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Linear stability analysis of a Level Set model of immersed elastic membrane

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Abstract

We perform a linear stability analysis of an elementary 1D model obtained from a Level Set formulation of the coupling between an immersed elastic interface and the surrounding fluid. Despite the striking simplicity of the studied model, relevant instability regimes are obtained, unifying results obtained in previous studies [5, 17]. We also present numerical tests on the full model which further illustrate the relevance of this simplified model.

1 Introduction

We study the numerical stability of an elastic membrane immersed in an incompressible 2D fluid. This coupled model is described by an Eulerian version of the Immersed Boundary Method of Peskin [13], proposed by Cottet & Maitre ([7],[8]), and using a Level Set formulation.
Indeed, the stability of the Immersed Boundary Method is well known to be affected by the application of the elastic force on the structure, as soon as it is treated explicitely. In order to remove the time step restrictions, many works have been dedicated to the study of implicit schemes ([16], [11], [14], [15], [12]). However these schemes seem to be unusable in practice because of the large computational cost related to the iterative resolution of the strongly non-linear coupling at each time step. The definition of semi-implicit or approximative implicit schemes allows to obtain more realistic methods. See [16], [14], [15] [9] et [2] and the references therein for details about such schemes.
There exists several ways to carry out the study of numerical stability. Among them is the control of discrete energy conservation ([2],[12]), which is based on the corresponding physical property. Another way is to perform a linear analysis near an equilibrium state of the system, in order to identify the modes associated to the membrane. In ([14], [15]) this approach enables the authors to show that the instability is in particular increased by the combination of a

small fluid viscosity and a strong elastic force.
The aim of this work is the derivation of stability conditions on the time step for schemes which treat the coupling explicitely, semi-implicitely or implicitely. We perform the analysis on a 1D linearized version of the model as its original form is not very suitable for numerical analysis. More precisely, the elastic force on the membrane is reduced to a surface tension. Despite of these simplifications, this *ad-hoc* model enables us to exhibit some clear relations between the fluid viscosity, the elastic force, and the numerical parameters. The results are in good accordance with some conditions of the litterature, in particular those proposed either by Brackbill et al. [5] or Vigneaux [17], depending on the physical parameters range.
In section 2 we briefly introduce the 2D Eulerian Immersed Boundary model and we derive the 1D linearized one used for the analysis. The time-stepping schemes studied are also presented. Section 3 is then dedicated to the computation of necessary stability conditions and to the comparison of these conditions with existing ones. Finally, numerical experiments on a relaxation test validate in section 4 the analytical results in the non-linear 2D case, for some range of parameters.

2 Immersed elastic membrane problem

Let Ω be an open bounded domain of $\mathbb{R}^2$, filled with a viscous incompressible and homogeneous fluid of density $\rho > 0$ and viscosity $\mu > 0$. Inside this domain, we consider an immersed elastic membrane $\Gamma_e(t)$ during a time interval $[0, T]$, $T > 0$. The membrane is supposed massless, and external forces are neglected. Let u and p be the fluid velocity and pressure fields. We introduce the Level Set function ϕ initialized as the signed distance to $\Gamma_e(0)$, and such that :

$$\Gamma_e(t) = \{x \in \Omega, \phi(t, x) = 0\}, \ \forall t \in [0, T].$$

2.1 Eulerian Immersed Boundary model in dimension 2

We study the following Eulerian version of the Immersed Boundary Method for this model [7],[8]:

$$\begin{cases} \rho\left(u_t + u.\nabla u\right) + \nabla p - \mu \Delta u = F_e \\ F_e = \left[\left(\nabla(E_e'(|\nabla\phi|)) \cdot \dfrac{\nabla \times \phi}{|\nabla\phi|}\right) \cdot \dfrac{\nabla \times \phi}{|\nabla\phi|} - E_e'(|\nabla\phi|)\kappa(\phi)\dfrac{\nabla\phi}{|\nabla\phi|}\right] \dfrac{|\nabla\phi|}{\varepsilon}\zeta\left(\dfrac{\phi}{\varepsilon}\right) \\ \operatorname{div} u = 0 \\ \phi_t + u \cdot \nabla\phi = 0 \end{cases} \tag{1}$$

with the linear stress-strain relationship

$$E_e'(r) = \nu_e(r - 1) \tag{2}$$

where ν_e and $\kappa(\phi)$ are the stiffness coefficient and the mean curvature of $\Gamma_e(t)$, ε is a numerical parameter related to interface smoothing, ζ is a cut-off function,

and F_e is finally the elastic force arising from the presence of the membrane in the fluid.
For a complete description this system has to be supplemented with initial and boundary conditions. One common choice is periodic or homogeneous boundary condition for the velocity, so that no boundary condition is needed for ϕ.

Remark 2.1. *When using this formulation, particular care must be taken of the spatial discretization of the elastic force term. Indeed, some derivatives have to be developed in order to avoid strong numerical instabilities, which must be kept apart from the instabilities related to the time-stepping choice. The study of these numerical instabilities is beyond the scope of this work and has been developed in [4].*

2.2 1D linearized model

We now introduce a 1D linearized version of the previous model. The elastic force is first reduced to a surface tension by taking $E'(r) = \nu_e$ in (1).
This results in the following 2D system:

$$\begin{cases} \dfrac{\partial u}{\partial t} + u.\nabla u - \mu \Delta u + \nabla p = -\nu_e \kappa(\phi) \nabla \phi \dfrac{1}{\varepsilon} \zeta\left(\dfrac{\phi}{\varepsilon}\right), \\ \operatorname{div} u = 0, \\ \dfrac{\partial \phi}{\partial t} + u.\nabla \phi = 0. \end{cases} \tag{3}$$

which next gives rise to the 1D model:

$$\begin{cases} \dfrac{\partial u}{\partial t} + u \dfrac{\partial u}{\partial x} - \mu \dfrac{\partial^2 u}{\partial x^2} = -\nu_e \dfrac{\partial^2 \phi}{\partial x^2} \dfrac{\partial \phi}{\partial x} \dfrac{1}{\varepsilon} \zeta\left(\dfrac{\phi}{\varepsilon}\right), \\ \dfrac{\partial \phi}{\partial t} + u \dfrac{\partial \phi}{\partial x} = 0. \end{cases} \tag{4}$$

