus $\Delta_i \cdot C_0 = 1$ for a unique index i . If C_0 is not a component of D , then $1 = D \cdot C_0 = \delta_i$, which contradicts Lemma 4.6. Similarly, if $C_0 = \Delta_j$, then $1 = D \cdot C_0 = \delta_i - \delta_j$. Hence $\delta_i = 1 + \delta_j > 1$. Once again, this contradicts Lemma 4.6. \square

Corollary 4.10. *The pencil \mathcal{L} is ample. Furthermore, for every irreducible component C of a member of the pencil \mathcal{L} on Y , we have $C \setminus \{P\} \cong \mathbb{A}^1$, and all these curves are pairwise disjoint.*

Proof. The first assertion follows immediately from Lemma 4.9 by the Nakai-Moishezon criterion. As for the second one, it follows from the well known fact that on an affine surface $V = Y \setminus L$, where L is a general fiber of \mathcal{L} , every degenerate fiber of the \mathbb{A}^1 -fibration $\varphi_{|\mathcal{L}|} : V \rightarrow \mathbb{A}^1$ is a disjoint union of affine lines, see e.g., [Mi], [Za]. \square

Lemma 4.11. *The pair (Y, D) is not log canonical at P .*

Proof. Let D_W denote the crepant pull-back of D on W , i.e., a \mathbb{Q} -divisor on W such that

$$K_W + D_W = p^*(K_Y + D) \quad \text{and} \quad p_* D_W = D,$$

see diagram (4). Then the exceptional (-1) -section S on W is the only q -horizontal component of D_W . For a general fiber L of q we have $2 = -K_W \cdot L = D_W \cdot L$. Therefore, the discrepancy $a(S, D)$ (i.e., the coefficient of S in D_W with the minus sign) equals -2 . This proves our assertion. \square

Corollary 4.12. $\text{mult}_P D > 1$.

Proof. If $\text{mult}_P D \leq 1$, then the pair (Y, D) is canonical at P , because $P \in Y$ is a smooth point. In particular, it is log canonical. This contradicts Lemma 4.11. \square

Lemma 4.13. *Any line l on Y through P is contained in $\text{supp } D$.*

Proof. Assuming the contrary we obtain $1 = -K_Y \cdot l = D \cdot l \geq \text{mult}_P D > 1$, a contradiction. \square

Lemma 4.14. *P cannot be an Eckardt point on Y .*

Proof. Suppose the contrary. Then by Lemma 4.13, up to a permutation we may assume that $\Delta_1, \Delta_2, \Delta_3$ are lines through P , where $\delta_1 \leq \delta_2 \leq \delta_3$. Letting

$$D' = \frac{1}{1 - \delta_1} (D - \delta_1 (\Delta_1 + \Delta_2 + \Delta_3)),$$

we get an effective \mathbb{Q} -divisor D' on Y with $D' \cdot \Delta_1 = 1$. Since $-K_Y \equiv \Delta_1 + \Delta_2 + \Delta_3$, we have $D' \equiv D$. Moreover, the proof of Lemma 4.11 works equally for D' . Hence the pair (Y, D') is not canonical at the point P and so $\text{mult}_P D' > 1$. This contradicts the inequality $\text{mult}_P D' \leq D' \cdot \Delta_1 = 1$. \square

Lemma 4.15. *The fibration $q : W \rightarrow \mathbb{P}^1$ has exactly two degenerate fibers. The general member L of \mathcal{L} is singular at P .*

Proof. The first assertion follows from Lemma 4.8. Let us show the second one. Assuming the contrary, for a smooth rational curve L on Y we have by adjunction $(K_Y + L) \cdot L = -2$. The Mori cone is spanned by the (-1) -curves E_1, \dots, E_{27} and so we have a decomposition

$$L \equiv \sum_{i=1}^{27} \alpha_i E_i,$$

where $\alpha_i \in \mathbb{Q}_{\geq 0}$. Hence for some i we have $(K_Y + L) \cdot E_i < 0$. Thus $L \cdot E_i < -K_Y \cdot E_i = 1$, and so L cannot be ample. This contradicts Corollary 4.10. \square

Remark 4.16. The minimal resolution $p : W \rightarrow Y$ of the base point P of \mathcal{L} dominates the embedded minimal resolution of the cusp on a general member L of \mathcal{L} . The exceptional divisor $E = p^{-1}(P) \subseteq W$ is a rational comb, such that the number of teeth equals the length of the Puiseux sequence of the cusp. The only (-1) -curve S_W in E is sitting on the handle of the comb. Hence $E = E^{(1)} + S_W + E^{(2)}$, where $E^{(k)} \subseteq p^{-1}(L^{(k)})$, $k = 1, 2$, and exactly one of the $E^{(k)}$ is a (negatively definite) linear chain of rational curves.

