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# GROUP ACTIONS ON AFFINE CONES

TAKASHI KISHIMOTO, YURI PROKHOROV, AND MIKHAIL ZAIDENBERG

*To Peter Russell on the occasion of his 70th birthday*

ABSTRACT. We address the following question:

*For which smooth projective varieties, the corresponding affine cone admits an action of a connected algebraic group different from the standard  $\mathbb{C}^*$ -action by scalar matrices and its inverse action?*

We show in particular that the affine cones over anticanonically embedded smooth del Pezzo surfaces of degree  $\geq 4$  possess such an action. Besides, we give some examples of rational Fano threefolds which have this property. A question in [FZ<sub>1</sub>] whether this property holds also for smooth cubic surfaces, occurs to be out of reach for our methods. Nevertheless, we provide a general geometric criterion that could be helpful in this case as well.

## CONTENTS

Introduction	2
1. Group actions on projective varieties	3
1.1. Automorphisms of smooth projective hypersurfaces	3
1.2. Automorphisms of smooth projective varieties	4
1.3. Homogeneous and almost homogeneous varieties	6
2. Groups acting on affine cones	7
2.1. Linear automorphisms of affine cones	7
2.2. Lifting $G$ -actions to affine cones	10
2.3. Groups acting on affine cones	12
2.4. Group actions on 2-dimensional affine cones	13
3. Group actions on 3-dimensional affine cones	15
3.1. Existence of $\mathbb{C}_+$ -actions on affine cones: a geometric criterion	15
3.2. $\mathbb{C}_+$ -actions on affine cones over del Pezzo surfaces	20
3.3. The Makar-Limanov invariant on affine cones over del Pezzo surfaces	22
4. On existence of $\mathbb{C}_+$ -actions on cones over cubic surfaces	24
4.1. The linear pencil on a cubic surface compatible with a cylinder	24
4.2. The inverse nef value	31
5. Cones over some rational Fano threefolds	34
References	36

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## INTRODUCTION

All varieties in this paper are defined over  $\mathbb{C}$ . By Corollary 1.13 in [FZ<sub>1</sub>], an isolated Cohen-Macaulay singularity  $(X, x)$  of a normal quasiprojective variety  $X$  is rational provided that  $X$  admits an effective action of the additive group  $\mathbb{C}_+$ , in particular of a connected non-abelian algebraic group. In the opposite direction, let us observe that, for instance, the singularity at the origin of the affine Fermat cubic in  $\mathbb{A}^4$

$$x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0$$

is rational. The question was raised [FZ<sub>1</sub>, Question 2.22] whether it also admits a non-diagonal action of a connected algebraic group, in particular, a  $\mathbb{C}_+$ -action. So far, we do not know the answer. However, we answer in affirmative a similar question for all del Pezzo surfaces of degree  $d \geq 4$ .

**Theorem 0.1.** *Let  $Y_d$  be a smooth del Pezzo surface of degree  $d$  anticanonically embedded into  $\mathbb{P}^d$ , and let  $X_d = \text{AffCone}(Y_d) \subseteq \mathbb{A}^{d+1}$  be the affine cone over  $Y_d$ . If  $4 \leq d \leq 9$  then  $X_d$  admits a nontrivial  $\mathbb{C}_+$ -action. Consequently, the automorphism group  $\text{Aut}(X_d)$  is infinite dimensional. Moreover, the Makar-Limanov invariant of  $X_d$  is trivial.*

Recall [Dol<sub>1</sub>, 10.1.1] that for  $d \leq 5$  the group  $\text{Aut}(Y_d)$  is finite. By definition, the Makar-Limanov invariant of an affine variety  $X$  is the subring of  $\mathcal{O}(X)$  of common invariants of all  $\mathbb{C}_+$ -actions on  $X$ . It is trivial when it consists of the constants.

One of our main results (Theorem 3.9) provides a necessary and sufficient condition for the existence of a nontrivial  $\mathbb{C}_+$ -action on an affine cone. As a corollary, for affine cones of dimension 3 we obtain the following geometric criterion.

**Theorem 0.2.** *Let  $Y$  be a smooth projective rational surface with a polarization  $\varphi_{|H|} : Y \hookrightarrow \mathbb{P}^n$ , and let  $X = \text{AffCone}_H(Y) \subseteq \mathbb{A}^{n+1}$  be the affine cone over  $Y \subseteq \mathbb{P}^n$ . Then  $X$  admits a nontrivial  $\mathbb{C}_+$ -action if and only if  $Y$  contains an  $H$ -polar cylinder i.e., a cylindrical Zariski open set*

$$U = Y \setminus \text{supp}(D) \simeq Z \times \mathbb{A}^1,$$

where  $Z$  is an affine curve and  $D \in |dH|$  is an effective divisor on  $\mathbb{P}^n$ .

Using this criterion, we show in Proposition 3.13 that for every smooth projective rational surface  $Y$  there exists a polarization  $\varphi_{|H|} : Y \hookrightarrow \mathbb{P}^n$  such that  $Y$  contains an  $H$ -polar cylinder and so the corresponding affine cone possesses an effective action of  $\mathbb{C}_+$ . It would be interesting to classify in any dimension all pairs  $(Y, H)$ , where  $Y$  is a smooth projective variety and  $H$  an ample divisor on  $Y$ , such that the affine cone  $X = \text{AffCone}_H(Y)$  admits an effective  $\mathbb{C}_+$ -action. We recover this classification for  $\dim_{\mathbb{C}}(Y) = 1$  and give some concrete examples in higher dimensions, especially in dimensions 2 and 3.

A theorem due to Matsumura, Monsky and Andreotti (see [MM], or [GH, §I.4], or Section 1 below) claims that any automorphism of a smooth hypersurface  $Y$  in  $\mathbb{P}^n$  of degree  $d$ , where  $d, n \geq 3$  and  $(d, n) \neq (4, 3)$ , is restriction of a unique projective linear transformation, and  $\text{Aut}(Y)$  is a finite group. In Corollary 2.4 we show that the automorphism group  $\text{Aut}(X)$  of the affine cone  $X$  over a smooth, non-birationally ruled projective variety  $Y$  is a linear group, and actually a central extension of a finite group by  $\mathbb{C}^*$ . Consequently, among the affine cones over smooth projective surfaces

in  $\mathbb{P}^3$ , only those of degree  $\leq 3$  can admit a nontrivial action of a connected algebraic group, and their automorphism groups can be infinite-dimensional. Actually, the 3-fold affine quadric cone possesses an effective linear action of the additive group  $\mathbb{C}_+^2$ , see Example 3.3 below or [AS], [Sh].

In Section 1 we give a short overview of the known results on the automorphism groups. In Section 2 we collect generalities on automorphisms of affine cones. Theorems 0.1 and 0.2 are proven in Section 3. In Section 4 we summarize some geometric facts that could be useful (in view of the criterion of Theorem 0.2) in order to answer Question 2.22 in [FZ<sub>1</sub>] cited above. In the final section 5 we describe two families of rational Fano threefolds such that the affine cones over their anti-canonical embeddings possess effective  $\mathbb{C}_+$ -actions.

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## 1. GROUP ACTIONS ON PROJECTIVE VARIETIES

In this section we recall some well known facts about the automorphism groups of projective or quasiprojective varieties; see e.g., [GH, §I.4], [LZ, §II.3]. For an algebraic variety  $Y$ , we let  $\text{Aut}(Y)$  denote the group of all biregular automorphisms of  $Y$  and  $\text{Bir}(Y)$  the group of all birational transformations of  $Y$  into itself. For a projective or an affine embedding  $Y \hookrightarrow \mathbb{P}^n$  ( $Y \hookrightarrow \mathbb{A}^n$ , respectively) we let  $\text{Lin}(Y)$  denote the group of all automorphisms of  $Y$  which extend linearly to the ambient space.

**1.1. Automorphisms of smooth projective hypersurfaces.** In the following theorems we gather some results concerning the groups  $\text{Lin}$ ,  $\text{Aut}$ , and  $\text{Bir}$  for projective hypersurfaces; see Matsumura and Monsky [MM], Iskovskikh and Manin [IM], Pukhlikov [Pu1]-[Pu3], Cheltsov [Chel], de Fernex, Ein and Mustăța [DEM].

**Theorem 1.1.** *Let  $Y$  be a smooth hypersurface in  $\mathbb{P}^n$  of degree  $d$ . Then for all  $d, n \geq 3$  except for  $(d, n) = (4, 3)$ ,*

$$\text{Aut}(Y) = \text{Lin}(Y)$$

*and this group is finite. It is trivial for a general hypersurface of degree  $d \geq 3$ .*

There is a similar result for Schubert hypersurfaces in flag varieties, see Theorem 8.8 in [Te]. For the group of birational transformations, the following hold.

**Theorem 1.2.** *For  $Y \subseteq \mathbb{P}^n$  as above and for all  $d > n \geq 2$  except for  $(d, n) = (3, 2)$ ,*

$$\text{Bir}(Y) = \text{Aut}(Y).$$

*This group is finite except in the case  $(d, n) = (4, 3)$  of a smooth quartic surface  $Y \subseteq \mathbb{P}^3$ , where it is discrete, but can be infinite and different from  $\text{Lin}(Y)$  which is finite. The group  $\text{Bir}(Y)$  of a very general quartic surface  $Y \subseteq \mathbb{P}^3$  is trivial.*

The case  $d \leq n$  is much more complicated. However, in this case there are deep partial results, see e.g., [IM, Pu1, DEM].

Let us indicate briefly some ideas used in the proofs. In case  $d \neq n + 1$  the proof is easy and exploits the fact that the canonical divisor  $K_Y = \mathcal{O}_Y(d - n - 1)$  is  $\text{Aut}(Y)$ -stable. In case  $d = n + 1$  the equalities  $\text{Bir}(Y) = \text{Aut}(Y) = \text{Lin}(Y)$  follow since such hypersurfaces represent Mori minimal models. Indeed a birational map between minimal models is an isomorphism in codimension 1, see e.g., [KM], hence it induces an isomorphism of the corresponding Picard groups. If  $n \geq 4$  then  $\text{Pic}(Y) \simeq \mathbb{Z}$  by the Lefschetz Hyperplane Section Theorem. Therefore any birational transformation  $\varphi$  of  $Y$  acts trivially on  $\text{Pic}(Y)$  and so preserves the complete linear system of hyperplane sections  $|\mathcal{O}_Y(1)|$ . Since  $Y$  is linearly normal<sup>1</sup>,  $\varphi$  is induced by a projective linear transformation of the ambient projective space  $\mathbb{P}^n$ . For the proof of finiteness of the group  $\text{Lin}(Y)$  and its triviality for general hypersurfaces, we refer to the classical paper of Matsumura and Monsky [MM].

By virtue of the Noether-Lefschetz Theorem, these arguments can be equally applied to very general smooth surfaces in  $\mathbb{P}^3$  of degree  $d \geq 4$ . For an arbitrary smooth surface  $Y$  in  $\mathbb{P}^3$ , the minimality of  $Y$  should be combined with the fact that  $\text{Pic}(Y)$  is torsion free. Indeed, any smooth surface in  $\mathbb{P}^3$  of degree  $d \geq 4$  represents a minimal model and so is not birationally ruled, hence its birational automorphisms are biregular; see e.g., [Mat, Theorem 1-8-6]. For  $d > 4$  the canonical class  $K_Y$  defines an equivariant polarization of  $Y$ , and  $\mathcal{O}_Y(1) \sim \frac{1}{d-4}K_Y$ . Since  $\text{Pic}(Y)$  is torsion free and  $H^0(\mathcal{O}_{\mathbb{P}^3}(1)) \rightarrow H^0(\mathcal{O}_Y(1))$  is a surjection, all automorphisms of  $Y$  are linear. This is not true, in general, in the case of a smooth quartic surface in  $\mathbb{P}^3$ . An example of such a surface with infinite automorphism group due to Fano and Severi is discussed in [MM, Theorem 4]. A non-linear biregular involution exists on any smooth quartic in  $\mathbb{P}^3$  containing skew lines, for instance, on the Fermat quartic  $x^4 + y^4 + z^4 + u^4 = 0$ ; see Takahashi [Ta].

For a quadric hypersurface  $X \subseteq \mathbb{A}^{n+1}$  of dimension  $n \geq 2$ , the group  $\text{Aut}(X)$  is infinite-dimensional [To, Lemma 1.1], cf. also example 3.3 below. For  $n = 2$  this group has an amalgamated product structure [DG]; cf. also [ML].

For a smooth cubic surface  $Y \subseteq \mathbb{P}^3$  the group  $\text{Aut}(Y) = \text{Lin}(Y)$  is finite, while the Cremona group  $\text{Bir}(Y) \simeq \text{Bir}(\mathbb{P}^2)$  is infinite-dimensional. The automorphism groups of such surfaces were listed by Hosoh [Ho<sub>1</sub>] who corrected an earlier classification by Segre [Se]; see also Manin [Man] and Dolgachev [Dol<sub>1</sub>]. The largest order of such a group is 648. This upper bound is achieved only for the Fermat cubic surface, see [Ho<sub>2</sub>]. The least common multiple of the orders of all these automorphism groups is  $3240 = 2^3 \cdot 3^4 \cdot 5$  (Gorinov [Gor]).

The Fermat quartic  $x^4 + y^4 + z^4 + u^4 = 0$  and the smooth quartic  $x^4 + y^4 + z^4 + u^4 + 12xyzu = 0$  in  $\mathbb{P}^3$  can be also characterized in terms of the orders of their automorphism groups, see Mukai [Mu], Kondo [Ko<sub>1</sub>], and Oguiso [Og].

**1.2. Automorphisms of smooth projective varieties.** According to the well known Matsumura Theorem<sup>2</sup> the group  $\text{Bir}(Y)$  of a smooth projective variety  $Y$  of general type is finite, hence also the group  $\text{Aut}(Y)$  is. In particular, this holds if  $c_1(Y) < 0$ . See e.g., Xiao [Xi<sub>1</sub>, Xi<sub>2</sub>] for effective bounds of orders of automorphism groups for the general type surfaces. A vast literature is devoted to automorphism groups of K3 and Enriques surfaces. These groups are discrete, and infinite in many cases. Finite groups

<sup>1</sup>I.e.,  $H^0(\mathcal{O}_{\mathbb{P}^n}(1)) \rightarrow H^0(\mathcal{O}_Y(1))$  is a surjection.

<sup>2</sup>Which generalizes earlier results by Andreotti for surfaces of general type; see Kobayashi-Ochiai [KO] and Noguchi-Sunada [NS] for further generalizations.

of automorphism of  $K3$ -surfaces were classified e.g., in [Dol<sub>2</sub>], [IS], [Ko<sub>2</sub>], [Mas], [Ni], [Xi<sub>2</sub>].

For varieties of non-general type we have the following result due to Kalka-Shiffman-Wong [KSW] and Lin [LZ, Theorem II.3.1.2].

**Theorem 1.3.** *Let  $Y$  be a smooth projective variety of dimension  $n$ . Suppose that not all Chern numbers of  $Y$  vanish and either  $c_1(Y) \leq 0$  or  $H^{n,0}(Y) \neq 0$ . Then  $\text{Aut}(Y)$  is a discrete group.*

The first assumption is fulfilled, for instance, if  $e(Y) = c_n(Y) \neq 0$ , or  $e(\mathcal{O}_Y) \neq 0$ , or  $c_1^n(Y) \neq 0$ , where  $e$  stands for the Euler characteristic. However, this assumption does not hold for an abelian variety  $Y = A$ . For any projective embedding  $A \hookrightarrow \mathbb{P}^N$ , the group  $\text{Lin}(A)$  is finite, see [GH, §II.6], while the group  $\text{Aut}(A) \supseteq A$  is infinite. Conversely, by Théorème Principale I of Blanchard [Bl] for any finite subgroup  $G \subseteq \text{Aut}(A)$  there exists a projective embedding  $A \hookrightarrow \mathbb{P}^N$  which linearizes  $G$ . A general form of Blanchard's Theorem is as follows (cf. [Ak<sub>3</sub>, Theorem 3.2.1]).

**Theorem 1.4.** *Let  $Y$  be a smooth projective variety and  $G \subseteq \text{Aut}(Y)$  a subgroup which acts finitely on  $\text{Pic}(Y)$ . Then there is a  $G$ -equivariant projective embedding  $Y \hookrightarrow \mathbb{P}^N$ .*

Indeed, such an embedding corresponds to a very ample  $G$ -invariant divisor class. However, if  $G$  acts finitely on  $\text{Pic}(Y)$  then the orbit of any ample class is an ample  $G$ -invariant class.

For instance, if  $\text{Pic}(Y)$  is discrete and  $G$  is connected then  $G$  acts trivially on  $\text{Pic}(Y)$ . Hence there exists a  $G$ -equivariant projective embedding  $Y \hookrightarrow \mathbb{P}^N$ .

As another example, consider a smooth Fano variety  $Y$  embedded by a pluri-anticanonical system  $\varphi_{|-mK_Y|} : Y \hookrightarrow \mathbb{P}^N$  for a suitable  $m > 0$ . The canonical bundle  $K_Y$  being stable under the action of the automorphism group  $\text{Aut}(Y)$  on  $Y$ , this embedding is equivariant and realizes  $\text{Aut}(Y)$  as a closed subgroup of  $\text{PGL}_{N+1}(\mathbb{C})$ . In particular, this applies to the anticanonical embeddings of del Pezzo surfaces  $Y_d \hookrightarrow \mathbb{P}^d$  of degree  $d \geq 3$ . A rough description of the automorphisms groups of these surfaces is as follows, see Proposition 10.1.1 in [Dol<sub>1</sub>] (cf. e.g., [De], [dF], [Ho<sub>3</sub>], [DI<sub>1</sub>, DI<sub>2</sub>], [BB], [Bla] for more delicate properties).

**Theorem 1.5.** *Let  $Y = Y_d$  be a del Pezzo surface of degree  $d \geq 3$ . Then the automorphism group  $\text{Aut}(Y)$  acts on the lattice  $Q = (\mathbb{Z}K_Y)^\perp \subseteq \text{Pic}(Y)$  preserving the intersection form. The image of the corresponding homomorphism  $\rho : \text{Aut}(Y) \rightarrow O(Q)$  is contained in the Weyl group  $W(Q)$ . The kernel of  $\rho$  is trivial for  $d \leq 5$  and is a connected linear algebraic group of dimension  $2d - 10$  for  $d \geq 6$ . More precisely, the following hold.*

- (1) *For  $d \leq 5$  the group  $\text{Aut}(Y)$  is finite.*
- (2) *For  $d \geq 6$  the identity component  $\text{Aut}_0(Y) = \ker(\rho)$  contains a 2-torus  $\mathbb{T}_2 \simeq (\mathbb{C}^*)^2$ , and  $\text{Aut}_0(Y) = \mathbb{T}_2$  for  $d = 6$ .*
- (3) *For  $d \geq 7$  besides the 2-torus  $\mathbb{T}_2$  the group  $\text{Aut}_0(Y)$  contains a subgroup isomorphic to  $\mathbb{A}_+^2 = (\mathbb{C}_+)^2$ . In particular, for  $d = 7$  there are a decomposition*

$$\text{Aut}(Y) \simeq (\mathbb{A}_+^2 \rtimes \mathbb{T}_2) \rtimes \mathbb{Z}/2\mathbb{Z}$$

and a faithful presentation  $\mathrm{Aut}_0(Y) \hookrightarrow \mathrm{GL}_3(\mathbb{C})$  with image

$$\begin{pmatrix} 1 & 0 & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}.$$

- (4) For  $d = 8$  either  $Y \rightarrow \mathbb{P}^2$  is a blowup at a point and then  $\mathrm{Aut}(Y) \simeq \mathrm{GL}_2 \ltimes \mathbb{A}_+^2$ , or  $Y \simeq \mathbb{P}^1 \times \mathbb{P}^1$  and then  $\mathrm{Aut}(Y) \simeq (\mathrm{PGL}_2(\mathbb{C}))^2 \rtimes \mathbb{Z}/2\mathbb{Z}$ .
- (5) Finally for  $d = 9$ ,  $Y \simeq \mathbb{P}^2$  and  $\mathrm{Aut}(Y) \simeq \mathrm{PGL}_3(\mathbb{C})$ .