We investigate the stability of (4) around the trivial stationary solution $(\overline{u}, \overline{\phi})$ given by:

$$\begin{cases} \overline{u}(x) = 0, \\ \overline{\phi}(x) = x. \end{cases} \tag{5}$$

Let $(u, \phi) = (\overline{u} + \widetilde{u}, \overline{\phi} + \widetilde{\phi})$ be a solution of (4), where $(\widetilde{u}, \widetilde{\phi})$ is a small perturbation. Plugging these expressions in (4) we get:

$$\begin{cases} \dfrac{\partial \widetilde{u}}{\partial t} + \widetilde{u} \dfrac{\partial \widetilde{u}}{\partial x} - \mu \dfrac{\partial^2 \widetilde{u}}{\partial x^2} = -\nu_e \dfrac{\partial^2 \widetilde{\phi}}{\partial x^2} \left(1 + \dfrac{\partial \widetilde{\phi}}{\partial x}\right) \dfrac{1}{\varepsilon} \zeta\left(\dfrac{\overline{\phi} + \widetilde{\phi}}{\varepsilon}\right), \\ \dfrac{\partial \widetilde{\phi}}{\partial t} + \widetilde{u} \dfrac{\partial \widetilde{\phi}}{\partial x} + \tilde{u} = 0. \end{cases} \tag{6}$$

By getting rid of the non-linear terms, we then obtain:

$$\begin{cases} \dfrac{\partial \widetilde{u}}{\partial t} - \mu \dfrac{\partial^2 \widetilde{u}}{\partial x^2} = -\nu_e \dfrac{\partial^2 \widetilde{\phi}}{\partial x^2} \dfrac{1}{\varepsilon} \zeta\left(\dfrac{\overline{\phi}}{\varepsilon}\right) + o(\widetilde{u}, \widetilde{\phi}), \\ \dfrac{\partial \widetilde{\phi}}{\partial t} + \widetilde{u} = o(\widetilde{u}, \widetilde{\phi}). \end{cases} \tag{7}$$

Actually,

$$\begin{aligned}
\frac{\partial^2\widetilde{\phi}}{\partial x^2}\left(1+\frac{\partial\widetilde{\phi}}{\partial x}\right)\frac{1}{\varepsilon}\zeta\left(\frac{\overline{\phi}+\widetilde{\phi}}{\varepsilon}\right) &= \frac{\partial^2\widetilde{\phi}}{\partial x^2}\frac{1}{\varepsilon}\zeta\left(\frac{\overline{\phi}+\widetilde{\phi}}{\varepsilon}\right)+o(\widetilde{\phi}) \\
&= \frac{\partial^2\widetilde{\phi}}{\partial x^2}\frac{1}{\varepsilon}\left[\zeta\left(\frac{\overline{\phi}}{\varepsilon}\right)+\frac{\widetilde{\phi}}{\varepsilon}\zeta'\left(\frac{\overline{\phi}}{\varepsilon}\right)\right]+o(\widetilde{\phi}) \\
&= \frac{\partial^2\widetilde{\phi}}{\partial x^2}\frac{1}{\varepsilon}\zeta\left(\frac{\overline{\phi}}{\varepsilon}\right)+o(\widetilde{\phi}).
\end{aligned}$$

The remaining simplification concerns the function ζ. We define it by:

$$\begin{cases}\zeta(r)=1 \text{ if } |r|\leq\dfrac{1}{2}, \\ \zeta(r)=0 \text{ else}\end{cases} \tag{8}$$

Let us point out that for $\zeta\left(\frac{\phi}{\varepsilon}\right)=0$, namely far from the fluid-membrane interface, the system (7) is no longer coupled. For the study below, we thus suppose that $\zeta\left(\frac{\phi}{\varepsilon}\right)=1$ everywhere, which means that we limit the study in a neighbourhood of the membrane. We are clearly conscious of the restriction brought by this hypothesis, since it is not clear how the presence of the membrane affects the stability in the whole computational domain, and conversely whether the computations in the remainder of the domain might bring stabilizing effects.

Finally, the 1D linearized model used for the stability analysis reads (we remove from now on the tilde on unknowns):

$$\begin{cases}\dfrac{\partial u}{\partial t}-\mu\dfrac{\partial^2 u}{\partial x^2}=-\dfrac{\nu_e}{\varepsilon}\dfrac{\partial^2\phi}{\partial x^2} \text{ on } [0,T]\times\mathbb{R} \\ \dfrac{\partial\phi}{\partial t}+u=0 \text{ on } [0,T]\times\mathbb{R} \\ u(0,x)=f(x),\ \phi(0,x)=g(x) \text{ on } \mathbb{R}\end{cases} \tag{9}$$

where $\varepsilon>0$, $\mu\geq 0$ and $\nu_e>0$ (we rule out the case $\nu_e=0$ in order to keep (9) in a coupled form).
This quite simplified model is not supposed to describe in a general way the fluid-membrane interactions, as the non-linearities, including in particular the inertial effects, have been neglected. However it enables us to perform a stability analysis and to draw some attractive conclusions.