The degenerate fibers $L_W^{(1)}, L_W^{(2)}$ of the induced linear system \mathcal{L}_W on W have the following structure: $L_W^{(k)}$ consists of $E^{(k)}$ and the proper transforms Δ'_i of the components $\Delta_i \subseteq L^{(k)}$, called *feathers*. The feathers are disjoint in W , and each of them meets $E^{(k)}$ at one point transversally.

4.17. Let $Q = (\rho \circ \sigma)(S_W)$, where as before S_W stands for a (-1) -section of q contained in $\text{Exc}(p)$, and let $\mathcal{L}_{\mathbb{P}^2} := \rho_*(\sigma_* \mathcal{L}_W)$. Then $\mathcal{L}_{\mathbb{P}^2}$ is a linear pencil on \mathbb{P}^2 generated by lines passing through Q . We let $\varphi := \rho \circ \sigma \circ p^{-1} : Y \dashrightarrow \mathbb{P}^2$, and we let \mathcal{H} denote the proper transform of $\mathcal{H}_{\mathbb{P}^2} := |\mathcal{O}_{\mathbb{P}^2}(1)|$ on Y via φ . With this notation, the following lemma holds.

Lemma 4.18. $\mathcal{L} \subseteq \mathcal{H}$.

Proof. We have

$$\mathcal{H}_W := (\rho \circ \sigma)_*^{-1} \mathcal{H}_{\mathbb{P}^2} = (\rho \circ \sigma)^* \mathcal{H}_{\mathbb{P}^2} = \sigma^*(\rho^* \mathcal{H}_{\mathbb{P}^2}) \supseteq \sigma^*(\rho^* \mathcal{L}_{\mathbb{P}^2}).$$

Indeed, $\mathcal{H}_{\mathbb{P}^2}$ is base point free. It is clear that $\rho^* \mathcal{L}_{\mathbb{P}^2} = \mathcal{L}_{\mathbb{F}_1} + S$, where S is the exceptional curve of ρ . Note that the centers of subsequent blowups in σ (including infinitesimally near centers) lie neither on the proper transform of S nor on that of general members of $\mathcal{L}_{\mathbb{P}^2}$. Hence,

$$\sigma^*(\rho^* \mathcal{L}_{\mathbb{P}^2}) = \sigma^*(\mathcal{L}_{\mathbb{F}_1} + S) = \mathcal{L}_W + S_W.$$

Since S_W is p -exceptional, applying p_* yields the assertion. \square

Corollary 4.19. \mathcal{L} cannot be contained in $|-mK_Y|$ for any $m \in \mathbb{N}$.

Proof. Assume to the contrary that $\mathcal{L} \subseteq |-mK_Y|$ for some $m \in \mathbb{N}$. Then the mobile linear system \mathcal{H} which defines the birational map $Y \dashrightarrow \mathbb{P}^2$ is also contained in $|-mK_Y|$. This contradicts the Segre-Manin theorem as it is stated in [KSC, Theorem 2.13]. \square

The following proposition provides a summary of previous results.

Proposition 4.20. *Let $Y \subset \mathbb{P}^3$ be a smooth cubic surface, and $X \subset \mathbb{A}^4$ be the affine cone over Y . Then X admits an effective \mathbb{C}_+ -action if and only if Y admits a linear pencil \mathcal{L} with the following properties:*

- (1) *The base locus $\text{Bs}(\mathcal{L})$ consists of a single point, say, P , which is not an Eckardt point on Y .*
- (2) *A general member of \mathcal{L} is singular at P .*
- (3) *\mathcal{L} has exactly two degenerate members, say $L^{(1)}$ and $L^{(2)}$, which have in total 8 irreducible components $\Delta_1, \dots, \Delta_8$.*
- (4) *All curves Δ_i , $i = 1, \dots, 8$, pass through P and are pairwise disjoint off P . Furthermore, $\Delta_i \setminus \{P\} \cong \mathbb{A}^1 \forall i$.*
- (5) *$-K_Y \equiv D := \sum_{i=1}^8 \delta_i \Delta_i$, where $\delta_i \in \mathbb{Q}$, $0 < \delta_i < 1 \forall i$.*
- (6) *The pair (Y, D) is not log canonical at P .*
- (7) *Every line on Y passing through P is one of the Δ_i .*
- (8) *For every $m > 0$, \mathcal{L} is not contained in $|-mK_Y|$.*

We don't know so far any example of a cubic surface with such a pencil \mathcal{L} . However we can construct a pencil with similar properties on any del Pezzo surface of degree 5. Note that in Examples 3.16-3.19, not all of these properties are fulfilled.