**Remark 1.6.** An effective  $\mathbb{A}_+^2$ -action on a del Pezzo surface  $Y$  of degree  $d = 7$  can be defined via the locally nilpotent derivations

$$(1) \quad \partial_{\alpha, \beta} = \alpha z \frac{\partial}{\partial x} + \beta z \frac{\partial}{\partial y}, \quad (\alpha, \beta) \in \mathbb{A}_+^2.$$

Indeed, the induced  $\mathbb{A}_+^2$ -action on  $\mathbb{A}^3$ :

$$(\alpha, \beta) \cdot (x, y, z) = (x + \alpha z, y + \beta z, z)$$

descends to an action on  $\mathbb{P}^2$  fixing the line  $z = 0$  pointwise. The blowup at two points on this line preserves the action. Likewise one defines an  $\mathbb{A}_+^2$ -action on  $Y$  for  $d = 8$  or  $d = 9$ .

**1.3. Homogeneous and almost homogeneous varieties.** By the Borel-Remmert Theorem [Ak<sub>3</sub>, 3.9] any connected, compact, homogeneous Kähler manifold  $V$  is bi-holomorphic to the product  $\mathrm{Alb}(V) \times Y$  of the Albanese torus and a (generalized) flag variety  $Y = G/P$  (i.e.,  $Y$  is the quotient of a connected semisimple linear algebraic group by a parabolic subgroup)<sup>3</sup>. It follows that every simply connected homogeneous compact Kähler manifold is a flag variety and the same is true for a rational projective homogeneous variety (for homogeneous compact complex manifolds satisfying both conditions this was established by Goto [Got]). Furthermore, Grauert and Remmert [GR] carried over a result of Chow [Cho] from abstract algebraic to Moishezon varieties. Namely, they proved that a homogeneous Moishezon variety is projective algebraic. Thus, if such a variety is simply connected or rational, it is a flag variety.

Every flag variety  $G/P$  is a projective rational Fano variety (see [Sn]). Every ample line bundle  $L$  on  $G/P$  is very ample (see e.g., [Chev<sub>2</sub>], [Ja], [La, §3.3.2], or [Te, Theorem 7.52]). The complete linear system  $|L|$  defines a  $G$ -equivariant embedding  $Y \hookrightarrow \mathbb{P}^n$  with a projectively normal image [RR, Theorem 1.iii].

For a maximal parabolic subgroup  $P_{\max} \subseteq G$ , the Picard group  $\mathrm{Pic}(G/P_{\max}) \cong \mathbb{Z}$  is generated by the class of a unique Schubert divisorial cycle in  $G/P_{\max}$ , and this class is very ample. In the case of a Grassmannian this class gives the Plücker embedding. For an arbitrary flag variety  $G/P$ , its Picard group  $\mathrm{Pic}(G/P)$  is also generated by the classes of the Schubert divisorial cycles; see e.g., [Chev<sub>2</sub>] or [Po<sub>2</sub>]. The set of maximal parabolic subgroups  $P_{\max}$  of  $G$  which contain  $P$  is finite. Every Schubert divisor class in  $\mathrm{Pic}(G/P)$  is lifted via a surjection  $G/P \rightarrow G/P_{\max}$ , see e.g. [LL] or [Sn]. A linear combination of these divisors is very ample if and only if its coefficients are all positive (see [Br] for the case of a full flag variety; the general case is similar [Te, Theorem 7.52]).

For a description of the automorphism groups of flag varieties see e.g., [Ak<sub>3</sub>, §3.3].

<sup>3</sup>See also [SDS] for a more general result in the projective case.

The following “cone theorem” describes certain *almost homogeneous* complex varieties. It is due to Akhiezer [Ak<sub>1</sub>, Theorem 3] in algebraic context and to Huckleberry and E. Oeljeklaus [HO<sub>1</sub>] in analytic one<sup>4</sup>.

**Theorem 1.7.** *Let  $X$  be an irreducible reduced complex space of dimension  $\geq 2$ . Suppose that a connected complex Lie group acts by biholomorphic transformations on  $X$  with an open orbit  $\Omega \subseteq X$  such that the complement  $E = X \setminus \Omega$  is a proper analytic subset with an isolated point, say,  $0 \in E$ . Then the normalization  $\nu: \tilde{X} \rightarrow X$  is one-to-one and  $\tilde{X}$  is biholomorphic to a projective or an affine cone over a flag variety  $G/P$  of some semisimple linear algebraic group  $G$  under a certain equivariant projective embedding. The isolated point  $0 \in E$  corresponds to the vertex of the cone. In particular, if  $(X, 0)$  is smooth then  $X \simeq \mathbb{A}^n$  or  $X \simeq \mathbb{P}^n$ .*

Thus the variety  $X$  as in the theorem equipped with an appropriate algebraic structure carries a regular almost transitive group action. If the initial group is a complex linear algebraic group, then  $G$  is its maximal semisimple subgroup [Ak<sub>1</sub>]. Given any  $G$ -equivariant projective embedding  $\varphi_{|H|}: Y = G/P \hookrightarrow \mathbb{P}^n$ , where  $\dim Y \geq 1$ , the affine cone  $\text{AffCone}_H(Y)$  over the image admits a regular action transitive off the vertex of a locally direct product  $\tilde{G} \cdot \mathbb{C}^*$ , with  $\mathbb{C}^*$  acting by homotheties, where  $\tilde{G} \rightarrow G$  is a finite group cover.

A similar description exists for the class of quasi-projective  $G$ -varieties  $X$ , where  $G$  is a connected linear algebraic group acting on  $X$  with an open orbit  $\Omega$ , provided that there is an equivariant completion  $\bar{X}$  of  $X$  with disconnected complement  $\bar{X} \setminus \Omega$  [Ak<sub>1</sub>, Theorem 2]. See also [Ak<sub>2</sub>] for the case that  $\bar{X} \setminus \Omega$  is a  $G$ -orbit of codimension 1 in  $\bar{X}$  (in this case it is connected).

An explicit description of almost homogeneous 2-dimensional affine cones over smooth projective curves is due to Popov [Po<sub>1</sub>] (see also [FZ<sub>2</sub>] for an alternative proof). We recall that a Veronese cone  $V_d$  is the affine cone over a smooth rational normal curve  $\Gamma_d \subset \mathbb{P}^d$  i.e., a linearly non-degenerate<sup>5</sup> smooth curve in  $\mathbb{P}^d$  of degree  $d$ . All such curves in  $\mathbb{P}^d$  are projectively equivalent and rational. For normal 2-dimensional cones, Popov’s Theorem can be stated as follows.

**Theorem 1.8** (V. Popov). *Let  $X$  be the affine cone over a smooth projective curve  $Y$ . If  $X$  is normal and admits an algebraic group action transitive in  $X \setminus \{0\}$ , then  $X$  is a Veronese cone  $V_d$  for some  $d \geq 1$ , and  $Y$  is a rational normal curve  $\Gamma_d$ .*

Popov [Po<sub>1</sub>] actually classified all almost homogeneous cones in dimension 2 with an isolated singularity (not necessarily normal). Every such cone possesses a linear  $\text{SL}(2, \mathbb{C})$ -action transitive off the vertex. The group  $\text{Aut}(X)$  of a Veronese cone is infinite dimensional and so cannot be linearized under an affine embedding; see Section 2.3 below.

## 2. GROUPS ACTING ON AFFINE CONES

**2.1. Linear automorphisms of affine cones.** Let us start with the following result.

<sup>4</sup>The smooth compact case was done first in [Oe]. See also [HO<sub>2</sub>, Ch. 2, §3, Theorem 1] for real groups.

<sup>5</sup>A projective variety  $Y \subseteq \mathbb{P}^n$  is linearly non-degenerate if it is not contained in any hyperplane.



**Proposition 2.1.** *Given two affine cones  $X_i = \text{AffCone}(Y_i) \subseteq \mathbb{A}^{n_i+1}$  over smooth, linearly non-degenerate, projective varieties  $Y_i \subseteq \mathbb{P}^{n_i}$  ( $i = 1, 2$ ) and an isomorphism  $\varphi : X_1 \xrightarrow{\sim} X_2$ , the differential  $d\varphi(0)$  provides a linear isomorphism  $\Phi : \mathbb{A}^{n_1+1} \xrightarrow{\sim} \mathbb{A}^{n_2+1}$  which restricts to an isomorphism  $\Phi|_{X_1} : X_1 \xrightarrow{\sim} X_2$ . In particular  $n_1 = n_2$ , and  $Y_1$  and  $Y_2$  are projectively equivalent.*

*Proof.* By the linear non-degeneracy assumption

$$T_0X_i = \mathbb{A}^{n_i+1}, \quad C_0X_i = X_i, \quad \text{and} \quad \mathbb{P}(C_0X_i) = Y_i, \quad i = 1, 2,$$

where  $T_0X_i$  is the Zariski tangent space to  $X_i$  at the vertex  $0 \in X_i$ , and  $C_0X_i$  is the tangent cone in 0 (see e.g., [CLS, §9.7]). Now the assertion follows since  $d\varphi(0)$  provides an isomorphism of the Zariski tangent spaces and sends the cone  $C_0X_1$  onto the cone  $C_0X_2$  [Da, §7.3]. In fact  $d\varphi(0)$  lifts to an isomorphism of blowups  $\text{Bl}_0(X_1) \xrightarrow{\sim} \text{Bl}_0(X_2)$  preserving the exceptional divisors. These divisors are isomorphic to  $Y_1$  and  $Y_2$ , respectively, and  $d\varphi(0)$  induces a linear isomorphism  $Y_1 \xrightarrow{\sim} Y_2$ .  $\square$

**Remark 2.2.** The isomorphism  $\varphi$  as in Proposition 2.1 does not need to be linear itself. However, this is the case under the additional assumption that  $Y_1$  is not birationally ruled (see Proposition 2.3 below). A birationally ruled projective variety is a variety birationally equivalent to a product  $Z \times \mathbb{P}^1$ . Recall also that a birational map  $\bar{f} : \bar{X}_1 \dashrightarrow \bar{X}_2$  is said to be *isomorphism in codimension one* if there are subsets  $B_i \subseteq \bar{X}_i$  of codimension at least 2 such that

$$\bar{f}|(\bar{X}_1 \setminus B_1) : \bar{X}_1 \setminus B_1 \rightarrow \bar{X}_2 \setminus B_2$$

is an isomorphism.

**Proposition 2.3.** *Consider the affine cones  $X_i = \text{AffCone}(Y_i) \subseteq \mathbb{A}^{n_i+1}$  over projective varieties  $Y_i \subsetneq \mathbb{P}^{n_i}$ ,  $i = 1, 2$ . Suppose that  $Y_1$  and  $Y_2$  are smooth, irreducible, and linearly non-degenerate. If  $Y_1$  is not birationally ruled then every isomorphism  $\varphi : X_1 \xrightarrow{\sim} X_2$  extends to a unique linear isomorphism  $\mathbb{A}^{n_1+1} \xrightarrow{\sim} \mathbb{A}^{n_2+1}$ . In particular  $n_1 = n_2$ , and  $Y_1$  and  $Y_2$  are projectively equivalent.*

The proposition follows immediately from Lemmas 2.7 and 2.8 below. Before passing to the lemmas, let us give two corollaries, which are the main results of this subsection.

**Corollary 2.4.** *Let  $X = \text{AffCone}(Y) \subseteq \mathbb{A}^{n+1}$  be the affine cone over a smooth projective variety  $Y \subseteq \mathbb{P}^n$ . If  $Y$  is not birationally ruled then  $\text{Aut}(X) = \text{Lin}(X)$ . Moreover,  $\text{Aut}(X)$  is a central extension of the group  $\text{Lin}(Y)$  by  $\mathbb{C}^*$ .*

Indeed, the exact sequence

$$0 \rightarrow \mathbb{C}^* \rightarrow \text{GL}(n+1, \mathbb{C}) \rightarrow \text{PGL}(n+1, \mathbb{C}) \rightarrow 0$$

yields the following one:

$$(2) \quad 0 \rightarrow \mathbb{C}^* \longrightarrow \text{Lin}(X) \xrightarrow{\pi} \text{Lin}(Y) \rightarrow 0.$$

**Corollary 2.5.** *Let  $X = \text{AffCone}(Y)$  be the affine cone over a smooth projective 3-fold  $Y$ . Suppose that  $Y$  is rationally connected and non-rational. Then  $\text{Aut}(X) = \text{Lin}(X)$ .*

*Proof.* Indeed if  $Y$  were birationally ruled i.e., birational to a product  $Z \times \mathbb{P}^1$ , then  $Z$  would be rationally connected and so a rational surface. Hence  $Y$  would be rational too, contrary to our assumption. Thus Corollary 2.4 applies and gives the assertion.  $\square$

**Example 2.6.** For instance, if  $Y \subseteq \mathbb{P}^n$  is a non-rational Fano 3-fold and  $X = \text{AffCone}(Y)$ , then  $\text{Aut}(X) = \text{Lin}(X)$ . As an example, one can consider any smooth cubic or quartic 3-fold  $Y \subseteq \mathbb{P}^4$ .

For the proof of the next lemma we refer the reader to [To, Theorem 2.2] or [Co, Proposition 2.7].

**Lemma 2.7.** *Let  $X_i = \bar{X}_i \setminus D_i$ ,  $i = 1, 2$ , where  $\bar{X}_i$  is a projective variety and  $D_i$  an irreducible divisor on  $\bar{X}_i$ . Suppose that  $X_i$  is regular near  $D_i$  for  $i = 1, 2$ . If  $D_1$  is not birationally ruled then any isomorphism  $f : X_1 \rightarrow X_2$  extends to a birational map  $\bar{f} : \bar{X}_1 \dashrightarrow \bar{X}_2$  which is an isomorphism in codimension 1. If in addition the divisors  $D_1$  and  $D_2$  are ample then  $\bar{f} : \bar{X}_1 \xrightarrow{\sim} \bar{X}_2$  is an isomorphism.*

Proposition 2.3 is now a direct consequence of the following lemma.

**Lemma 2.8.** *Consider two projective varieties  $Y_i \subsetneq \mathbb{P}^{n_i}$ , where  $n_i \geq 2$ ,  $i = 1, 2$ . Suppose that  $Y_1$  and  $Y_2$  are smooth, irreducible, and linearly non-degenerate. Consider also the affine cones  $X_i = \text{AffCone}(Y_i) \subseteq \mathbb{A}^{n_i+1}$  over  $Y_i$  and the projective cones  $\bar{X}_i \subseteq \mathbb{P}^{n_i+1}$ ,  $i = 1, 2$ . Let  $\varphi : X_1 \xrightarrow{\sim} X_2$  be an isomorphism such that the induced birational map  $\bar{\varphi} : \bar{X}_1 \dashrightarrow \bar{X}_2$  is an isomorphism in codimension 1. Then  $\varphi$  extends to a unique linear isomorphism  $\Phi : \mathbb{A}^{n_1+1} \xrightarrow{\sim} \mathbb{A}^{n_2+1}$ . In particular  $n_1 = n_2$ , and  $Y_1$  and  $Y_2$  ( $\bar{X}_1$  and  $\bar{X}_2$ , respectively) are projectively equivalent.*

*Proof.* We let  $D_i = \bar{X}_i \setminus X_i$  denote the divisor at infinity; it is a scheme-theoretic hyperplane section. Since  $D_1$  and  $D_2$  are ample then (similarly as in Lemma 2.7)  $\varphi$  extends to an isomorphism  $\bar{\varphi} : \bar{X}_1 \rightarrow \bar{X}_2$ , which sends  $0 \in \mathbb{A}^{n_1+1}$  to  $0 \in \mathbb{A}^{n_2+1}$ . Indeed, these points are the only singular points of the projective cones  $\bar{X}_1$  and  $\bar{X}_2$ . Moreover,  $\bar{\varphi}$  sends the generators of the cone<sup>6</sup>  $\bar{X}_1$  into generators of  $\bar{X}_2$ . Indeed, every generator  $l_1$  of  $\bar{X}_1$  meets  $D_1$  transversally in one point. The image  $l_2 = \varphi(l_1) \subseteq \bar{X}_2$  possesses similar properties, hence  $l_2$  is again a projective line through the origin i.e., a generator of the cone  $\bar{X}_2$ .

It follows that the orbits of the  $\mathbb{C}^*$ -action on  $\bar{X}_1$  are sent to the orbits of the  $\mathbb{C}^*$ -action on  $\bar{X}_2$ . Furthermore  $\bar{\varphi}$  is  $\mathbb{C}^*$ -equivariant, hence it induces an isomorphism  $\varphi^* : \mathcal{O}(Y_2) \xrightarrow{\sim} \mathcal{O}(Y_1)$  of the homogeneous coordinate rings. These graded rings are the coordinate rings of the affine cones  $X_1$  and  $X_2$ , respectively, generated by their first graded pieces<sup>7</sup>. The graded isomorphism  $\varphi^*$  restricts to a linear isomorphism, say,  $\Psi : \mathbb{A}^{n_2+1} \xrightarrow{\sim} \mathbb{A}^{n_1+1}$  between these first graded pieces. The dual isomorphism  $\Phi = \Psi^\vee : \mathbb{A}^{n_1+1} \xrightarrow{\sim} \mathbb{A}^{n_2+1}$  provides a desired linear extension of  $\varphi$ . The uniqueness of such an extension follows immediately, since  $Y_1$  and  $Y_2$  are assumed to be linearly non-degenerate.  $\square$

For a projective variety  $Y \subseteq \mathbb{P}^n$  with affine cone  $X = \text{AffCone}(Y)$  it can happen that  $\text{Aut}(Y) \neq \text{Lin}(Y)$ , while  $\text{Aut}(X) = \text{Lin}(X)$ , as in the following examples.

**Examples 2.9.** 1. Let  $A$  be an abelian variety. Consider a projective embedding  $A \hookrightarrow \mathbb{P}^n$  (for instance, a smooth cubic in  $\mathbb{P}^2$ ) with affine cone  $X = \text{AffCone}(A)$ . By Corollary 2.4  $\text{Aut}(X) = \text{Lin}(X)$ . While  $\text{Lin}(A)$  is a finite group (see [GH, §II.6] or

<sup>6</sup>That is the projective lines on  $\bar{X}_1$  passing through the origin.

<sup>7</sup>Consisting of the restrictions to  $X_i$  of linear functions on  $\mathbb{A}^{n_i+1}$ ,  $i = 1, 2$ .

Section 1 above), the group  $\text{Aut}(A)$  contains the subgroup of translations and so is infinite. Thus  $\text{Aut}(Y) \neq \text{Lin}(Y)$  (cf. Blanchard's Theorem 1.4).

2. A smooth quartic  $Y \subseteq \mathbb{P}^3$  is a K3-surface and so is not birationally ruled. Hence again  $\text{Aut}(X) = \text{Lin}(X)$ , where  $X = \text{AffCone}(Y) \subseteq \mathbb{A}^4$ . Moreover,  $\text{Lin}(Y)$  is a finite group, while the group  $\text{Aut}(Y)$  can be infinite, see the discussion in §1.1. Clearly, non-linear automorphisms of  $Y$  are not induced by automorphisms of  $X$ .

**2.2. Lifting  $G$ -actions to affine cones.** In this subsection we address the following questions.

- (1) *When a  $G$ -action on  $Y$  is induced by a  $G$ -action on  $X$ ?*
- (2) *When a  $G$ -action on  $Y$  is induced by a  $\tilde{G}$ -action on  $X$ ?*

A related question is:

*Which projective representations can be lifted to linear ones?*

Simple examples show that one needs some restrictions on such a projective representation. In the first example below the group  $G$  is finite, and is connected algebraic in the second.

**Examples 2.10.** 1. The standard representation on  $\mathbb{A}^2$  of the group of quaternions  $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$  induces a faithful representation of  $Q_8$  on any Veronese cone  $V_d \simeq \mathbb{A}^2/\mathbb{Z}_d$  with  $d$  odd (cf. Subsection 2.4 below). The latter representation descends to an effective linear action on  $\mathbb{P}^1$  of the dihedral group

$$D_2 = Q_8/Z(Q_8) \simeq (\mathbb{Z}/2\mathbb{Z})^2.$$

However, this  $D_2$ -action on  $\mathbb{P}^1$  cannot be lifted to a  $D_2$ -action on  $\mathbb{A}^2$  or on any of the Veronese cones  $V_d$  with  $d$  odd. Indeed, otherwise the exact sequence

$$0 \longrightarrow Z(Q_8) \longrightarrow Q_8 \longrightarrow D_2 \longrightarrow 0$$

would split, which is not the case. In other words, the faithful projective representation  $D_2 \rightarrow \text{PGL}_{n+1}(\mathbb{C})$  induced by the Veronese embedding  $\varphi_{|\mathcal{O}_{\mathbb{P}^1}(n)|} : \mathbb{P}^1 \rightarrow \mathbb{P}^n$  lifts to a linear representation  $D_2 \rightarrow \text{GL}_{n+1}(\mathbb{C})$  if and only if  $n = 2k > 0$  is even and so  $\mathcal{O}_{\mathbb{P}^1}(n) = -kK_{\mathbb{P}^1}$ .