2.3 Time-stepping schemes

Let $\Delta t>0$ and $\Delta x>0$ be the time and space discretization steps, $x_j=j\Delta x$, $j\in\mathbb{Z}$, and $t_n=n\Delta t$, $n\in\mathbb{N}$. We denote by $u_j^n\approx u(t_n,x_j)$ and $\phi_j^n\approx\phi(t_n,x_j)$, $j=0,...,M+1$, the numerical approximation of solutions of (9), by some finite difference scheme to be described. Discretizing diffusion implicitely, so that the

stability analysis will focus on the coupling, we define the following schemes to compute these numerical values:

- Scheme 1: Explicit (u, ϕ)-coupling:

$$\begin{cases} \dfrac{u_j^{n+1} - u_j^n}{\Delta t} - \mu \dfrac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{(\Delta x)^2} = -\dfrac{\nu_e}{\varepsilon} \dfrac{\phi_{j+1}^n - 2\phi_j^n + \phi_{j-1}^n}{(\Delta x)^2} \\ \dfrac{\phi_j^{n+1} - \phi_j^n}{\Delta t} + u_j^n = 0 \\ u_j^0 = f_j,\ \phi_j^0 = g_j \end{cases} \tag{10}$$

- Scheme 2: Semi-implicit (u, ϕ)-coupling:

$$\begin{cases} \dfrac{u_j^{n+1} - u_j^n}{\Delta t} - \mu \dfrac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{(\Delta x)^2} = -\dfrac{\nu_e}{\varepsilon} \dfrac{\phi_{j+1}^n - 2\phi_j^n + \phi_{j-1}^n}{(\Delta x)^2} \\ \dfrac{\phi_j^{n+1} - \phi_j^n}{\Delta t} + u_j^{n+1} = 0 \\ u_j^0 = f_j,\ \phi_j^0 = g_j \end{cases} \tag{11}$$

- Scheme 3: Implicit (u, ϕ)-coupling:

$$\begin{cases} \dfrac{u_j^{n+1} - u_j^n}{\Delta t} - \mu \dfrac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{(\Delta x)^2} = -\dfrac{\nu_e}{\varepsilon} \dfrac{\phi_{j+1}^{n+1} - 2\phi_j^{n+1} + \phi_{j-1}^{n+1}}{(\Delta x)^2} \\ \dfrac{\phi_j^{n+1} - \phi_j^n}{\Delta t} + u_j^{n+1} = 0 \\ u_j^0 = f_j,\ \phi_j^0 = g_j \end{cases} \tag{12}$$

In practice Scheme 1 is not used, as the new value of u is naturally available for the resolution of ϕ. Scheme 2 is rather implemented, with no additional cost. However Scheme 1 is still studied for the purpose of making comparison with litterature conditions.

3 Linear stability analysis

This section is concerned with the numerical stability of the time-stepping schemes defined above, which means that we expect to detect unbounded oscillations of the numerical solutions as the physical and numerical parameters vary. We look for solutions u_j^n and ϕ_j^n that have the form of Fourier modes sums:

$$u_j^n = \sum_{k \in \mathbb{Z}} \xi^n(k) e^{2i\pi kj\Delta x}, \quad \phi_j^n = \sum_{k \in \mathbb{Z}} \chi^n(k) e^{2i\pi kj\Delta x}$$

By substituting these expressions into each of the former schemes, we can compute an amplification matrix $A(k)$ defined by:

$$\begin{pmatrix} \xi^n(k) \\ \chi^n(k) \end{pmatrix} = (A(k))^n \begin{pmatrix} \xi^0(k) \\ \chi^0(k) \end{pmatrix},$$

whose spectral radius allows to specify necessary stability conditions, through the Von Neumann condition.
The aim of this section is the proof of the following lemma, concerning the schemes 1, 2, and 3:

Lemma 3.1. *A necessary condition for the stability of the numerical scheme defined by (10) is*

$$\Delta t \leq \frac{\mu\varepsilon}{\nu_e} \tag{13}$$

Lemma 3.2. *A necessary condition for the stability of the numerical scheme defined by (11) is*

$$\Delta t < \frac{(\mu\varepsilon + \max(\mu\varepsilon, \sqrt{\nu_e\varepsilon}\Delta x))}{\nu_e} \tag{14}$$

Lemma 3.3. *The scheme defined by (12) is unconditionally stable.*

3.1 Proof of Lemma 3.1

Let $k \in \mathbb{Z}$. By replacing u_j^n by $\xi^n(k)e^{2i\pi kj\Delta x}$ and ϕ_j^n by $\chi^n(k)e^{2i\pi kj\Delta x}$ in (10), and by removing the k-dependence of χ^n and ξ^n for the sake of readability, we get

$$\begin{cases} \left(1 + \dfrac{4\mu\Delta t}{(\Delta x)^2}\sin^2(\pi k\Delta x)\right)\xi^{n+1} = \xi^n + \dfrac{4\nu_e\Delta t}{\varepsilon(\Delta x)^2}\sin^2(\pi k\Delta x)\,\chi^n \\ \chi^{n+1} = \chi^n - \Delta t\xi^n \end{cases} \tag{15}$$

We define $\beta(k) = \dfrac{4\nu_e\Delta t}{\varepsilon(\Delta x)^2}\sin^2(\pi k\Delta x)$ and $\delta(k) = 1 + \dfrac{4\mu\Delta t}{(\Delta x)^2}\sin^2(\pi k\Delta x)$ to obtain

$$\begin{pmatrix} \xi^{n+1} \\ \chi^{n+1} \end{pmatrix} = \begin{pmatrix} \dfrac{1}{\delta(k)} & \dfrac{\beta(k)}{\delta(k)} \\ -\Delta t & 1 \end{pmatrix} \begin{pmatrix} \xi^n \\ \chi^n \end{pmatrix} = A(k) \begin{pmatrix} \xi^n \\ \chi^n \end{pmatrix}$$

The eigenvalues $m^{\pm}$ of $A(k)$ are the solutions of

$$\frac{\delta(k)}{1 + \Delta t\beta(k)}m^2 - \frac{\delta(k) + 1}{1 + \Delta t\beta(k)}m + 1 = 0$$

As $\delta(k) = 1 + \dfrac{\mu\varepsilon}{\nu_e}\beta(k)$, this equation is equivalent to

$$\frac{1 + \frac{\mu\varepsilon}{\nu_e}\beta(k)}{1 + \Delta t\beta(k)}m^2 - \frac{2 + \frac{\mu\varepsilon}{\nu_e}\beta(k)}{1 + \Delta t\beta(k)}m + 1 = 0$$

whose discriminant is given by:

$$\Delta(k) \quad = \quad \frac{\beta(k)}{(1+\Delta t\beta(k))^2}\left[\left(\frac{\mu\varepsilon}{\nu_e}-4\Delta t\right)\frac{\mu\varepsilon}{\nu_e}\beta(k)-4\Delta t\right]$$

Let us note that for $k = 0$, $\beta(k) = 0$. In this case we solve the equation $m^2 - 2m + 1 = 0$, which admits the double root 1. We suppose thereafter that $k \neq 0$, which means $\beta(k) > 0$ as $\nu_e > 0$.

