Homogenization of accelerated Frenkel-Kontorova models with n types of particles

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Abstract

We consider systems of ODEs that describe the dynamics of particles. Each particle satisfies a Newton law (including the acceleration term) where the force is created by the interactions with the other particles and with a periodic potential. The presence of a damping term allows the system to be monotone. Our study takes into account the fact that the particles can be different.

After a proper hyperbolic rescaling, we show that the solutions to this system of ODEs converge to the solution of a macroscopic homogenized Hamilton-Jacobi equation.

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1 Introduction

The goal of this paper is to show homogenization results for the dynamics of accelerated Frenkel-Kontorova type systems with n types of particles. In this introduction, we start with the simplest accelerated Frenkel-Kontorova model where there is only one type of particle (see Eq. (1.2)). We then explain how to deal with n types of particles (see Eq. (1.6)). We finally present the general case, namely systems of ODEs of the following form (for a fixed $m \in \mathbb{N}$)

$$m_0 \frac{d^2 U_i}{d\tau^2} + \frac{dU_i}{d\tau} = F_i(\tau, U_{i-m}, \dots, U_{i+m})\,. \tag{1.1}$$

where $U_i(\tau)$ denotes the position of the particle $i \in \mathbb{Z}$ at the time τ. Here, m_0 is the mass of the particle and F_i is the force acting on the particle i, which will be precised later. Remark the presence of the friction term $\frac{dU_i}{d\tau}$ on the left hand side of the equation. This term makes the dynamics overdamped and allow to see this system as a monotone system where the mass m_0 is choose small enough. Recall that the case of fully overdamped dynamics, i.e. for $m_0 = 0$, has already been treated in [9] (for only one type of particles).

An important assumption is that this system is monotone. See below for details and precise assumptions.

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1.1 The classical overdamped Frenkel-Kontorova model

The classical Frenkel-Kontorova model describes a chain of classical particles evolving in a one dimensional space, coupled with their neighbours and subjected to a periodic potential. If τ denotes time and $U_i(\tau)$ denotes the position of the particle $i \in \mathbb{Z}$, one of the simplest FK models is given by the following dynamics

$$m_0 \frac{d^2 U_i}{d\tau^2} + \frac{dU_i}{d\tau} = U_{i+1} - 2U_i + U_{i-1} + \sin(2\pi U_i) + L \tag{1.2}$$

where m_0 denotes the mass of the particle, L is a constant driving force which can make the whole "train of particles" move and the term $\sin(2\pi U_i)$ describes the force created by a periodic potential whose period is assumed to be 1. Notice that in the previous equation, we set to one physical constants in front of the elastic and the exterior forces (friction and periodic potential). The goal of our work is to say what is the macroscopic behaviour of the solution U of (1.2) as the number of particles per length unit goes to infinity. As mentioned above, the particular case where $m_0 = 0$ is referred to as the fully overdamped one and is studied in [9].

We would like next to give the flavour of the results we obtain in this paper. In order to do so, let us assume that at initial time, particles satisfy

$$\begin{aligned} U_i(0) &= \varepsilon^{-1} u_0(i\varepsilon) \\ \frac{dU_i}{d\tau}(0) &= 0 \end{aligned}$$

for some $\varepsilon > 0$ and some Lipschitz continuous function $u_0(x)$ which satisfies the following assumption

Initial gradient bounded from above and below

$$0 < 1/K_0 \leq (u_0)_x \leq K_0 \quad \text{on} \quad \mathbb{R} \tag{1.3}$$

for some fixed $K_0 > 0$.

Such an assumption can be interpreted by saying that at initial time, the number of particles per length unit lies in $(K_0^{-1}\varepsilon^{-1}, K_0\varepsilon^{-1})$.

It is then natural to ask what is the macroscopic behaviour of the solution U of (1.2) as ε goes to zero, *i.e.* as the number of particles per length unit goes to infinity. To this end, we define the following function which describes the rescaled positions of the particles

$$\overline{u}^\varepsilon(t,x) = \varepsilon U_{\lfloor \varepsilon^{-1} x \rfloor}(\varepsilon^{-1} t) \tag{1.4}$$

where $\lfloor \cdot \rfloor$ denotes the floor integer part. One of our main results states that the limiting dynamics as ε goes to 0 of (1.2) is determined by a first order Hamilton-Jacobi equation of the form

$$\begin{cases} u^0_t = \overline{F}(u^0_x) & \text{for} \quad (t,x) \in (0,+\infty) \times \mathbb{R}, \\ u^0(0,x) = u_0(x) & \text{for} \quad x \in \mathbb{R} \end{cases} \tag{1.5}$$

where $\overline{F}$ is a continuous function to be determined. More precisely, we have the following homogenization result

Theorem 1.1 (**Homogenization of the accelerated FK model**). *There exists a critical value m_0^c such that for all $m_0 \in]0, m_0^c]$ and all $L \in \mathbb{R}$, there exists a continuous function $\overline{F} : \mathbb{R} \to \mathbb{R}$ such that, under assumption* (1.3)*, the function $\overline{u}^\varepsilon$ converges locally uniformly towards the unique viscosity solution u^0 of* (1.5).

Remark 1.2. The critical mass m_0^c is made precise in Assumption (A3) below.

1.2 Example of systems with n types of particles

We now present the case of system with n types of particles. Let us start with the typical problem we have in mind. Let $n \in \mathbb{N}\setminus\{0\}$ be some integer and let us consider a sequence of real numbers $(\theta_i)_{i\in\mathbb{Z}}$ such that

$$\theta_{i+n} = \theta_i > 0 \quad \text{for all} \quad i \in \mathbb{Z}\,.$$

It is then natural to consider the generalized FK model with n different types of particles that stay ordered on the real line. Then, instead of satisfying (1.2), we can assume that U_i satisfies for $\tau \in (0,+\infty)$ and $i \in \mathbb{Z}$

$$m_0 \frac{d^2U_i}{d^2\tau} + \frac{dU_i}{d\tau} = \theta_{i+1}(U_{i+1}-U_i) - \theta_i(U_i - U_{i-1}) + \sin(2\pi U_i) + L \tag{1.6}$$

Such a model is sketched on figure 1.

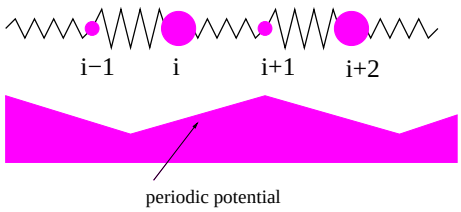

Figure 1: The FK model with $n = 2$ type of particles (and of springs) and an interaction up to the $m = 1$ neighbours

For this model, we can prove the same kind of homogenization result as Theorem 1.1.

For all $i \in \mathbb{Z}$, let us introduce the following function

$$\Xi_i(\tau) = U_i(\tau) + 2m_0 \frac{dU_i}{d\tau}(\tau)\,.$$

The idea to introduce this function, which allows us to obtain a monotone system, is inspired of the work of Baesens and MacKay [2] and of Hu, Qin and Zheng [10]. Here we will simplify the proof of the monotonicity of the system.

Using, this new function, the system of ODEs (1.6) can be rewritten in the following form: for $\tau \in (0,+\infty)$ and $i \in \mathbb{Z}$,

$$\begin{cases} \frac{dU_i}{d\tau} = \frac{1}{2m_0}(\Xi_i - U_i) \\ \\ \frac{d\Xi_i}{d\tau} = 2\theta_{i+1}(U_{i+1}-U_i) - 2\theta_i(U_i-U_{i-1}) + 2\sin(2\pi U_i) + 2L + \frac{1}{2m_0}(U_i - \Xi_i)\,. \end{cases}$$

It is convenient to introduce the following notation

$$\alpha_0 = \frac{1}{2m_0}\,.$$

Remark 1.3. It would be also possible to consider more generally: $\Xi_i(\tau) = U_i(\tau) + \frac{1}{\alpha}\frac{dU_i}{d\tau}(\tau)$ with $\frac{1}{\alpha} > m_0$. In order to simplify here the presentation, we choose $\alpha = 1/(2m_0)$. Moreover, for the classical Frenkel-Kontorova model (1.2), the choice $\alpha = 1/(2m_0)$ is optimal in the sense that the critical value m_0^c for which the system is monotone is the best we can get.

1.3 General systems with n types of particles

More generally, we would like to study the generalized Frenkel-Kontorova model (1.1) with n types of particles. In order to do so, let us consider a general sequence of functions $v = (v_j(y))_{j\in\mathbb{Z}}$ satisfying

$$v_{j+n}(y) = v_j(y+1)\,.$$

For $m \in \mathbb{N}$, we set

$$[v]_{j,m}(y) = (v_{j-m}(y), \dots, v_{j+m}(y))\,.$$

We are going to study a function

$$(u, \xi) = ((u_j(\tau, y))_{j\in\mathbb{Z}}, (\xi_j(\tau, y))_{j\in\mathbb{Z}})$$

satisfying the following system of equations: for all $(\tau, y) \in (0, +\infty) \times \mathbb{R}$ and all $j \in \mathbb{Z}$,

$$\left\{ \begin{array}{l} \left\{ \begin{array}{l} (u_j)_\tau = \alpha_0(\xi_j - u_j) \\ (\xi_j)_\tau = 2F_j(\tau, [u(\tau,\cdot)]_{j,m}) + \alpha_0(u_j - \xi_j)\,, \end{array} \right. \\ \\ \left\{ \begin{array}{l} u_{j+n}(\tau, y) = u_j(\tau, y+1) \\ \xi_{j+n}(\tau, y) = \xi_j(\tau, y+1)\,. \end{array} \right. \end{array} \right. \tag{1.7}$$

This system is referred to as *the generalized Frenkel-Kontorova (FK for short) model*. It is satisfied in the viscosity sense (see Definition 2.1). Moreover, we will consider viscosity solutions which are possibly discontinuous.

Let us now make precise the assumptions we make on the functions $F_j : \mathbb{R} \times \mathbb{R}^{2m+1} \to \mathbb{R}$ that maps (τ, V) to $F_j(\tau, V)$. It is convenient to write $V \in \mathbb{R}^{2m+1}$ as $(V_{-m}, \dots, V_m)$.

(A1) **(Regularity)**

$$\left\{ \begin{array}{l} F_j \text{ is continuous}\,, \\ F_j \text{ is Lipschitz continuous in } V \text{ uniformly in } \tau \text{ and } j\,. \end{array} \right.$$

(A2) **(Monotonicity in V_i, $i \neq 0$)**

$$F_j(\tau, V_{-m}, ..., V_m) \text{ is non-decreasing in } V_i \quad \text{for } i \neq 0\,.$$

(A3) **(Monotonicity in V_0)**

$$\alpha_0 + 2\frac{\partial F_j}{\partial V_0} \geq 0 \quad \text{for all } j \in \mathbb{Z}\,.$$

Keeping in mind the notation we chose above ($\alpha_0 = (2m_0)^{-1}$), this assumption can be interpreted as follows: the mass has to be small in comparison with the variations of the nonlinearity, which means that the system is sufficiently overdamped. This assumption guarantees that $2F_j(\tau, V) + \alpha_0 V_0$ is non-decreasing in V_0 for all $j \in \mathbb{Z}$.

(A4) **(Periodicity)**

$$\left\{ \begin{array}{l} F_j(\tau, V_{-m} + 1, ..., V_m + 1) = F_j(\tau, V_{-m}, ..., V_m)\,, \\ F_j(\tau + 1, V) = F_j(\tau, V)\,. \end{array} \right.$$

(A5) **(Periodicity of the type of particles)**

$$F_{j+n} = F_j \quad \text{for all} \quad j \in \mathbb{Z}\,.$$

When $n = 1$, we explained in [9] that the system of ODEs can be embedded into a single partial differential equation (more precisely, in a single ordinary differential equation with a real parameter x). Here, taking into account the "n-periodicity" of the indices j, it can be embedded into n coupled systems of equations.

The next assumption allows to guarantee that the ordering property of the particles, *i.e.* $u_j \le u_{j+1}$, is preserved for all time.

(A6) **(Ordering)** For all $(V_{-m}, \dots, V_m, V_{m+1}) \in \mathbb{R}^{2m+2}$ such that $V_{i+1} \ge V_i$ for all $1 \le |i| \le m$, we have

$$2F_{j+1}(\tau, V_{-m+1}, \dots, V_{m+1}) + \alpha_0 V_1 \ge 2F_j(\tau, V_{-m}, \dots, V_m) + \alpha_0 V_0 \,.$$

Remark 1.4. If, for all $j \in \{1, \dots, n-1\}$, we have $F_{j+1} = F_j$ then assumption (A6) is a direct consequence of assumptions (A2) and (A3).

Example 1. *We see that assumptions (A1)-(A5) are in particular satisfied for the FK system (1.6) with n types of particles ($\theta_{n+j} = \theta_j$), $m = 1$ and $F_j(\tau, V_{-1}, V_0, V_1) = \theta_{j+1}(V_1 - V_0) - \theta_j(V_0 - V_{-1}) + \sin(2\pi V_0) + L$ for $\alpha_0 \ge 2(\theta_j + \theta_{j+1}) + 4\pi$. To get (A6) we have to assume furthemore that $\alpha_0 \ge 4\theta_j + 4\pi$.*

We next rescale the generalized FK model and set for $\varepsilon > 0$:

$$\begin{cases} u_j^\varepsilon(t,x) = \varepsilon u_j\left(\dfrac{t}{\varepsilon}, \dfrac{x}{\varepsilon}\right) \\ \\ \xi_j^\varepsilon(t,x) = \varepsilon \xi_j\left(\dfrac{t}{\varepsilon}, \dfrac{x}{\varepsilon}\right) \end{cases}$$

Precisely, the function $(u^\varepsilon, \xi^\varepsilon) = \left(\left(u_j^\varepsilon(t,x)\right)_{j\in\mathbb{Z}}, \left(\xi_j^\varepsilon(t,x)\right)_{j\in\mathbb{Z}}\right)$ satisfies the following problem for all $j \in \mathbb{Z}$, $t > 0$, $x \in \mathbb{R}$,

$$\begin{cases} \begin{cases} (u_j^\varepsilon)_t = \alpha_0 \frac{\xi_j^\varepsilon - u_j^\varepsilon}{\varepsilon} \\ \\ (\xi_j^\varepsilon)_t = 2F_j\left(\frac{t}{\varepsilon}, \left[\frac{u^\varepsilon(t,\cdot)}{\varepsilon}\right]_{j,m}\right) + \alpha_0 \frac{u_j^\varepsilon - \xi_j^\varepsilon}{\varepsilon} \end{cases} \\ \\ \begin{cases} u_{j+n}^\varepsilon(t,x) = u_j^\varepsilon(t, x+\varepsilon) \\ \xi_{j+n}^\varepsilon(t,x) = \xi_j^\varepsilon(t, x+\varepsilon) \end{cases} \end{cases} \tag{1.8}$$

We assume the following initial conditions

$$\begin{cases} u_j^\varepsilon(0,x) = u_0\left(x + \frac{j\varepsilon}{n}\right) \\ \xi_j^\varepsilon(0,x) = u_0\left(x + \frac{j\varepsilon}{n}\right) + \frac{\varepsilon}{\alpha_0} v_0\left(x + \frac{j\varepsilon}{n}\right) \end{cases} \tag{1.9}$$

where v_0 is a bounded function. In the sequel, we denote $\xi_0^\varepsilon = u_0 + \frac{\varepsilon}{\alpha_0} v_0$. Finally, we assume that u_0 and ξ_0^ε satisfy

(A0) (Gradient bound from below) There exist $K_0 > 0$ and $M_0 > 0$ such that

$$\begin{aligned} 0 < 1/K_0 \le (u_0)_x \le K_0 \quad &\text{on} \quad \mathbb{R}\,, \\ 0 < 1/K_0 \le (\xi_0^\varepsilon)_x \le K_0 \quad &\text{on} \quad \mathbb{R}\,, \\ \|u_0 - \xi_0^\varepsilon\|_\infty &\le M_0 \varepsilon\,. \end{aligned}$$

Then we have the following homogenization result

Theorem 1.5 (**Homogenization of systems with n types of particles**). *Assume that $(F_j)_j$ satisfies* (A1)-(A6)*, and assume that the initial data u_0, ξ_0^ε satisfy* (A0)*. Consider the solution $((u_j^\varepsilon)_{j\in\mathbb{Z}}, (\xi_j^\varepsilon)_{j\in\mathbb{Z}})$ of* (1.8)-(1.9)*. Then, there exists a continuous function $\overline{F} : \mathbb{R} \mapsto \mathbb{R}$ such that, for all integer $j \in \mathbb{Z}$, the functions u_j^ε and ξ_j^ε converge uniformly on compact sets of $(0, +\infty) \times \mathbb{R}$ to the unique viscosity solution u^0 of (1.5).*

Remark 1.6. The fact that we obtain, at the limit, just one equation to describe the evolution of the system can seem surprising. In fact, this essentially comes from assumption (A6) and the definition of ξ_j^ε. Indeed, it could be shown that assumption (A6) implies the order $u_{j+1}^\varepsilon \geq u_j^\varepsilon$ and $\xi_{j+1}^\varepsilon \geq \xi_j^\varepsilon$. Then, the system can be essentially sketched by just two equations (one for the evolution of the u and one for the ξ). But by microscopic definition of ξ_j^ε, we have $\xi_j^\varepsilon = u_j^\varepsilon + O(\varepsilon)$ and so just one equation is sufficient to describe the macroscopic evolution of all the system.