2. The standard projective representation of  $G = \text{PGL}_2(\mathbb{C})$  on  $\mathbb{P}^1$  induces a linear  $G$ -action on the rational normal curve  $\Gamma_d \subseteq \mathbb{P}^d$ . Suppose that the latter action can be lifted to the Veronese cone  $V_d = \text{AffCone}(\Gamma_d) \subseteq \mathbb{A}^{d+1}$ . This would give an irreducible representation of  $G = \text{PGL}_2(\mathbb{C})$  of dimension  $d + 1$ . However, such a representation does exist only for  $d$  even. Indeed, every irreducible representation of  $\text{PGL}_2(\mathbb{C})$  yields an irreducible representation of  $\text{SL}_2(\mathbb{C})$  trivial on the center, and vice versa.

**Remark 2.11.** Concerning question (2), recall that for any perfect group  $G$  there exists a unique universal central extension (or Schur cover)  $G'$  of  $G$  such that every projective representation of  $G$  is induced by a linear representation of  $G'$  (see [St, §7]). For a finite perfect group  $G$ , the Schur cover  $G'$  is again finite. For a perfect (e.g., semi-simple) connected linear algebraic group  $G$  over  $\mathbb{C}$ , the Schur cover is just the simply connected universal covering group  $G'$  of  $G$ .

Any linear action  $G \rightarrow \text{Lin}(X)$  on the affine cone  $X = \text{AffCone}(Y)$  induces (via the exact sequence (2)) a linear action  $G \rightarrow \text{Lin}(Y)$  on  $Y$ . The latter factorizes through the action on  $Y$  of the quotient group  $G/(G \cap \mathbb{C}^*)$ . Answering question (2) above,

in the following proposition we provide a simple criterion as to when a  $G$ -action on  $Y$  is induced by a  $\tilde{G}$ -action on  $X = \text{AffCone}(Y)$ , where  $\tilde{G}$  is a central extension of  $G$  (which is not a Schur cover). The proof is straightforward. Let us remind that any linear action  $G \rightarrow \text{Lin}(Y)$  of a group  $G$  on a projective variety  $Y \subseteq \mathbb{P}^n$  stabilizes the very ample divisor class  $[\mathcal{O}_Y(1)] \in \text{Pic}(Y)$ .

**Proposition 2.12.** (a) *Let  $Y$  be a smooth projective variety and  $G \rightarrow \text{Aut}(Y)$  be a group action on  $Y$ . If this action stabilizes a very ample divisor class  $|H| \in \text{Pic}(Y)$ , then it extends linearly to the ambient projective space  $\mathbb{P}^n = \mathbb{P}H^0(Y, \mathcal{O}_Y(H))$ .*  
 (b) *Furthermore, let  $X = \text{AffCone}_H(Y)$  be the affine cone over  $\varphi_{|H|}(Y)$ . Consider the central extension  $\tilde{G} = \pi^{-1}(G) \subseteq \text{Lin}(X)$  of  $G$  by  $\mathbb{C}^*$ , where  $\pi : \text{Lin}(X) \rightarrow \text{Lin}(Y)$  is as in (2). Then the group  $\tilde{G}$  acts linearly on  $X$  inducing the given  $G$ -action on  $Y$ .*

**Corollary 2.13.** *Let  $G$  be a connected linear algebraic group. Then any regular  $G$ -action on a smooth projective variety  $Y \subseteq \mathbb{P}^n$  is induced by a regular  $\tilde{G}$ -action on the affine cone  $\text{AffCone}(Y)$ , where  $\tilde{G} = \pi^{-1}(G) \subseteq \text{Lin}(X)$  is a central extension of  $G$  by  $\mathbb{C}^*$ .*

*Proof.* Since  $G$  is connected,  $G$  acts on  $\text{Pic}_0(Y)$ . The group  $G$  being a rational variety [Chev<sub>1</sub>], every morphism of  $G$  to the abelian variety  $\text{Pic}_0(Y)$  is constant. Hence the  $G$ -action on  $\text{Pic}_0(Y)$  is trivial, and so is the induced action on the Neron-Severi group  $NS(Y) = \text{Pic}(Y)/\text{Pic}_0(Y)$ . Thus  $G$  acts trivially on  $\text{Pic}(Y)$ . By Proposition 2.12(a) the  $G$ -action on  $Y$  extends linearly to  $\mathbb{P}^n$ . Now the result follows.  $\square$

**Remark 2.14.** Instead of referring to [Chev<sub>1</sub>] one can show directly that every morphism  $f : G \rightarrow A$  to an abelian variety  $A$  is constant. Clearly,  $f$  is constant on any abelian subgroup of  $G$  and on its cosets. Hence  $f$  is also constant on any solvable subgroup. In particular, it is constant on  $\text{Rad}(G)$  and on its cosets. Thus  $f$  induces a morphism  $G/\text{Rad}(G) \rightarrow A$ . So we may assume that  $G$  is semisimple. Consider a maximal torus  $\mathbb{T} \subseteq G$  and the collection of its root vectors  $(H_\alpha)_\alpha \subseteq T_e G = \text{lie}(G)$ . The subset  $T_e \mathbb{T} \cup (H_\alpha)_\alpha$  consists of the tangent vectors of algebraic one-parameter subgroups of  $G$  and spans the tangent space  $T_e G$ . Hence the differential  $df(e)$  vanishes. Now the assertion follows. Indeed, applying left shifts one can produce a similar situation in any point  $g$  of  $G$ .

A stronger statement holds for pluri-canonical or pluri-anticanonical embeddings.

**Proposition 2.15.** *Let  $Y$  be a smooth projective variety. Suppose that for some  $m \in \mathbb{Z}$  there is an embedding  $\varphi = \varphi_{|mK_Y|} : Y \hookrightarrow \mathbb{P}^n$ , and let  $X = \text{AffCone}(\varphi(Y))$ . Then*

$$\text{Lin}(X) = \mathbb{C}^* \times \text{Lin}(\varphi(Y)) \simeq \mathbb{C}^* \times \text{Aut}(Y),$$

*where  $\mathbb{C}^*$  acts on the cone  $X$  by scalar matrices.*

*Proof.* Indeed, the group  $\text{Aut}(Y)$  acts on the linear system  $|mK_Y|$  yielding an isomorphism  $\text{Aut}(Y) \simeq \text{Lin}(\varphi(Y))$ . Moreover,  $\text{Aut}(Y)$  acts on the linear bundle  $\mathcal{O}(mK_Y)$ . Hence it acts linearly on  $H^0(Y, \mathcal{O}(mK_Y))$ . The dual action on  $H^0(Y, \mathcal{O}(mK_Y))^\vee$  preserves the cone  $X$ . This gives an embedding  $\text{Aut}(Y) \hookrightarrow \text{Lin}(X)$  and a splitting of the exact sequence

$$0 \rightarrow \mathbb{C}^* \rightarrow \text{Lin}(X) \rightarrow \text{Lin}(\varphi(Y)) \simeq \text{Aut}(Y) \rightarrow 0.$$

Since the subgroup  $\mathbb{C}^* \subseteq \text{Lin}(X)$  is central, the assertions follow.  $\square$

This proposition can be applied to the anticanonical embeddings of del Pezzo surfaces. In the case where there is a  $\mathbb{C}_+$ -action on  $X$  the group  $\text{Aut}(X)$  is infinite dimensional. For instance, this is so for the cones over del Pezzo surfaces of degree  $d \geq 4$ . For  $d \geq 7$  there exists a linear  $\mathbb{A}_+^2$ -action on  $X$ . While for  $6 \geq d \geq 4$  the group  $\text{Aut}(Y)$  is finite or toric, hence any  $\mathbb{C}_+$ -action on  $X$  is non-linear; cf. Theorem 0.1 in the Introduction and also Theorems 1.5 and 3.19.

**2.3. Groups acting on affine cones.** Similarly as in Proposition 2.1, in the case of a reductive group action a weaker analog of Corollary 2.4 holds without the assumption of birational non-ruledness.

**Lemma 2.16.** *Suppose that a connected reductive group  $G$  acts effectively on the affine cone  $X \subseteq \mathbb{A}^{n+1}$  over a smooth linearly non-degenerate projective variety  $Y \subsetneq \mathbb{P}^n$ . Then there is a faithful representation  $\rho : G \rightarrow \text{GL}(n+1, \mathbb{C})$ , which restricts to an effective linear  $G$ -action on  $X$  inducing a linear action of  $G$  on  $Y$ .*

*Proof.* The vertex  $0 \in X$  is an isolated singular point of  $X$ , hence a fixed point of  $G$ . Since  $G$  is reductive, the induced representation  $\rho$  of  $G$  on the Zariski tangent space  $T_0X$  is faithful (see e.g., [Ak<sub>3</sub>] or [FZ<sub>2</sub>, Lemma 2.7(b)]) and descends to  $Y$  via the projective representation  $\bar{\rho} : G \rightarrow \text{PGL}(n+1, \mathbb{C}) = \text{GL}(n+1, \mathbb{C})/\mathbb{C}^*$ .  $\square$

Let us note that for a non-reductive group action,  $\rho$  as above can be trivial. For instance, this is the case for the  $\mathbb{C}_+$ -action  $t.(x, y) = (x + ty^2, y)$  on  $X = \mathbb{A}^2$ .

The following theorem is complementary to Corollary 2.4; cf. also [HO<sub>1</sub>] for (a).

**Theorem 2.17.** *We let  $X \subseteq \mathbb{A}^n$  ( $n \geq 2$ ) be the affine cone over a smooth projective variety  $Y \subseteq \mathbb{P}^{n-1}$ . Suppose that*

- *The group  $\text{Aut}(Y)$  is finite.*
- *A connected algebraic group  $G$  of dimension  $\geq 2$  acts effectively on  $X$  and contains a 1-dimensional torus  $\mathbb{T} \simeq \mathbb{C}^*$  acting on  $\mathbb{A}^n$  via scalar matrices.*

*Then the following hold.*

- (a)  *$G$  is a solvable group of rank 1.*
- (b) *There exists an  $\mathbb{A}^1$ -fibration  $\theta : X \rightarrow Z$ , where  $Z$  is an affine variety equipped with a good  $\mathbb{C}^*$ -action and  $\theta$  is equivariant with respect to the standard  $\mathbb{C}^*$ -action on  $X$ . Furthermore,  $Z$  is normal if  $X$  is.*
- (c)  *$Y$  is uniruled via a family of rational curves parameterized by  $(Z \setminus \{\theta(0)\})/\mathbb{C}^*$ .*

*Proof.* Consider a Levi decomposition  $G = \text{Rad}_u(G) \rtimes L$ , where  $L \subseteq G$  is a Levi subgroup (i.e., a maximal connected reductive subgroup) containing  $\mathbb{T}$ . By Lemma 2.16 the induced representation  $\rho$  of  $L$  on the Zariski tangent space  $T_0X$  is faithful. Moreover  $\mathbb{T}$  (which acts on  $T_0X$  by scalar matrices) is a central subgroup of  $L$ . Since the group  $\text{Aut}(Y)$  is finite and  $L$  is connected, the induced action of the quotient group  $L/\mathbb{T}$  on  $Y$  is trivial. Thus  $L = \mathbb{T}$  is a maximal torus of  $G = \text{Rad}_u(G) \rtimes \mathbb{T}$ , and so  $G$  is solvable of rank 1.

By our assumption  $\dim_{\mathbb{C}}(G) \geq 2$ . Hence the unipotent radical  $\text{Rad}_u(G)$  is non-trivial and contains a one-parameter subgroup  $U \simeq G_a$ . All orbits of  $U$  are closed in  $X$ , and the one-dimensional orbits are isomorphic to the affine line  $\mathbb{A}^1$ . Therefore  $X$  is affine

uniruled. Its coordinate ring  $A = \mathcal{O}_X$  is graded by the dual lattice  $\mathbb{T}^\vee \simeq \mathbb{Z}$ . This grading is actually positive:

$$A = \bigoplus_{k \geq 0} A_k.$$

The infinitesimal generator  $\partial$  of the induced  $G_a$ -action on  $A$  is a homogeneous locally nilpotent derivation of  $A$  (see e.g., [Re] or [FZ<sub>2</sub>]). The ring of invariants  $B = \ker(\partial) = A^{G_a}$  is a graded subalgebra of  $A$  with  $B_0 = A_0 = \mathbb{C}$ . Therefore the affine variety  $Z = \operatorname{spec}(B)$  is endowed by a  $\mathbb{T}$ -action with a unique attractive fixed point  $0' = \theta(0)$ , where  $\theta : X \rightarrow Z$  is the orbit map of the  $G_a$ -action on  $X$ . Thus  $\theta$  is a  $\mathbb{T}$ -equivariant surjection induced by the inclusion  $B \subseteq A$  of graded rings. If  $A$  is integrally closed in  $\operatorname{Frac}(A)$  then also  $B$  is. Indeed, let  $Z' = \operatorname{spec}(\bar{B})$  be the normalization of  $Z$ , where  $\bar{B}$  is the integral closure of  $B$  in  $\operatorname{Frac}(A)$ . Since  $X$  is normal the morphism  $X \rightarrow Z$  factorizes as  $X \rightarrow Z' \xrightarrow{\nu} Z$ . The locally nilpotent derivation  $\partial$  stabilizes  $\bar{B}$  (see e.g., [Sei], [Vas], or [FZ<sub>1</sub>, Lemma 2.15]) and so the morphisms  $X \rightarrow Z' \xrightarrow{\nu} Z$  are equivariant with respect to the induced  $\mathbb{C}_+$ -actions. The  $\mathbb{C}_+$ -action is trivial on  $Z$ , hence also on  $Z'$  since  $Z' \rightarrow Z$  is finite. Thus  $B \subseteq \bar{B} \subseteq \ker \partial = B$ , so  $B = \bar{B}$  is normal as soon as  $A$  is.

Since a general one-dimensional orbit of  $U \simeq G_a$  in  $X$  does not pass through the vertex  $0 \in X$  and is not contained in an orbit closure of  $T$  (i.e., in a generator of the cone), there is a Zariski open subset, say,  $\Omega$  of  $Y$  covered by the images of these orbits. Taking Zariski closures yields a family of rational curves parameterized by  $(Z \setminus \{0'\})/\mathbb{C}^*$ . Thus  $Y$  is uniruled, as claimed.  $\square$

**2.4. Group actions on 2-dimensional affine cones.** The following corollary is immediate from Proposition 2.1.

**Corollary 2.18.** *Consider two smooth linearly non-degenerate curves  $Y_i \subseteq \mathbb{P}^{n_i}$  ( $i = 1, 2$ ) of degrees  $d_i$ , and let  $X_i = \operatorname{AffCone}(Y_i) \subseteq \mathbb{A}^{n_i+1}$  be the corresponding affine cones. Then  $X_1 \simeq X_2$  if and only if these cones are linearly isomorphic, if and only if  $n_1 = n_2$ ,  $d_1 = d_2$  and  $Y_1$  and  $Y_2$  are projectively equivalent.*

Similarly, from Corollary 2.4 we deduce the following one.

**Corollary 2.19.** *Let  $X = \operatorname{AffCone}(Y) \subseteq \mathbb{A}^{n+1}$  be the affine cone over a smooth, non-rational projective curve  $Y \subseteq \mathbb{P}^n$ . Then  $\operatorname{Aut}(X) = \operatorname{Lin}(X)$ , and this group is a central extension of the finite group  $\operatorname{Lin}(Y)$  by  $\mathbb{C}^*$ .*

**Remarks 2.20.** 1. However,  $\operatorname{Aut}(Y) \neq \operatorname{Lin}(Y)$  for an elliptic curve  $Y \subseteq \mathbb{P}^n$ , see Example 2.9(1). Consider further a smooth rational curve  $Y \subseteq \mathbb{P}^n$  of degree  $d > n$ . Then  $Y$  is neither linearly nor projectively normal. Indeed,  $Y$  is a linear projection of the rational normal curve  $\Gamma_d \subseteq \mathbb{P}^d$ , and  $X = \operatorname{AffCone}(Y)$  is a linear projection of the Veronese cone  $V_d = \operatorname{AffCone}(\Gamma_d)$ . The latter projection gives a normalization of  $X$ . This is not an isomorphism as it diminishes the dimension of the Zariski tangent space at the vertex.

2. The normalizations of the affine cones  $X_1$  and  $X_2$  over two smooth rational curves  $Y_1$  and  $Y_2$ , respectively, are isomorphic if and only if  $\deg(Y_1) = \deg(Y_2)$ . While in general the (non-normal) affine surface  $X = \operatorname{AffCone}(Y)$  admits non-trivial equisingular deformations arising from deformations of the projective embedding  $Y \hookrightarrow \mathbb{P}^n$ . For instance, smooth rational curves  $Y$  of type  $(1, a)$  on a quadric  $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$  vary in

a family of projective dimension  $2a + 1$ . Hence for any  $a \geq 3$  the group  $\mathrm{PSO}(4, \mathbb{C})$  cannot act transitively on this family.

3. Since any group action on an affine cone  $X$  lifts to the normalization, it is enough to restrict to normal cones. For the normal Veronese cone  $X = V_d \subseteq \mathbb{A}^{d+1}$  over a rational normal curve  $Y = \Gamma_d \subseteq \mathbb{P}^d$  we have  $\mathrm{Aut}(X) \neq \mathrm{Lin}(X)$ . Moreover,  $V_d \simeq \mathbb{A}^2/\mathbb{Z}_d$  being a toric surface, for every  $d \geq 1$  the group  $\mathrm{Aut}(V_d)$  is infinite dimensional. In particular this is not an algebraic group. Indeed, the graded coordinate ring  $\mathcal{O}(V_d)$  admits a nonzero locally nilpotent derivation  $\partial$  corresponding to an effective  $\mathbb{C}_+$ -action on  $V_d$  [FZ<sub>2</sub>]. The kernel  $\ker(\partial) \subseteq \mathcal{O}(V_d)$  is isomorphic to the polynomial ring  $\mathbb{C}[t]$ . For any  $p \in \mathbb{C}[t]$ , the derivation  $p \cdot \partial$  is again locally nilpotent. Thus  $\mathbb{C}[t] \cdot \partial$  is the Lie algebra of an infinite dimensional abelian subgroup  $G \subseteq \mathrm{Aut}(V_d)$ .

4. There are actually two independent  $\mathbb{C}_+$ -actions on  $V_d$  with different orbits, and even a continuous family of such actions; see e.g., [FZ<sub>2</sub>]. Danilov and Gizatullin [DG] studied the structure of an amalgamated product on the group  $\mathrm{Aut}(V_d)$ , while Makar-Limanov [ML] provided an explicit description of this group.

5. Similarly, independent  $\mathbb{C}_+$ -actions, and an amalgamated product structure, exist on any normal affine toric surface different from  $\mathbb{A}_*^1 \times \mathbb{A}^1$  or  $\mathbb{A}_*^1 \times \mathbb{A}_*^1$ . Every such surface is of the form  $V_{d,e} = \mathbb{A}^2/\mathbb{Z}_d$  for an appropriate diagonal action

$$\zeta.(x, y) = (\zeta x, \zeta^e y), \quad \text{where } \zeta^d = 1 \quad \text{and} \quad \gcd(e, d) = 1,$$

of the cyclic group  $\mathbb{Z}_d = \langle \zeta \rangle$  on the affine plane  $\mathbb{A}^2$ . Choosing a system of homogeneous generators in the graded coordinate ring  $\mathcal{O}(V_{d,e})$  yields an embedding  $V_{d,e} \hookrightarrow \mathbb{A}^{N+1}$ , which is equivariant with respect to a suitable diagonal  $\mathbb{C}^*$ -action on  $\mathbb{A}^{N+1}$  with positive weights  $w = (w_0, \dots, w_N)$ . In this way  $V_{d,e}$  can be realized as the affine cone over a smooth rational curve in the corresponding weighted projective space  $\mathbb{P}_w^N = (\mathbb{A}^{N+1} \setminus \{0\})/\mathbb{C}^*$ .

6. By Popov's Theorem [Po<sub>1</sub>]<sup>8</sup> the Veronese cones  $V_d = V_{d,1}$  can be characterized as normal affine surfaces on which an algebraic group acts with an open orbit and a fixed point.