Using that $\dfrac{1+\Delta t\beta(k)}{1+\frac{\mu\varepsilon}{\nu_e}\beta(k)}$ stands for the square of the complex root's module when $\Delta(k) < 0$, we distinguish between three cases:

- For $\Delta t > \dfrac{\mu\varepsilon}{\nu_e}$, i.e. $\dfrac{1+\Delta t\beta(k)}{1+\frac{\mu\varepsilon}{\nu_e}\beta(k)} > 1$, there must hold $\Delta(k) < 0$ and we get two complex conjugate eigenvalues of module $\sqrt{\dfrac{1+\Delta t\beta(k)}{1+\frac{\mu\varepsilon}{\nu_e}\beta(k)}} > 1$. Therefore the Von Neumann condition is not satisfied.

- For $\Delta t < \dfrac{\mu\varepsilon}{\nu_e}$, i.e. $\dfrac{1+\Delta t\beta(k)}{1+\frac{\mu\varepsilon}{\nu_e}\beta(k)} < 1$,
  - If $\Delta(k) < 0$, we get two complex conjugate eigenvalues of module $\sqrt{\dfrac{1+\Delta t\beta(k)}{1+\frac{\mu\varepsilon}{\nu_e}\beta(k)}} < 1$.
  - If $\Delta(k) > 0$, we get two real and distinct eigenvalues, which read

$$m^{\pm} \quad = \quad \frac{2+\dfrac{\mu\varepsilon}{\nu_e}\beta(k) \pm \sqrt{\beta(k)}\sqrt{\left(\dfrac{\mu\varepsilon}{\nu_e}-4\Delta t\right)\dfrac{\mu\varepsilon}{\nu_e}\beta(k)-4\Delta t}}{2+2\dfrac{\mu\varepsilon}{\nu_e}\beta(k)}$$

Thus

$$\frac{2+\dfrac{\mu\varepsilon}{\nu_e}\beta(k) - \sqrt{\beta(k)}\sqrt{\left(\dfrac{\mu\varepsilon}{\nu_e}\right)^2\beta(k)}}{2+2\dfrac{\mu\varepsilon}{\nu_e}\beta(k)} \leq m^{\pm}$$

and

$$m^{\pm} \leq \frac{2+\dfrac{\mu\varepsilon}{\nu_e}\beta(k) + \sqrt{\beta(k)}\sqrt{\left(\dfrac{\mu\varepsilon}{\nu_e}\right)^2\beta(k)}}{2+2\dfrac{\mu\varepsilon}{\nu_e}\beta(k)}$$

i.e.

$$0 < \frac{2}{2+2\frac{\mu\varepsilon}{\nu_e}\beta(k)} \leq m^{\pm} \leq \frac{2+2\frac{\mu\varepsilon}{\nu_e}\beta(k)}{2+2\frac{\mu\varepsilon}{\nu_e}\beta(k)} = 1$$

Therefore $|m^{\pm}| \leq 1$.

– Finally if $\Delta(k) = 0$ we get the double root

$$m = \frac{2+\frac{\mu\varepsilon}{\nu_e}\beta(k)}{2+2\frac{\mu\varepsilon}{\nu_e}\beta(k)}$$

Which verifies $|m| < 1$.

For $\Delta t < \frac{\mu\varepsilon}{\nu_e}$ the Von Neumann condition is therefore always satisfied.

- Finally for $\Delta t = \frac{\mu\varepsilon}{\nu_e}$, i.e. $\frac{1+\Delta t\beta(k)}{1+\frac{\mu\varepsilon}{\nu_e}\beta(k)} = 1$, then

$$\Delta(k) = \frac{\beta(k)}{(1+\Delta t\beta(k))^2}\left[-3\left(\frac{\mu\varepsilon}{\nu_e}\right)^2\beta(k) - 4\Delta t\right] < 0,$$

and we get two complex conjugate eigenvalues of module $\sqrt{\frac{1+\Delta t\beta(k)}{1+\frac{\mu\varepsilon}{\nu_e}\beta(k)}} = 1$. Thus the Von Neumann condition is still satisfied.

In conclusion, a necessary condition for the numerical stability of Scheme 1 is

$$\boxed{\Delta t \leq \frac{\mu\varepsilon}{\nu_e}} \tag{16}$$

We point out that for $\mu = 0$, this scheme is then unconditionally unstable.

3.2 Proof of Lemma 3.2

By an analogous argument as in the former section, the stability analysis reduces to the computation of the solutions $m^{\pm}$ of the following equation:

$$\left(1+\frac{\mu\varepsilon}{\nu_e}\beta(k)\right)m^2 - \left(2+\left(\frac{\mu\varepsilon}{\nu_e}-\Delta t\right)\beta(k)\right)m + 1 = 0$$

whose discriminant is given by:

$$\Delta(k) = 4\Delta t\beta(k)\left(\left(\frac{\mu\varepsilon}{\nu_e} - \Delta t\right)^2 \frac{\nu_e}{\varepsilon(\Delta x)^2}\sin^2(\pi k\Delta x) - 1\right)$$

We rule out the case $k = 0$ (double root 1), and we point out that

$$\frac{1}{1 + \frac{\mu\varepsilon}{\nu_e}\beta(k)} < 1$$

Let us note $w = \left(\frac{\mu\varepsilon}{\nu_e} - \Delta t\right)^2 \frac{\nu_e}{\varepsilon(\Delta x)^2}$. As before we distinguish between three cases:

- If $w \leq 1$, i.e. $\Delta t \in \left[\frac{\mu\varepsilon}{\nu_e} - \frac{\sqrt{\varepsilon}\Delta x}{\sqrt{\nu_e}}; \frac{\mu\varepsilon}{\nu_e} + \frac{\sqrt{\varepsilon}\Delta x}{\sqrt{\nu_e}}\right]$, then $\Delta(k) \leq 0$.