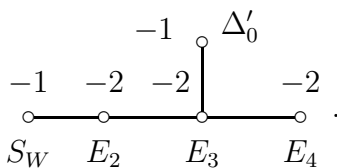
Example 4.21. Consider the following pencil $\mathcal{L}_{\mathbb{P}^2}$ of degree 6 curves on \mathbb{P}^2 :

$$\alpha(y^2z - x^3)^2 + \beta(y^2 - xz)(y^4 - x^4) = 0.$$

The base locus of $\mathcal{L}_{\mathbb{P}^2}$ consists of 5 points P_0, \dots, P_4 , where $P_0 = (0 : 0 : 1)$ and $\{P_1, \dots, P_4\} = (x^2 = z^2, \quad xz = y^2)$. Furthermore, $\mathcal{L}_{\mathbb{P}^2}$ has no fixed component and so its general member L is irreducible, having $\text{mult}_{P_0} L = 4$ and $\text{mult}_{P_i} L = 2$ for $i = 1, \dots, 4$. Therefore L is rational curve. An easy computation shows that any of the singular points P_i of L is resolved by one blowup, and the singularity of L at P_0 is cuspidal. Note that no three of the four singular points P_1, \dots, P_4 of L are collinear. Therefore, the blowup of these four points $\sigma : Y \rightarrow \mathbb{P}^2$ yields a del Pezzo surface Y of degree 5. Let \mathcal{L} be the birational transform of $\mathcal{L}_{\mathbb{P}^2}$ on Y and let $P = \sigma^{-1}(P_0)$. Then \mathcal{L} is a pencil of rational curves having a cuspidal singularity at P , smooth and disjoint outside P .

There are exactly two degenerate members of \mathcal{L} in \mathbb{P}^2 , namely the double cuspidal cubic $C' = 2(y^2z - x^3)$ and the union of the conic $C'' = (y^2 - xz)$ and the four lines $(y^4 - x^4)$. We have $-K_{\mathbb{P}^2} \equiv D_{\mathbb{P}^2} := \frac{1}{4}C' + \frac{1}{4}C''$. Let D be the proper transform of $D_{\mathbb{P}^2}$ on Y . Since $\text{mult}_{P_i} D_{\mathbb{P}^2} = 1$ for $i = 1, \dots, 4$, we have $K_Y + D = \sigma^*(K_{\mathbb{P}^2} + D_{\mathbb{P}^2}) \equiv 0$. Hence \mathcal{L} possesses properties (1)-(5) in Proposition 4.20.

The degenerate fibers $L_W^{(1)}, L_W^{(2)}$ of the induced linear system \mathcal{L}_W on W have the following structure: $L_W^{(1)}$ consists of a curve E_1 with self-intersection -5 joint with the (-1) -section S_W , and besides E_1 it contains the (-1) -feathers $\Delta'_1, \dots, \Delta'_5$ meeting E_1 . While $L_W^{(2)}$ contains the only feather Δ'_0 of multiplicity 2. The weighted dual graph of $L_W^{(2)}$ looks as follows:

(9) 

The diagram shows a weighted dual graph with nodes S_W, E_2, E_3, E_4 on a horizontal line and Δ'_0 above E_3 . Weights are: $S_W: -1, E_2: -2, E_3: -2, E_4: -2, \Delta'_0: -1$. Edges connect S_W to E_2 , E_2 to E_3 , E_3 to E_4 , and E_3 to Δ'_0 .

The exceptional divisor of $p : W \rightarrow Y$ is $E = E_1 + \dots + E_4$, while $\Delta_i = p(\Delta'_i)$, $i = 0, \dots, 5$, are the components of the degenerate fibers $L^{(j)} = p(L_W^{(j)})$ ($j = 1, 2$) of the pencil \mathcal{L} on Y . More precisely, Δ_0 is the proper transform in Y of the cuspidal cubic C' , Δ_1 is that of the conic C'' , while $\Delta_2, \dots, \Delta_5 \subseteq Y$ are the proper transforms of the four lines ($y^4 = x^4$) in \mathbb{P}^2 .

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