Remark 1.7. In the case $m_0 = 0$, i.e. $\alpha_0 = +\infty$, and $u^\varepsilon \equiv \xi^\varepsilon$ in (1.8), Theorem 1.5 still holds. Indeed, this is a special case that is contained in our analysis.

We will explain in the next subsection how the non-linearity $\bar{F}$, known as the effective Hamiltonian, is determined. We will see that this has to do with the existence of solutions of (1.8), (1.9) with a special form. More precisely, the solutions are constructed thanks to functions known as hull functions.

1.4 Hull functions

In this subsection, we introduce the notion of *hull function* for system (1.7). More precisely, we look for particular functions $((h_j(\tau, z))_{j\in\mathbb{Z}}, (g_j(\tau, z))_{j\in\mathbb{Z}}$ such that $(u_j(\tau, y) = h_j(\tau, py + \lambda\tau), \xi_j(\tau, y) = g_j(\tau, py + \lambda\tau))$ is a solution of (1.7) on $\Omega = (-\infty, +\infty) \times \mathbb{R} = \mathbb{R}^2$. Here is a precise definition.

Definition 1.8 (**Hull function for systems of n types of particles**).
Given $(F_j)_j$ satisfying (A1)-(A6)*, $p \in (0, +\infty)$ and a number $\lambda \in \mathbb{R}$, we say that a family of functions $((h_j)_j, (g_j)_j)$ is* a hull function *for* (1.7) *if it satisfies for all $(\tau, z) \in \mathbb{R}^2$, $j \in \mathbb{Z}$*

$$
\left\{
\begin{array}{l}
(h_j)_\tau + \lambda (h_j)_z = \alpha_0 (g_j - h_j) \\
\\
h_j(\tau + 1, z) = h_j(\tau, z) \\
h_j(\tau, z + 1) = h_j(\tau, z) + 1 \\
h_{j+n}(\tau, z) = h_j(\tau, z + p) \\
h_{j+1}(\tau, z) \geq h_j(\tau, z) \\
(h_j)_z(\tau, z) \geq 0 \\
\exists C \text{ s.t. } |h_j(\tau, z) - z| \leq C
\end{array}
\right.
\qquad
\left\{
\begin{array}{l}
(g_j)_\tau + \lambda (g_j)_z = 2F_j(\tau, [h(\tau, \cdot)]_{j,m}(z)) + \alpha_0 (h_j - g_j) \\
\\
g_j(\tau + 1, z) = g_j(\tau, z) \\
g_j(\tau, z + 1) = g_j(\tau, z) + 1 \\
g_{j+n}(\tau, z) = g_j(\tau, z + p) \\
g_{j+1}(\tau, z) \geq g_j(\tau, z) \\
(g_j)_z(\tau, z) \geq 0 \\
\exists C \text{ s.t. } |g_j(\tau, z) - z| \leq C\,.
\end{array}
\right. \tag{1.10}
$$

In the case where $(F_j)_j$ do not depend on τ, we also require that the hull function $((h_j)_j, (g_j)_j)$ are independent on τ and we denote it by $((h_j(z))_j, (g_j(z))_j)$.

Remark 1.9. The last line of (1.10) implies in particular that $\varepsilon h_j(\tau, \frac{z}{\varepsilon}) \to z$ and $\varepsilon g_j(\tau, \frac{z}{\varepsilon}) \to z$ as $\varepsilon \to 0$.

Given $p > 0$, the following theorem explains how the effective Hamiltonian $\overline{F}(p)$ is determined by an existence/non-existence result of hull functions as $\lambda \in \mathbb{R}$ varies.

Theorem 1.10 (**Effective Hamiltonian and hull function**). *Given $(F_j)_j$ satisfying* (A1)-(A6) *and $p \in (0, +\infty)$, there exists a unique real number λ for which there exists a hull function $((h_j)_j, (g_j)_j)$ (depending on p) satisfying* (1.10)*. Moreover the real number $\lambda = \overline{F}(p)$, seen as a function of p, is continuous on $(0, +\infty)$.*

1.5 Qualitative properties of the effective Hamiltonian

We have moreover the following result

Theorem 1.11 (Qualitative properties of $\overline{F}$). *Let $(F_j)_j$ satisfying* (A1)-(A6). *For any constant $L \in \mathbb{R}$, let $\overline{F}(L,p)$ denote the effective Hamiltonian given in Theorem 1.10 for $p \in (0,+\infty)$, associated with $(F_j)_j$ replaced by $(L+F_j)_j$.*

Then $(L,p) \mapsto \overline{F}(L,p)$ is continuous and we have the following properties

(i) (Bound) *we have*

$$|\overline{F}(L,p) - L| \leq C_p\,.$$

(ii) (Monotonicity in L)

$$\overline{F}(L,p) \quad \textit{is non-decreasing in} \quad L\,.$$

1.6 Organization of the article

In Section 2, we give some useful results concerning viscosity solutions for systems. In Section 3, we prove the convergence result assuming the existence of hull functions. The construction of hull functions is given in Sections 4 and 5. Finally, Section 6 is devoted to the proof of the qualitative properties of the effective Hamiltonian.

1.7 Notation

Given $r, R > 0$, $t \in \mathbb{R}$ and $x \in \mathbb{R}$, $Q_{r,R}(t,x)$ denotes the following neighbourhood of (t,x)

$$Q_{r,R}(t,x) = (t-r, t+r) \times (x-R, x+R)\,.$$

For $V = (V_1, \dots, V_N) \in \mathbb{R}^N$, $|V|_\infty$ denotes $\max_j |V_j|$. Given a family of function $(v_j(\cdot))_{j\in\mathbb{Z}}$ and two integers $j, m \in \mathbb{Z}$, $[v]_{j,m}$ denotes the function $(v_{j-m}(\cdot), \dots, v_{j+m}(\cdot))$.

2 Viscosity solutions

This section is devoted to the definition of viscosity solutions for systems of equations such as (1.7), (1.8) and (1.10). In order to construct hull functions when proving Theorem 1.10, we will also need to consider a perturbation of (1.7) with linear plus bounded initial data. For all these reasons, we define a viscosity solution for a generic equation whose Hamiltonian $(G_j)_j$ satisfies proper assumptions.

Before making precise assumptions, definitions and fundamental results we will need later (such as stability, comparison principle, existence), we refer the reader to the user's guide of Crandall, Ishii, Lions [5] and the book of Barles [3] for an introduction to viscosity solutions and [4, 18, 14, 15] and references therein for results concerning viscosity solutions for systems of weakly coupled partial differential equations.

2.1 Main assumptions and definitions

For $0 < T \leq +\infty$, consider the following Cauchy problem (for general non-linearities $(G_j)_j$): for $j \in \mathbb{Z}$, $\tau > 0$ and $y \in \mathbb{R}$,

$$\left\{ \begin{array}{l} \left\{ \begin{array}{l} (u_j)_\tau = \alpha_0(\xi_j - u_j) \\ (\xi_j)_\tau = G_j(\tau, [u(\tau,\cdot)]_{j,m}, \xi_j, py - \xi_j(\tau,y) - \inf_{y'\in\mathbb{R}} \left(\xi_j(\tau,y') - py'\right), (\xi_j)_y) \end{array} \right. \\ \\ \left\{ \begin{array}{l} u_{j+n}(\tau,y) = u_j(\tau, y+1) \\ \xi_{j+n}(\tau,y) = \xi_j(\tau,y+1) \end{array} \right. \end{array} \right. \tag{2.1}$$

submitted to the initial conditions

$$\left\{ \begin{array}{l} u_j(0,y) = u_0(y + \frac{j}{n}) := u_{0,j}(y) \\ \xi_j(0,y) = \xi_0(y + \frac{j}{n}) := \xi_{0,j}(y)\,. \end{array} \right. \tag{2.2}$$

Example 2. *The most important example we have in mind is the following one*

$$G_j(\tau, V_{-m}, \cdots, V_m, r, a, q) = 2F_j(\tau, V) + \alpha_0(V_0 - r) + \delta(a_0 + a)q^+$$

for some constants $\delta \geq 0$, $a_0, a, q \in \mathbb{R}$ *and where* F_j *appears in* (1.7),(1.8), (1.10).

We make the following assumptions on (G_j).

(A1') **(Regularity)**

(i) G_j is continuous

(ii) For all $R > 0$, there exists $L_0 > 0$ such that for all $\tau, V, W, r, s, a, q_1, q_2, j$, with $a \in [-R, R]$, we have

$$|G_j(\tau, V, r, a, q_1) - G_j(\tau, W, s, a, q_2)| \leq L_0|V - W|_\infty + L_0|r - s| + L_0|q_1 - q_2|\,.$$

(iii) There exists $L_1 > 0$ such that for all V, a, b, τ, r, q,

$$|G_j(\tau, V, r, a, q) - G_j(\tau, V, r, b, q)| \leq L_1|a - b||q|\,.$$

(A2') **(Monotonicity in V_i, $i \neq 0$)**

$$G_j(\tau, V_{-m}, ..., V_m, r, a, q) \text{ is non-decreasing in } V_i \text{ for } i \neq 0.$$

(A3') **(Monotonicity in a and V_0)**

$$G_j(\tau, V_{-m}, ..., V_m, r, a, q) \text{ is non-decreasing in } a \text{ and in } V_0.$$

(A4') **(Periodicity)** For all $(\tau, V, r, a, q) \in \mathbb{R} \times \mathbb{R}^{2m+1} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ and $j \in \{1, \ldots, n\}$

$$\begin{cases} G_j(\tau, V_{-m} + 1, ..., V_m + 1, r + 1, a, q) = G_j(\tau, V_{-m}, ..., V_m, r, a, q)\,, \\ G_j(\tau + 1, V, r, a, q) = G_j(\tau, V, r, a, q)\,. \end{cases}$$

(A5') **(Periodicity of the type of particles)**

$$G_{j+n} = G_j \quad \text{for all} \quad j \in \mathbb{Z}\,.$$

(A6') **(Ordering)** For all $(V_{-m}, \ldots, V_m, V_{m+1}) \in \mathbb{R}^{2m+2}$ such that $\forall i, V_{i+1} \geq V_i$, we have

$$G_{j+1}(\tau, V_{-m+1}, \ldots, V_{m+1}, r, a, q) \geq G_j(\tau, V_{-m}, \ldots, V_m, r, a, q)\,.$$

In view of (2.1), it is clear that, if G_j effectively depends on the variable a, solutions must be such that the infimum of $\xi_j(\tau, y) - p \cdot y$ is finite for all time τ. In this case, we will only consider solutions ξ_j satisfying for some $C_0(T) > 0$: for all $\tau \in [0, T)$ and all $y, y' \in \mathbb{R}$

$$|\xi_j(\tau, y + y') - \xi_j(\tau, y) - py'| \leq C_0\,. \tag{2.3}$$

When $T = +\infty$, we may assume that (2.3) holds true for all time $T_0 > 0$ for a family of constants $C_0 > 0$.

Since we have to solve a Cauchy problem, we have to assume that the initial datum satisfies the assumption

(A0') **(Initial condition)**

(u_0, ξ_0) satisfies (A0) (with $\varepsilon = 1$); it also satisfies (2.3) if G_j depends on a for some j.

Finally, we recall the definition of the upper and lower semi-continuous envelopes, u^* and u_*, of a locally bounded function u.

$$u^*(\tau,y)=\limsup_{(t,x)\to(\tau,y)} u(t,x) \quad \text{and} \quad u_*(\tau,y)=\liminf_{(t,x)\to(\tau,y)} u(t,x)\,.$$

We can now define viscosity solutions for (2.1).

Definition 2.1 (Viscosity solutions). *Let $T>0$ and $u_0:\mathbb{R}\to\mathbb{R}$ and $\xi_0:\mathbb{R}\to\mathbb{R}$ be such that* (A0') *is satisfied. Let $u_j:\mathbb{R}^+\times\mathbb{R}\to\mathbb{R}$ and $\xi_j:\mathbb{R}^+\times\mathbb{R}\to\mathbb{R}$ be locally bounded. We denote by $\Omega=(0,T]\times\mathbb{R}$.*

– *The function $((u_j)_j,(\xi_j)_j)$ is a* subsolution *(resp. a* supersolution*) of* (2.1) *on Ω if* (2.3) *holds true for ξ_j in the case where G_j depends on a,*

$$\forall j,n,\forall(\tau,y),\quad u_{j+n}(\tau,y)=u_j(\tau,y+1),\quad \xi_{j+n}(\tau,y)=\xi_j(\tau,y+1)$$

and for all $j\in\{1,\dots,n\}$, u_j and ξ_j are upper semi-continuous (resp. lower semi-continuous), and for all $(\tau,y)\in\Omega$ and all test function $\phi\in C^1(\Omega)$ such that $u_j-\phi$ attains a local maximum (resp. a local minimum) at the point (τ,y), then we have

$$\phi_\tau(\tau,y)\le\alpha_0(\xi_j(\tau,y)-u_j(\tau,y)) \quad (\textit{resp. } \ge) \tag{2.4}$$

and for all $(\tau,y)\in\Omega$ and all test function $\phi\in C^1(\Omega)$ such that $\xi_j-\phi$ attains a local maximum (resp. a local minimum) at the point (τ,y), then we have

$$\phi_\tau(\tau,y)\le G_j(\tau,[u(\tau,\cdot)]_{j,m}(y),\xi_j(\tau,y),\inf_{y'\in\mathbb{R}}(\xi_j(\tau,y')-py')+py-\xi_j(\tau,u),\phi_y(\tau,y)) \tag{2.5}$$

$$(\textit{resp. } \ge).$$

– *The function $((u_j)_j,(\xi_j)_j)$ is a* subsolution *(resp.* supersolution*) of* (2.1),(2.2) *if $((u_j)_j,(\xi_j)_j)$ is a subsolution (resp. supersolution) on Ω and if it satisfies moreover for all $y\in\mathbb{R}$, $j\in\{1,\dots,n\}$*

$$u_j(0,y)\le u_0(y+\frac{j}{n}) \quad (\textit{resp. } \ge)\,,$$
$$\xi_j(0,y)\le \xi_0(y+\frac{j}{n}) \quad (\textit{resp. } \ge)\,.$$

– *A function $((u_j)_j,(\xi_j)_j)$ is a* viscosity solution *of (2.1) (resp. of* (2.1),(2.2)*) if $((u_j^*)_j,(\xi_j^*)_j)$ is a subsolution and $(((u_j)_*)_j,((\xi_j)_*)_j)$ is a supersolution of (2.1) (resp. of* (2.1),(2.2)*).*

Sub- and supersolutions satisfy the following comparison principle which is a key property of the equation.