Let us construct an explicit example of a non-linear biregular automorphism of a Veronese cone  $V_d$  for every  $d \geq 1$ .

**Example 2.21.** Consider as before the Veronese cone  $V_d \subseteq \mathbb{A}^{d+1}$  over a rational normal curve  $\Gamma_d \subseteq \mathbb{P}^d$ . Let  $\bar{V}_d \subset \mathbb{P}^{d+1}$  be the Zariski closure, and  $\tilde{V}_d$  be the blowup of  $\bar{V}_d$  at the vertex  $0 \in V_d$ . It is well known [DG] that  $\tilde{V}_d \simeq \Sigma_d$ , where  $\Sigma_d$  denotes a Hirzebruch surface with the exceptional section  $S_0$  and a disjoint section at infinity, say,  $S_\infty$  with  $S_0^2 = -d$  and  $S_\infty^2 = d$ . Therefore

$$V_d \simeq (\Sigma_d \setminus S_\infty)/S_0.$$

To exhibit a non-linear automorphism of  $V_d$  is the same as to exhibit an automorphism of  $\Sigma_d \setminus S_\infty$ , which extends to a birational transformation of  $\Sigma_d$  preserving the exceptional section  $S_0$  but not the ruling  $\pi : \Sigma_d \rightarrow \mathbb{P}^1$  (or, equivalently, which blows down the curve  $S_\infty$ ). On the level of dual graphs, such a birational transformation consists e.g., in the following sequence of blowups and blowdowns [FKZ]:

<sup>8</sup>This is actually an earlier version of the Cone Theorem in dimension 2; see Section 1.

$$\begin{array}{ccccccc}
d & & -1 & -1 & A_d & & 0 & A_d & & -1 & -1 & A_d & & d \\
\circ & \dashrightarrow & \circ & \circ & \square & \dashrightarrow & \circ & \square & \dashrightarrow & \circ & \circ & \square & \dashrightarrow & \circ \\
S_\infty & & S_\infty & v_{d+1} & & & v_{d+1} & & & S'_\infty & v_{d+1} & & & S'_\infty
\end{array}$$

Here a box marked  $A_d$  represents the linear chain  $[[ -2, \dots, -2 ]]$  of length  $d$ . The centers of blowups on the curves  $S_\infty$  and  $v_{d+1}$  can vary. Anyhow, the section  $S_\infty$  being contracted, the resulting biregular transformation of the Veronese cone  $V_d$  is non-linear.

### 3. GROUP ACTIONS ON 3-DIMENSIONAL AFFINE CONES

The main result of this section is the existence of a  $\mathbb{C}_+$ -actions on the affine cones over every smooth del Pezzo surface of degree  $\geq 4$ . The proof exploits a general geometric criterion for the existence of such an action.

#### 3.1. Existence of $\mathbb{C}_+$ -actions on affine cones: a geometric criterion.

**3.1.** Let  $Y$  be a smooth projective variety, and let  $H \in \text{Div}(Y)$  be an ample polarization of  $Y$ . Consider the total space  $\hat{X}$  of the line bundle  $\mathcal{O}_Y(H)$  with the zero section  $S_0 \subseteq \hat{X}$ . Under the natural identification  $S_0 \simeq Y$ , we have  $\mathcal{O}_{S_0}(S_0) = \mathcal{O}_Y(-H)$ . Hence  $S_0$  is contractible, i.e., there is a birational contraction  $v : \hat{X} \rightarrow X$ , where  $X$  is a normal affine variety and  $v(S_0)$  is a point. In this situation, we call  $X$  a *generalized cone* over  $(Y, H)$ . If  $H$  is very ample, then  $X$  coincides with the normalization of the usual affine  $\text{AffCone}(Y)$  cone over  $Y \hookrightarrow \mathbb{P}^n$ , where the embedding is given by the linear system  $|H|$ . So we write  $X = \text{AffCone}_H(Y)_{\text{norm}}$ . In this section we provide a criterion of existence of a  $\mathbb{C}_+$ -action on a generalized cone.

Let us note that  $X$  can be compactified to the projective cone  $\bar{X}$  over  $Y$  by adding a divisor at infinity  $S_\infty \simeq Y$ . The divisor  $S_\infty$  on  $\bar{X}$  being ample, the variety  $X = \bar{X} \setminus S_\infty$  is affine.

**3.2.** For instance, the affine cone over  $\mathbb{P}^2$  in  $\mathbb{A}^3$  coincides with  $\mathbb{A}^3$  and so admits a transitive action of the additive group  $\mathbb{C}_+^3$ . In the following example we exhibit an effective  $\mathbb{C}_+^2$ -action on the affine cone  $X \subseteq \mathbb{A}^4$  over a smooth quadric  $Y \subseteq \mathbb{P}^3$  (cf. another constructions in [Sh]). The automorphism groups of affine quadrics were studied e.g., in [DG, Doe, To]. Over a general base field, this group is infinite dimensional as soon as the corresponding quadratic form is isotropic [To, Lemma 1.1]. The proof of Lemma 1.1 in [To] provides a nontrivial linear  $\mathbb{C}_+$ -action on any quadric over  $\mathbb{C}$ . In the following example we exhibit an explicit effective  $\mathbb{C}_+^2$ -action on the affine cone over a smooth quadric in  $\mathbb{P}^3$ .

**Example 3.3.** All smooth quadrics in  $\mathbb{P}^3$  are projectively equivalent. Choosing for instance the quadric

$$Y = \{xy = zu\}$$

we can define a linear  $\mathbb{C}_+^2$ -action on  $X = \text{AffCone}(Y) \subseteq \mathbb{A}^4$  by the following pair of commuting locally nilpotent derivations on the ring  $A = \mathcal{O}(X)$ :

$$\partial_1 = u\partial/\partial x + y\partial/\partial z \quad \text{and} \quad \partial_2 = u\partial/\partial y + x\partial/\partial z.$$

See [AS, Sh] for a more thorough treatment on the subject.

Let us introduce the following notion.



**Definition 3.4.** Let  $X$  be an affine variety. For a function  $f \in \mathcal{O}(X)$  we let

$$\mathbb{D}_+(f) = X \setminus \mathbb{V}_+(f), \quad \text{where } \mathbb{V}_+(f) := f^{-1}(0).$$

We say that  $X$  is *cylindrical* if  $X$  contains a dense principal Zariski open subset  $U = \mathbb{D}_+(f)$  isomorphic to the cylinder  $Z \times \mathbb{A}^1$  over an affine variety  $Z$ .

The following proposition generalizes Lemma 1.6 in [FZ<sub>2</sub>].

**Proposition 3.5.** *For an irreducible affine variety  $X$ , the following conditions are equivalent:*

- (i)  $X$  possesses an effective  $\mathbb{C}_+$ -action.
- (ii)  $X$  is cylindrical.

*Proof.* First we suppose that  $X$  possesses an effective  $\mathbb{C}_+$ -action  $\psi$  with the associate locally nilpotent derivation  $\partial \neq 0$ . The filtration

$$(3) \quad 0 \in \ker \partial \subsetneq \ker \partial^{(2)} \subsetneq \ker \partial^{(3)} \dots$$

being strictly increasing, we can find  $g \in \mathcal{O}(X)$  such that  $\partial^{(2)}g = 0$  but  $h := \partial g \neq 0$ . Thus  $\partial h = 0$  and so  $h \in \mathcal{O}(X)$  is  $\psi$ -invariant. Letting  $s = g/h$  and  $U = \mathbb{D}_+(h)$  the function  $s \in \mathcal{O}(U)$  gives a *slice* of  $\partial$  that is,  $\partial(s) = 1$ . Consequently, the restriction of  $s$  to any 1-dimensional orbit  $O$  of  $\psi$  in  $U$  is an affine coordinate on  $O \simeq \mathbb{A}^1$ . By the Slice Theorem ([Fr, Cor. 1.22]),  $\mathcal{O}(U) \simeq \ker(\partial)[s]$  and  $\partial = \partial/\partial s$ . Therefore  $U \simeq Z \times \mathbb{A}^1$ , where  $Z = \text{Spec}(\ker \partial) \simeq s^{-1}(0)$ . This yields (ii).

To show the converse, assume that  $X$  is cylindrical. Let  $U = \mathbb{D}_+(f) \simeq Z \times \mathbb{A}^1$  be a principal cylinder in  $X$  as in Definition 3.4. We consider the natural  $\mathbb{C}_+$ -action  $\phi$  on  $U$  by translations along the second factor. Since  $f|_U$  does not vanish it is constant along any orbit of  $\phi$  and so  $\phi$ -invariant. Letting  $\partial$  denote the locally nilpotent derivation on  $\mathcal{O}(U)$  associated to  $\phi$ , the derivation  $\partial_n := f^n \partial \in \text{Der}(\mathcal{O}(U))$  is again locally nilpotent for any  $n \in \mathbb{N}$ . Let  $a_1, \dots, a_k$  be a system of generators of  $\mathcal{O}(X)$ , and let  $N \in \mathbb{N}$  be sufficiently large so that  $f^N \partial a_i \in \mathcal{O}(X)$  for any  $i = 1, \dots, k$ . Then  $\partial_N(a_i) \in \mathcal{O}(X)$  for any  $i = 1, \dots, k$ , hence  $\partial_N(\mathcal{O}(X)) \subseteq \mathcal{O}(X)$ . Thus the derivation  $\partial_N|_{\mathcal{O}(X)} \in \text{Der}(\mathcal{O}(X))$  is locally nilpotent and so generates an effective  $\mathbb{C}_+$ -action  $\psi$  on  $X$ . Therefore (i) holds.  $\square$

**Remark 3.6.** Clearly the  $\mathbb{C}_+$ -actions  $\phi$  and  $\psi|_U$  as in the proof have the same orbits, and  $\mathbb{V}_+(f) = \{f = 0\}$  consists of fixed points of  $\psi$ .

In the case of affine cones, Theorem 3.9 below gives a more practical criterion. We need the following definition.

**Definition 3.7.** For a projective variety  $Y$  with a (very ample) polarization  $\varphi_{|H|} : Y \hookrightarrow \mathbb{P}^n$ , we call an  $H$ -polar subset any Zariski open subset of the form  $U = Y \setminus \text{supp } D$ , where  $D \in |dH|$  is an effective divisor on  $\mathbb{P}^n$ .

**3.8.** Recall that an affine ruling on a variety  $U$  is a morphism  $\pi : U \rightarrow Z$  such that every scheme theoretic fiber of  $\pi$  is isomorphic to the affine line  $\mathbb{A}^1$ . By a theorem of Kambayashi and Miyanishi [KaMi] (see also [KaWr, RS, Du]), every affine ruling  $\pi : U \rightarrow Z$  on a normal variety  $U$  over a normal base  $Z$  is a locally trivial  $\mathbb{A}^1$ -bundle.

**Theorem 3.9.** *Let  $Y$  be a smooth projective variety with a very ample polarization  $\varphi_{|H|} : Y \hookrightarrow \mathbb{P}^n$ . Then the following hold.*

- (a) If the affine cone  $X = \text{AffCone}_H(Y)$  admits an effective  $\mathbb{C}_+$ -action, then  $Y$  possesses an  $H$ -polar open subset  $U$ , which is the total space of a line bundle  $U \rightarrow Z$ .
- (b) Conversely, if  $Y$  possesses an  $H$ -polar open subset  $U$  equipped with an affine ruling  $U \xrightarrow{\mathbb{A}^1} Z$  and a section  $Z \rightarrow U$ , where  $Z$  is smooth and  $\text{Pic}(Z) = 0$ , then the affine cone  $X = \text{AffCone}_H(Y)$  admits an effective  $\mathbb{C}_+$ -action.

*Proof.* (a) Let  $\psi'$  be an effective  $\mathbb{C}_+$ -action on  $X$  with associate locally nilpotent derivation  $\partial' \neq 0$ . Using the natural grading of the coordinate ring

$$A = \mathcal{O}(X) = \bigoplus_{i \geq 0} A_i,$$

$\partial'$  can be decomposed into a finite sum of homogeneous derivations  $\partial' = \sum_{i=1}^n \partial'_i$ , where the principal component  $\partial := \partial'_n \neq 0$  is again locally nilpotent. The  $\mathbb{C}_+$ -action  $\psi$  on  $X$  generated by  $\partial$  extends to an effective action of a semi-direct product  $G = \mathbb{C}_+ \rtimes \mathbb{C}^*$  on  $X$ .

The filtration (3) from the proof of Proposition 3.5 consists now of graded subrings. Hence we can find homogeneous elements  $\hat{g}, \hat{h} \in A$  such that  $\partial \hat{g} = \hat{h}$  and  $\partial \hat{h} = 0$ . In the notation of 3.4 we let

$$\hat{U} = \mathbb{D}_+(\hat{h}) \subseteq X \quad \text{and} \quad \hat{Z} = \mathbb{V}_+(\hat{g}) \setminus \mathbb{V}_+(\hat{h}) \subseteq \hat{U}.$$

Likewise in the proof of Proposition 3.5, we obtain a decomposition  $\hat{U} \simeq \hat{Z} \times \mathbb{A}^1$ .

Furthermore,  $G$  acts on  $\hat{U} \simeq \hat{Z} \times \mathbb{A}^1$  respecting the product structure. More precisely,  $\mathbb{C}_+$  acts by shifts on the second factor i.e., along the fibers of the morphism  $\hat{\pi} : \hat{U} \rightarrow \hat{Z}$ . Since  $\hat{g}, \hat{h}$ , and  $\partial$  are homogeneous,  $\mathbb{C}^*$  acts on  $\hat{U}$  stabilizing  $\hat{Z}$  and sending the fibers of  $\hat{\pi}$  into fibers. The factorization by the  $\mathbb{C}^*$ -action on  $\hat{U}$  yields a Zariski open subset  $U = \hat{U}/\mathbb{C}^* \subseteq Y$  and a divisor  $Z = \hat{Z}/\mathbb{C}^*$  on  $U$  so that

$$(4) \quad \hat{U} = \text{AffCone}(U) \setminus \{0\} \quad \text{and} \quad \hat{Z} = \text{AffCone}(Z) \setminus \{0\}.$$

The map  $\hat{\pi}$  defines an affine ruling  $\pi : U \xrightarrow{\mathbb{A}^1} Z$  with a section  $Z \hookrightarrow U$ . Each fiber of  $\pi$  is the quotient of a  $G$ -orbit in  $U$  by the  $\mathbb{C}^*$ -action. Since  $Y$  and  $Z$  are smooth, by the Kambayashi-Miyayashi Theorem cited in 3.8,  $\pi$  is an  $\mathbb{A}^1$ -bundle and, moreover, a vector bundle since it possesses a section.

Finally, since  $D := \hat{h}^*(0) \in |dH|$ , where  $d = \deg(\hat{h})$ , the open set  $U \subseteq Y$  is  $H$ -polar (see Definition 3.7). This shows (a).

To show (b), suppose that  $Y$  possesses an  $H$ -polar open subset  $U$  with an affine ruling  $\pi : U \xrightarrow{\mathbb{A}^1} Z$  and a section  $Z \rightarrow U$ , where  $Z$  is smooth and  $\text{Pic}(Z) = 0$ . Since both  $U$  and  $Z$  are smooth,  $\pi$  is locally trivial by the Kambayashi-Miyayashi Theorem. Since  $\pi$  has a section  $Z$ ,  $\pi : U \rightarrow Z$  is a line bundle and  $Z$  is the zero section. This bundle is trivial since  $\text{Pic}(Z) = 0$ . Thus  $U \simeq Z \times \mathbb{A}^1$ . In particular  $\text{Pic}(U) = 0$ .

Let further  $\sigma : \tilde{X} \rightarrow X$  be the blowup of the vertex  $0 \in X$ . The induced morphism  $\rho : \tilde{X} \rightarrow Y$  has a natural structure of a line bundle with the exceptional divisor  $E = \sigma^{-1}(0)$  as the zero section. Since  $\text{Pic}(U) = 0$ , the restriction  $\rho|_{\tilde{U}} : \tilde{U} \rightarrow U$  to  $\tilde{U} := \rho^{-1}(U) \subseteq \tilde{X}$  yields a trivial line bundle. Hence

$$\tilde{U} \simeq U \times \mathbb{A}^1 \simeq Z \times \mathbb{A}^1 \times \mathbb{A}^1 \quad \text{and, similarly,} \quad \tilde{Z} := \rho^{-1}(Z) \simeq Z \times \mathbb{A}^1.$$

Under this isomorphism  $E \cap \tilde{U}$  is sent to  $U \times \{0\} \simeq Z \times \mathbb{A}^1 \times \{0\}$  and  $E \cap \tilde{Z}$  to  $Z \times \{0\}$ . For  $\hat{U} = \tilde{U} \setminus E$  and  $\hat{Z} = \tilde{Z} \setminus E$  as in (4) we obtain

$$(5) \quad \hat{U} \simeq Z \times \mathbb{A}^1 \times \mathbb{C}^* \quad \text{and} \quad \hat{Z} \simeq Z \times \{0\} \times \mathbb{C}^*.$$

Thus  $\hat{U} \simeq \hat{Z} \times \mathbb{A}^1$  is a cylinder in  $X$ . Since  $U \subseteq Y$  is  $H$ -polar,  $\hat{U} \subseteq X$  is a principal Zariski open subset and so  $X$  is cylindrical. Now (b) follows by Proposition 3.5.  $\square$

**Remarks 3.10.** 1. This theorem, with the same proof, holds also for generalized cones (see 3.1). In particular, we may assume that  $H$  is just an ample divisor.

2. It is easily seen that if a cone  $X = \text{AffCone}_H(Y)$  admits an effective  $\mathbb{C}_+$ -action, then also the cone  $X_k = \text{AffCone}_{kH}(Y)$  admits such an action for any  $k \geq 1$ . Moreover, this cone  $X_k$  is normal for  $k \gg 1$ , see [Ha, Ch. II, Ex5.14].

**Remark 3.11.** The construction of a  $\mathbb{C}_+$ -action on  $X$  as in the proof of (b) can be made more explicite. The product  $G = \mathbb{C}_+ \times \mathbb{C}^*$  acts on  $\hat{U}$  preserving the product structure in (5):

$$G \ni (a, \lambda) : \hat{U} \rightarrow \hat{U}, \quad (z, x, y) \mapsto (z, x + a, \lambda y).$$

The generators  $\partial/\partial x$  and  $y\partial/\partial y$  of the  $\mathbb{C}_+$ - and  $\mathbb{C}^*$ -actions commute. Letting  $D = X \setminus \hat{U}$ , there is a regular function  $f \in \mathcal{O}(X)$  such that  $\text{div}(f) = nD$ . Moreover, we can choose  $f$  of the form  $f = y^k g(z)$ , where  $g \neq 0$ . For  $N \gg 1$  the  $\mathbb{C}_+$ -action generated by  $\partial = f^N \partial/\partial x$  extends to the cone  $X$ , see the proof of Proposition 3.5. With this new  $\mathbb{C}_+$ -action, a semidirect product  $\mathbb{C}_+ \rtimes \mathbb{C}^*$  acts effectively on  $X$ . However, the factors do not commute any more.

Theorem 3.9 yields the following criterion of existence of a  $\mathbb{C}_+$ -action on certain 3-dimensional affine cones.

**Corollary 3.12.** *Let  $Y$  be a rational smooth projective surface with a polarization  $\varphi_{|H|} : Y \hookrightarrow \mathbb{P}^n$ , and let  $X = \text{AffCone}_H(Y) \subseteq \mathbb{A}^{n+1}$  be the affine cone over  $Y$ . Then  $X$  admits a nontrivial  $\mathbb{C}_+$ -action if and only if  $Y$  possesses an  $H$ -polar cylinder  $U \simeq Z \times \mathbb{A}^1$ , where  $Z$  is a smooth affine curve.*

*Proof.* Since  $Y$  is smooth and rational,  $Z$  as in Theorem 3.9 is a non-complete smooth rational curve. Thus  $\text{Pic}(Z) = 0$ . Hence the affine rulings from Theorem 3.9 and its proof are actually direct products. Our assertion can be easily deduced now from Theorem 3.9.  $\square$

Using this criterion, we show next that for an arbitrary smooth rational surface  $Y$ , some affine cone over  $Y$  admits a nontrivial  $\mathbb{C}_+$ -action.