  For $\Delta(k) < 0$ we get two complex conjugate eigenvalues of module $\sqrt{\frac{1}{1 + \frac{\mu\varepsilon}{\nu_e}\beta(k)}} < 1$.

  For $\Delta(k) = 0$ the double root reads $m = \frac{2 + \left(\frac{\mu\varepsilon}{\nu_e} - \Delta t\right)\beta(k)}{2 + 2\frac{\mu\varepsilon}{\nu_e}\beta(k)}$. So $0 \leq m \leq 1$.
  Thus the Von Neumann condition is satisfied in this case.

- If $w > 1$ and $\Delta t < \frac{\mu\varepsilon}{\nu_e} - \frac{\sqrt{\varepsilon}\Delta x}{\sqrt{\nu_e}}$, then there exists some k values such that $\Delta(k) > 0$.
  For k such that $\Delta(k) \leq 0$ we obtain stability by a similar argument as before.
  If $\Delta(k) > 0$ instead, the product and the sum of the two eigenvalues are positive, so $m^{\pm} \geq 0 > -1$. As $m^- \leq m^+$, it suffices to prove that $m^+ \leq 1$. We have

  $$m^+ \leq \frac{1 + \left(\frac{\mu\varepsilon}{\nu_e} - \Delta t\right)\beta(k)}{1 + \frac{\mu\varepsilon}{\nu_e}\beta(k)} \leq 1$$

  Hence the Von Neumann condition is satisfied for $w > 1$ and $\Delta t < \frac{\mu\varepsilon}{\nu_e} - \frac{\sqrt{\varepsilon}\Delta x}{\sqrt{\nu_e}}$.

- Finally, if $w > 1$ and $\Delta t > \frac{\mu\varepsilon}{\nu_e} + \frac{\sqrt{\varepsilon}\Delta x}{\sqrt{\nu_e}}$, as $m^- \leq m^+$ it remains to prove that $m^+ \leq 1$ and $m^- \geq -1$.

We get readily $m^+ \leq \dfrac{2}{2+2\dfrac{\mu\varepsilon}{\nu_e}\beta(k)} \leq 1$.

Besides,

$$\begin{aligned}
& m^- \geq -1 \\
\Leftrightarrow \quad & 2 + \left(\frac{\mu\varepsilon}{\nu_e} - \Delta t\right)\beta(k) - \sqrt{\beta(k)}\sqrt{\left(\frac{\mu\varepsilon}{\nu_e} - \Delta t\right)^2 \beta(k) - 4\Delta t} \\
& \geq -2 - 2\frac{\mu\varepsilon}{\nu_e}\beta(k) \\
\Leftrightarrow \quad & 4 + \left(\frac{3\mu\varepsilon}{\nu_e} - \Delta t\right)\beta(k) > \sqrt{\beta(k)}\sqrt{\left(\frac{\mu\varepsilon}{\nu_e} - \Delta t\right)^2 \beta(k) - 4\Delta t} \geq 0
\end{aligned}$$

By taking the square of each side, we get

$$m^- \geq -1 \quad \Rightarrow \quad \frac{\mu\varepsilon}{\nu_e}\left(\frac{2\mu\varepsilon}{\nu_e} - \Delta t\right)\beta^2(k) + \left(\frac{6\mu\varepsilon}{\nu_e} - \Delta t\right)\beta(k) + 4 \geq 0 \tag{17}$$

Let's recall that $\beta(k) \geq 0$ and study the polynomial in $\beta(k)$. Its discriminant is:

$$\Delta = \left(\frac{2\mu\varepsilon}{\nu_e} + \Delta t\right)^2 > 0$$

Thus its roots are $\beta_1 = -\dfrac{\nu_e}{\mu\varepsilon}$ and $\beta_2 = \dfrac{4}{\Delta t - \frac{2\mu\varepsilon}{\nu_e}}$. For $\Delta t < \dfrac{2\mu\varepsilon}{\nu_e}$ the two roots are negative, so (17) is always satisfied.
On the contrary if $\Delta t > \dfrac{2\mu\varepsilon}{\nu_e}$ then $\beta_1 < 0$ et $\beta_2 > 0$. The polynomial may be positive for $\beta(k) < \beta_2$, which is impossible.
At last for $\Delta t = \dfrac{2\mu\varepsilon}{\nu_e}$, the right hand side of (17) reduces to $\dfrac{4\mu\varepsilon}{\nu_e}\beta(k) + 4 \geq 0$, which is always satisfied.
Collecting conditions above we are able to conclude that (17) is verified if and only if $\Delta t \leq \dfrac{2\mu\varepsilon}{\nu_e}$.
Therefore if $w > 1$ and $\Delta t > \dfrac{\mu\varepsilon}{\nu_e} + \dfrac{\sqrt{\varepsilon}\Delta x}{\sqrt{\nu_e}}$, the Von Neumann condition is satisfied if and only if $\Delta t \leq \dfrac{2\mu\varepsilon}{\nu_e}$.