Proposition 2.2 (Comparison principle).
Assume (A0') *and that $(G_j)_j$ satisfies* (A1')-(A5')*. Let (u_j,ξ_j) (resp. (v_j,ζ_j)) be a subsolution (resp. a supersolution) of* (2.1)*,* (2.2) *such that* (2.3) *holds true for ξ_j and ζ_j in the case where G_j depends on a. We also assume that there exists a constant $K>0$ such that for all $j\in\{1,\dots,n\}$ and $(t,x)\in[0,T]\times\mathbb{R}$, we have:*

$$u_j(t,x)\le u_{0,j}(x)+K(1+t),\quad \xi_j(t,x)\le\xi_{0,j}(x)+K(1+t) \tag{2.6}$$
$$(\text{resp. } -v_j(t,x)\le -u_{0,j}(x)+K(1+t),\quad -\zeta_j(t,x)\le-\xi_{0,j}(x)+K(1+t))$$

If

$$u_j(0,x)\le v_j(0,x)\quad\text{and}\quad \xi_j(0,x)\le\zeta_j(0,x)\quad \textit{for all } j\in\mathbb{Z},\ x\in\mathbb{R}\,,$$

then

$$u_j(t,x)\le v_j(t,x)\quad\text{and}\quad \xi_j(t,x)\le\zeta_j(t,x)\quad \textit{for all } j\in\mathbb{Z},\ (t,x)\in[0,T]\times\mathbb{R}\,.$$

Remark 2.3. Even if it was not specified in [9], the Lipschitz continuity in q of G_j is necessary to obtain a general comparison principle.

Proof. In view of assumption (A1')(i) and using the change of variable $\bar{u}_j(t,x) = e^{-\lambda t}u_j(t,x)$ and $\bar{\xi}_j(t,x) = e^{-\lambda t}\xi_j(t,x)$, it is classical that without loss of generality, we can assume that for all $r \geq s$

$$G_j(\tau, V, r, a, q) - G_j(\tau, V, s, a, q) \leq -L'(r-s) \tag{2.7}$$

for $L' \geq L_0 > 0$.

We define

$$M = \sup_{(t,x)\in(0,T)\times\mathbb{R}} \max_{j\in\{1,\dots,n\}} \max\left(u_j(t,x) - v_j(t,x), \xi_j(t,x) - \zeta_j(t,x)\right) .$$

The proof is decomposed in several steps.

Step 1: The test function

We argue by contradiction by assuming that $M > 0$. Classically, we duplicate the space variable by considering for ε, α and η "small" positive parameters, the functions

$$\varphi(t,x,y,j) = u_j(t,x) - v_j(t,y) - e^{At}\frac{|x-y|^2}{2\varepsilon} - \alpha|x|^2 - \frac{\eta}{T-t}$$
$$\phi(t,x,y,j) = \xi_j(t,x) - \zeta_j(t,y) - e^{At}\frac{|x-y|^2}{2\varepsilon} - \alpha|x|^2 - \frac{\eta}{T-t}$$

where A is a positive constant which will be chosen later. We also consider

$$\Psi(t,x,y,j) = \max(\varphi(t,x,y,j), \phi(t,x,y,j)),$$

Using inequalities (2.6) and assumption (A0'), we get

$$u_j(t,x) - v_j(t,y) \leq u_{0,j}(x) - u_{0,j}(y) + 2K(1+T) \leq K_0|x-y| + 2K(1+T)$$

and

$$\xi_j(t,x) - \zeta_j(t,y) \leq K_0|x-y| + 2K(1+T)$$

We then deduce that

$$\lim_{|x|,|y|\to\infty} \varphi(t,x,y,j) = -\infty = \lim_{|x|,|y|\to\infty} \phi(t,x,y,j)\,,$$

Using also the fact that φ and ϕ are u.s.c, we deduce that Ψ reaches its maximum at some point $(\bar{t},\bar{x},\bar{y},\bar{j})$.

Let us assume that $\Psi(\bar{t},\bar{x},\bar{y},\bar{j}) = \phi(\bar{t},\bar{x},\bar{y},\bar{j})$ (the other case being similar and even simpler). We first remark that for α and η small enough, we have

$$\Psi(\bar{t},\bar{x},\bar{y},\bar{j}) = M_{\varepsilon,\alpha,\eta} \geq \frac{M}{2} > 0\,.$$

In particular,

$$\xi_{\bar{j}}(\bar{t},\bar{x}) - \zeta_{\bar{j}}(\bar{t},\bar{y}) > 0\,.$$

Step 2: Viscosity inequalities for $\bar{t} > 0$

By duplicating the time variable and passing to the limit [5, 3], we classically get that there are real numbers $a, b, \bar{p} \in \mathbb{R}$ such that

$$a - b = \frac{\eta}{(T-\bar{t})^2} + Ae^{A\bar{t}}\frac{|\bar{x}-\bar{y}|^2}{2\varepsilon}, \quad \bar{p} = e^{A\bar{t}}\frac{\bar{x}-\bar{y}}{\varepsilon}$$

and

$$\begin{aligned} a &\leq G_{\bar{j}}(\bar{t}, [u(\bar{t},\cdot)]_{\bar{j},m}(\bar{x}), \xi_{\bar{j}}(\bar{t},\bar{x}), \inf(\xi_{\bar{j}}(\bar{t},y') - py') + p\bar{x} - \xi_{\bar{j}}(\bar{t},\bar{x}), \bar{p} + 2\alpha\bar{x}) \\ b &\geq G_{\bar{j}}(\bar{t}, [v(\bar{t},\cdot)]_{\bar{j},m}(\bar{y}), \zeta_{\bar{j}}(\bar{t},\bar{y}), \inf(\zeta_{\bar{j}}(\bar{t},y') - py') + p\bar{y} - \zeta_{\bar{j}}(\bar{t},\bar{y}), \bar{p}). \end{aligned}$$

Subtracting the two above inequalities, we get

$$\begin{aligned} \frac{\eta}{T^2} + Ae^{A\bar{t}}\frac{|\bar{x}-\bar{y}|^2}{2\varepsilon} \leq & G_{\bar{j}}(\bar{t}, [u(\bar{t},\cdot)]_{\bar{j},m}(\bar{x}), \xi_{\bar{j}}(\bar{t},\bar{x}), \inf(\xi_{\bar{j}}(\bar{t},y') - py') + p\bar{x} - \xi_{\bar{j}}(\bar{t},\bar{x}), \bar{p} + 2\alpha\bar{x}) \\ & - G_{\bar{j}}(\bar{t}, [v(\bar{t},\cdot)]_{\bar{j},m}(\bar{y}), \zeta_{\bar{j}}(\bar{t},\bar{y}), \inf(\zeta_{\bar{j}}(\bar{t},y') - py') + p\bar{y} - \zeta_{\bar{j}}(\bar{t},\bar{y}), \bar{p}) \end{aligned} \tag{2.8}$$

Step 3: Estimate on $u_k(\bar{t},\bar{x}) - v_k(\bar{t},\bar{y})$
If $k \in \{1,\dots,n\}$, by the inequality $\varphi(\bar{t},\bar{x},\bar{y},k) \leq \phi(\bar{t},\bar{x},\bar{y},\bar{j})$, we directly get that

$$u_k(\bar{t},\bar{x}) - v_k(\bar{t},\bar{y}) \leq \xi_{\bar{j}}(\bar{t},\bar{x}) - \zeta_{\bar{j}}(\bar{t},\bar{y})\,.$$

If $k \notin \{1,\dots,n\}$, let us define $l_k \in \mathbb{Z}$ such that $k - l_k n = \tilde{k} \in \{1,\dots,n\}$. By periodicity, we then have

$$\begin{aligned} u_k(\bar{t},\bar{x}) - v_k(\bar{t},\bar{y}) =& u_{\tilde{k}+l_k n}(\bar{t},\bar{x}) - v_{\tilde{k}+l_k n}(\bar{t},\bar{y}) \\ =& u_{\tilde{k}}(\bar{t},\bar{x}+l_k) - v_{\tilde{k}}(\bar{t},\bar{y}+l_k) \\ \leq& \xi_{\bar{j}}(\bar{t},\bar{x}) - \zeta_{\bar{j}}(\bar{t},\bar{y}) - \alpha(|\bar{x}|^2 - |\bar{x}+l_k|^2) \end{aligned}$$

where we have used the inequality $\varphi(\bar{t},\bar{x}+l_k,\bar{y}+l_k,\tilde{k}) \leq \phi(\bar{t},\bar{x},\bar{y},\bar{j})$ for the last line. So, for all $k \in \mathbb{Z}$ (and in particular for $k \in \{\bar{j}-m,\dots,\bar{j}+m\}$), we finally deduce that

$$u_k(\bar{t},\bar{x}) - v_k(\bar{t},\bar{y}) \leq \xi_{\bar{j}}(\bar{t},\bar{x}) - \zeta_{\bar{j}}(\bar{t},\bar{y}) + \alpha\left||\bar{x}|^2 - |\bar{x}+l_k|^2\right|. \tag{2.9}$$

Step 4: Estimate on the right hand side of (2.8)

We now want to estimate the right hand side of (2.8). We have

$$
\begin{aligned}
&G_{\bar{j}}\left(\bar{t},[u(\bar{t},\cdot)]_{\bar{j},m}(\bar{x}),\xi_{\bar{j}}(\bar{t},\bar{x}),\inf(\xi_{\bar{j}}(\bar{t},y')-py')+p\bar{x}-\xi_{\bar{j}}(\bar{t},\bar{x}),\bar{p}+2\alpha\bar{x}\right)\\
&-G_{\bar{j}}\left(\bar{t},[v(\bar{t},\cdot)]_{\bar{j},m}(\bar{y}),\zeta_{\bar{j}}(\bar{t},\bar{y}),\inf(\zeta_{\bar{j}}(\bar{t},y')-py')+p\bar{y}-\zeta_{\bar{j}}(\bar{t},\bar{y}),\bar{p}\right)\\
\\
\leq& G_{\bar{j}}\Bigg(\bar{t},\left[v(\bar{t},\cdot)+\xi_{\bar{j}}(\bar{t},\bar{x})-\zeta_{\bar{j}}(\bar{t},\bar{y})+\alpha\left||\bar{x}|^2-|\bar{x}+l.|^2\right|\right]_{\bar{j},m}(\bar{y}),\xi_{\bar{j}}(\bar{t},\bar{x}),\\
&\qquad\inf(\xi_{\bar{j}}(\bar{t},y')-py')+p\bar{x}-\xi_{\bar{j}}(\bar{t},\bar{x}),\bar{p}+2\alpha\bar{x}\Bigg)\\
&-G_{\bar{j}}\left(\bar{t},[v(\bar{t},\cdot)]_{\bar{j},m}(\bar{y}),\zeta_{\bar{j}}(\bar{t},\bar{y}),\inf(\zeta_{\bar{j}}(\bar{t},y')-py')+p\bar{y}-\zeta_{\bar{j}}(\bar{t},\bar{y}),\bar{p}\right)\\
\\
\leq& L_0(\xi_{\bar{j}}(\bar{t},\bar{x})-\zeta_{\bar{j}}(\bar{t},\bar{y}))+L_0\alpha\max_{k\in\{\bar{j}-m,\dots,\bar{j}+m\}}\left||\bar{x}|^2-|\bar{x}+l_k|^2\right|\\
&+G_{\bar{j}}\left(\bar{t},[v(\bar{t},\cdot)]_{\bar{j},m}(\bar{y}),\xi_{\bar{j}}(\bar{t},\bar{x}),\inf(\xi_{\bar{j}}(\bar{t},y')-py')+p\bar{x}-\xi_{\bar{j}}(\bar{t},\bar{x}),\bar{p}+2\alpha\bar{x}\right)\\
&-G_{\bar{j}}\left(\bar{t},[v(\bar{t},\cdot)]_{\bar{j},m}(\bar{y}),\zeta_{\bar{j}}(\bar{t},\bar{y}),\inf(\zeta_{\bar{j}}(\bar{t},y')-py')+p\bar{y}-\zeta_{\bar{j}}(\bar{t},\bar{y}),\bar{p}\right)\\
\\
\leq& L_0(\xi_{\bar{j}}(\bar{t},\bar{x})-\zeta_{\bar{j}}(\bar{t},\bar{y}))+L_0\alpha\max_{k\in\{\bar{j}-m,\dots,\bar{j}+m\}}\left||\bar{x}|^2-|\bar{x}+l_k|^2\right|-L'(\xi_{\bar{j}}(\bar{t},\bar{x})-\zeta_{\bar{j}}(\bar{t},\bar{y}))\\
&+G_{\bar{j}}\left(\bar{t},[v(\bar{t},\cdot)]_{\bar{j},m}(\bar{y}),\zeta_{\bar{j}}(\bar{t},\bar{y}),\inf(\xi_{\bar{j}}(\bar{t},y')-py')+p\bar{x}-\xi_{\bar{j}}(\bar{t},\bar{x}),\bar{p}+2\alpha\bar{x}\right)\\
&-G_{\bar{j}}\left(\bar{t},[v(\bar{t},\cdot)]_{\bar{j},m}(\bar{y}),\zeta_{\bar{j}}(\bar{t},\bar{y}),\inf(\zeta_{\bar{j}}(\bar{t},y')-py')+p\bar{y}-\zeta_{\bar{j}}(\bar{t},\bar{y}),\bar{p}\right)\\
\\
\leq& L\alpha\max_{k\in\{\bar{j}-m,\dots,\bar{j}+m\}}(2|l_k\bar{x}|+l_k^2)\\
&+L_1\left(\inf(\xi_{\bar{j}}(\bar{t},y')-py')+p\bar{x}-\xi_{\bar{j}}(\bar{t},\bar{x})-\inf(\zeta_{\bar{j}}(\bar{t},y')-py')-p\bar{y}+\zeta_{\bar{j}}(\bar{t},\bar{y})\right)|\bar{p}|\\
&+G_{\bar{j}}\left(\bar{t},[v(\bar{t},\cdot)]_{\bar{j},m}(\bar{y}),\zeta_{\bar{j}}(\bar{t},\bar{y}),\inf(\xi_{\bar{j}}(\bar{t},y')-py')+p\bar{y}-\xi_{\bar{j}}(\bar{t},\bar{x}),\bar{p}+2\alpha\bar{x}\right)\\
&-G_{\bar{j}}\left(\bar{t},[v(\bar{t},\cdot)]_{\bar{j},m}(\bar{y}),\zeta_{\bar{j}}(\bar{t},\bar{y}),\inf(\xi_{\bar{j}}(\bar{t},y')-py')+p\bar{y}-\xi_{\bar{j}}(\bar{t},\bar{x}),\bar{p}\right)
\end{aligned}
\tag{2.10}
$$

where we have used (2.9) for the first inequality, assumption (A1')(ii) for the second one, (2.7) for the third one and assumption (A1')(iii) for the last one.