**Proposition 3.13.** *Let  $Y$  be a rational smooth projective surface. Then there is an embedding  $\varphi : Y \hookrightarrow \mathbb{P}^n$  such that the affine cone  $X = \text{AffCone}(\varphi(Y))$  is normal and admits an effective  $\mathbb{C}_+$ -action.*

*Proof.* Any point  $Q \in Y$  possesses an affine neighborhood  $U \simeq \mathbb{A}^2$ . An argument from [Fu, (2.5)] shows that  $Y \setminus U$  supports an ample divisor. Indeed,  $\text{Pic}(Y)$  is a free abelian group generated by the components  $\Delta_i$  of the divisor  $Y \setminus U$ . Hence  $\Delta_j^2 > 0$  for some  $j$ .

Choose a nef and big effective divisor  $D = \sum \delta_i \Delta_i$  such that  $D \cdot \Delta_i > 0$  whenever  $\delta_i > 0$ , with a maximal possible value of  $\lambda(D) := \text{card}\{i \mid \delta_i > 0\}$ . Assume on the contrary that  $\text{supp}(D) \neq \text{supp}(\sum \Delta_i)$ , i.e.,  $\delta_i = 0$  for some  $i$ . Since  $\text{supp}(\sum \Delta_i)$  is

connected, there is a component  $\Delta_k \not\subseteq \text{supp}(D)$  with  $D \cdot \Delta_k > 0$ . Then for  $t \gg 0$  the divisor  $tD + \Delta_k$  is again nef and big. This contradicts our maximality assumption for  $\lambda(D)$ . Therefore  $\text{supp}(D) = \text{supp}(\sum \Delta_i)$  is ample. So for  $m \gg 1$  the linear system  $|mD|$  gives an embedding  $Y \hookrightarrow \mathbb{P}^n$  with a projectively normal image, see Exercise 5.14 in [Ha, Ch. II]. Since  $Y$  admits an  $|mD|$ -polar cylinder,  $X$  is normal and cylindrical. By Corollary 3.12,  $X$  admits an effective  $\mathbb{C}_+$ -action, as required.  $\square$

The following question arises.

**3.14. Question.** *Does there exist a polarized smooth rational surface  $(Y, H)$  without any  $H$ -polar cylinder?*

**Remark 3.15.** If  $U$  is an  $H$ -polar cylinder on  $Y$  then it is also  $kH$ -polar for any  $k \in \mathbb{N}$ , and vice versa. Thus the existence of an  $H$ -polar cylinder depends only on the ray of  $H$  in the ample cone of  $Y$ . Moreover, since the irreducible components of the divisor  $D = Y \setminus U$  span the Picard group  $\text{Pic}(Y)$  and the ample cone is open, the property of a cylinder  $U$  to be  $H$ -polar is stable under small perturbation of  $H$ .

For any smooth rational projective surface, the  $H$ -polar cylinder from Proposition 3.13 can be chosen to be isomorphic to the affine plane. Let us provide similar examples in higher dimensions.

**Example 3.16.** Consider a flag variety  $G/P$  with an ample polarization  $H$  (see §1.3). By Corollary 2.13 the  $G$ -action on  $G/P$  lifts to a  $\tilde{G}$ -action on the cone  $\text{AffCone}_H(G/P)$ , where  $\tilde{G}$  is the universal cover of  $G$ . The actions of one-parameter unipotent subgroups of  $\tilde{G}$  yield effective  $\mathbb{C}_+$ -actions on the cone. Actually  $G/P$  contains an  $H$ -polar open cylinder  $U$  isomorphic to an affine space  $\mathbb{A}^n$  (cf. Theorem 3.9(a)).

Indeed, let  $B_+ \subseteq P$  be a Borel subgroup of  $G$ , and let  $B_-$  be the opposite Borel subgroup so that  $B_+ \cap B_-$  is a Cartan subgroup. Then  $B_- \cdot P$  is open in  $G$  and so the  $B_-$ -orbit  $U$  of  $e \cdot P$  is open in  $G/P$ . Thus  $U$  is a big Schubert cell. Since  $U$  is also an orbit of the maximal unipotent subgroup  $B_u \subseteq B_-$ , it is isomorphic to  $\mathbb{A}^n$ . In particular,  $U$  is a cylinder in  $Y$ . Letting  $D = Y \setminus U = \bigcup_i D_i$ , the Schubert divisors  $D_i$  form a basis in  $\text{Pic}(G/P)$ . In this basis  $H = \sum_i \alpha_i D_i$ , where  $\alpha_i > 0$  for all  $i$  since the divisor  $H$  is ample, see [Sn] or [Te, Theorem 7.53]. Hence  $U$  is an  $H$ -polar cylinder in  $G/P$ .

We note that the action of  $\tilde{G}$  on the affine cone  $\tilde{X} := \text{AffCone}_H(G/P)$  is transitive off the vertex  $0 \in \tilde{X}$ . Indeed, we may suppose that  $X = \tilde{G}/\tilde{P}$ , where  $\tilde{G}$  is semisimple, simply connected, and  $\tilde{P} \subseteq \tilde{G}$  is parabolic. Since  $\tilde{X}$  is affine, the stabiliser  $\text{Stab}_{\tilde{G}}(x)$  of a point  $x \in \tilde{X} \setminus \{0\}$  cannot contain a parabolic subgroup. Hence the stabilizer  $\text{Stab}_{\tilde{G}}([x])$  (conjugate to  $\tilde{P}$ ) acts non-trivially on the generator of the cone through  $x$ . By Corollary 1.5 in [Po<sub>4</sub>] the Makar-Limanov invariant  $ML(\tilde{X})$  is trivial (cf. Theorem 3.26; see Section 3.3 below for the definition of the Makar-Limanov invariant).

**Remark 3.17.** The existence of a cylinder in a projective variety isomorphic to an affine space is rather exceptional. For instance, none of smooth rational cubic 4-folds in  $\mathbb{P}^5$ , and none of smooth 3-fold intersections of a pair of quadrics in  $\mathbb{P}^5$  contains a Zariski open set isomorphic to an affine space, see [PS] and [Pr<sub>2</sub>]. At the same time, every smooth intersection  $Y$  of a pair of quadrics in  $\mathbb{P}^5$  contains a  $(-K_Y)$ -polar cylinder, see Proposition 5.1 below and its proof.

**3.2.  $\mathbb{C}_+$ -actions on affine cones over del Pezzo surfaces.** Let us explain the reason why are we interested in the affine cones over del Pezzo surfaces.

**3.18.** A normal variety  $X$  is  $\mathbb{Q}$ -Gorenstein if some multiple  $nK_X$  of the canonical Weil divisor  $K_X$  is Cartier. This notion is important in the Mori minimal model program (MMP). It is easily seen that the generalized cone  $X = \text{AffCone}_H(Y)$  over a smooth polarized variety  $(Y, H)$  is  $\mathbb{Q}$ -Gorenstein if and only if  $aH \sim -bK_Y$  for some  $a \in \mathbb{N}$ ,  $b \in \mathbb{Z}$  ([Kol<sub>1</sub>, Example 3.8]). (If, moreover,  $H \sim -K_Y$ , then  $X$  is Gorenstein and has at most canonical singularity at the origin.)

On the other hand, if  $\text{Aut}(X) \neq \text{Lin}(X)$  then by Corollary 2.4  $Y$  is birationally ruled, hence the Kodaira dimension of  $Y$  is negative, see [Kol<sub>2</sub>]. Thus  $b > 0$ , i.e.,  $-K_Y$  is ample. Consequently,  $Y$  is a Fano variety.

Therefore, if the affine cone  $X$  over  $(Y, H)$  is  $\mathbb{Q}$ -Gorenstein and admits an effective non-linear  $\mathbb{C}_+$ -action, then  $Y$  is a Fano variety and  $H \in \mathbb{Q}_{>0}[-K_Y]$ . In particular, if  $\dim(Y) = 2$  then  $Y$  is a del Pezzo surface with its pluri-anticanonical embedding.<sup>9</sup>

From now on we assume that  $Y$  is a del Pezzo surface of degree  $d \geq 3$  and  $H = -K_Y$  is the anti-canonical polarization. Thus the linear system  $|-K_Y|$  is very ample and provides an embedding  $Y \hookrightarrow \mathbb{P}^d$  onto a projectively normal smooth surface of degree  $d$ , see e.g., [Dol<sub>1</sub>]. The affine cone  $X = \text{AffCone}_{-K_Y}(Y)$  has a normal, canonical, Gorenstein (hence also Cohen-Macaulay) singularity at the vertex.

The following theorem is the main result of this subsection (see Theorem 0.1 in the Introduction).

**Theorem 3.19.** *Let  $Y_d$  be a smooth del Pezzo surface of degree  $d$  anticanonically embedded into  $\mathbb{P}^d$ , where  $4 \leq d \leq 9$ , and let  $X_d \subseteq \mathbb{A}^{d+1}$  be the affine cone over  $Y_d$ . Then  $X_d$  admits a nontrivial  $\mathbb{C}_+$ -action.*

*Proof.* Consider a pencil  $\mathcal{L}_{\mathbb{P}^2} = \langle C_1, C_2 \rangle$  on  $\mathbb{P}^2$  generated by a smooth conic  $C_1$  and a double line  $C_2 = 2l$ , where  $l$  is tangent to  $C_1$  at a point  $P_0 \in C_1$ . Then  $L \setminus \{P_0\} \simeq \mathbb{A}^1$ , where  $L$  is a general member of  $\mathcal{L}_{\mathbb{P}^2}$ . Moreover,  $U = \mathbb{P}^2 \setminus (C_1 \cup C_2)$  is a cylinder over  $\mathbb{A}_*^1$ . Blowing up at  $9 - d$  distinct points  $Q_i$  on  $C_1 \setminus \{P_0\}$ , where  $9 \geq d \geq 4$ , we obtain a del Pezzo surface  $Y$  of degree  $d$  with a contraction  $\sigma : Y \rightarrow \mathbb{P}^2$ , and any such surface can be obtained in this way, except for  $\mathbb{P}^1 \times \mathbb{P}^1$ .

The cylinder  $U' = \sigma^{-1}(U) \simeq U$  is  $(-K_Y)$ -polar (see Definition 3.7). Indeed, let  $E_i = \sigma^{-1}(Q_i)$ ,  $i = 1, \dots, 9 - d$ . For any  $1 \gg \varepsilon > 0$  we have

$$-K_{\mathbb{P}^2} \equiv (1 + \varepsilon)C_1 + (1 - 2\varepsilon)l.$$

Hence,

$$-K_Y = \sigma^*(-K_{\mathbb{P}^2}) - \sum_{i=1}^{9-d} E_i \equiv (1 + \varepsilon)\Delta_1 + (1 - 2\varepsilon)\Delta_2 + \varepsilon \sum_{i=1}^{9-d} E_i,$$

where  $\Delta_1$  and  $\Delta_2$  are the proper transforms in  $Y$  of  $C_1$  and  $l$ , respectively. Thus  $U'$  is a  $(-K_Y)$ -polar cylinder on  $Y$ .

In the remaining case where  $Y = \mathbb{P}^1 \times \mathbb{P}^1$ , the natural embedding  $\mathbb{A}^2 = \mathbb{A}^1 \times \mathbb{A}^1$  into  $Y$  yields a  $(-K_Y)$ -polar cylinder on  $Y$ . Applying now Corollary 3.12 ends the proof.  $\square$

<sup>9</sup>Cf. Remark 2.9(3).

The proof exploits a  $(-K_{\mathbb{P}^2})$ -polar cylinder on  $\mathbb{P}^2$  made of a pencil of conics with a common tangent line. Based on the same idea, we give below some alternative constructions of polar cylinders on anticanonically polarized del Pezzo surfaces of degrees  $\geq 4$ . Due to Corollary 3.12, this leads to new  $\mathbb{C}_+$ -actions on the cones over del Pezzo surfaces of degree  $\geq 4$  under their anticanonical embeddings. These examples will be useful in the sequel.

**Example 3.20.** Consider a pencil of rational curves on  $\mathbb{P}^2$  with a unique base point  $P$ . (Similarly, one can find such a pencil on the quadric  $\mathbb{P}^1 \times \mathbb{P}^1$ .) Then the complement of the union of its degenerate members (or of a general one, if all members are non-degenerate) is a  $(-K_{\mathbb{P}^2})$ -polar cylinder on  $\mathbb{P}^2$ . In [MiSu] an example was proposed of such a pencil of quintic curves. Moreover, there is a smooth conic  $C_1$  and a rational unicuspidal quintic  $C_2$  from the pencil as in [MiSu] that meet in one point, the cuspidal point of the quintic.

These two curves generate a pencil  $\mathcal{L}_{\mathbb{P}^2} = \langle 5C_1, 2C_2 \rangle$  of rational curves of degree 10 with a unique base point such that  $\mathbb{P}^2 \setminus (C_1 \cup C_2)$  is a cylinder. Similarly as in the proof above, every del Pezzo surface  $Y$  of degree  $d \geq 4$  can be obtained, along with a  $(-K_Y)$ -polar cylinder, by blowing up a certain set of  $9 - d$  points on  $C_1$ . Indeed, we can write

$$-K_{\mathbb{P}^2} \equiv \left(\frac{3}{2} - \varepsilon\right)C_1 + \frac{2}{5}\varepsilon C_2$$

with an appropriate  $\varepsilon > 0$ , and then proceed in the same fashion as in the proof.

**Example 3.21.** Picking up four points  $P_1, \dots, P_4$  in  $\mathbb{P}^2$  in general position, we consider the pencil of lines  $\mathcal{L}_{\mathbb{P}^2}$  on  $\mathbb{P}^2$  generated by  $l_1 = (P_1P_2)$  and  $l_2 = (P_3P_4)$ . The blowup  $\sigma : Y \rightarrow \mathbb{P}^2$  of these points yields a del Pezzo surface  $Y$  of degree 5. We have

$$-K_{\mathbb{P}^2} \equiv \frac{3}{2}l_1 + \frac{3}{2}l_2 \quad \text{and so} \quad -K_Y \equiv \frac{3}{2}l'_1 + \frac{3}{2}l'_2 + \frac{1}{2} \sum_{i=1}^4 E_i,$$

where  $l'_i$  is the proper transform of  $l_i$ ,  $i = 1, 2$ , and  $E_i$  is the exceptional  $(-1)$ -curve over  $P_i$ ,  $i = 1, \dots, 4$ . Then  $L^{(1)} = l'_1 + E_1 + E_2$  and  $L^{(2)} = l'_2 + E_3 + E_4$  are the only degenerate fibers of the pencil  $\mathcal{L} = \sigma_*^{-1}(\mathcal{L}_{\mathbb{P}^2})$  on  $Y$ . Since

$$D := \frac{1}{2}(L^{(1)} + L^{(2)}) + (l'_1 + l'_2) \equiv -K_Y,$$

the open set

$$Y \setminus \text{supp}(D) = \mathbb{P}^2 \setminus (l_1 \cup l_2) \simeq \mathbb{A}_*^1 \times \mathbb{A}^1$$

is a  $(-K_Y)$ -polar cylinder on  $Y$ . A similar construction can be applied to any del Pezzo surface of degree  $d \geq 5$ .

**Example 3.22.** Consider the pencil  $\mathcal{L}_{\mathbb{P}^2}$  of unicuspidal rational curves  $\alpha y z^{n-1} + \beta x^n = 0$  in  $\mathbb{P}^2$ , where  $n \geq 1$ . Blowing up  $k \leq 4$  points in  $\mathbb{P}^2$ , at most two on each of the lines  $x = 0$  and  $y = 0$  off their common point  $(0 : 0 : 1)$  we obtain examples of  $(-K_Y)$ -polar cylinders on an arbitrary del Pezzo surface of degree  $d \geq 5$ . For  $n = 1$  and  $k = 4$  this gives again the cylinder from Example 3.21.

**Remark 3.23.** The idea to start with a  $(-K_{\mathbb{P}^2})$ -polar cylinder on  $\mathbb{P}^2$  cannot be carried out any more in case of a smooth cubic surface  $Y \subseteq \mathbb{P}^3$ . Indeed, suppose we are given a cylinder  $\mathbb{P}^2 \setminus C \simeq Z \times \mathbb{A}^1$ , where  $C$  is a reduced plane curve of degree  $d$ , not necessarily smooth or irreducible, and let  $\mathcal{L}_{\mathbb{P}^2}$  be the corresponding pencil. Then  $\text{Bs}(\mathcal{L}_{\mathbb{P}^2})$  consists

of one point  $P_0 \in C$ , and  $C \setminus \{P_0\}$  is a disjoint union of components isomorphic to  $\mathbb{A}^1$ . Performing a blowup  $\sigma : Y \rightarrow \mathbb{P}^2$  of  $m$  points  $P_i \in C \setminus \{P_0\}$  with exceptional curves  $E_j = \sigma^{-1}(p_j)$ ,  $i = 1, \dots, m$ , from the equalities

$$3\sigma^*(C) \sim d\sigma^*(-K_{\mathbb{P}^2}) = -dK_Y + d \sum_{j=1}^m E_j \quad \text{and} \quad 3\sigma^*(C) = 3C' + 3 \sum_{j=1}^m E_j,$$

where  $C'$  is the proper transform of  $C$  on  $Y$ , we obtain

$$-dK_Y \sim 3C' + (3-d) \sum_{j=1}^m E_j =: D.$$

Here  $D$  is an effective divisor with  $\text{supp}(D) = C' + \sum_{j=1}^m E_j$  if and only if  $d \leq 2$  i.e.,  $C$  is a line or a conic. Since the centers of blowup  $P_i$ ,  $i = 1, \dots, m$ , are situated on  $C$  and  $Y$  must be del Pezzo, we have  $m \leq 5$  and so  $\deg(Y) \geq 4$ .

In the next example, starting with a pencil on  $\mathbb{P}^2$  with five base points, we construct a  $(-K_Y)$ -polar cylinder of different type on arbitrary del Pezzo surface  $Y$  of degree  $d = 5$ , by resolving all the base points but one.

**Example 3.24.** Consider the following pencil  $\mathcal{L}_{\mathbb{P}^2}$  of rational plane sextics:

$$\alpha(y^2z - x^3)^2 + \beta(y^2 - xz)(y^4 - x^4) = 0.$$

The base locus of  $\mathcal{L}_{\mathbb{P}^2}$  consists of the points  $P_0, \dots, P_4$ , where  $P_0 = (0 : 0 : 1)$  and  $\{P_1, \dots, P_4\} = (x^2 = z^2, \quad xz = y^2)$ . Furthermore,  $\mathcal{L}_{\mathbb{P}^2}$  has no fixed component and so its general member  $L$  is irreducible. Since  $\text{mult}_{P_0}(L) = 4$  and  $\text{mult}_{P_i}(L) = 2$  for  $i = 1, \dots, 4$ , the curve  $L$  is rational. Any singular point  $P_i$  of  $L$  is resolved by one blowup, and the singularity of  $L$  at  $P_0$  is cuspidal. No three of the points  $P_1, \dots, P_4$  are collinear. Therefore the blowup  $\sigma : Y \rightarrow \mathbb{P}^2$  of the latter points yields a del Pezzo surface  $Y$  of degree 5, and any such surface arises in this way. Let  $\mathcal{L}$  be the proper transform of  $\mathcal{L}_{\mathbb{P}^2}$  on  $Y$ , and let  $P = \sigma^{-1}(P_0)$ . Then  $\mathcal{L}$  is a pencil of rational curves with a cuspidal singularity at the unique base point  $P$ , smooth and disjoint outside  $P$ .

There are exactly two degenerate members of  $\mathcal{L}_{\mathbb{P}^2}$ , namely the double cuspidal cubic  $C' = 2(y^2z - x^3)$  and the union  $C''$  of the conic  $(y^2 = xz)$  and the four lines  $(y^4 = x^4)$ . We have  $-K_{\mathbb{P}^2} \equiv D_{\mathbb{P}^2} := \frac{1}{4}C' + \frac{1}{4}C''$ . Let  $D$  be the proper transform of  $D_{\mathbb{P}^2}$  on  $Y$ . Since  $\text{mult}_{P_i}(D_{\mathbb{P}^2}) = 1$  for  $i = 1, \dots, 4$ , we have  $K_Y + D = \sigma^*(K_{\mathbb{P}^2} + D_{\mathbb{P}^2}) \equiv 0$ . Hence  $U = Y \setminus (C' \cup C'')$  is a  $(-K_Y)$ -polar cylinder on  $Y$  (see also Example 4.17 below).