To conclude, a necessary condition for the stability of Scheme 2 is

$$\boxed{\Delta t < \frac{\mu\varepsilon + \max(\mu\varepsilon, \sqrt{\nu_e\varepsilon}\Delta x)}{\nu_e}} \tag{18}$$

Remark 3.4. *If $\mu = 0$, (11) identically reads:*

$$\begin{cases} \dfrac{\phi_j^{n+1} - 2\phi_j^n + \phi_j^{n-1}}{(\Delta t)^2} - \dfrac{\nu_e}{\varepsilon} \dfrac{\phi_{j+1}^n - 2\phi_j^n + \phi_{j-1}^n}{(\Delta x)^2} = 0 \\ \phi_j^0 = g_j,\ \phi_j^1 = g_j - \Delta t f_j \end{cases} \tag{19}$$

This is a discretization of the wave equation

$$\begin{cases} \dfrac{\partial^2 \phi}{\partial t^2} - \dfrac{\nu_e}{\varepsilon} \dfrac{\partial^2 \phi}{\partial x^2} = 0 \\ \phi(0,x) = g(x),\ \dfrac{\partial \phi(0,x)}{\partial t} = -f(x) \end{cases} \tag{20}$$

with an explicit scheme centered in space and time. The time step stability condition for such a scheme is known to reads:

$$\Delta t < \frac{\sqrt{\varepsilon}\Delta x}{\sqrt{\nu_e}} \tag{21}$$

We can check that taking $\mu = 0$ in (18) immediately gives (21).

3.3 Proof of Lemma 3.3

In order to determine a stability condition for the Scheme 3, we solve the following equation:

$$\left(1 + \left(\frac{\mu\varepsilon}{\nu_e} + \Delta t\right)\beta(k)\right) m^2 - \left(2 + \frac{\mu\varepsilon}{\nu_e}\beta(k)\right) m + 1 = 0$$

whose discriminant is given by:

$$\Delta(k) = \beta(k)\left(\left(\frac{\mu\varepsilon}{\nu_e}\right)^2 \beta(k) - 4\Delta t\right)$$

As before, we rule out the case $k = 0$ (double root 1), and we point out that

$$\frac{1}{1 + \left(\frac{\mu\varepsilon}{\nu_e} + \Delta t\right)\beta(k)} < 1$$

For $\Delta(k) < 0$ we get two complex conjugate eigenvalues of module

$$\sqrt{\frac{1}{1 + \left(\frac{\mu\varepsilon}{\nu_e} + \Delta t\right)\beta(k)}} < 1$$

When $\Delta(k) > 0$, the two real eigenvalues read

$$m^{\pm} = \frac{2 + \frac{\mu\varepsilon}{\nu_e}\beta(k) \pm \sqrt{\beta(k)}\sqrt{\left(\frac{\mu\varepsilon}{\nu_e}\right)^2 \beta(k) - 4\Delta t}}{2 + 2\left(\frac{\mu\varepsilon}{\nu_e} + \Delta t\right)\beta(k)}$$

So

$$\frac{2+\frac{\mu\varepsilon}{\nu_e}\beta(k)-\sqrt{\beta(k)}\sqrt{\left(\frac{\mu\varepsilon}{\nu_e}\right)^2\beta(k)}}{2+2\left(\frac{\mu\varepsilon}{\nu_e}+\Delta t\right)\beta(k)} \leq m^{\pm} \leq \frac{2+\frac{\mu\varepsilon}{\nu_e}\beta(k)+\sqrt{\beta(k)}\sqrt{\left(\frac{\mu\varepsilon}{\nu_e}\right)^2\beta(k)}}{2+2\left(\frac{\mu\varepsilon}{\nu_e}+\Delta t\right)\beta(k)}$$

Thus

$$0 \leq \frac{2}{2+2\left(\frac{\mu\varepsilon}{\nu_e}+\Delta t\right)\beta(k)} \leq m^{\pm} \leq \frac{2+2\frac{\mu\varepsilon}{\nu_e}\beta(k)}{2+2\left(\frac{\mu\varepsilon}{\nu_e}+\Delta t\right)\beta(k)} \leq 1$$

Therefore $|m^{\pm}| \leq 1$.

Finally, if $\Delta(k) = 0$ the double root is

$$m = \frac{2+\frac{\mu\varepsilon}{\nu_e}\beta(k)}{2+2\left(\frac{\mu\varepsilon}{\nu_e}+\Delta t\right)\beta(k)}$$

Thus $|m| \leq 1$.

In conclusion, the Von Neumann necessary condition is always satisfied for Scheme 3.

3.4 Comments

According to the Lemma 3.1, 3.2, and 3.3, we claim that Scheme 1 and 2 are conditionally stable, whereas Scheme 3 is unconditionally stable.
Despite the conditions (16) and (18) are only necessary ones, the relation between viscous and elastic effects clearly appears. Indeed, the strongest time step restriction occurs for a small fluid viscosity and a large elastic force. In this respect these conditions are in good accordance with the condition computed in [17] with a mathematical analysis of the continuous model.
More precisely, the parameter ε in (16) and (18) stands for the size of an interface's neighbourhood on which the elastic force is applied. In practice, this parameter is taken in the order of Δx. By taking $\varepsilon = \Delta x$ in (16) and (18) we obtain

$$\boxed{\Delta t \leq \frac{\mu\Delta x}{\nu_e}} \tag{22}$$

for Scheme 1, and

$$\boxed{\Delta t < \frac{\mu\Delta x + \max(\mu, \sqrt{\nu_e\Delta x})\Delta x}{\nu_e}} \tag{23}$$

for Scheme 2.
We recognize for Scheme 1 the condition of [17]. Moreover, the semi-implicit treatment of coupling in Scheme 2 allows us to relax easily the former condition. Indeed, for Δx and μ fixed, beyond a certain value of ν_e, we reach a $1/\sqrt{\nu_e}$-dependence for the time step, instead of the $1/\nu_e$-dependence related to the explicit coupling. Therefore this condition is by far less restrictive when ν_e is large. As the scheme studied in [17] is the same as Scheme 2, we may explain this relaxation on the condition by the fact that the mathematical study performed in [17] did not benefit fully from the semi-implicit coupling between u and ϕ. We can confirm this argument by the fact that the condition of [17] matches (22) instead.

Moreover, when the fluid viscosity is small, (23) reads approximately

$$\Delta t < \sqrt{\frac{\Delta x^3}{\nu_e}} \tag{24}$$

We then retrieve a condition similar to that computed in [5] by an heuristic method, with a dependence in $\Delta x^{\frac{3}{2}}$.