Using the fact that $\alpha|\bar{x}|\to 0$ as $\alpha\to 0$, we deduce that

$$
\begin{aligned}
&2mL\alpha\max_k(2|l_k\bar{x}|+l_k^2)\\
&+G_{\bar{j}}\left(\bar{t},[v(\bar{t},\cdot)]_{\bar{j},m}(\bar{y}),\zeta_{\bar{j}}(\bar{t},\bar{y}),\inf(\xi_{\bar{j}}(\bar{t},y')-py')+p\bar{y}-\xi_{\bar{j}}(\bar{t},\bar{x}),\bar{p}+2\alpha\bar{x}\right)\\
&-G_{\bar{j}}\left(\bar{t},[v(\bar{t},\cdot)]_{\bar{j},m}(\bar{y}),\zeta_{\bar{j}}(\bar{t},\bar{y}),\inf(\xi_{\bar{j}}(\bar{t},y')-py')+p\bar{y}-\xi_{\bar{j}}(\bar{t},\bar{y}),\bar{p}\right)\\
=&o_\alpha(1)
\end{aligned}
$$

where we have used (2.3) to bound $\inf(\xi_{\bar{j}}(\bar{t},y')-py')+p\bar{y}-\xi_{\bar{j}}(\bar{t},\bar{y})$ uniformly in a.

Step 5: Passing to the limit
Using the fact that $\phi(\bar{t},z,z,\bar{j})\leq\phi(\bar{t},\bar{x},\bar{y},\bar{j})$, we deduce that

$$
\xi_{\bar{j}}(\bar{t},z)-\xi_{\bar{j}}(\bar{t},\bar{x})\leq\zeta_{\bar{j}}(\bar{t},z)-\zeta_{\bar{j}}(\bar{t},\bar{y})+\alpha|z|^2.
$$

Combining this with the previous step, we get

$$\begin{aligned}
&\frac{\eta}{T^2} + Ae^{A\bar{t}}\frac{|\bar{x}-\bar{y}|^2}{2\varepsilon} \\
\leq& o_\alpha(1) + L_1\left(\inf(\zeta_{\bar{j}}(\bar{t},y') - py' - \zeta_{\bar{j}}(\bar{t},\bar{y}) + \alpha|y'|^2) - \inf(\zeta_{\bar{j}}(\bar{t},y') - py' - \zeta_{\bar{j}}(\bar{t},\bar{y}))\right)|\bar{p}| + p(\bar{x}-\bar{y})|\bar{p}| \\
\leq& o_\alpha(1) + L_1\left(\inf(\zeta_{\bar{j}}(\bar{t},y') - py' + \alpha|y'|^2) - \inf(\zeta_{\bar{j}}(\bar{t},y') - py')\right)|\bar{p}| + pe^{A\bar{t}}\frac{|\bar{x}-\bar{y}|^2}{\varepsilon} \qquad (2.11)
\end{aligned}$$

Choosing $A = 2p$, we finally get

$$\frac{\eta}{T^2} \leq o_\alpha(1) + \left(\inf(\zeta_{\bar{j}}(\bar{t},y') - py' + \alpha|y'|^2) - \inf(\zeta_{\bar{j}}(\bar{t},y') - py')\right)|\bar{p}|.$$

Using the fact that for $\bar{p} = O(1)$ when $\alpha \to 0$ (in fact the $O(1)$ depends on ε which is fixed) and using classical arguments about inf convolution, we get that

$$\left(\inf(\zeta_{\bar{j}}(\bar{t},y') - py' + \alpha|y'|^2) - \inf(\zeta_{\bar{j}}(\bar{t},y') - py')\right)|\bar{p}| = o_\alpha(1)$$

and so

$$\frac{\eta}{T^2} \leq o_\alpha(1)$$

which is a contradiction for α small enough.

Step 6: Case $\bar{t} = 0$
We assume that there exists a sequence $\varepsilon_n \to 0$ such that $\bar{t} = 0$. In this case, we have

$$0 < \frac{M}{2} \leq M_{\varepsilon_n,\alpha,\eta} \leq \xi_0(\bar{x}) - \xi_0(\bar{y}) - \frac{|\bar{x}-\bar{y}|^2}{2\varepsilon_n} - \alpha|x|^2 \leq \xi_0(\bar{x}) - \xi_0(\bar{y}) \leq \|D\xi_0\|_{L^\infty}|\bar{x}-\bar{y}|\,.$$

Using the fact that $|\bar{x}-\bar{y}| \to 0$ as $\varepsilon_n \to 0$ yields a contradiction. □

Let us now give a comparison principle on bounded sets. To this end, for a given point $(\tau_0, y_0) \in (0,T)\times\mathbb{R}$ and for all $r, R > 0$, let us set

$$Q_{r,R} = (\tau_0 - r, \tau_0 + r) \times (y_0 - R, y_0 + R).$$

We then have the following result which proof is similar to the one of Proposition 2.2

Proposition 2.4 (Comparison principle on bounded sets).
Assume (A1')-(A5') and that $G_j(\tau, V, r, a, q)$ does not depend on the variable a for each j. Assume that $((u_j)_j, (\xi_j)_j)$ is a subsolution (resp. $((v_j)_j, (\zeta)_j)$ a supersolution) of (2.1) on the open set $Q_{r,R} \subset (0,T)\times\mathbb{R}$. Assume also that for all $j \in \{1,\dots,n\}$

$$u_j \leq v_j \quad \text{and} \quad \xi_j \leq \zeta_j \quad \text{on } (\overline{Q}_{r,R+m}\backslash Q_{r,R}).$$

Then $u_j \leq v_j$ and $\xi_j \leq \zeta_j$ on $Q_{r,R}$ for $j \in \{1,\dots,n\}$.

We now turn to the question of the existence. Classically, we need to construct barriers for (2.1). In view of Assumptions (A1')(ii) and (A4'), for K_0 given in (A0), the following quantity

$$\overline{G} = \sup_{\tau\in\mathbb{R},\ |q|\leq K_0,\ j\in\{1,\dots,n\}} |G_j(\tau,0,0,0,q)| \qquad (2.12)$$

is finite. Let us also denote $L_2 := L_1K_0$. Hence, for all $\tau, a, b, r \in \mathbb{R}$, $V \in \mathbb{R}^{2m+1}$, $q \in [-K_0, K_0]$ and $j \in \{1,\dots,n\}$,

$$|G_j(\tau,V,r,a,q) - G_j(\tau,V,r,b,q)| \leq L_2|a-b|. \qquad (2.13)$$

Then we have the following lemma:

Lemma 2.5 (**Existence of barriers**). *Assume* (A0')-(A5'). *There exists a constant* $C > 0$ *such that*

$$((u_j^+(\tau,y))_j,(\xi_j^+(\tau,y))_j) = ((u_0(y+\frac{j}{n})+K_1\tau)_j,(\xi_0(y+\frac{j}{n})+K_1\tau)_j)$$

and

$$((u_j^-(\tau,y))_j,(\xi_j^-(\tau,y))_j) = ((u_0(y+\frac{j}{n})-K_1\tau)_j,(\xi_0(y+\frac{j}{n})-K_1\tau)_j)$$

are respectively super and sub-solution of (2.1), (2.2) *for all* $T > 0$. *Moreover, we can choose*

$$K_1 = \max\left(L_2C_0 + L_0\left(2 + K_0\frac{m}{n} + M_0\right) + \overline{G}, \alpha_0 M_0\right) \tag{2.14}$$

where C_0 *and* M_0 *are given in* (A0').

Proof. We prove that $((u_j^+(\tau,y))_j,(\xi_j^+(\tau,y))_j)$ is a supersolution of (2.1), (2.2). In view of (A0) with $\varepsilon = 1$, we have for all $j \in \{1,\dots,n\}$

$$\alpha_0(\xi_j^+(\tau,y) - u_j^+(\tau,y)) = \alpha_0(u_0(y+\frac{j}{n}) - \xi_0(y+\frac{j}{n})) \le \alpha_0 M_0 \le K_1$$

and

$$\begin{aligned}
&G_j\left(\tau,[u^+(\tau,\cdot)]_{j,m}(y),\xi_j^+(\tau,y),\inf_{y'\in\mathbb{R}}\left(\xi_j^+(\tau,y') - py'\right) + py - \xi_j^+(\tau,y),(\xi_j^+)_y(\tau,y)\right)\\
=&G_j\Bigg(\tau,[u^+(\tau,\cdot) - \lfloor u_j^+(\tau,y)\rfloor]_{j,m}(y),\xi_j^+(\tau,y) - \lfloor u_j^+(\tau,y)\rfloor,\\
&\qquad\inf_{y'\in\mathbb{R}}\left(\xi_0(y'+\frac{j}{n}) - py'\right) + py - \xi_0(y+\frac{j}{n}),(\xi_0)_y(y+\frac{j}{n})\Bigg)\\
\le&L_2C_0 + L_0 + L_0 + G_j\left(\tau,[u^+(\tau,\cdot) - u_j^+(\tau,y)]_{j,m}(y),\xi_j^+(\tau,y) - u_j^+(\tau,y),0,(\xi_0)_y(y+\frac{j}{n})\right)\\
\le&L_2C_0 + L_0 + L_0 + L_0K_0\frac{m}{n} + L_0M_0 + G_j\left(\tau,0,\dots 0,0,0,(\xi_0)_y(y+\frac{j}{n})\right)\\
\le&L_2C_0 + 2L_0 + L_0K_0\frac{m}{n} + L_0M_0 + \overline{G}
\end{aligned}$$

where we have used the periodicity assumption (A4') for the second line, assumptions (A0') and (A1')(ii) for the third line, the fact that $|u_0(y+\frac{j+k}{n}) - u_0(y+\frac{j}{n})| \le K_0\frac{m}{n}$ for $|k| \le m$ and assumption (A0') for the forth line and $|(\xi_j^+)_y| \le K_0$ for the last line.

When $G_j(\tau,V,r,a,q)$ is independent on a, we can simply choose $L_2 = 0$. This ends the proof of the Lemma. □

By applying Perron's method together with the comparison principle, we immediately get from the existence of barriers the following result:

Theorem 2.6 (**Existence and uniqueness for** (2.1)**).** *Assume* (A0')-(A5'). *Then there exists a unique solution* $((u_j)_j,(\xi_j)_j)$ *of* (2.1), (2.2). *Moreover the functions* u_j,ξ_j *are continuous.*

We now claim that particles are ordered.

Proposition 2.7 (**Ordering of the particles**). *Assume* (A0') *and that* $(G_j)_j$ *satisfies* (A1')-(A6'). *Let* (u_j,ξ_j) *be a solution of* (2.1)-(2.2) *such that* (2.3) *holds true for* ξ_j *if* G_j *depends on* a. *Assume also that* u_j *are Lipschitz continuous in space* (L_u *denotes the constant of Lipschitz). Then* u_j *and* ξ_j *are non-decreasing with respect to* j.

Proof of Proposition 2.7. The idea of the proof is to define $(v_j, \zeta_j) = (u_{j+1}, \xi_{j+1})$. In particular, we have

$$(v_j(0,y), \zeta_j(0,y)) \geq (u_j(0,y), \xi_j(0,y)).$$

Moreover, $((v_j)_j, (\zeta_j)_j)$ is a solution of

$$\begin{cases} \begin{cases} (v_j)_\tau = \alpha_0(\zeta_j - v_j), \\ (\zeta_j)_\tau = G_{j+1}(\tau, [v(\tau,\cdot)]_{j,m}, \zeta_j, \inf_{y'\in\mathbb{R}} (\zeta_j(\tau,y') - py') + py - \zeta_j(\tau,y), (\zeta_j)_y), \end{cases} \\ \begin{cases} v_{j+n}(\tau,y) = v_j(\tau,y+1), \\ \zeta_{j+n}(\tau,y) = \zeta_j(\tau,y+1) \end{cases} \\ \begin{cases} v_j(0,y) = u_0(y + \frac{j}{n}), \\ \zeta_j(0,y) = \xi_0(y + \frac{j}{n}) \end{cases} \end{cases}$$

Now the goal is to obtain $u_j \leq v_j$ and $\xi_j \leq \zeta_j$. The arguments are essentially the same as those used in the proof of the comparison principle. The main difference is that (2.8) is replaced by

$$\begin{aligned} \frac{\eta}{T^2} + Ae^{A\bar{t}}\frac{|\bar{x}-\bar{y}|^2}{2\varepsilon} \leq & G_{\bar{j}}(\bar{t}, [u(\bar{t},\cdot)]_{\bar{j},m}(\bar{x}), \xi_{\bar{j}}(\bar{t},\bar{x}), \inf(\xi_{\bar{j}}(\bar{t},y') - py') + p\bar{x} - \xi_{\bar{j}}(\bar{t},\bar{x}), \bar{p} + 2\alpha\bar{x}) \\ & - G_{\bar{j}+1}(\bar{t}, [v(\bar{t},\cdot)]_{\bar{j},m}(\bar{y}), \zeta_{\bar{j}}(\bar{t},\bar{y}), \inf(\zeta_{\bar{j}}(\bar{t},y') - py') + p\bar{y} - \zeta_{\bar{j}}(\bar{t},\bar{y}), \bar{p}) \\ \leq & G_{\bar{j}}(\bar{t}, [u(\bar{t},\cdot)]_{\bar{j},m}(\bar{y}), \xi_{\bar{j}}(\bar{t},\bar{x}), \inf(\xi_{\bar{j}}(\bar{t},y') - py') + p\bar{x} - \xi_{\bar{j}}(\bar{t},\bar{x}), \bar{p} + 2\alpha\bar{x}) \\ & - G_{\bar{j}+1}(\bar{t}, [v(\bar{t},\cdot)]_{\bar{j},m}(\bar{y}), \zeta_{\bar{j}}(\bar{t},\bar{y}), \inf(\zeta_{\bar{j}}(\bar{t},y') - py') + p\bar{y} - \zeta_{\bar{j}}(\bar{t},\bar{y}), \bar{p}) + L_0L_u|\bar{x}-\bar{y}| \end{aligned}$$

where we have used the Lipschitz continuity of u and assumption (A1').