### 3.3. The Makar-Limanov invariant on affine cones over del Pezzo surfaces.

**3.25.** For an algebra  $A$  over a field  $k$ , its Makar-Limanov invariant  $\text{ML}(A)$  is defined as the intersection of the kernels of all locally nilpotent derivations on  $A$ . It is trivial if  $\text{ML}(A) = k$ . Following [MiMa] we say that  $A$  is of class  $\text{ML}_i$  if the quotient field  $\text{Frac}(\text{ML}(A))$  has transcendence degree  $i$ . If  $\text{ML}(A)$  is finitely generated then  $i = \dim(Z)$ , where  $Z = \text{spec ML}(A)$ . Thus  $A \in \text{ML}_0$  whenever  $A$  has trivial Makar-Limanov invariant. For instance,  $\mathbb{A}^3 \in \text{ML}_0$  (regarded as the affine cone over  $\mathbb{P}^2$ ).

For  $A$  graded there are graded versions  $\text{ML}^{(h)}(A)$  and  $\text{ML}_i^{(h)}(A)$  of  $\text{ML}(A)$  and  $\text{ML}_i$ , respectively [FZ<sub>2</sub>], where one restricts to homogeneous locally nilpotent derivations. Clearly,  $\text{ML}(A) \subseteq \text{ML}^{(h)}(A)$ . Hence the usual ML invariant is trivial if the homogeneous is.

**Theorem 3.26.** *Let  $X$  be the affine cone over a smooth, anticanonically embedded del Pezzo surface  $Y \subseteq \mathbb{P}^d$  of degree  $d \geq 4$ . Then the homogeneous Makar-Limanov invariant  $\text{ML}^{(h)}(X)$  is trivial i.e.,  $X \in \text{ML}_0^{(h)}$ .*

*Proof.* By Theorem 1.5 and Proposition 2.15, for  $d \geq 6$  the surface  $Y$  and the cone  $X = \text{AffCone}_{-K_Y}(Y)$  are toric. Since  $X$  is not isomorphic to a product  $X' \times \mathbb{A}_*^1$ , by Lemma 4.5 in [Li] the homogeneous Makar-Limanov invariant of  $X$  is trivial.

It remains to show that  $X \in \text{ML}_0^{(h)}$  for  $d = 4, 5$ . Note that for an arbitrary graded algebra  $A = \bigoplus_i A_i$ , the graded subalgebra  $\text{ML}^{(h)}(A)$  is non-trivial if and only if there exists a non-constant homogeneous element  $h \in A_n \cap \text{ML}^{(h)}(A)$  (so  $h$  is annihilated by all homogeneous locally nilpotent derivations on  $A$ ). In the case of an affine cone  $X$ , the degree  $n = \deg(h)$  is positive. Hence  $\Gamma = \mathbb{V}(h) \in |n(-K_Y)|$  is an effective ample divisor on  $Y$ .

Let as before  $Y \subseteq \mathbb{P}^d$  be a del Pezzo surface of degree  $d \geq 4$ . Suppose on the contrary that  $\text{ML}^{(h)}(X) \neq \mathbb{C}$ , and let  $h \in A_n \cap \text{ML}^{(h)}(A)$  be nonconstant. Then the affine cone over the curve  $\text{supp}(\Gamma)$  is a divisor on  $X$  stable under any  $\mathbb{C}_+$ -action defined by a homogeneous locally nilpotent derivation on  $\mathcal{O}(X)$ . Hence for every  $(-K_Y)$ -polar cylinder  $U$  on  $Y$ , the curve  $\text{supp}(\Gamma)$  consists of components of the members of the linear pencil  $\mathcal{L}$  on  $Y$  associated with  $U$ . In particular, for all  $(-K_Y)$ -polar cylinders on  $Y$  there must be a common component of the associated linear pencils  $\mathcal{L}$ .

For  $d = 5$ , we let  $\sigma : Y \rightarrow \mathbb{P}^2$  be the blowup of four points  $P_1, \dots, P_4$  in  $\mathbb{P}^2$  with exceptional curves  $E_i = \sigma^{-1}(P_i)$ . There are exactly ten lines on  $Y$ . Besides  $E_1, \dots, E_4$  these are the proper transforms  $l_{ij}$  of the lines  $(P_i P_j)$  on  $\mathbb{P}^2$ , where  $1 \leq i < j \leq 4$ . For every pair of lines  $(l_{ij}, l_{i'j'})$  with distinct indices  $i, j, i', j'$ , the curves

$$L_1 := l_{ij} + E_i + E_j \quad \text{and} \quad L_2 := l_{i'j'} + E_{i'} + E_{j'}$$

are the only degenerate members of a cylindrical linear pencil on  $Y$  (cf. Example 3.21). The 3 such pencils have no common component except for the lines  $E_1, \dots, E_4$ .

Let us replace the lines  $E_1, \dots, E_4$  on  $Y$  by some other four disjoint lines, e.g. by  $l_{12}, l_{13}, l_{23}, E_4$ . We consider also the three associated cylindrical pencils on  $Y$  e.g., that with degenerate members

$$L'_1 := E_2 + l_{12} + l_{23} \quad \text{and} \quad L'_2 := E_4 + l_{13} + l_{24}.$$

These pencils have no common component except for the lines  $l_{12}, l_{13}, l_{23}, E_4$ . The line  $E_4$  is the only common component of all six above pencils. With yet further choice of a pencil, we can eliminate also this latter line. Thus the homogeneous Makar-Limanov invariant of  $Y$  is trivial, as stated.

Let further  $d = 4$ , and let  $\sigma_0 : Y \rightarrow \mathbb{P}^2$  be the blowup of five points  $P_1, \dots, P_5$  in general position in  $\mathbb{P}^2$ , with exceptional curves  $E_i = \sigma_0^{-1}(P_i)$ ,  $i = 1, \dots, 5$ . We let  $C$  denote the unique smooth conic through the points  $P_i$ . Given a point  $Q \in C$  different from the  $P_i$ , similarly as in the proof of Theorem 3.19 we consider the pencil of conics on  $\mathbb{P}^2$  generated by  $C$  and  $2l_Q$ , where  $l_Q$  is the tangent line to  $C$  at  $Q$ . Two different such pencils on  $\mathbb{P}^2$  have no common member except for the conic  $C$  itself. Thus  $C'$  and the lines  $E_i$ ,  $i = 1, \dots, 5$ , are the only common components of the induced cylindrical pencils on the del Pezzo surface  $Y$  of degree 4.

Consider next the contraction  $\sigma_1 : Y \rightarrow \mathbb{P}^2$  of the five disjoint lines  $C', l_{12}, l_{13}, l_{14}, l_{15}$  on  $Y$ . Then  $\sigma_1(E_1)$  is a conic in  $\mathbb{P}^2$ , which plays now the role of  $C$ . Once again, the



only common components of the induced cylindrical pencils on  $Y$  are  $E_1$  and the five disjoint lines above meeting  $E_1$ .

Likewise, for the six different contractions  $\sigma_i : Y \rightarrow \mathbb{P}^2$ ,  $i = 0, \dots, 5$ , the only common component of the induced cylindrical pencils on  $Y$  is  $C'$ . However, the ample divisor  $\Gamma$  as at the beginning of the proof cannot be supported by  $C'$ . This contradiction finishes the proof.  $\square$

**Problem.** *Describe all affine cones whose homogeneous Makar-Limanov invariant is trivial.*

#### 4. ON EXISTENCE OF $\mathbb{C}_+$ -ACTIONS ON CONES OVER CUBIC SURFACES

In this section we analyze in detail the case of a smooth cubic surface  $Y \subseteq \mathbb{P}^3$ . We do not know whether the affine cone  $X = \text{AffCone}(Y)$  carries a  $\mathbb{C}_+$ -action. However, we obtain in Proposition 4.21 and Theorem 4.23 below a detailed information on an eventual anticanonical polar cylinder in  $Y$ . This makes the criterion 3.12 of the existence of a  $\mathbb{C}_+$ -action much more concrete in our particular case. We adopt the following convention.

**4.1. Convention.** Let  $Y$  be a smooth cubic surface in  $\mathbb{P}^3$ . Suppose that the affine cone  $X = \text{AffCone}(Y) \subseteq \mathbb{A}^4$  admits an effective  $\mathbb{C}_+$ -action. Then by Corollary 3.12  $Y$  possesses a  $(-K_Y)$ -polar cylinder  $U \simeq \mathbb{A}^1 \times Z$ , where  $Z$  is an affine smooth rational curve. In other words  $Y \setminus U = \text{supp}(D)$ , where

$$(6) \quad D = \sum_{i=1}^n \delta_i \Delta_i \equiv -K_Y$$

is an effective  $\mathbb{Q}$ -divisor on  $Y$  with  $\delta_i \in \mathbb{Q}_{>0} \forall i = 1, \dots, n$ .

**4.1. The linear pencil on a cubic surface compatible with a cylinder.** Let  $\mathcal{L}$  be the pencil on  $Y$  with general member  $L_z = \overline{\text{pr}_2^{-1}(z)}$  for  $z \in Z$ . It is easily seen that  $\mathcal{L}$  has at least one degenerate member. In what follows we suppose that  $\text{supp } D$  does not contain a non-degenerate member of  $\mathcal{L}$  (otherwise, up to numerical equivalence, we replace such a member by a degenerate one). Under these assumptions, the following hold.

**Lemma 4.2.** *The support of  $D$  is connected and simply connected, and contains at least 7 irreducible components.*

*Proof.* The projection  $\text{pr}_2 : U \rightarrow Z$  extends to a rational map  $Y \dashrightarrow \mathbb{P}^1$  defined by the pencil  $\mathcal{L}$  as above. A general member  $L$  of  $\mathcal{L}$  is a rational curve smooth off a unique point  $P$ , where  $\{P\} = L \cap \text{supp}(D)$ , and  $L \setminus \{P\} = L \cap U \simeq \mathbb{A}^1$ . Thus  $L$  is unibranch at  $P$ . The only base point of  $\mathcal{L}$  (if exists) is contained in  $\text{supp}(D)$ .

Since  $D$  is ample, by the Lefschetz Hyperplane Section Theorem  $\text{supp}(D)$  is connected. Resolving, if necessary, the base point of  $\mathcal{L}$  by a modification  $p : W \rightarrow Y$  yields a rational surface  $W$  with a pencil of rational curves  $\mathcal{L}_W = p_*^{-1}\mathcal{L}$  that are fibers of  $q = q|_{\mathcal{L}_W} : W \rightarrow \mathbb{P}^1$ . By Zariski's Main Theorem, the total transform  $p^{-1}(\text{supp } D)$  is still connected, and is a union of a  $(-1)$ -section, say,  $S$  of  $\mathcal{L}_W$  and of some number of rational trees contained in fibers of  $\mathcal{L}_W$ . Hence  $p^{-1}(\text{supp } D)$  is also a tree of rational curves i.e., is connected and simply connected. The exceptional divisor  $E$  of  $p$  being a

subtree of  $p^{-1}(\text{supp } D)$ , the contraction of  $E$  does not affect the simply-connectedness. This proves the first assertion.

Since  $Z$  is a rational smooth affine curve, we have  $\text{Pic}(U) = \text{Pic}(\mathbb{A}^1) \times \text{Pic}(Z) = 0$ . By virtue of the exact sequence

$$(7) \quad G := \sum_{i=1}^n \mathbb{Z}\Delta_i \rightarrow \text{Pic}(Y) \rightarrow \text{Pic}(U) = 0$$

the free abelian group  $G$  generated by the components  $\Delta_i$  of  $D$  surjects onto  $\text{Pic}(Y) \simeq \mathbb{Z}^7$ . Therefore  $\text{rk}(G) \geq 7$ , which proves the second assertion.  $\square$

**Lemma 4.3.** *The pencil  $\mathcal{L}$  has a unique base point, say,  $P$ , and  $\deg(\mathcal{L}) \geq 3$ .*

*Proof.* If on the contrary  $\text{Bs}(\mathcal{L}) = \emptyset$ , then the pencil of conics  $\mathcal{L}$  on  $Y$  with a section, say,  $S = \Delta_0$  defines a morphism  $\varphi_{|C|} : Y \rightarrow \mathbb{P}^1$  (extending the projection  $\text{pr}_2$  of the cylinder) with exactly five degenerate fibers  $L_1, \dots, L_5$ . Each degenerate fiber consists of two lines on  $Y$  intersecting transversally at one point. At most one of these two lines, say,  $l_i$  meets the cylinder  $U$ , while the other one, say,  $\Delta_i$  is a components of  $D$ . Since  $D$  is connected we have  $\Delta_i \cdot S = 1$  and  $l_i \cdot S = 0$ ,  $i = 1, \dots, 5$ . By the Adjunction Formula we get

$$1 = (-K_Y) \cdot l_i = D \cdot l_i = \delta_i - x,$$

where  $x = 0$  if  $l_i \neq \Delta_j \forall j$  and  $x = \delta_j > 0$  otherwise. Hence  $\delta_i = 1 + x \geq 1$ . Similarly, for a general fiber  $L$  of  $\mathcal{L}$ ,

$$2 = (-K_Y) \cdot L = D \cdot L = \delta_0 S \cdot L = \delta_0$$

and so  $\delta_0 = 2$ .

On the other hand, by the Adjunction Formula

$$-K_Y \cdot S = 2 + S^2 = D \cdot S = 2S^2 + \sum_{i=1}^5 \delta_i.$$

Therefore,

$$2 = S^2 + \sum_{i=1}^5 \delta_i = S^2 + \sum_{i=1}^5 \delta_i \geq -1 + 5 = 4,$$

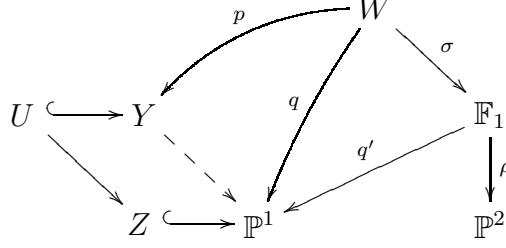
a contradiction. The inequality  $(-K_Y) \cdot \mathcal{L} \geq 3$  is now immediate.  $\square$

**Remarks 4.4.** 1. Actually the degree of  $\mathcal{L}$  must be essentially higher, since by Lemma 4.9 below  $D$  has 8 irreducible components.

2. The assertion of the lemma holds also for any del Pezzo surface of degree 4 or 5, with a similar proof. However, it fails for degree 6. Indeed, pick 3 points  $P_0, P_1, P_2$  in general position in  $\mathbb{P}^2$ , and consider the pencil generated by the lines  $l_i = (P_0 P_i)$ ,  $i = 1, 2$ . Blowing up these points we get a del Pezzo surface  $Y$  of degree 6 with a base point free pencil. Then the complement in  $Y$  of the total transform of  $l_1 \cup l_2$  is a  $(-K_Y)$ -polar cylinder.

**4.5.** In the sequel we frequently use the following commutative diagram:

(8)



where  $p : W \rightarrow Y$  is the minimal resolution of the base point  $P$  of  $\mathcal{L}$ ,  $q : W \rightarrow \mathbb{P}^1$  is the induced pencil,  $\sigma : W \rightarrow \mathbb{F}_1$  is composed of the contraction of all components of degenerate fibers of  $q$  except those which meet the exceptional  $(-1)$ -section  $S_W$  of  $q$ , and  $\rho : \mathbb{F}_1 \rightarrow \mathbb{P}^2$  is the contraction of this exceptional section.

**Lemma 4.6.**  $\delta_i < 1$  in (6) for all  $i = 1, \dots, n$ .

*Proof.* It is enough to show that  $\delta_1 < 1$ . By symmetry then  $\delta_i < 1 \forall i$ . Since the anticanonical divisor of  $Y$  is ample, we have  $(-K_Y) \cdot \Delta_i > 0 \forall i = 1, \dots, n$ . We distinguish between the following 3 cases :

- (i)  $(-K_Y) \cdot \Delta_1 \geq 3$ ,
- (ii)  $(-K_Y) \cdot \Delta_1 = 2$ , and
- (iii)  $(-K_Y) \cdot \Delta_1 = 1$ .

In case (i) suppose on the contrary that  $\delta_1 \geq 1$ . Since  $n \geq 7$  by Lemma 4.2 and the divisor  $-K_Y \equiv D$  is ample, we obtain

$$3 = (-K_Y) \cdot D = \sum_{i=1}^n \delta_i (-K_Y) \cdot \Delta_i > \delta_1 (-K_Y) \cdot \Delta_1 \geq 3,$$

which gives a contradiction.

In case (ii)  $\Delta_1 \subseteq Y$  is a conic,  $\Delta_1^2 = 0$ , and  $-K_Y \equiv \Delta_1 + E$ , where  $E$  is the residual line cut out on  $Y$  by a plane in  $\mathbb{P}^3$  through  $\Delta_1$ .

Let  $E = \Delta_i$  for some  $i > 1$ ; we may assume that  $i = 2$ . Then  $-K_Y \equiv \Delta_1 + \Delta_2$ ,  $\Delta_1^2 = 0$  and  $\Delta_1 \cdot \Delta_2 = 2$ . Thus

$$(9) \quad 2 = (-K_Y) \cdot \Delta_1 = D \cdot \Delta_1 \geq \delta_2 \Delta_1 \cdot \Delta_2 = 2\delta_2,$$

so  $\delta_2 \leq 1$ . Furthermore,

$$1 = \Delta_2 \cdot D = \sum_{i=1}^n \delta_i \Delta_2 \cdot \Delta_i \geq 2\delta_1 - \delta_2.$$

Hence  $\delta_1 \leq \frac{1}{2}(\delta_2 + 1) \leq 1$ .

If  $\delta_1 = 1$  then also  $\delta_2 = 1$ . Since  $\Delta_1 + \Delta_2 \equiv -K_Y \equiv D$  we get  $D = \Delta_1 + \Delta_2$ . Thus  $n = 2$ , which contradicts Lemma 4.2. Therefore in this case  $\delta_1 < 1$ , as stated.

If further  $E \neq \Delta_i \forall i$  then

$$(10) \quad 1 = (-K_Y) \cdot E = D \cdot E \geq \delta_1 \Delta_1 \cdot E = 2\delta_1,$$

hence  $\delta_1 \leq 1/2$ . Thus anyway,  $\delta_1 < 1$  in case (ii).

In case (iii)  $\Delta_1$  is a line on  $Y$ . Let  $C$  be a residual conic of  $\Delta_1$ , so that  $\Delta_1 + C \equiv -K_Y$  is a hyperplane section. We have as before

$$2 = (-K_Y) \cdot C = D \cdot C \geq \delta_1 \Delta_1 \cdot C = 2\delta_1,$$

hence  $\delta_1 \leq 1$ .

If  $\delta_1 = 1$  then  $D \cdot C = \Delta_1 \cdot C = 2$  and so  $(D - \Delta_1) \cdot C = 0$ . Therefore the divisor  $\text{supp}(D - \Delta_1)$  is supported on the members of the pencil of conics  $|C|$  on  $Y$ . The curve  $\Delta_1$  meets each fiber twice, and so the morphism  $\varphi_{|C|}$  restricted to  $\Delta_1$  has 2 branch points.

By Lemma 4.2 the curve  $\text{supp } D$  is simply connected, hence it cannot contain the whole fiber of  $\varphi_{|C|}$  which meets the component  $\Delta_1$  of  $D$  at two distinct points. We claim however that if a degenerate fiber  $l_1 + l_2$  of the pencil  $|C|$  contains a component, say,  $\Delta_i = l_1$  of  $D$ , then its second component  $l_2 = \Delta_j$  is also contained in  $\text{supp } D$  and, moreover,  $\delta_i = \delta_j$ . Indeed, since  $\delta_1 = 1$  and  $\Delta_1$  is a line on  $Y$  we have

$$(11) \quad 1 = \Delta_i \cdot D = \Delta_i \cdot \Delta_1 + \Delta_i \cdot (D - \Delta_1) = 1 - \delta_i + \sum_{k \neq 1, i} \delta_k \Delta_i \cdot \Delta_k.$$

The only component of the latter sum that meets  $\Delta_i$  can be the line  $l_2$ . Hence  $l_2 = \Delta_j$  for some  $j \neq 1, i$ . Now (11) shows that  $\delta_i = \delta_j$ , as claimed.