In conclusion the stability condition relative to Scheme 2 is unifying the conditions of [17], valid when viscous effects are large and the one computed in [5] when inertial effect are dominating.

4 Numerical tests

In this section we numerically study the stability of the following scheme:

$$\begin{cases} \dfrac{u^{n+1}-u^n}{\Delta t} + u^n.\nabla u^n - \mu \Delta u^{n+1} + \nabla p^{n+1} - F(\phi^n) = 0 \\ F(\phi^n) = \left[\left(E''(|\nabla \phi^n|) \dfrac{\nabla^2 \phi^n \nabla \phi^n}{|\nabla \phi^n|} \dfrac{\nabla \times \phi^n}{|\nabla \phi^n|} \right) \dfrac{\nabla \times \phi^n}{|\nabla \phi^n|} \right. \\ \left. -E'(|\nabla \phi^n|)\kappa(\phi^n)\nabla \phi^n \right] \dfrac{1}{\varepsilon} \zeta \left(\dfrac{\phi^n}{\varepsilon |\nabla \phi^n|} \right) \\ \operatorname{div} u^{n+1} = 0 \\ \dfrac{\phi^{n+1}-\phi^n}{\Delta t} + u^{n+1} \nabla \phi^n = 0 \end{cases}$$

This is a 2D non linear version of Scheme 2. For this model we enforce the following condition on time step:

$$\Delta t \leq \Delta t_A, \quad \Delta t_A = C \frac{(\mu + \max(\mu, \sqrt{\nu_e \Delta x}))\Delta x}{\nu_e} \tag{25}$$

with C a positive constant determined numerically.

We validate numerically the condition (23) on a simple test case currently used in the litterature ([10], [2], [12]): an elastic membrane, which has initially an elliptic shape, relaxes toward a circular equilibrium state. The membrane is immerged in an incompressible fluid so that the interior area it delimits remains constant. The magnitude and frequency of oscillations around the equilibrium form depends on the fluid viscosity and the membrane's stiffness.

The initial membrane is an ellipse with major and minor axes $\tilde{a} = 0.65$ and $\tilde{b} = 0.575$ respectively. Its stretching, constant along the curve, is $e = 1.2526$. This configuration corresponds to an equilibrium circle of radius $R \simeq 0.6113$. The initial and equilibrium states are then quite close as the gap between the final radius and the initial axes is up to 6%. The initial velocity and pressure are set to zero. The computational domain is $[0, 2] \times [0, 2]$, and we set $\rho = 1$ and $\mu = 1$ throughout the domain. We define $M = 2/\Delta x$.

Our aim is to show that our stability condition is sharper than the two other conditions on the classical test described above when we vary coefficients. Thus in a first test, we fix Δx and test the dependence in ν_e of the condition (23), for ν_e from 1 to 10^6, with $\Delta x = 0.03125$ ($M = 64$). We control the time evolution of the ellipse axes. Each simulation is displayed for $t \in [0, 200/\sqrt{\nu_e}]$. The time steps Δt_A used in the simulations, obtained from (23) are presented in table 1, in comparison with the maximum time steps recommended by the conditions of [17] and [5]. The same constant C is used for Δt_A and Δt_V even if we could adjust it. However, the goal of the comparison is mostly to show the discrepancy between the time steps Δt_A and Δt_V as ν_e grows up. The maximum time step Δt_V, valid for small ν_e (with respect to viscosity), becomes far too much restrictive for high ν_e.

ν_e	Δt_A with $C = 0.2$	Δt_V with $C = 0.2$	Δt_B
1	1.25×10^{-2}	6.25×10^{-3}	$C \times 5.52427 \times 10^{-3}$
10^2	1.72985×10^{-4}	6.25×10^{-5}	$C \times 5.52427 \times 10^{-4}$
10^4	1.16735×10^{-5}	6.25×10^{-7}	$C \times 5.52427 \times 10^{-5}$
10^6	1.1111×10^{-6}	6.25×10^{-9}	$C \times 5.52427 \times 10^{-6}$

Table 1: The largest time steps allowed for stability, for $\mu = 1$, $M = 64$ and ν_e from 1 to 10^6: Δt_A computed with the condition (23) and used in the simulations, Δt_V and Δt_B determined with the conditions of [17] and [5].

From the pictures 1 and 2 we can deduce that the condition (23), less restrictive than the condition of [17] as ν_e grows up, enables to ensure stability for a certain range of stiffness ($\nu_e \leq 10^4$). Note that for $\nu_e = 10^6$, the equilibrium state is not reached, but for this kind of stiffness, strong non-linear instabilities are expected, and thus the conclusions of the linear analysis are no longer significant (this is in contrast with an unstable case with a time-step higher that Δt_A for which the simulation blows up in a few steps). Moreover, the oscillations size decreases quite rapidly, and becomes smaller that the grid size $\Delta x = 0.03125$, which may be another cause of instability.

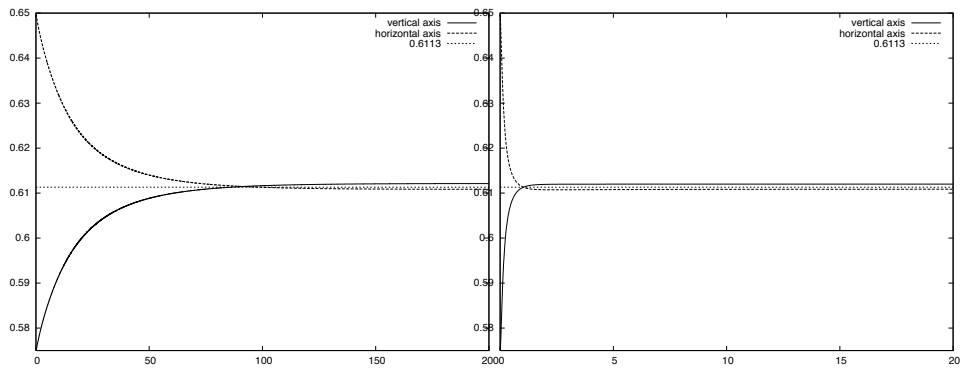

Figure 1: The time evolution of the membrane axes for $\nu_e = 1$ (left) and $\nu_e = 10^2$ (right), $\mu = 1$, $M = 64$.