To obtain the contradiction, we have to estimate the right hand side of this inequality. First, using step 3 of the proof of the comparison principle (and with the same notation), we can define

$$\delta = \xi_{\bar{j}}(\bar{t},\bar{x}) - \zeta_{\bar{j}}(\bar{t},\bar{y}) + L_u|\bar{x}-\bar{y}| + \alpha \max_{k\in\{\bar{j}-m,\dots,\bar{j}+m\}} (2|l_k\bar{x}| + l_k^2) \geq 0$$

such that for $k \in \{\bar{j}-m, \dots, \bar{j}+m\}$,

$$u_k(\bar{t},\bar{y}) - v_k(\bar{t},\bar{y}) \leq \delta. \tag{2.15}$$

We now can compute

$$\begin{aligned} & G_{\bar{j}}(\bar{t}, [u(\bar{t},\cdot)]_{\bar{j},m}(\bar{y}), \xi_{\bar{j}}(\bar{t},\bar{x}), \inf(\xi_{\bar{j}}(\bar{t},y') - py') + p\bar{x} - \xi_{\bar{j}}(\bar{t},\bar{x}), \bar{p} + 2\alpha\bar{x}) \\ & - G_{\bar{j}+1}(\bar{t}, [v(\bar{t},\cdot)]_{\bar{j},m}(\bar{y}), \zeta_{\bar{j}}(\bar{t},\bar{y}), \inf(\zeta_{\bar{j}}(\bar{t},y') - py') + p\bar{y} - \zeta_{\bar{j}}(\bar{t},\bar{y}), \bar{p}) + L_0L_u|\bar{x}-\bar{y}| \\ \leq & G_{\bar{j}}(\bar{t}, (u_{\bar{j}+k}(\bar{t},\bar{y}) + (m+k)\delta)_{k=-m,\dots,m}, \xi_{\bar{j}}(\bar{t},\bar{x}), \inf(\xi_{\bar{j}}(\bar{t},y') - py') + p\bar{x} - \xi_{\bar{j}}(\bar{t},\bar{x}), \bar{p} + 2\alpha\bar{x}) \\ & - G_{\bar{j}+1}(\bar{t}, (v_{\bar{j}+k}(\bar{t},\bar{y}) + (m+k+1)\delta)_{k=-m,\dots,m}, \zeta_{\bar{j}}(\bar{t},\bar{y}), \inf(\zeta_{\bar{j}}(\bar{t},y') - py') + p\bar{y} - \zeta_{\bar{j}}(\bar{t},\bar{y}), \bar{p}) \\ & + L_0(2m+1)\delta + L_0L_u|\bar{x}-\bar{y}| \end{aligned}$$

where we have used the monotonicity assumption (A2')-(A3') and the regularity (A1'). Now we can use assumption (A6'). Indeed, we have for all $k \in \{-m, m-1\}$

$$v_{\bar{j}+k}(\bar{t},\bar{y}) + (m+k+1)\delta = u_{\bar{j}+k+1}(\bar{t},\bar{y}) + (m+k+1)\delta$$

and for $k \in \{-m, m\}$

$$v_{\bar{j}+k}(\bar{t},\bar{y}) + (m+k+1)\delta - u_{\bar{j}+k}(\bar{t},\bar{y}) + (m+k)\delta \geq 0$$

where we have used (2.15). This implies that

$$\begin{aligned}&G_{\bar{j}}(\bar{t},[u(\bar{t},\cdot)]^{-}_{\bar{j},m}(\bar{y}),\xi^{-}_{\bar{j}}(\bar{t},\bar{x}),\inf(\xi^{-}_{\bar{j}}(\bar{t},y')-py')+p\bar{x}-\xi^{-}_{\bar{j}}(\bar{t},\bar{x}),\bar{p}+2\alpha\bar{x})\\ \leq&G_{\bar{j}+1}(\bar{t},(v^{-}_{\bar{j}+k}(\bar{t},\bar{y})+(m+k+1)\delta)_{k=-m,\dots,m},\xi^{-}_{\bar{j}}(\bar{t},\bar{x}),\inf(\xi^{-}_{\bar{j}}(\bar{t},y')-py')+p\bar{x}-\xi^{-}_{\bar{j}}(\bar{t},\bar{x}),\bar{p}+2\alpha\bar{x})\end{aligned}$$

Now, to obtain the contradiction, it suffices to follow the computation from the third inequality of (2.10) (choose $L' \geq (2m+1)L_0$ in (2.7)) to obtain

$$\frac{\eta}{T^2} \leq o_\alpha(1) + 2(m+1)L_0L_u|\bar{x}-\bar{y}|$$

which is absurd for α and ε small enough (since $|\bar{x}-\bar{y}| \to 0$ as $\varepsilon \to 0$) □

3 Convergence

This section is devoted to the proof of the main homogenization result (Theorem 1.5). The proof relies on the existence of hull functions (Theorem 1.10) and qualitative properties of the effective Hamiltonian (Theorem 1.11). As a matter of fact, we will use the existence of Lipschitz continuous sub- and super-hull functions (see Proposition 5.2). All these results are proved in the next sections.

We start with some preliminary results. The following result is a straightforward corollary of Lemma 2.5 and the comparison principle by a change of variables.

Lemma 3.1 (**Barriers uniform in ε**). *Assume* (A0)-(A5). *Then there is a constant $C > 0$, such that for all $\varepsilon > 0$, the solution $((u^\varepsilon_j)_j, (\xi^\varepsilon_j))$ of* (1.8), (1.9) *satisfies for all $t > 0$ and $x \in \mathbb{R}$*

$$|u^\varepsilon_j(t,x) - u_0(x+\frac{j\varepsilon}{n})| \leq Ct \quad \text{and} \quad |\xi^\varepsilon_j(t,x) - \xi^\varepsilon_0(x+\frac{j\varepsilon}{n})| \leq Ct.$$

We also have the following preliminary lemma.

Lemma 3.2 (ε**-bounds on the gradient**). *Assume* (A0)-(A5). *Then the solution $((u^\varepsilon)_j, (\xi^\varepsilon_j)_j)$ of* (1.8), (1.9) *satisfies for all $t > 0$, $x \in \mathbb{R}$, $z > 0$ and $j \in \mathbb{Z}$*

$$\varepsilon\left\lfloor \frac{z}{\varepsilon K_0}\right\rfloor \leq u^\varepsilon_j(t,x+z) - u^\varepsilon_j(t,x) \leq \varepsilon\left\lceil \frac{zK_0}{\varepsilon}\right\rceil \tag{3.1}$$

and

$$\varepsilon\left\lfloor \frac{z}{\varepsilon K_0}\right\rfloor \leq \xi^\varepsilon_j(t,x+z) - \xi^\varepsilon_j(t,x) \leq \varepsilon\left\lceil \frac{zK_0}{\varepsilon}\right\rceil .$$

Remark 3.3. In particular we obtain that functions $u^\varepsilon_j(t,x)$ and $\xi^\varepsilon_j(t,x)$ are non-decreasing in x.

Proof of Lemma 3.2. We prove the bound from below (the proof is similar for the bound from above). We first remark that (A0) implies that the initial condition satisfies for all $j \in \mathbb{Z}$

$$u^\varepsilon_j(0,x+z) = u_0(x+z+\frac{j\varepsilon}{n}) \geq u_0(x+\frac{j\varepsilon}{n}) + z/K_0 \geq u^\varepsilon_j(0,x) + k\varepsilon \quad \text{with} \quad k = \left\lfloor \frac{z}{\varepsilon K_0}\right\rfloor \tag{3.2}$$

and

$$\xi^\varepsilon_j(0,x+z) \geq \xi^\varepsilon_j(0,x) + k\varepsilon\,.$$

From (A4), we know that for $\varepsilon = 1$, the equation is invariant by addition of integers to solutions. After rescaling it, Equation (1.8) is invariant by addition of constants of the form $k\varepsilon$, $k \in \mathbb{Z}$. For this reason the solution of (1.8) associated with initial data $((u^\varepsilon_j(0,x)+k\varepsilon)_j, (\xi^\varepsilon_j(0,x)+k\varepsilon)_j)$ is $((u^\varepsilon_j+k\varepsilon)_j, (\xi^\varepsilon_j+k\varepsilon)_j)$. Similarly the equation is invariant by translations. Therefore the solution with initial data $((u^\varepsilon_j(0,x+$

$z))_j, (\xi^\varepsilon_j(0,x+z)_j)$ is $((u^\varepsilon_j(t,x+z))_j, (\xi^\varepsilon_j(t,x+z))_j)$. Finally, from (3.2) and the comparison principle (Proposition 2.2), we get

$$u^\varepsilon_j(t,x+z) \geq u^\varepsilon_j(t,x) + k\varepsilon \quad \text{and} \quad \xi^\varepsilon_j(t,x+z) \geq \xi^\varepsilon_j(t,x) + k\varepsilon$$

which proves the bound from below. This ends the proof of the lemma. □

We now turn to the proof of Theorem 1.5.

Proof of Theorem 1.5. We only have to prove the result for all $j \in \{1,\dots,n\}$. Indeed, using the fact that $u^\varepsilon_{j+n}(t,x) = u^\varepsilon_j(t,x+\varepsilon)$ and $\xi^\varepsilon_{j+n}(t,x) = \xi^\varepsilon_j(t,x+\varepsilon)$, we will get the complete result.

For all $j \in \{1,\dots,n\}$, we introduce the following half-relaxed limits

$$\overline{u}_j = \limsup_{\varepsilon\to 0}{}^* u^\varepsilon_j, \quad \overline{\xi}_j = \limsup_{\varepsilon\to 0}{}^* \xi^\varepsilon_j$$

$$\underline{u}_j = \liminf_{\varepsilon\to 0}{}_* u^\varepsilon_j, \quad \underline{\xi}_j = \liminf_{\varepsilon\to 0}{}_* \xi^\varepsilon_j\,.$$

These functions are well defined thanks to Lemma 3.1. We then define

$$\overline{v} = \max_{j\in\{1,\dots,n\}} \max(\overline{u}_j, \overline{\xi}_j), \quad \underline{v} = \min_{j\in\{1,\dots,n\}} \min(\underline{u}_j, \underline{\xi}_j)\,.$$

We get from Lemmata 3.1 and 3.2 that both functions $w = \overline{v}, \underline{v}$ satisfy for all $t > 0$, $x, x' \in \mathbb{R}$ (recall that $\xi^\varepsilon_0 \to u_0$ as $\varepsilon \to 0$)

$$\begin{aligned} |w(t,x) - u_0(x)| &\leq Ct\,, \\ K_0^{-1}|x-x'| &\leq w(t,x) - w(t,x') \leq K_0|x-x'|\,. \end{aligned} \tag{3.3}$$

We are going to prove that $\overline{v}$ is a subsolution of (1.5). Similarly, we can prove that $\underline{v}$ is a supersolution of the same equation. Therefore, from the comparison principle for (1.5), we get that $u^0 \leq \underline{v} \leq \overline{v} \leq u^0$. And then $\overline{v} = \underline{v} = u^0$, which shows the expected convergence of the full sequence u^ε_j and ξ^ε_j towards u^0 for all $j \in \{1,\dots,n\}$.

We now prove in several steps that $\overline{v}$ is a subsolution of (1.5). We classically argue by contradiction: then there exists $(\overline{t},\overline{x}) \in (0,+\infty)\times\mathbb{R}$ and a test function $\phi \in C^1$ such that

$$\left\{\begin{array}{lll} \overline{v}(\overline{t},\overline{x}) = \phi(\overline{t},\overline{x}) & & \\ \overline{v} \leq \phi & \text{on} \quad Q_{r,2r}(\overline{t},\overline{x}), & \text{with} \quad r>0 \\ \overline{v} \leq \phi - 2\eta & \text{on} \quad \overline{Q}_{r,2r}(\overline{t},\overline{x}) \setminus Q_{r,r}(\overline{t},\overline{x}), & \text{with} \quad \eta>0 \\ \phi_t(\overline{t},\overline{x}) = \overline{F}(\phi_x(\overline{t},\overline{x})) + \theta, & \text{with} \quad \theta>0 & \end{array}\right. \tag{3.4}$$

Let p denote $\phi_x(\overline{t},\overline{x})$. From (3.3), we get

$$0 < 1/K_0 \leq p \leq K_0\,. \tag{3.5}$$

Combining Theorems 1.10 and 1.11, we get the existence of a hull function $((h_i)_i, (g_i)_i)$ associated with p such that

$$\lambda = \overline{F}(p) + \frac{\theta}{2} = \overline{F}(\overline{L}, p) \quad \text{with} \quad \overline{L} > 0\,.$$

Indeed, we know from these results that the effective Hamiltonian is non-decreasing in L, continuous and goes to $\pm\infty$ as $L \to \pm\infty$.

We now apply the perturbed test function method introduced by Evans [7] in terms here of hull functions instead of correctors. Precisely, let us consider the following twisted perturbed test functions for $i \in \{1,\dots,n\}$

$$\phi^\varepsilon_i(t,x) = \varepsilon h_i\left(\frac{t}{\varepsilon}, \frac{\phi(t,x)}{\varepsilon}\right), \quad \psi^\varepsilon_i(t,x) = \varepsilon g_i\left(\frac{t}{\varepsilon}, \frac{\phi(t,x)}{\varepsilon}\right)\,.$$

We then define the family of perturbed test functions $(\phi_i^\varepsilon)_{i\in\mathbb{Z}}, ((\psi_i^\varepsilon)_{i\in\mathbb{Z}})$ by using the following relation

$$\phi_{i+kn}^\varepsilon(t,x) = \phi_i^\varepsilon(t, x+\varepsilon k), \quad \psi_{i+kn}^\varepsilon(t,x) = \psi_i^\varepsilon(t,x+\varepsilon k).$$

Here the test functions are twisted in the same way as in [12]. In order to get a contradiction, we first assume that the functions h_i and g_i are C^1 and continuous in z uniformly in $\tau \in \mathbb{R},\ i \in \{1,\dots,n\}$. In view of the third line of (1.10), we see that this implies that h_i and g_i are uniformly continuous in z (uniformly in $\tau \in \mathbb{R},\ i \in \{1,\dots,n\}$). For simplicity, and since we will construct approximate hull functions with such a regularity, we even assume that h_i and g_i are globally Lipschitz continuous in z (uniformly in $\tau \in \mathbb{R},\ i \in \{1,\dots,n\}$). We will next see how to treat the general case.

Case 1: h_i and g_i are C^1 and globally Lipschitz continuous in z

Step 1.1: $((\phi_i^\varepsilon)_i, (\psi_i^\varepsilon)_i)$ is a supersolution of (1.8) in a neighbourhood of $(\overline{t},\overline{x})$
When h_i and g_i are C^1, it is sufficient to check directly the supersolution property of $(\phi_i^\varepsilon, \psi_i^\varepsilon)$ for $(t,x) \in Q_{r,r}(\overline{t},\overline{x})$. We begin by the equation satisfied by ϕ_i^ε. We have, with $\tau = t/\varepsilon$ and $z = \phi(t,x)/\varepsilon$,

$$\begin{aligned}
(\phi_i^\varepsilon)_t =& (h_i)_\tau(\tau,z) + (h_i)_z(\tau,z)\phi_t(t,x)\\
=& (\phi_t(t,x)-\lambda)(h_i)_z(\tau,z) + \alpha_0(g_i(\tau,z)-h_i(\tau,z))\\
=& (\phi_t(t,x) - \phi_t(\overline{t},\overline{x}) + \frac{\theta}{2})(h_i)_z(\tau,z) + \frac{\alpha_0}{\varepsilon}(\psi_i^\varepsilon(\tau,z) - \phi_i^\varepsilon(\tau,z))\\
\geq& \frac{\alpha_0}{\varepsilon}(\psi_i^\varepsilon(\tau,z) - \phi_i^\varepsilon(\tau,z)) \qquad (3.6)
\end{aligned}$$

where we have used the equation satisfied by h_i to get the second line and the non-negativity of h_z, the fact that $\theta > 0$ and the fact that ϕ is C^1, to get the last line on $Q_{r,r}(\overline{t},\overline{x})$ for $r > 0$ small enough.

We now turn to equation satisfied by ψ_i. With the same notation, we have

$$\begin{aligned}
&(\psi_i^\varepsilon)_t(t,x) - 2F_i\left(\tau, \left[\frac{\phi^\varepsilon(t,\cdot)}{\varepsilon}\right]_{i,m}(x)\right) - \frac{\alpha_0}{\varepsilon}(\phi_i^\varepsilon - \psi_i^\varepsilon) \qquad (3.7)\\
=&(g_i)_\tau(\tau,z) + \phi_t(t,x)(g_i)_z(\tau,z) - 2F_i\left(\tau, \left[\frac{\phi^\varepsilon(t,\cdot)}{\varepsilon}\right]_{i,m}(x)\right) - \alpha_0(h_i(\tau,z) - g_i(\tau,z))\\
=&(\phi_t(t,x)-\lambda)\ (g_i)_z(\tau,z) + 2\overline{L} + 2\left(F_i\left(\tau, [h(\tau,\cdot)]_{i,m}(z)\right) - F_i\left(\tau, \left[\frac{\phi^\varepsilon(t,\cdot)}{\varepsilon}\right]_{i,m}(x)\right)\right)\\
\geq&(\phi_t(t,x)-\lambda)\ (g_i)_z(\tau,z) + 2\overline{L} - 2L_F\left|[h(\tau,\cdot)]_{i,m}(z) - \left[\frac{\phi^\varepsilon(t,\cdot)}{\varepsilon}\right]_{i,m}(x)\right|_\infty
\end{aligned}$$

where we have used that Equation (1.10) is satisfied by $(g_i)_i$ to get the third line and (A1) to get the fourth one; here, L_F denotes the maximum of the Lipschitz constant of F_i (for $i \in \{1,\dots,n\}$) with respect to V for the norm $|\cdot|_\infty$ on $\mathbb{R}^{2m+1}$.