Furthermore, since  $\Delta_i \cup \Delta_j$  meets  $\Delta_1$  twice and  $\text{supp } D$  is a tree, the line  $\Delta_1$  passes through the intersection point  $\Delta_i \cap \Delta_j$ . On the other hand,  $\Delta_1$  is tangent to exactly two members of the pencil  $|C|$ , which are either smooth or consist of two lines  $\Delta_i$  and  $\Delta_j$  meeting  $\Delta_1$  at their common point (an Eckardt point of  $Y$ ). By the simply connectedness of  $\text{supp } D$ , none of the other components of members of  $|C|$  can be contained in  $\text{supp } D$ . Hence  $\text{supp } D$  can contain at most 5 components, namely,  $\Delta_1$  and the components of two degenerate members tangent to  $\Delta_1$ . However, this contradicts Lemma 4.2, since by this lemma  $\text{supp } D$  consists of at least 7 components. Now the proof is completed.  $\square$

**Lemma 4.7.** *Every component of the degenerate members of the pencil  $\mathcal{L}$  on  $Y$  passes through the base point  $P$  of  $\mathcal{L}$ .*

*Proof.* Assume on the contrary that there is a component  $C_0$  of a degenerate member  $L^{(0)}$  of  $\mathcal{L}$  such that  $P \notin C_0$ . By Zariski's Lemma  $C_0^2 < 0$ . Hence  $C_0$  is a  $(-1)$ -curve on  $Y$  and so  $D \cdot C_0 = (-K_Y) \cdot C_0 = 1$ . By an easy argument (cf. the proof of Lemma 4.2) the curve  $\text{supp } L^{(0)}$  is connected and simply connected (this is a tree of rational curves outside  $P$ ). If there is another component  $C_1$  of  $L^{(0)}$  which meets  $C_0$  and does not pass through  $P$ , then  $L^{(0)}$  contains the configuration of two crossing lines  $C_0 + C_1$  which do not pass through  $P$ . Then  $\mathcal{L}$  must be the linear system of conics  $|C_0 + C_1|$ . By Lemma 4.3 this leads to a contradiction. Hence  $C_0$  cannot separate  $\text{supp } L^{(0)}$  and so  $C_0$  meets the complement  $\text{supp}(L^{(0)} - C_0)$  at one point transversally.

It follows that  $\Delta_i \cdot C_0 = 1$  for a unique index  $i$ . If  $C_0$  is not a component of  $D$ , then  $1 = D \cdot C_0 = \delta_i$ , which contradicts Lemma 4.6. Similarly, if  $C_0 = \Delta_j$ , then  $1 = D \cdot C_0 = \delta_i - \delta_j$ . Hence  $\delta_i = 1 + \delta_j > 1$ . Once again, this contradicts Lemma 4.6.  $\square$

**Lemma 4.8.** *The pencil  $\mathcal{L}$  has at most two degenerate members.*

*Proof.* Recall (see 4.5) that  $p : W \rightarrow Y$  stands for the minimal resolution of the base locus of  $\mathcal{L}$  and  $q : W \rightarrow \mathbb{P}^1$  for the fibration given by  $p_*^{-1}\mathcal{L}$ . Write  $p$  as a composition of blowups of points over  $P$ :

$$(12) \quad p : W \xrightarrow{p_1} W_1 \xrightarrow{p_2} \dots \xrightarrow{p_N} W_N = Y,$$

where the exceptional divisor  $S_W$  of  $p_1$  is a  $q$ -horizontal  $(-1)$ -curve on  $W$ . A general fiber  $L$  of  $q$  is a smooth rational curve meeting  $S_W$  at one point. Indeed,  $L \setminus S_W \simeq p(L) \setminus P \simeq \mathbb{A}^1$ . Therefore  $S_W$  is a section of  $q$ .

Let  $C_1, \dots, C_m$  be the components of degenerate fibers  $F_1, \dots, F_m$  of  $q$  meeting  $S_W$ . We claim that all the curves  $C_i$  are  $p$ -exceptional. Indeed, otherwise for some  $i$ , the image  $p((F_i \setminus C_i))$  would be a component of a degenerate member of  $\mathcal{L}$  which does not pass through  $P$ . The latter contradicts Lemma 4.7.

Note that, on each step, the exceptional divisor of  $p_k \circ \dots \circ p_N$  is an SNC tree of rational curves. On the other hand, all the curves  $p_1(C_i)$  on  $W_1$  pass through the point  $p_1(S_W)$ . Therefore  $m \leq 2$ .  $\square$

**Lemma 4.9.** *The pencil  $\mathcal{L}$  has exactly two degenerate members, say,  $L^{(1)}$  and  $L^{(2)}$ . Furthermore,  $\text{supp } D = \text{supp}(L^{(1)} + L^{(2)})$  consists of 8 irreducible components i.e.,  $n = 8$  in (6).*

*Proof.* Assume that the only degenerate member of  $\mathcal{L}$  is  $L^{(1)}$ . In this case,  $\text{supp } D \subseteq \text{supp } L^{(1)}$ ,  $Z \supseteq \mathbb{A}^1$ , and  $U \supseteq \mathbb{A}^2$ .

If  $Z \simeq \mathbb{A}^1$  then  $\text{Pic}(U) = 0$  and  $H^0(U, \mathcal{O}_U) = \mathbb{C}$ , hence  $\text{Pic}(Y) \simeq \sum_{i=1}^n \mathbb{Z} \cdot \Delta_i$ . Thus in this case  $n = 7$  and  $-K_Y = \sum_{i=1}^7 m_i \Delta_i$  for some  $m_i \in \mathbb{Z}$ . On the other hand,  $-K_Y \equiv D = \sum_{i=1}^7 \delta_i \Delta_i$ , where  $0 < \delta_i < 1 \ \forall i$  according to Lemma 4.6. Since the decomposition of  $-K_Y$  in  $\text{Pic}(Y) \otimes \mathbb{Q}$  is unique, this yields a contradiction.

If further  $Z \simeq \mathbb{P}^1$  then  $\text{Pic}(U) \simeq \mathbb{Z}$  and  $H^0(U, \mathcal{O}_U) = \mathbb{C}$ . From the exact sequence (7), where  $\text{Pic}(U) = 0$  is replaced by  $\text{Pic}(U) \simeq \mathbb{Z}$ , we obtain  $n = 6$ . By Lemma 4.2, this leads again to a contradiction. Therefore  $\mathcal{L}$  has indeed two degenerate members.

As for the second assertion, assuming on the contrary that  $\text{supp } D \neq \text{supp}(L^{(1)} + L^{(2)})$  we would have  $Z \supseteq \mathbb{A}^1$ . Now the same argument as before yields a contradiction. Since the Picard group  $\text{Pic}(Y) \cong \mathbb{Z}^7$  is generated by the irreducible components of  $L^{(1)} + L^{(2)}$  and  $L^{(1)} \equiv L^{(2)}$  is the only relation between these components, we obtain that  $n = 8$ .  $\square$

**Corollary 4.10.** *The pencil  $\mathcal{L}$  is ample. Furthermore, for every irreducible component  $C$  of a member of the pencil  $\mathcal{L}$  on  $Y$  we have  $C \setminus \{P\} \simeq \mathbb{A}^1$ , and two such components have just the point  $P$  in common.*

*Proof.* The first assertion follows immediately from Lemma 4.7 by the Nakai-Moishezon criterion. As for the second one, it follows from the well known fact that on an affine surface  $V = Y \setminus L$ , where  $L$  is a general fiber of  $\mathcal{L}$ , every degenerate fiber of the  $\mathbb{A}^1$ -fibration  $\varphi|_{\mathcal{L}} : V \rightarrow \mathbb{A}^1$  is a disjoint union of affine lines, see e.g., [Mi], [Za].  $\square$

**Lemma 4.11.** *The pair  $(Y, D)$  is not log canonical at  $P$ .*

*Proof.* Let  $D_W$  denote the crepant pull-back of  $D$  on  $W$  as in (8) i.e., a  $\mathbb{Q}$ -divisor on  $W$  such that

$$K_W + D_W = p^*(K_Y + D) \quad \text{and} \quad p_* D_W = D.$$

The exceptional  $(-1)$ -section  $S$  on  $W$  is the only  $q$ -horizontal component of  $D_W$ . For a general fiber  $L$  of  $q$  we have  $2 = (-K_W) \cdot L = D_W \cdot L$ . Therefore, the discrepancy  $a(S, D)$  (i.e., the coefficient of  $S$  in  $D_W$  with the opposite sign) equals  $-2$ . This proves our assertion.  $\square$

**Corollary 4.12.**  $\text{mult}_P(D) > 1$ .

*Proof.* If  $\text{mult}_P(D) \leq 1$ , then the pair  $(Y, D)$  is canonical at  $P$ , because  $P \in Y$  is a smooth point. In particular, it is log canonical. This contradicts Lemma 4.11.  $\square$

**Lemma 4.13.** *Any line  $l$  on  $Y$  through  $P$  is contained in  $\text{supp } D$ .*

*Proof.* Assuming the contrary we obtain  $1 = (-K_Y) \cdot l = D \cdot l \geq \text{mult}_P(D) > 1$ , a contradiction.  $\square$

**Lemma 4.14.**  *$P$  cannot be an Eckardt point on  $Y$ .*

*Proof.* Suppose the contrary. Then by Lemma 4.13, up to a permutation we may assume that  $\Delta_1, \Delta_2, \Delta_3$  are lines through  $P$ , where  $\delta_1 \leq \delta_2 \leq \delta_3$ . Since  $D \cdot \Delta_1 = (-K_Y) \cdot \Delta_1 = 1$ , for an effective  $\mathbb{Q}$ -divisor on  $Y$

$$(13) \quad D' := \frac{1}{1 - \delta_1}(D - \delta_1(\Delta_1 + \Delta_2 + \Delta_3))$$

we obtain  $D' \cdot \Delta_1 = 1$  and by (13)

$$D \equiv -K_Y \equiv \Delta_1 + \Delta_2 + \Delta_3 \equiv D'.$$

Now the proof of Lemma 4.11 works equally for  $D'$ . Hence the pair  $(Y, D')$  is not canonical at  $P$  and so  $\text{mult}_P(D') > 1$ . This contradicts the inequality  $\text{mult}_P(D') \leq D' \cdot \Delta_1 = 1$ .  $\square$

**Lemma 4.15.** *The fibration  $q : W \rightarrow \mathbb{P}^1$  has exactly two degenerate fibers. The general member  $L$  of  $\mathcal{L}$  is singular at  $P$ .*

*Proof.* The first assertion follows from Lemma 4.9. Let us show the second one. Assuming the contrary, for a smooth rational curve  $L$  on  $Y$  we have by adjunction  $(K_Y + L) \cdot L = -2$ . The Mori cone of  $Y$  being spanned by the  $(-1)$ -curves  $E_1, \dots, E_{27}$ , there is a decomposition

$$L \equiv \sum_{i=1}^{27} \alpha_i E_i,$$

where  $\alpha_i \in \mathbb{Q}_{\geq 0}$ . Hence  $(K_Y + L) \cdot E_i < 0$  for some  $i$ . Thus  $L \cdot E_i < (-K_Y) \cdot E_i = 1$ , and so  $L$  cannot be ample. This contradicts Corollary 4.10.  $\square$

**Remark 4.16.** The minimal resolution  $p : W \rightarrow Y$  of the base point  $P$  of  $\mathcal{L}$  dominates the embedded minimal resolution of the cusp at  $P$  of a general member  $L$  of  $\mathcal{L}$ . The exceptional divisor  $E = p^{-1}(P) \subseteq W$  is a rational comb with the number of teeth equal the length of the Puiseux sequence of the cusp. The only  $(-1)$ -curve  $S_W$  in  $E$  is sitting on the handle of the comb. Hence  $E = E^{(1)} + S_W + E^{(2)}$ , where  $E^{(k)} \subseteq p^{-1}(L^{(k)})$ ,  $k = 1, 2$ , and exactly one of the  $E^{(k)}$  is a negatively definite linear chain of rational curves.

The degenerate members  $L_W^{(1)}, L_W^{(2)}$  of the induced linear system  $\mathcal{L}_W$  on  $W$  have the following structure:  $L_W^{(k)}$  consists of  $E^{(k)}$  and the proper transforms  $\Delta'_i$  of the

components  $\Delta_i \subseteq L^{(k)}$ , called *feathers*. The feathers are disjoint in  $W$ , and each of them meets  $E^{(k)}$  at one point transversally. Let us illustrate this for the surface  $Y$  from Example 3.24, with the pullback  $\mathcal{L}_W$  of the pencil  $\mathcal{L}$  on  $Y$ .

**Example 4.17.** Up to interchanging the degenerate fibers  $L_W^{(1)}$  and  $L_W^{(2)}$  of  $\mathcal{L}_W$ , their structure is as follows. The fiber  $L_W^{(1)}$  contains a component  $E_1$  with self-intersection  $-5$  joint with the  $(-1)$ -section  $S_W$ , and also the  $(-1)$ -feathers  $\Delta'_1, \dots, \Delta'_5$  meeting  $E_1$ . The second fiber  $L_W^{(2)}$  contains the only feather  $\Delta'_0$  of multiplicity 2. The weighted dual graph of the configuration  $L_W^{(1)} + S_W + L_W^{(2)}$  is as follows:

$$(14) \quad \begin{array}{ccccccc} & (-1)_5 & (\Delta'_i)_{i=1,\dots,5} & & -1 & & \Delta'_0 \\ & \boxed{\phantom{0}} & & & \circ & & \\ & -5 & & -1 & -2 & -2 & -2 \\ & \circ & \circ & \circ & \circ & \circ & \\ & E_1 & S_W & E_2 & E_3 & E_4 & \end{array}$$

where the box denotes a disjoint union of five  $(-1)$ -feathers  $\Delta'_1, \dots, \Delta'_5$  joint with  $E_1$ . The exceptional divisor of  $p : W \rightarrow Y$  is  $E = E_1 + \dots + E_4$ . While  $\Delta_i = p(\Delta'_i)$  ( $i = 0, \dots, 5$ ) are the components of the degenerate fibers  $L^{(j)} = p(L_W^{(j)})$  ( $j = 1, 2$ ) of the pencil  $\mathcal{L}$  on  $Y$ . More precisely,  $\Delta_0$  is the proper transform in  $Y$  of the cuspidal cubic  $C'$ ,  $\Delta_1$  is that of the conic  $C''$ , while  $\Delta_2, \dots, \Delta_5 \subseteq Y$  are the proper transforms of the four lines ( $y^4 = x^4$ ) in  $\mathbb{P}^2$ .

**4.18.** Let  $Q = (\rho \circ \sigma)(S_W) \in \mathbb{P}^2$ , where as in 4.5  $S_W$  stands for a  $(-1)$ -section of  $q$  contained in  $\text{Exc}(p)$ . Then  $\mathcal{L}_{\mathbb{P}^2} := \rho_*(\sigma_*\mathcal{L}_W)$  is the linear pencil of lines through  $Q$  on  $\mathbb{P}^2$ . We let  $\varphi := \rho \circ \sigma \circ p^{-1} : Y \dashrightarrow \mathbb{P}^2$ , and we let  $\mathcal{H}$  denote the proper transform of  $\mathcal{H}_{\mathbb{P}^2} := |\mathcal{O}_{\mathbb{P}^2}(1)|$  on  $Y$  via  $\varphi$ . With this notation, the following holds.

**Lemma 4.19.**  $\mathcal{L} \subseteq \mathcal{H}$ .

*Proof.* We have

$$\mathcal{H}_W := (\rho \circ \sigma)_*^{-1} \mathcal{H}_{\mathbb{P}^2} = (\rho \circ \sigma)^* \mathcal{H}_{\mathbb{P}^2} = \sigma^*(\rho^* \mathcal{H}_{\mathbb{P}^2}) \supseteq \sigma^*(\rho^* \mathcal{L}_{\mathbb{P}^2}).$$

Indeed,  $\mathcal{H}_{\mathbb{P}^2}$  is base point free. It is clear that  $\rho^* \mathcal{L}_{\mathbb{P}^2} = \mathcal{L}_{\mathbb{F}_1} + S$ , where  $S$  is the exceptional curve of  $\rho$ . Note that the centers of subsequent blowups in  $\sigma$  (including infinitesimally near centers) lie neither on the proper transform of  $S$  nor on that of general members of  $\mathcal{L}_{\mathbb{P}^2}$ . Hence,

$$\sigma^*(\rho^* \mathcal{L}_{\mathbb{P}^2}) = \sigma^*(\mathcal{L}_{\mathbb{F}_1} + S) = \mathcal{L}_W + S_W.$$

Since  $S_W$  is  $p$ -exceptional, applying  $p_*$  yields the assertion.  $\square$

**Corollary 4.20.**  $\mathcal{L}$  cannot be contained in  $|-mK_Y|$  for any  $m \in \mathbb{N}$ .

*Proof.* Assume to the contrary that  $\mathcal{L} \subseteq |-mK_Y|$  for some  $m \in \mathbb{N}$ . Then the mobile linear system  $\mathcal{H}$ , which defines the birational map  $\varphi : Y \dashrightarrow \mathbb{P}^2$  (see 4.18), is also contained in  $|-mK_Y|$ . This contradicts the Segre-Manin theorem as stated in [KSC, Theorem 2.13].  $\square$

The results of this subsection can be summarized as follows.

**Proposition 4.21.** *Let  $Y \subseteq \mathbb{P}^3$  be a smooth cubic surface, and  $X \subseteq \mathbb{A}^4$  be the affine cone over  $Y$ . Then  $X$  admits an effective  $\mathbb{C}_+$ -action if and only if  $Y$  admits a linear pencil  $\mathcal{L}$  with the following properties:*

- (1) *The base locus  $\text{Bs}(\mathcal{L})$  consists of a single point, say,  $P$ , which is not an Eckardt point on  $Y$ .*
- (2) *A general member  $L$  of  $\mathcal{L}$  is singular at  $P$ , and  $L \setminus \{P\} \simeq \mathbb{A}^1$ .*
- (3)  *$\mathcal{L}$  has exactly two degenerate members, say  $L^{(1)}$  and  $L^{(2)}$ , where the curve  $L^{(1)} \cup L^{(2)}$  consist of 8 irreducible components  $\Delta_1, \dots, \Delta_8$ .*
- (4) *All curves  $\Delta_i$ ,  $i = 1, \dots, 8$ , pass through  $P$  and are pairwise disjoint off  $P$ . Furthermore,  $\Delta_i \setminus \{P\} \simeq \mathbb{A}^1 \forall i$ .*
- (5) *Every line on  $Y$  passing through  $P$  is one of the  $\Delta_i$ .*
- (6)  *$-K_Y \equiv D := \sum_{i=1}^8 \delta_i \Delta_i$ , where  $\delta_i \in \mathbb{Q}$  and  $0 < \delta_i < 1 \forall i$ .*
- (7) *The pair  $(Y, D)$  is not log canonical at  $P$ .*
- (8) *For every  $m > 0$ ,  $\mathcal{L}$  is not contained in  $|-mK_Y|$ .*

We do not know so far any example of a cubic surface with such a pencil  $\mathcal{L}$ . For the pencils on del Pezzo surfaces from Examples 3.20-3.24, not all of the properties (1)-(8) are fulfilled. For instance, the pencil of Example 3.24 satisfies (1)-(7), however (8) fails, since  $\mathcal{L} \sim -2K_Y$ .

## 4.2. The inverse nef value.

**4.2.2.** The nef value plays an important role in the adjunction-theoretic classification of polarized varieties. For a projective variety  $Y$  polarized by a nef divisor  $H$  we define the inverse nef value  $t_0 = t_0(Y, H)$  to be the supremum of  $t$  such that the divisor  $H + tK_Y$  is nef i.e.,

$$(15) \quad H \cdot C \geq t(-K_Y) \cdot C = t \deg(C)$$

for every curve  $C$  on  $Y$ . By the Kawamata rationality theorem [Mat, Thm. 7.1.1],  $t_0$  is achieved and is rational. By the Kawamata-Shokurov base-point-free theorem [Mat, Thm. 6.2.1], the divisor  $H + t_0K_Y$  is semiample i.e., the complete linear system  $|m(H + t_0K_Y)|$  has no base point for all  $m \gg 0$  and defines a surjective morphism  $\varphi: Y \rightarrow Y'$  with connected fibers onto a normal projective variety  $Y'$ . In particular,  $\kappa(H + t_0K_Y) \geq 0$ , where  $\kappa$  stands for the Iitaka-Kodaira dimension.

For a smooth cubic surface  $Y$  in  $\mathbb{P}^3$  satisfying Convention 4.1, we let  $\mathcal{H} = \varphi_*^{-1}(\mathcal{O}_{\mathbb{P}^2}(1))$  be the mobile linear system on  $Y$  constructed in 4.18. In this setting the inverse nef value  $t_0 = t_0(Y, \mathcal{H})$  is a positive integer (indeed, for  $t = t_0$  the equality in (15) is achieved on a  $(-1)$ -curve). Moreover,  $\kappa(\mathcal{H} + t_0K_Y) = 0$  if and only if  $\mathcal{H} + t_0K_Y \equiv 0$ . However, in the latter case by Corollary 4.19 a  $(-K_Y)$ -polar cylinder on  $Y$  cannot exist.