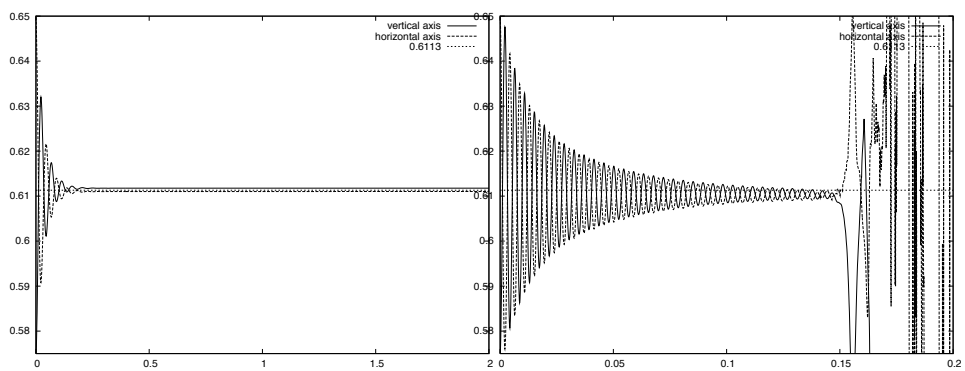

Figure 2: The time evolution of the membrane axes for $\nu_e = 10^4$ (left) and $\nu_e = 10^6$ (right), $\mu = 1$, $M = 64$.

A last clue that the instability for $\nu_e = 10^6$ has nothing to do with linear instability is that it persists when the time step is divided by e.g. 10. This suggests that an implicit coupling may fail to guarantee stability, in spite of

an increased computational cost. Indeed, computations are performed with a massless membrane, which means that the fluid density is set everywhere. Thus we are typically in the case of a large added mass effect, with strong interactions between pressure field variations and the membrane. It is shown in [3] that in this case the convergence of the iterative method in order to solve each time step may be hard to be reached, unless an under-relaxation scheme with a problem dependent coefficient is used.

Next we study the Δx-dependence of the condition (23), with $\nu_e = 1$ fixed. As before we present in table 2 the time steps Δt_A, Δt_V, and Δt_B, for $M = 64$, 128, 256. This is to advocate that the $\Delta x^{\frac{3}{2}}$ dependence of Δt_B is not sharp when surface tension effects are small.

M	Δt_A with $C = 0.2$	Δt_V with $C = 0.2$	Δt_B
64	1.25×10^{-2}	6.25×10^{-3}	$C \times 1.10485 \times 10^{-3}$
128	6.25×10^{-3}	3.125×10^{-3}	$C \times 3.90625 \times 10^{-4}$
256	3.125×10^{-3}	1.5625×10^{-3}	$C \times 1.38107 \times 10^{-4}$

Table 2: The largest time steps allowed for stability, for $\mu = 1$, $\nu_e = 1$ and M from 64 to 256: Δt_A computed with the condition (23) and used in the simulations, Δt_V and Δt_B determined with the conditions of [17] and [5].

The time evolution of the membrane axes displayed on picture 3 allows to confirm the Δx-dependence of the condition (23).

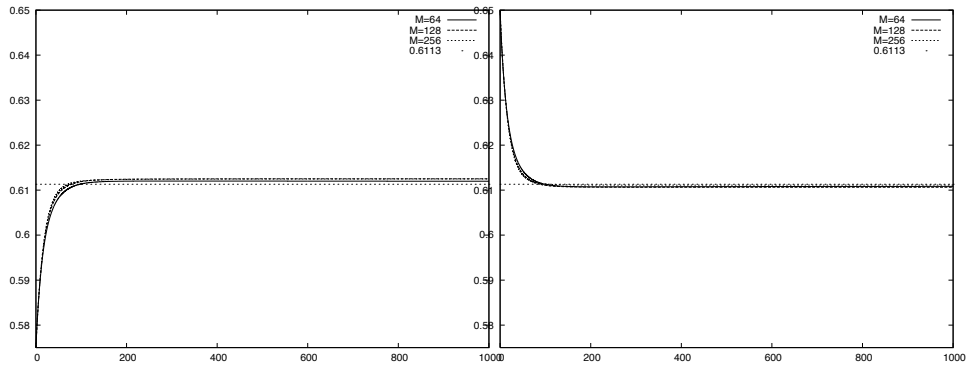

Figure 3: The time evolution of the membrane axes for $\nu_e = 1$, $\mu = 1$ and Δx variable. Left: vertical axe. Right: horizontal axe.

In conclusion, the dependence of the condition (23) in Δx and $1/\sqrt{\nu_e}$ is numerically validated on a non-linear 2D test case, for a certain stiffness range. As far as the ν_e-dependence is concerned, the stability of the naturally implemented semi-implicit scheme seems to be less conditioned than it was anounced in [17]. Furthermore, the condition (23) and that proposed in [5] give rise to time steps of the same magnitude when the viscous effects are small with respect to the surface tension effects.

5 Summary and conclusions

In this work we studied the numerical stability of three time-stepping schemes for a fluid-elastic membrane coupling model. The linear stability analysis based on a simplified 1D model enabled us to obtain stability conditions for explicit and semi-implicit coupling. According to the flow regime, these conditions can be compared to the conditions proposed either in [17] or in [5] with different methods. On the one hand, the dependence in $1/\sqrt{\nu_e}$ of the condition for semi-implicit coupling accounts for a relaxation of the condition of [17], especially for problems with strong membrane stiffness. On the other hand, the dependence in $\Delta x^{\frac{1}{2}}$ of the same condition allows to compare it to that of [5].
The numerical tests on a non-linear 2D case validate the analysis results for a certain range of membrane's stiffness.
The study of the added mass effect and the implementation of an efficient implicit scheme are under consideration for further work.

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