Let us next estimate, for $i \in \{1,\dots,n\},\ j \in \{-m,\dots,m\}$ and $\varepsilon > 0$,

$$\mathcal{I}_{i,j} = h_{i+j}(\tau,z) - \frac{\phi_{i+j}^\varepsilon(t,x)}{\varepsilon}$$

If $i+j \in \{1,\dots,n\}$, then, by definition of ϕ_{i+j}, we have

$$\mathcal{I}_{i,j} = h_{i+j}\left(\frac{t}{\varepsilon}, \frac{\phi(t,x)}{\varepsilon}\right) - \frac{\phi_{i+j}^\varepsilon(t,x)}{\varepsilon} = 0.$$

If $i+j \notin \{1,\dots,n\}$, let us define l such that $1 \le i+j-ln \le n$. We then have

$$\begin{aligned}\mathcal{I}_{i,j} =& h_{i+j-ln}(\tau, z+lp) - \frac{\phi^\varepsilon_{i+j-ln}(t, x+\varepsilon l)}{\varepsilon} \\ =& h_{i+j-ln}\left(\frac{t}{\varepsilon}, \frac{\phi(t,x)}{\varepsilon} + lp\right) - h_{i+j-ln}\left(\frac{t}{\varepsilon}, \frac{\phi(t,x+\varepsilon l)}{\varepsilon}\right) \\ =& h_{i+j-ln}\left(\frac{t}{\varepsilon}, \frac{\phi(t,x)}{\varepsilon} + lp\right) - h_{i+j-ln}\left(\frac{t}{\varepsilon}, \frac{\phi(t,x)}{\varepsilon} + lp + o_r(1)\right)\end{aligned}$$

where $o_r(1)$ only depends on the modulus of continuity of ϕ_x on $Q_{r,r}(\overline{t},\overline{x})$ (for ε small enough such that $\varepsilon l \le r$ with l uniformly bounded and then $(t, x+\varepsilon l) \in Q_{r,2r}(\overline{t},\overline{x})$). Hence, if h_i are Lipschitz continuous with respect to z uniformly in τ and i, we conclude that we can choose ε small enough so that

$$\overline{L} - L_F \left| [h(\tau,\cdot)]_{i,m}(z) - \left[\frac{\phi^\varepsilon(t,\cdot)}{\varepsilon}\right]_{i,m}(x)\right|_\infty \ge 0\,. \tag{3.8}$$

Combining (3.7) and (3.8), we obtain

$$\begin{aligned}(\psi^\varepsilon_i)_t(t,x) - 2F_i\left(\tau, \left[\frac{\phi^\varepsilon(t,x)}{\varepsilon}\right]_{i,m}(x)\right) + \frac{\alpha_0}{\varepsilon}(\phi^\varepsilon_i - \psi^\varepsilon_i) &\ge (\phi_t(t,x) - \lambda)\ (g_i)_z(\tau,z) \\ &\ge \left(\frac{\theta}{2} + \phi_t(t,x) - \phi_t(\overline{t},\overline{x})\right)\ (g_i)_z(\tau,z) \\ &= \left(\frac{\theta}{2} + o_r(1)\right)\ (g_i)_z(\tau,z) \ge 0\,.\end{aligned}$$

We used the non-negativity of $(g_i)_z$, the fact that $\theta > 0$ and again the fact that ϕ is C^1, to get the result on $Q_{r,r}(\overline{t},\overline{x})$ for $r > 0$ small enough. Therefore, when the h_i and g_i are C^1 and Lipschitz continuous on z uniformly in τ and i, $((\phi^\varepsilon_i)_i, (\psi^\varepsilon_i)_i)$ is a viscosity supersolution of (1.8) on $Q_{r,r}(\overline{t},\overline{x})$.

Step 1.2: getting the contradiction
By construction (see Remark 1.9), we have $\phi^\varepsilon_i \to \phi$ and $\psi^\varepsilon_i \to \phi$ as $\varepsilon \to 0$ for all $i \in \{1,\dots,n\}$, and therefore from the fact that $\overline{u}_j \le \bar{v} \le \phi - 2\eta$ on $\overline{Q}_{r,2r}(\overline{t},\overline{x}) \setminus Q_{r,r}(\overline{t},\overline{x})$ (see (3.4)), we get for ε small enough

$$u^\varepsilon_i \le \phi^\varepsilon_i - \eta \le \phi^\varepsilon_i - \varepsilon k_\varepsilon \quad \text{on} \quad \overline{Q}_{r,2r}(\overline{t},\overline{x}) \setminus Q_{r,r}(\overline{t},\overline{x})$$

with the integer

$$k_\varepsilon = \lfloor \eta/\varepsilon \rfloor\,.$$

In the same way, we have

$$\xi^\varepsilon_i \le \psi^\varepsilon_i - \eta \le \psi^\varepsilon_i - \varepsilon k_\varepsilon \quad \text{on} \quad \overline{Q}_{r,2r}(\overline{t},\overline{x}) \setminus Q_{r,r}(\overline{t},\overline{x})\,.$$

Therefore, for $m\varepsilon \le r$, we can apply the comparison principle on bounded sets to get

$$u^\varepsilon_i \le \phi^\varepsilon_i - \varepsilon k_\varepsilon, \quad \xi^\varepsilon_i \le \psi^\varepsilon_i - \varepsilon k_\varepsilon \quad \text{on} \quad Q_{r,r}(\overline{t},\overline{x})\,. \tag{3.9}$$

Passing to the limit as ε goes to zero, we get

$$\overline{u}_i \le \phi - \eta, \quad \overline{\xi}_i \le \phi - \eta \quad \text{on} \quad Q_{r,r}(\overline{t},\overline{x})$$

which implies that

$$\overline{v} \le \phi - \eta \quad \text{on} \quad Q_{r,r}(\overline{t},\overline{x}).$$

This gives a contradiction with $\overline{v}(\overline{t},\overline{x}) = \phi(\overline{t},\overline{x})$ in (3.4). Therefore $\overline{v}$ is a subsolution of (1.5) on $(0,+\infty)\times\mathbb{R}$ and we get that u_j^ε and ξ_j^ε converges locally uniformly to u^0 for $j \in \{1,\dots,n\}$. This ends the proof of the theorem.

Case 2: general case for h
In the general case, we can not check by a direct computation that $((\phi_i^\varepsilon)_i, (\psi_i^\varepsilon)_i)$ is a supersolution on $Q_{r,r}(\overline{t},\overline{x})$. The difficulty is due to the fact that the h_i and the g_i may not be Lipschitz continuous in the variable z.

This kind of difficulties was overcomed in [12] by using Lipschitz super-hull functions, *i.e.* functions satisfying (1.10) with $\geq$ instead of $=$ in the first line. Indeed, it is clear from the previous computations that it is enough to conclude. In [12], such regular super-hull functions (as a matter of fact, regular super-correctors) were built as exact solutions of an approximate Hamilton-Jacobi equation. Moreover this Lipschitz continuous hull function is a supersolution for the exact Hamiltonian with a slightly bigger λ.

Here we conclude using a similar result, namely Proposition 5.2. Notice that in Proposition 5.2 h_i and g_i are only Lipschitz continuous and not C^1. This is not a restriction, because the result of Step 1.1 can be checked in the viscosity sense using test function (see [7] for further details). Comparing with [12], notice that we do not have to introduce an additional dimension because here $p > 0$ (see (3.5)). This ends the proof of the theorem. □

4 Ergodicity and construction of hull functions

In this section, we first study the ergodicity of the equation (2.1) by studying the associated Cauchy problem (Subsection 4.1). We then construct hull functions (Subsection 4.2).

4.1 Ergodicity

In this subsection, we study the Cauchy problem associated with (2.1) with

$$G_j(\tau,V,r,a,q) = G_j^\delta(\tau,V,r,a,q) = 2F_j(\tau,V) + \alpha_0(V_0 - r) + \delta(a_0 + a)q^+ \tag{4.1}$$

with $\delta \geq 0$, $a_0 \in \mathbb{R}$ and with initial data $y \mapsto py$. We prove that there exists a real number λ (called the "slope in time" or "rotation number") such that the solution (u_j, ξ_j) stays at a finite distance of the linear function $\lambda\tau + py$. We also estimate this distance and give qualitative properties of the solution.

We begin by a regularity result concerning the solution of (2.1). Using (A1), we know that there exists a constant $L_F > 0$ such that for all $V, W \in \mathbb{R}^{2m+1}$, $\tau \in \mathbb{R}$,

$$|F_j(\tau, V+W) - F_j(\tau,V)| \leq L_F|W|_\infty \tag{4.2}$$

with $|W|_\infty = \max_{k=-m,\dots,m}|W_k|$.

We have the following regularity result

Proposition 4.1 (Bound on the gradient). *Assume* (A1)-(A5) *and* $p > 0$. *Let* $\delta > 0$, $a_0 \in \mathbb{R}$ *and* $(u_j,\xi_j)_j$ *be the solution of* (2.1), (2.2) *with* $G_j = G_j^\delta$ *defined by* (4.1) *and* $u_0(y) = py$. *Assume that* (2.3) *holds true for* ξ_j. *Then* $(u_j,\xi_j)_j$ *satisfies*

$$0 \leq (u_j)_y \leq p + \frac{2L_F}{\delta} \quad \text{and} \quad 0 \leq (\xi_j)_y \leq p + \frac{2L_F}{\delta}\,. \tag{4.3}$$

Proof. We first show that u_j and ξ_j are non-decreasing with respect to y. Since the equation (2.1) is invariant by translations in y and using the fact that for all $b \geq 0$, we have

$$u_0(y + b + \frac{j}{n}) \geq u_0(y + \frac{j}{n})\,.$$

we deduce from the comparison principle that

$$u_j(\tau, y+b) \geq u_j(\tau, y) \quad \text{and} \quad \xi_j(\tau, y+b) \geq \xi_j(\tau, y)$$

which shows that u_j and ξ_j are non-decreasing in y.

We now explain how to get the Lipschitz estimate. We would like to prove that $\overline{M} \leq 0$ where

$$\overline{M} = \sup_{\tau\in(0,T),x,y\in\mathbb{R},j\in\{1,\dots,n\}} \max\Big\{ u_j(\tau,x) - u_j(\tau,y) - L|x-y| - \frac{\eta}{T-\tau} - \alpha|x|^2, \\ \xi_j(\tau,x) - \xi_j(\tau,y) - L|x-y| - \frac{\eta}{T-\tau} - \alpha|x|^2 \Big\}$$

as soon as $L > p + \frac{2L_F}{\delta} > 0$ for any $\eta, \alpha > 0$. We argue by contradiction by assuming that $\overline{M} > 0$ for such an L. We next exhibit a contradiction. The supremum defining $\overline{M}$ is attained since ξ_j satisfies (2.3) and u_j can be explicitly computed.

Case 1. Assume that the supremum is attained for the function u_j at $\tau \in [0,T)$, $j \in \{1,\dots,n\}$, $x, y \in \mathbb{R}$. Since we have by assumption $\overline{M} > 0$, this implies that $\tau > 0$, $x \neq y$. Hence we can obtain the two following viscosity inequalities (by doubling the time variable and passing to the limit)

$$\begin{aligned} a &\leq \alpha_0(\xi_j(\tau,x) - u_j(\tau,x)) \\ b &\geq \alpha_0(\xi_j(\tau,y) - u_j(\tau,y)) \end{aligned}$$

with $a - b = \frac{\eta}{(T-\tau)^2}$. Subtracting these inequalities, we obtain

$$\frac{\eta}{(T-\tau)^2} \leq \alpha_0([\xi_j(\tau,x) - \xi_j(\tau,y)] - [u_j(\tau,x) - u_j(\tau,y)]) \leq 0\,.$$

We thus get $\eta \leq 0$ which is a contradiction in Case 1.

Case 2. Assume next that the supremum is attained for the function ξ_j. By using the same notation and by arguing similarly, we obtain the following inequality

$$\begin{aligned} \frac{\eta}{(T-\tau)^2} \quad \leq \quad & 2F_j(\tau, u_{j-m}(\tau,x), \dots, u_{j+m}(\tau,x)) - 2F_j(\tau, u_{j-m}(\tau,y), \dots, u_{j+m}(\tau,y)) \\ & +\alpha_0([u_j(\tau,x) - u_j(\tau,y)] - [\xi_j(\tau,x) - \xi_j(\tau,y)]) \\ & +\delta[p(x-y) - (\xi_j(\tau,x) - \xi_j(\tau,y))]L\mathrm{sign}^+(x-y) + 2\alpha\delta(a_0 + C_0)|x| \end{aligned}$$

where sign^+ is the Heaviside function and where we have used assumptions (2.3). We now use

- the fact that the supremum is attained for the function ξ_j
- the fact that $\xi_j(\tau,x) > \xi_j(\tau,y)$ implies that $x > y$ (remember that we already proved that ξ_j is non-decreasing with respect to y)
- Assumption (4.2)
- the fact that $\alpha\delta(a_0 + C_0)(x)^+ = o_\alpha(1)$

in order to get from the previous inequality the following one

$$\frac{\eta}{(T-\tau)^2} \quad \leq \quad 2L_F \sup_{l\in\{-m,\dots,m\}} |u_{j+l}(\tau,x) - u_{j+l}(\tau,y)| + \delta pL|x-y| - L\delta(\xi_j(\tau,x) - \xi_j(\tau,y)) + o_\alpha(1)\,.$$

Using the same computation as the one of the proof of Proposition 2.2 Step 3, we get

$$\sup_{l\in\{-m,\dots,m\}} |u_{j+l}(\tau,x)-u_{j+l}(\tau,y)| \le |\xi_j(\tau,x)-\xi_j(\tau,y)| + C\alpha(1+|x|)$$

where C is a constant. Since $C\alpha(1+|x|)=o_\alpha(1)$ and $\overline{M}>0$, we finally deduce that

$$\frac{\eta}{T^2} \le 2L_F|\xi_j(\tau,x)-\xi_j(\tau,y)| + \delta p(\xi_j(\tau,x)-\xi_j(\tau,y)) - L\delta(\xi_j(\tau,x)-\xi_j(\tau,y)) + o_\alpha(1)$$

For α small enough, it is now sufficient to use once again that $\xi_j(\tau,x)>\xi_j(\tau,y)$ and the fact that $L>p+\frac{2L_F}{\delta}$ in order to get the desired contradiction in Case 2. The proof is now complete. □

We now claim that particles are ordered.

Proposition 4.2 (**Ordering of the particles**). *Assume* (A0'), *(A1)-(A6) and let* $\delta\ge 0$, $a_0\in\mathbb{R}$ *and* $(u_j^\delta,\xi_j^\delta)_j$ *be the solution of* (2.1), (2.2) *with* $G_j=G_j^\delta$ *defined by* (4.1). *Assume that* (2.3) *holds true for* ξ_j *if* $\delta>0$. *Then* u_j^δ *and* ξ_j^δ *are non-decreasing with respect to* j.