By virtue of Theorem 4.23 below, the same conclusion holds in the case where  $\kappa(\mathcal{H} + t_0K_Y) = 1$ . In the latter case the linear system  $|m(\mathcal{H} + t_0K_Y)|$  defines for  $m \gg 1$  a conic bundle  $Y \rightarrow \mathbb{P}^1$ . Indeed, the image curve is rational since  $Y$  is, and an irreducible general fiber  $F$  with  $F^2 = 0$  and  $-K_F = -K_Y|_F$  ample is a smooth conic. Actually  $|\mathcal{H} + t_0K_Y|$  defines already a conic bundle. For assuming that  $\mathcal{H} + t_0K_Y \equiv \beta F$ , where  $\beta \in \mathbb{Q}$ , and taking intersection with a line  $l$  on  $Y$  such that  $F \cdot l = 1$ , we obtain  $\beta \in \mathbb{N}$ .



**Theorem 4.23.** *Let  $\chi : Y \dashrightarrow \mathbb{P}^2$  be a birational map and  $\mathcal{H} = \chi_*^{-1}(|\mathcal{O}_{\mathbb{P}^2}(1)|)$  be the proper transform on  $Y$  of the complete linear system of lines on  $\mathbb{P}^2$ . Then there is no conic  $F$  on  $Y$  such that*

$$(16) \quad \mathcal{H} \sim -aK_Y + bF \quad \text{for some } a \in \mathbb{N} \text{ and } b \in \mathbb{Z}.$$

*Proof.* We use the methods developed in [Is<sub>1</sub>, Is<sub>2</sub>]. Consider a resolution of indeterminacies of  $\chi$ :

$$\begin{array}{ccc} & \tilde{Y} & \\ p \swarrow & & \searrow q \\ Y & \overset{\chi}{\dashrightarrow} & \mathbb{P}^2 \end{array}$$

Decomposing  $p$  into a sequence of blowups with exceptional curves  $E_1, \dots, E_n$ , the linear system  $\tilde{\mathcal{H}} = q^*(|\mathcal{O}_{\mathbb{P}^2}(1)|)$  on  $\tilde{Y}$  and the line bundle  $K_{\tilde{Y}}$  can be written in  $\text{Pic}(\tilde{Y})$  as

$$(17) \quad \tilde{\mathcal{H}} = p^*(\mathcal{H}) - \sum_{i=1}^n m_i E_i^* \quad \text{and} \quad K_{\tilde{Y}} = p^*(K_Y) + \sum_{i=1}^n E_i^*.$$

Computing the intersection numbers  $\tilde{\mathcal{H}}^2$  and  $\tilde{\mathcal{H}} \cdot K_{\tilde{Y}}$ , by (17) we obtain

$$(18) \quad 1 = \mathcal{H}^2 - \sum_{i=1}^n m_i^2 \quad \text{and} \quad -3 = K_Y \cdot \mathcal{H} + \sum_{i=1}^n m_i.$$

Suppose on the contrary that (16) holds for some conic  $F$  on  $Y$ . We choose the minimal possible value of  $a > 0$ . Since  $F^2 = 0$  on  $Y$ , from (16) and (18) we deduce

$$(19) \quad \sum_{i=1}^n m_i^2 = 3a^2 + 4ab - 1 \quad \text{and} \quad \sum_{i=1}^n m_i = 3a + 2b - 3.$$

In the rest of the proof we use the following Claims 1-4.

*Claim 1.* *There is a birational transformation  $Y \dashrightarrow Y'$ , where  $Y'$  is again a smooth cubic surface in  $\mathbb{P}^3$ , such that for the proper transforms  $\mathcal{H}'$  of  $\mathcal{H}$  and  $F'$  of  $F$  on  $Y'$  we have*

$$\mathcal{H}' \sim -aK_{Y'} + b'F'$$

*with the same  $a$  as in (16), and additionally with*

$$m'_i := \max_i \{m'_i\} \leq a \quad \forall i,$$

*where the integers  $m'_i$  have the same meaning on  $Y'$  as the  $m_i$  have on  $Y$ .*

*Proof of Claim 1.* Suppose that  $m_i > a$  for some value of  $i$ . Consider the conic bundle  $\varphi = \varphi|_F : Y \rightarrow \mathbb{P}^1$ . Let us perform an elementary transformation at the point  $P = p(E_i) \in Y$ . First we apply the blowup  $\sigma : \hat{Y} \rightarrow Y$  of  $P$  with exceptional divisor  $E$ . Assuming that  $m_i = \text{mult}_P(\mathcal{H})$ , on the new surface  $\hat{Y}$  we have

$$(20) \quad \hat{\mathcal{H}} := \sigma_*^{-1}(\mathcal{H}) = \sigma^*(\mathcal{H}) - m_i E \quad \text{and} \quad K_{\hat{Y}} = \sigma^*(K_Y) + E.$$

Modulo linear equivalence we may choose the conic  $F$  passing through the point  $P$ . Then  $F$  is irreducible. Indeed, otherwise  $F = F_1 + F_2$ , where  $F_1, F_2$  are two lines on  $Y$  and  $F_1$  passes through  $P$ . So

$$a < m_i \leq (F_1 \cdot \mathcal{H})_P \leq F_1 \cdot \mathcal{H} = F_1 \cdot (-aK_Y + b(F_1 + F_2)) = a,$$

which is impossible. Thus the proper transform  $\widehat{F} \sim \sigma^*(F) - E$  of  $F$  on  $\widehat{Y}$  is a  $(-1)$ -curve.

The contraction  $\sigma' : \widehat{Y} \rightarrow Y'$  of  $\widehat{F}$  to a point  $P' \in Y'$  yields a smooth conic  $F' := \sigma'_*(E)$  passing through  $P'$  on the resulting cubic surface  $Y'$ , such that

$$\mathcal{H}' := \sigma'_*(\widehat{\mathcal{H}}) \sim -aK_{Y'} + b'F' \quad \text{for some } b' \in \mathbb{Z}.$$

Using (16) and (20), on  $Y'$  we obtain

$$\text{mult}_{P'}(\mathcal{H}') = \widehat{\mathcal{H}} \cdot \widehat{F} = \sigma^*(\mathcal{H}) \cdot \widehat{F} - m_i E \cdot \widehat{F} = \mathcal{H} \cdot F - m_i = -aK_Y \cdot F - m_i = 2a - m_i < a.$$

Iterating this procedure we achieve finally that  $m'_i \leq a$  for all values of  $i$ , as required.  $\square$

So we assume in the sequel that

$$(21) \quad m = \max_i \{m_i\} \leq a \quad \forall i.$$

*Claim 2.* Under the assumption (21) we have  $b < 0$ .

*Proof of Claim 2.* From (19) and (21) we obtain

$$(22) \quad 3a^2 + 4ab = 1 + \sum_{i=1}^n m_i^2 \leq 1 + m \sum_{i=1}^n m_i \leq 1 + m(3a + 2b - 3) \leq 1 + a(3a + 2b - 3).$$

It follows by (22) that  $2ab \leq 1 - 3a$ . Since  $a \geq 1$  then  $b \leq \frac{1}{2a} - \frac{3}{2} \leq -1$ .  $\square$

*Claim 3 (the Noether-Fano Inequality).* For  $m$  as in (21) we have

$$a \geq m > a + b.$$

*Proof of Claim 3.* The first inequality follows by (21). To show the second, suppose on the contrary that

$$(23) \quad m \leq a + b.$$

From (22) and (23) we obtain

$$(24) \quad 3a^2 + 4ab \leq 1 + m(3a + 2b - 3) \leq 1 + (a + b)(3a + 2b - 3).$$

Thus by (23) and (24)

$$(25) \quad 3 \leq 3m \leq 3(a + b) \leq 1 + b(a + 2b).$$

We claim that  $a + 2b \geq 0$ . Indeed, let  $C$  be the residual line of the conic  $F$  on  $Y$  so that  $F + C \sim -K_Y$ . Then by (16),

$$0 \leq \mathcal{H} \cdot C = (-aK_Y + bF) \cdot C = a + 2b.$$

Now (25) leads to a contradiction, since  $b < 0$  by Claim 2.  $\square$

*Claim 4.* Consider the morphism  $\varphi : Y \rightarrow \mathbb{P}^1$  defined by the pencil of conics  $|F|$  on  $Y$ . Let  $m = m_i$ , and let  $Q = p(E_i) \in Y$ . If a line  $l$  on  $Y$  passes through  $Q$ , then  $l$  is a component of the fiber of  $\varphi$  through  $Q$ .

*Proof of Claim 4.* We have

$$(26) \quad (\mathcal{H} \cdot l)_Q \geq \text{mult}_Q(\mathcal{H}) = m > a + b.$$

On the other hand,

$$(27) \quad (\mathcal{H} \cdot l)_Q \leq \mathcal{H} \cdot l = (-aK_Y + bF) \cdot l = a + bF \cdot l.$$

By (26) and (27),  $b < bF \cdot l$ , where  $b < 0$  by Claim 2. Therefore  $F \cdot l = 0$ , and the claim follows.  $\square$

Let again  $C$  be the residual  $(-1)$ -curve of the conic  $F$  on  $Y$  so that  $-K_Y \sim F + C$ . We let  $V$  denote a del Pezzo surface of degree 4 obtained by the contraction  $\pi : Y \rightarrow V$  of  $C$ , and  $\mathcal{H}_V$ ,  $F_V$ , etc. denote the images on  $V$  of  $\mathcal{H}$ ,  $F$ , etc. Due to (16),

$$(28) \quad -K_V \sim F_V \quad \text{and} \quad \mathcal{H}_V \sim -aK_V + bF_V \sim -(a+b)K_V.$$

By Claim 4 there is no line on  $V$  through the point  $Q_V := \pi(Q)$ . The blowup  $\sigma : Y' \rightarrow V$  at  $Q_V$  yields yet another cubic surface  $Y'$  with the exceptional  $(-1)$ -curve  $E = \sigma^{-1}(Q_V)$ . For the proper transform  $\mathcal{H}' = \sigma_*^{-1}(\mathcal{H}_V)$  on  $Y'$  we obtain by (28):

$$(29) \quad \mathcal{H}' \sim \sigma^*(\mathcal{H}_V) - mE \sim (a+b)\sigma^*(-K_V) - mE \sim (a+b)(-K_{Y'}) + (a+b-m)E.$$

The linear system

$$|F'| = |-K_{Y'} - E|$$

defines a conic bundle on  $Y'$ . Plugging  $E \sim -K_{Y'} - F'$  into (29) we deduce:

$$(30) \quad \mathcal{H}' \sim (2a+2b-m)(-K_{Y'}) - (a+b-m)F'.$$

Using Claims 2 and 3,

$$(31) \quad 2a+2b-m = (a+b) + (a+b-m) < a.$$

By virtue of (30) the latter inequality contradicts the minimality of  $a$ . Now the proof of Theorem 4.23 is completed.  $\square$

**Corollary 4.24.** *Under Convention 4.1 the divisor  $\mathcal{H} + t_0K_Y$  in 4.22 is big i.e.,*

$$\kappa(\mathcal{H} + t_0K_Y) = 2.$$

## 5. CONES OVER SOME RATIONAL FANO THREEFOLDS

In this section we provide examples of two families of rational Fano 3-folds such that the affine cones over their anti-canonical embeddings admit nontrivial  $\mathbb{C}_+$ -actions.

**Proposition 5.1.** *Consider a smooth intersection  $Y = Y_{2,2}$  of two quadric hypersurfaces in  $\mathbb{P}^5$ . Then the affine cone  $X$  over  $Y$  admits an effective  $\mathbb{C}_+$ -action.*

*Proof.* According to the criterion of Theorem 3.9, it is enough to construct a  $(-K_Y)$ -polar open cylinder on  $Y$ . Fix a line  $l \subseteq Y$ . Consider the diagram

$$\begin{array}{ccc} & \tilde{Y} & \\ \sigma \swarrow & & \searrow \varphi \\ Y & \overset{\psi}{\dashrightarrow} & \mathbb{P}^3 \end{array}$$

where  $\psi$  is the projection with center  $l$ ,  $\sigma$  is the blowup of  $l$ , and  $\varphi$  is the blowdown of the divisor  $D$  which is swept out by lines meeting  $l$  (i.e.,  $\sigma(D)$  is the union of lines meeting  $l$ ; see [GH, Ch. 6]). It is easily seen that  $\Gamma = \varphi(D)$  is a smooth quintic curve

in  $\mathbb{P}^3$  of genus 2. The image  $Q = \varphi(E)$  of the exceptional divisor  $E$  of  $\sigma$  is a quadric in  $\mathbb{P}^3$ . For a line  $l \subseteq Y$ , the following alternative holds: either

- (1)  $N_{l/X} = \mathcal{O}_l \oplus \mathcal{O}_l$  and  $Q$  is smooth,

or

- (2)  $N_{l/X} = \mathcal{O}_l(1) \oplus \mathcal{O}_l(-1)$  and  $Q$  is singular.

Anyhow,

$$Y \setminus \sigma(D) \simeq \mathbb{P}^3 \setminus Q.$$

Suppose that  $Q$  is singular; then  $Q$  is a quadratic cone. Let  $\Pi$  be a plane in  $\mathbb{P}^3$  passing through the vertex  $P$  of  $Q$ . We claim that  $\mathbb{P}^3 \setminus (Q \cup \Pi)$  is a principal cylinder. Indeed, consider the projection  $\pi_P$  with center  $P$  and its resolution:

$$\begin{array}{ccc} & \tilde{\mathbb{P}}^3 & \\ \sigma' \swarrow & & \searrow \varphi' \\ \mathbb{P}^3 & \xrightarrow{\pi_P} & \mathbb{P}^2 \end{array}$$

Let  $E' \subseteq \tilde{\mathbb{P}}^3$  be the exceptional divisor of  $\sigma'$ , and let  $Q' \subseteq \tilde{\mathbb{P}}^3$  be the proper transform of  $Q$ . Then  $C = \varphi'(Q') \subseteq \mathbb{P}^2$  is a conic, and  $E'$  is a section of the  $\mathbb{P}^1$ -bundle  $\tilde{\mathbb{P}}^3 \rightarrow \mathbb{P}^2$ . Furthermore,

$$\tilde{\mathbb{P}}^3 \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1)).$$

Letting  $\Pi' \subseteq \tilde{\mathbb{P}}^3$  be the proper transform of  $\Pi$ , the image  $\varphi'(\Pi') = H \subseteq \mathbb{P}^2$  is a line. We have

$$\mathbb{P}^3 \setminus (Q \cup \Pi) \simeq \tilde{\mathbb{P}}^3 \setminus (Q' \cup \Pi' \cup E')$$

is an  $\mathbb{A}^1$ -bundle over  $\mathbb{P}^2 \setminus (C \cup H)$ . Since  $\text{Pic}(\mathbb{P}^2 \setminus (C \cup H)) = 0$  we obtain

$$\mathbb{P}^3 \setminus (Q \cup \Pi) \simeq \mathbb{A}^1 \times (\mathbb{P}^2 \setminus (C \cup H)),$$

as required.  $\square$

Let us exhibit yet another family of Fano threefolds with Picard rank 1. Their moduli space is 6-dimensional. Every member  $Y$  of this family admits a  $(-K_Y)$ -polar cylinder, whereas the subfamily of completions of  $\mathbb{A}^3$  is only 4-dimensional [Fur], [Pr<sub>1</sub>].

**Proposition 5.2.** *Let  $Y = Y_{22} \subseteq \mathbb{P}^{13}$  be a Fano variety of genus 12 with  $\text{Pic}(Y) = \mathbb{Z} \cdot [-K_Y]$ , anticanonically embedded into  $\mathbb{P}^{13}$ . Then the affine cone  $X$  over  $Y$  admits an effective  $\mathbb{C}_+$ -action.*

*Proof.* Again, it is enough to construct a  $(-K_Y)$ -polar open cylinder on  $Y$ . Then the result follows by applying Theorem 3.9. Picking a line  $l_1 \subseteq Y$  we consider the commutative diagram

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{\chi} & \tilde{Y}^+ \\ \sigma_1 \downarrow & & \downarrow \varphi_1 \\ \mathbb{P}^{13} \supseteq Y = Y_{22} & \xrightarrow{\psi_1} & Y_5 \subseteq \mathbb{P}^6 \end{array}$$

where  $\sigma_1$  is the blowup of  $l_1$ ,  $\psi_1$  is the double projection with center  $l_1$ <sup>10</sup> onto a Fano threefold  $Y_5$  of degree 5 and of Fano index 2, anticanonically embedded into  $\mathbb{P}^6$ ,  $\varphi_1$  is the blowup of a smooth rational curve  $\Gamma \subseteq Y_5$  of degree 5, and  $\chi$  is a flop; see [IP, §4.3]. We have

$$Y \setminus H_1 \simeq Y_5 \setminus H_2,$$

where  $H_1 \subseteq Y$  is a hyperplane section with  $\text{mult}_{l_1}(H_1) = 3$ , and  $H_2 \subseteq Y_5$  is a hyperplane section passing through  $\Gamma$ . Thus it suffices to show that  $Y_5 \setminus H_2$  contains an  $H_2$ -polar cylinder.

Let further  $l_2 \subseteq Y_5$  be a line. Recall that the family of all lines on  $Y_5$  is parameterized by  $\mathbb{P}^2$ , and either  $N_{l_2/Y_5} \simeq \mathcal{O}_{l_2} \oplus \mathcal{O}_{l_2}$ , or  $N_{l_2/Y_5} \simeq \mathcal{O}_{l_2}(1) \oplus \mathcal{O}_{l_2}(-1)$ . The lines of second type are parameterized by a smooth conic on  $\mathbb{P}^2$ ; see [FN]. There exists a line  $l_2$  on  $Y_5$  of second type contained in  $H_2$ . Consider the projection  $\psi_2$  with center  $l_2$  and its resolution:

$$\begin{array}{ccc} & \tilde{Y}_5 & \\ \sigma_2 \swarrow & & \searrow \varphi_2 \\ \mathbb{P}^6 \supseteq Y_5 & \xrightarrow{\psi_2} & Q \subseteq \mathbb{P}^4 \end{array}$$

where  $\sigma_2$  is the blowup of  $l_2$ , and  $Q \subseteq \mathbb{P}^4$  is a smooth quadric. We have

$$Y_5 \setminus H'_2 \simeq Q \setminus H_3,$$

where  $H'_2 \subseteq Y_5$  and  $H_3 \subseteq Q$  are hyperplane sections such that  $\text{mult}_{l_2}(H'_2) = 2$ , and  $H'_2$  is swept out by lines meeting  $l_2$ . Since  $\psi_2$  is a projection,

$$Y_5 \setminus (H_2 \cup H'_2) \simeq Q \setminus (H_3 \cup H'_3),$$

where  $H'_3 \subseteq Q$  is another hyperplane section (possibly  $H'_3 = H_3$ ). It remains to show that the complement  $Q \setminus (H_3 \cup H'_3)$  contains an  $H_3$ -polar cylinder.

We may assume that  $H'_3 \neq H_3$ . The projection  $\pi_P : \mathbb{P}^4 \dashrightarrow \mathbb{P}^3$  with center at a general point  $P \in H_3 \cap H'_3$  yields an isomorphism

$$Q \setminus (H_3 \cup H'_3) \simeq \mathbb{P}^3 \setminus (\Pi_1 \cup \Pi_2 \cup \Pi_3),$$

where  $\Pi_1, \Pi_2, \Pi_3$  are three planes in  $\mathbb{P}^3$ . So the existence of an  $H_3$ -polar cylinder on  $Q \setminus (H_3 \cup H'_3)$  is equivalent to the existence of a  $\Pi_1$ -polar cylinder on  $\mathbb{P}^3 \setminus (\Pi_1 \cup \Pi_2 \cup \Pi_3)$ . Now the assertion easily follows.  $\square$

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<sup>10</sup>That is,  $\psi_1 : Y \dashrightarrow \mathbb{P}^6$  is defined by the linear system  $|\sigma_1^* \mathcal{O}_Y(1) - 2\tilde{E}|$ , where  $\tilde{E} \subseteq \tilde{Y}$  is the exceptional divisor of  $\sigma_1$ .

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