Proof. If $\delta>0$, the results is a straightforward consequence of Propositions 2.7 and 4.1. If $\delta=0$, the result is obtained by stability of viscosity solution (i.e. $u_j^\delta\to u_j^0$ and $\xi_j^\delta\to\xi_j^0$ as $\delta\to 0$). □

Proposition 4.3 (**Ergodicity**). *Let* $0\le\delta\le 1$ *and* $a_0\in\mathbb{R}$. *Assume* (A0)-(A6) *and let* $(u_j,\xi_j)_j$ *be a solution of* (2.1), (2.2) *with* G_j *defined in* (4.1) *and with initial data* $u_0(y)=\xi_0(y)=py$ *with some* $p>0$. *Then there exists* $\lambda\in\mathbb{R}$ *such that for all* $(\tau,y)\in[0,+\infty)\times\mathbb{R}$, $j\in\{1,\dots,n\}$

$$|u_j(\tau,y)-py-\lambda\tau|\le C_3 \quad \text{and} \quad |\xi_j(\tau,y)-py-\lambda\tau|\le C_3 \tag{4.4}$$

and

$$|\lambda|\le C_4 \tag{4.5}$$

where

$$\begin{aligned} C_3 &= 14+\frac{6C_4}{\alpha_0}+6p+2K_1 \\ C_4 &= \frac{1}{2}\left(2L_F(2+p(m+n))+2\sup_\tau F(\tau,0,\dots,0)+(p+2L_F)(a_0+C_0)\right) \end{aligned} \tag{4.6}$$

(where a_0 *is chosen equal to zero for* $\delta=0$*). Moreover we have for all* $\tau\ge 0$, $y,y'\in\mathbb{R}$, $j\in\{1,\dots,n\}$

$$\begin{cases} u_j(\tau,y+1/p)=u_j(\tau,y)+1 \\ (u_j)_y(\tau,y)\ge 0 \\ |u_j(\tau,y+y')-u_j(\tau,y)-py'|\le 1 \\ u_{j+1}(\tau,y)\ge u_j(\tau,y) \end{cases} \qquad \begin{cases} \xi_j(\tau,y+1/p)=\xi_j(\tau,y)+1 \\ (\xi_j)_y(\tau,y)\ge 0 \\ |\xi_j(\tau,y+y')-\xi_j(\tau,y)-py'|\le 1 \\ \xi_{j+1}(\tau,y)\ge \xi_j(\tau,y)\,. \end{cases} \tag{4.7}$$

In order to prove Proposition 4.3, we will need the following Lemma:

Lemma 4.4. *Consider* $\Lambda:\mathbb{R}^+\to\mathbb{R}$ *a continuous function which is sub-additive, that is to say: for all* $t,s\ge 0$,

$$\Lambda(t+s)\le\Lambda(t)+\Lambda(s)\,.$$

Then $\frac{\Lambda(t)}{t}$ *has a limit* l *as* $t\to+\infty$ *and*

$$l=\inf_{t>0}\frac{\Lambda(t)}{t}\,.$$

Proof of Proposition 4.3. We perform the proof in three steps. We first recall that the fact that u_j and ξ_j are non-decreasing in y and j follows from Proposition 4.1 and Proposition 4.2.

Step 1: control of the space oscillations. We are going to prove the following estimate.

Lemma 4.5. *For all $\tau > 0$, all $y, y' \in \mathbb{R}$ and all $j \in \{1, \dots, n\}$,*

$$|u_j(\tau, y+y') - u_j(\tau, y) - py'| \le 1 \quad \text{and} \quad |\xi_j(\tau, y+y') - \xi_j(\tau, y) - py'| \le 1\,. \tag{4.8}$$

Proof. We have

$$u_j(0, y+1/p) = \xi_j(0, y+1/p) = \xi_j(0,y) + 1 = u_j(0,y) + 1\,.$$

Therefore from the comparison principle and from the integer periodicity of the Hamiltonian (see (A3')), we get that

$$u_j(\tau, y+1/p) = u_j(\tau, y) + 1 \quad \text{and} \quad \xi_j(\tau, y+1/p) = \xi_j(\tau, y) + 1\,.$$

Since $u_j(\tau, y)$ is non-decreasing in y, we deduce that for all $b \in [0, 1/p]$

$$0 \le u_j(\tau, b) - u_j(\tau, 0) \le 1$$

Let now $y \in \mathbb{R}$, that we write $py = k + a$ with $k \in \mathbb{Z}$ and $a \in [0, 1)$. Then we have

$$u_j(\tau, y) - u_j(\tau, 0) = k + u_j(\tau, a/p) - u_j(\tau, 0)$$

which implies, for some $b \in [0, 1/p)$,

$$u_j(\tau, y) - u_j(\tau, 0) - py = -a + u_j(\tau, b) - u_j(\tau, 0)$$

and then for all $\tau > 0$ and all $y \in \mathbb{R}$,

$$|u_j(\tau, y) - u_j(\tau, 0) - py| \le 1\,.$$

In the same way, we get

$$|\xi_j(\tau, y) - \xi_j(\tau, 0) - py| \le 1\,.$$

Finally, we obtain (4.8) by using the invariance by translations in y of the problem. □

Step 2: estimate on $|u_j(\tau, y) - \xi_j(\tau, y)|$.

Lemma 4.6. *For all $j \in \{1, \dots, n\}$ and $0 \le \delta \le 1$,*

$$\|u_j - \xi_j\|_{L^\infty} \le \frac{C_4}{\alpha_0} \tag{4.9}$$

where

$$C_4 = \frac{1}{2}\left(2L_F(2 + p(m+n)) + 2\sup_\tau F(\tau, 0, \dots, 0) + (p + 2L_F)(a_0 + C_0)\right)$$

(where a_0 is chosen equal to zero for $\delta = 0$)

Proof. We recall that $((u_j), (\xi_j))$ is solution of

$$\begin{cases} (u_j)_\tau = \alpha_0(\xi_j - u_j) \\ (\xi_j)_\tau \le 2F_j(\tau, [u_j(\tau, \cdot)]_{j,m}) + \alpha_0(u_j - \xi_j) + \delta(a_0 + C_0)((\xi_j)_y)^+ \end{cases} \tag{4.10}$$

where we have used (2.3). Using Proposition 4.1, we deduce that (for $\delta \le 1$)

$$\delta(a_0 + C_0)((\xi_j)_y)^+ \le (p + 2L_F)(a_0 + C_0). \tag{4.11}$$

We now want to bound $F_j(\tau, [u_j(\tau,\cdot)]_{j,m})$. We have

$$\begin{aligned} F_j(\tau,[u_j(\tau,\cdot)]_{j,m}(y)) =& F_j(\tau,[u_j(\tau,\cdot)-\lfloor u_j(\tau,y)\rfloor]_{j,m}(y)) \\ \leq& L_F + F_j(\tau,[u_j(\tau,\cdot)-u_j(\tau,y)]_{j,m}(y)) \\ \leq& L_F + L_F \sup_{k\in\{-m,\dots,m\}} |u_{j+k}(\tau,y)-u_j(\tau,y)| + \sup_\tau F(\tau,0,\dots 0) \end{aligned} \tag{4.12}$$

where we have used the periodicity assumption (A4) for the first line, the Lipschitz regularity of F for the second and third one. Moreover for all $i \in \{1,\dots,n\},\ k \in \{0,\dots m\}$, we have that

$$\begin{aligned} 0 \leq u_{i+k}(\tau,y) - u_i(\tau,y) =& u_{i+k-\lceil \frac{k}{n}\rceil n}(\tau, y + \left\lceil \frac{k}{n} \right\rceil n) - u_i(\tau,y) \\ \leq& u_i(\tau, y + \left\lceil \frac{k}{n} \right\rceil n) - u_i(\tau,y) \\ \leq& 1 + p \left\lceil \frac{k}{n} \right\rceil n \\ \leq& 1 + p(m+n) \end{aligned} \tag{4.13}$$

where we have used the periodicity of u_i for the first line, the monotonicity in i of u_i for the second one and the control of the oscillation (4.8) for the third one. We then deduce that

$$F_j(\tau,[u_j(\tau,\cdot)]_{j,m}(y)) \leq L_F(2+p(m+n)) + \sup_\tau F(\tau,0,\dots 0).$$

Combining this with (4.10) and (4.11), we deduce that

$$\begin{cases} (u_j)_\tau \geq \alpha_0(\xi_j - u_j) \\ (\xi_j)_\tau \leq 2C_4 + \alpha_0(u_j - \xi_j) \end{cases}$$

We now define for all $j \in \mathbb{Z}$ $v_j = \xi_j - u_j$. Classical arguments in viscosity solution show that

$$(v_j)_\tau \leq 2(C_4 - \alpha_0 v_j).$$

We then deduce that

$$v_j \leq \frac{C_4}{\alpha_0}.$$

Using the same arguments with super-solution for ξ_j, we get the desired result. □

Step 3: control of the time oscillations.
We now explain how to control the time oscillations. The proof is inspired of [12]. Let us introduce the following continuous functions defined for $T > 0$

$$\begin{aligned} \lambda^u_+(T) &= \sup_{j\in\{1,\dots,n\}} \sup_{\tau\geq 0} \frac{u_j(\tau+T,0)-u_j(\tau,0)}{T} \\ \lambda^u_-(T) &= \inf_{j\in\{1,\dots,n\}} \inf_{\tau\geq 0} \frac{u_j(\tau+T,0)-u_j(\tau,0)}{T} \end{aligned}$$

and

$$\begin{aligned} \lambda^\xi_+(T) &= \sup_{j\in\{1,\dots,n\}} \sup_{\tau\geq 0} \frac{\xi_j(\tau+T,0)-\xi_j(\tau,0)}{T} \\ \lambda^\xi_-(T) &= \inf_{j\in\{1,\dots,n\}} \inf_{\tau\geq 0} \frac{\xi_j(\tau+T,0)-\xi_j(\tau,0)}{T} \end{aligned}$$

and

$$\lambda_+(T) = \sup(\lambda_+^u(T), \lambda_+^\xi(T)) \quad \text{and} \quad \lambda_-(T) = \inf(\lambda_-^u(T), \lambda_-^\xi(T)).$$

In particular, these functions satisfy $-\infty \leq \lambda_-(T) \leq \lambda_+(T) \leq +\infty$.

The goal is to prove that $\lambda_+(T)$ and $\lambda_-(T)$ have a common limit as $T \to \infty$. We would like to apply Lemma 4.4.

In view of the definition of λ_+^u and λ_+^ξ, we see that $T \mapsto T\lambda_+^u(T)$ and $T \mapsto T\lambda_+^\xi(T)$ are sub-additive. Analogously, $T \mapsto -T\lambda_-^u(T)$ and $T \mapsto -T\lambda_-^\xi(T)$ are also sub-additive. Hence, if we can prove that these quantities $\lambda_\pm^u(T)$, $\lambda_\pm^\xi(T)$ are finite, we will know that they converge. We will then have to prove that the limit of λ_+ and λ_- are the same.

We first prove that $\lambda_\pm$ are finite.

Step 3.1: first control on the time oscillations

Lemma 4.7. *For all $T > 0$,*

$$-K_1 - \frac{C_1}{T} \leq \lambda_-(T) \leq \lambda_+(T) \leq K_1 + \frac{C_1}{T} \tag{4.14}$$

where $C_1 = \frac{C_4}{\alpha_0} + 3 + p$ and K_1 is defined in (2.14).

Proof. Consider $j \in \{1, \dots, n\}$. Using the control of the space oscillations (4.8), we get that

$$u_j(\tau, y) \geq \lfloor \Delta \rfloor + py - 1 \quad \text{and} \quad \xi_j(\tau, y) \geq \lfloor \Delta \rfloor + py - 1$$

where

$$\Delta = \inf_{j \in \{1,\dots,n\}} \inf(u_j(\tau, 0), \xi_j(\tau, 0))\,.$$

Recalling (from Lemma 2.5) that $\lfloor \Delta \rfloor + py + \frac{pj}{n} - 1 - K_1 t$ is a subsolution and using the comparison principle on the time interval $[\tau, \tau + t)$, we deduce that

$$u_j(\tau + t, y) \geq \lfloor \Delta \rfloor + py + \frac{pj}{n} - 1 - K_1 t \quad \text{and} \quad \xi_j(\tau + t, y) \geq \lfloor \Delta \rfloor + py + \frac{pj}{n} - 1 - K_1 t \tag{4.15}$$

We now want to estimate Δ from below. Let us assume that the infimum in Δ is reached for the index $\bar{j} \in \{1, \dots, n\}$. Then $\bar{j} \geq j - n$ since $j \in \{1, \dots, n\}$. We then deduce that

$$\begin{aligned}
\lfloor \Delta \rfloor \geq & \Delta - 1 \\
\geq & u_{\bar{j}}(\tau, 0) - \frac{C_4}{\alpha_0} - 1 \\
\geq & u_{j-n}(\tau, 0) - \frac{C_4}{\alpha_0} - 1 \\
\geq & u_j(\tau, -1) - \frac{C_4}{\alpha_0} - 1 \\
\geq & u_j(\tau, 0) - \frac{C_4}{\alpha_0} - 2 - p
\end{aligned}$$

where we have used (4.9) for the second line, the fact that $(u_j)_j$ is non-decreasing in j for the third line, the periodicity of u_j for the fourth line and (4.8) for the last one. In the same way, we get that

$$\lfloor \Delta \rfloor \geq \xi_j(\tau, 0) - \frac{C_4}{\alpha_0} - 2 - p.$$

Injecting this in (4.15), we get that

$$u_j(\tau + t, y) \geq u_j(\tau, 0) - \frac{C_4}{\alpha_0} - 3 - p + py - K_1 t \tag{4.16}$$

and

$$\xi_j(\tau+t,y) \geq \xi_j(\tau,0) - \frac{C_4}{\alpha_0} - 3 - p + py - K_1 t.$$

In the same way, we also get

$$u_j(\tau+t,y) \leq u_j(\tau,0) + \frac{C_4}{\alpha_0} + 3 + p + py + K_1 t \tag{4.17}$$

and

$$\xi_j(\tau+t,y) \leq \xi_j(\tau,0) + \frac{C_4}{\alpha_0} + 3 + p + py + K_1 t.$$

Taking $y = 0$, we finally get (4.14) □

Step 3.2: Refined control on the time oscillations
We now estimate $\lambda_+ - \lambda_-$ in order to prove that they have the same limit.

Lemma 4.8. *For all $T > 0$,*

$$|\lambda_+(T) - \lambda_-(T)| \leq \frac{C_2}{T}$$

where $C_2 = 7 + \frac{4C_4}{\alpha_0} + 4p + 2C_1 + 2K_1$.

Proof. By definition of $\lambda_\pm(T)$, for all $\varepsilon > 0$, there exists $\tau^\pm \geq 0$ and $v^\pm \in \{u_1, \dots u_n, \xi_1, \dots \xi_n\}$ such that

$$\left| \lambda_\pm(T) - \frac{v^\pm(\tau^\pm + T, 0) - v^\pm(\tau^\pm, 0)}{T} \right| \leq \varepsilon.$$

Consider $j \in \{1, \dots, n\}$. We choose $\beta \in [0,1)$ such that $\tau^+ - \tau^- - \beta = k \in \mathbb{Z}$ and we set

$$\Delta_j^u = u_j(\tau^+, 0) - u_j(\tau^- + \beta, 0), \quad \Delta_j^\xi = \xi_j(\tau^+, 0) - \xi_j(\tau^- + \beta, 0)$$

and

$$\Delta = \sup_{j \in \{1,\dots,n\}} \sup(\Delta_j^u, \Delta_j^\xi).$$

Using (4.8), we get that

$$u_j(\tau^+, y) \leq u_j(\tau^- + \beta, y) + 2 + \lceil \Delta \rceil \quad \text{and} \quad \xi_j(\tau^+, y) \leq \xi_j(\tau^- + \beta, y) + 2 + \lceil \Delta \rceil.$$

Using the comparison principle, we then deduce that

$$u_j(\tau^+ + T, y) \leq u_j(\tau^- + \beta + T, y) + 2 + \lceil \Delta \rceil \quad \text{and} \quad \xi_j(\tau^+ + T, y) \leq \xi_j(\tau^- + \beta + T, y) + 2 + \lceil \Delta \rceil. \tag{4.18}$$

We now want to estimate $\lceil \Delta \rceil$ from above. Let us assume that the maximum in Δ is reached for the index $\bar{j}$. We then have for all $j \in \{1, \dots, n\}$

$$\begin{aligned}
\lceil \Delta \rceil \leq & u_{\bar{j}}(\tau^+, 0) - u_{\bar{j}}(\tau^- + \beta, 0) + \frac{2C_4}{\alpha_0} + 1 \\
\leq & u_{j+n}(\tau^+, 0) - u_{j-n}(\tau^- + \beta, 0) + \frac{2C_4}{\alpha_0} + 1 \\
\leq & u_j(\tau^+, 1) - u_j(\tau^- + \beta, -1) + \frac{2C_4}{\alpha_0} + 1 \\
\leq & u_j(\tau^+, 0) - u_j(\tau^- + \beta, 0) + \frac{2C_4}{\alpha_0} + 3 + 2p
\end{aligned}$$

where we have used (4.9) for the first line, the fact that $(u_j)_j$ is non-decreasing in j for the second line, the periodicity of u_j for the third line and (4.8) for the last one. In the same way, we also get

$$\lceil \Delta \rceil \leq \xi_j(\tau^+, 0) - \xi_j(\tau^- + \beta, 0) + \frac{2C_4}{\alpha_0} + 3 + 2p$$

Injecting this in (4.18), we get

$$u_j(\tau^+ + T, y) \leq u_j(\tau^- + \beta + T, y) + 5 + \frac{2C_4}{\alpha_0} + 2p + \Delta_j^u$$

and

$$\xi_j(\tau^+ + T, y) \leq \xi_j(\tau^- + \beta + T, y) + 5 + \frac{2C_4}{\alpha_0} + 2p + \Delta_j^\xi$$

Taking $y = 0$ and using (4.16) (with $\tau = \tau^-$ and $t = \beta$) and (4.17) (with $\tau = \tau^- + T$ and $t = \beta$), we get

$$u_j(\tau^+ + T, 0) - u_j(\tau^+, 0) \leq u_j(\tau^- + T, 0) - u_j(\tau^-, 0) + 5 + \frac{2C_4}{\alpha_0} + 2p + 2C_1 + 2K_1\,.$$

In the same way, we get

$$\xi_j(\tau^+ + T, 0) - \xi_j(\tau^+, 0) \leq \xi_j(\tau^- + T, 0) - \xi_j(\tau^-, 0) + 5 + \frac{2C_4}{\alpha_0} + 2p + 2C_1 + 2K_1\,.$$

Using also (4.9), (4.8) and the fact that $(u_j)_j$ and $(\xi_j)_j$ are non-decreasing in j, we finally get

$$v^+(\tau^+ + T, 0) - v^+(\tau^+, 0) \leq v^-(\tau^- + T, 0) - v^-(\tau^-, 0) + C_2\,.$$

The comparison of u_j and ξ_j makes appear the additional constant $2C_4/\alpha_0$, and the comparison between u_j and u_k creates an additional constant $2 + 2p$ (like in the bound on $\lceil \Delta \rceil$ above). This explains the value of the new constant C_2.

This implies that

$$T\lambda^+(T) \leq T\lambda^-(T) + 2\varepsilon + C_2\,.$$

Since this is true for all $\varepsilon > 0$, the proof of the lemma is complete. □

Step 3.3: Conclusion

We now can conclude that $\lim_{T\to+\infty} \lambda_\pm(T)$ are equal. If λ denotes the common limit, we also have, by Lemma 4.4, that for every $T > 0$,

$$\lambda_-(T) \leq \lambda \leq \lambda_+(T).$$

Moreover, by Lemma 4.8, we have

$$\lambda_+(T) \leq \lambda_-(T) + \frac{C_2}{T}$$

and so

$$\lambda_-(T) \leq \lambda \leq \lambda_-(T) + \frac{C_2}{T}$$

We finally deduce (using a similar argument for λ_+) that

$$|\lambda_\pm(T) - \lambda| \leq \frac{C_2}{T}.$$

Combining this estimate and (4.8), we get with $T = \tau$

$$|u_j(\tau, y) - u_j(0, 0) - py - \lambda\tau| \leq C_2 + 1$$

and

$$|\xi_j(\tau, y) - \xi_j(0, 0) - py - \lambda\tau| \leq C_2 + 1\,.$$

This finally implies (4.4) with $C_3 = C_2 + 1$. □

4.2 Construction of hull functions for general Hamiltonians

In this subsection, we construct hull functions for a general Hamiltonian G_j. As we shall see, this is a straightforward consequence of the construction of time-space periodic solutions of (4.19); see Proposition 4.9 and Corollary 4.10 below. We will then prove that the time slope we constructed in Proposition 4.3 is unique and that the map $p \mapsto \lambda$ is continuous; see Proposition 4.11 below.

Given $p > 0$, we consider the equation in $\mathbb{R} \times \mathbb{R}$

$$\left\{ \begin{array}{l} \left\{ \begin{array}{l} (u_j)_\tau = \alpha_0(\xi_j - u_j) \\ (\xi_j)_\tau = G_j(\tau, [u(\tau,\cdot)]_{j,m}, \xi_j, \inf_{y' \in \mathbb{R}} (u_j(\tau,y') - py') + py - u_j(\tau,y), (u_j)_y) \end{array} \right. \\ \\ \left\{ \begin{array}{l} u_{j+n}(\tau,y) = u_j(\tau,y+1) \\ \xi_{j+n}(\tau,y) = \xi_j(\tau,y+1)\,, \end{array} \right. \end{array} \right. \tag{4.19}$$

where $G_j = G_j^\delta$ is given in (4.1) for $\delta \geq 0$. Then we have the following result

Proposition 4.9. (Existence of time-space periodic solutions of (4.19)**)**
Let $0 \leq \delta \leq 1$, $a_0 \in \mathbb{R}$ and $p > 0$. Assume (A1)-(A6)*. Then there exist functions $((u_j^\infty)_j, (\xi_j^\infty)_j$ solving* (4.19) *on $\mathbb{R} \times \mathbb{R}$ and a real number $\lambda \in \mathbb{R}$ satisfying for all $\tau, y \in \mathbb{R}$, $j \in \{1, \dots, n\}$*

$$\begin{array}{rcl} |u_j^\infty(\tau,y) - py - \lambda\tau| & \leq & 2\lceil C_3 \rceil \\ |\xi_j^\infty(\tau,y) - py - \lambda\tau| & \leq & 2\lceil C_3 \rceil\,. \end{array} \tag{4.20}$$

Moreover $((u_j^\infty)_j, (\xi_j^\infty)_j)$ satisfies for $j \in \{1, \dots, n\}$

$$\left\{ \begin{array}{l} u_j^\infty(\tau, y+1/p) = u_j^\infty(\tau,y) + 1 \\ u_j^\infty(\tau+1, y) = u_j^\infty(\tau, y + \lambda/p) \\ (u_j^\infty)_y(\tau,y) \geq 0 \\ u_{j+1}(\tau,y) \geq u_j(\tau,y)\,. \end{array} \right. \qquad \left\{ \begin{array}{l} \xi_j^\infty(\tau, y+1/p) = \xi_j^\infty(\tau,y) + 1 \\ \xi_j^\infty(\tau+1, y) = \xi_j^\infty(\tau, y + \lambda/p) \\ (\xi_j^\infty)_y(\tau,y) \geq 0 \\ \xi_{j+1}(\tau,y) \geq \xi_j(\tau,y)\,. \end{array} \right. \tag{4.21}$$

Eventually, when the Hamiltonians G_j are independent on τ, we can choose u_j^∞ and ξ_j^∞ independent on τ.

By considering for all $\tau, z \in \mathbb{R}$

$$\left\{ \begin{array}{l} h_j(\tau,z) = u_j^\infty(\tau, (z - \lambda\tau)/p) \quad \text{if } j \in \{1, \dots, n\} \\ h_{j+n}(\tau,z) = h_j(\tau, z+p) \quad \text{otherwise} \end{array} \right. \tag{4.22}$$

and for all $\tau, z \in \mathbb{R}$,

$$\left\{ \begin{array}{l} g_j(\tau,z) = \xi_j^\infty(\tau, (z - \lambda\tau)/p) \quad \text{if } j \in \{1, \dots, n\} \\ g_{j+n}(\tau,z) = g_j(\tau, z+p) \quad \text{otherwise} \end{array} \right. \tag{4.23}$$

we immediately get the following corollary

Corollary 4.10. (Existence of hull functions)
Assume (A1)-(A6)*. There exists a hull function $((h_j)_j, (g_j)_j)$ in the sense of Definition 1.8 satisfying*

$$|h_j(\tau,z) - z| \leq 2\lceil C_3 \rceil$$

and

$$|g_j(\tau,z) - z| \leq 2\lceil C_3 \rceil$$

We now turn to the proof of Proposition 4.9.

Proof of Proposition 4.9. The proof is performed in three steps. In the first one, we construct sub- and supersolutions of (4.19) in $\mathbb{R}\times\mathbb{R}$ with good translation invariance properties (see the first two lines of (4.21)). We next apply Perron's method in order to get a (possibly discontinuous) solution satisfying the same properties. Finally, in step 3, we prove that if the functions G_j do not depend on τ, then we can construct a solution in such a way that it does not depend on τ either.

Step 1: global sub- and supersolution
By Proposition 4.3, we know that the solution (u_j,ξ_j) of (2.1), (2.2) with initial data $u_0(y)=py=\xi_0(y)$ satisfies on $[0,+\infty)\times\mathbb{R}$

$$\text{(4.24)}\qquad \begin{cases} (u_j)_y\geq 0,\\ |u_j(\tau,y)-py-\lambda\tau|\leq C_3,\\ |u_j(\tau,y+y')-u_j(\tau,y)-py'|\leq 1\\ u_{j+1}(\tau,y)\geq u_j(\tau,y) \end{cases} \qquad \begin{cases} (\xi_j)_y\geq 0,\\ |\xi_j(\tau,y)-py-\lambda\tau|\leq C_3,\\ |\xi_j(\tau,y+y')-\xi_j(\tau,y)-py'|\leq 1\\ \xi_{j+1}(\tau,y)\geq \xi_j(\tau,y)\,. \end{cases}$$

We first construct a subsolution and a supersolution of (4.19) for $\tau\in\mathbb{R}$ (and not only $\tau\geq 0$) that also satisfy the first two lines of (4.21), *i.e.* satisfy for all $k,l\in\mathbb{Z}$,

$$\text{(4.25)}\qquad U(\tau+k,y)=U(\tau,y+\lambda\frac{k}{p}) \quad\text{and}\quad U(\tau,y+\frac{l}{p})=U(\tau,y)+l\,.$$

To do so, we consider the sequence, for $m\in\mathbb{N}$ and $j\in\{1,\dots,n\}$,

$$u_j^m(\tau,y)=u_j(\tau+m,y)-\lambda m,\quad \xi_j^m(\tau,y)=\xi_j(\tau+m,y)-\lambda m$$

and consider

$$\begin{aligned} \overline{u}_j&=\limsup_{m\to+\infty}{}^{*}u_j^m,\quad \overline{\xi}_j=\limsup_{m\to+\infty}{}^{*}\xi_j^m\\ \underline{u}_j&=\liminf_{m\to+\infty_*}u_j^m,\quad \underline{\xi}_j=\liminf_{m\to+\infty_*}\xi_j^m\,. \end{aligned}$$

We first remark that thanks to (4.4), all these semi-limits are finite. We also remark that for all $k,l\in\mathbb{Z}$,

$$(\overline{u}_j(\tau+k,y-k\lambda/p+l/p)-l,\overline{\xi}_j(\tau+k,y-k\lambda/p+l/p)-l)$$

is a subsolution of (2.1). A similar remark can be done for the supersolutions $(\underline{u}_j,\underline{\xi}_j)_j$.

Now a way to construct subsolution (resp. a supersolution) of (2.1) satisfying (4.25) is to consider

$$\text{(4.26)}\qquad \begin{cases} \overline{u}_j^{\infty}(\tau,y)=\sup_{k,l\in\mathbb{Z}}\left(\overline{u}_j(\tau+k,y-k\lambda/p+l/p)-l\right),\\ \overline{\xi}_j^{\infty}(\tau,y)=\sup_{k,l\in\mathbb{Z}}\left(\overline{\xi}_j(\tau+k,y-k\lambda/p+l/p)-l\right) \end{cases}$$

and

$$\text{(4.27)}\qquad \left(\text{resp.}\quad \begin{cases} \underline{u}_j^{\infty}(\tau,y)=\inf_{k,l\in\mathbb{Z}}\left(\underline{u}_j(\tau+k,y-k\lambda/p+l/p)-l\right),\\ \underline{\xi}_j^{\infty}(\tau,y)=\inf_{k,l\in\mathbb{Z}}\left(\underline{\xi}_j(\tau+k,y-k\lambda/p+l/p)-l\right)\,. \end{cases}\right)$$

Notice that $\overline{u}_j^{\infty}$, $\underline{u}_j^{\infty}$, $\overline{\xi}_j^{\infty}$ and $\underline{\xi}_j^{\infty}$ satisfy moreover (4.24) on $\mathbb{R}\times\mathbb{R}$. Therefore we have in particular

$$\overline{u}_j^{\infty}\leq\underline{u}_j^{\infty}+2\lceil C_3\rceil\quad\text{and}\quad\overline{\xi}_j^{\infty}\leq\underline{\xi}_j^{\infty}+2\lceil C_3\rceil\,.$$

Step 2: existence by Perron's method

Applying Perron's method we see that the lowest supersolution $((u_j^\infty)_j, (\xi_j^\infty)_j)$ lying above $((\overline{u}_j^\infty)_j, (\overline{\xi}_j^\infty)_j)$ is a (possibly discontinuous) solution of (4.24) on $\mathbb{R}\times\mathbb{R}$ and satisfies

$$\overline{u}_j^\infty \le u_j^\infty \le \underline{u}_j^\infty + 2\lceil C_3\rceil \quad \text{and} \quad \overline{\xi}_j^\infty \le \xi_j^\infty \le \underline{\xi}_j^\infty + 2\lceil C_3\rceil\,.$$

We next prove that u^∞ satisfies (4.21). For $j\in\{1,\dots,n\}$, let us consider

$$\tilde{u}_j^\infty(\tau,y) = \left(\inf_{k,l\in\mathbb{Z}}\left(u_j^\infty(\tau+k, y-k\lambda/p+l/p)-l\right)\right)_* \tag{4.28}$$

$$\tilde{\xi}_j^\infty(\tau,y) = \left(\inf_{k,l\in\mathbb{Z}}\left(\xi_j^\infty(\tau+k, y-k\lambda/p+l/p)-l\right)\right)_*$$

By construction the family $((\tilde{u}_j^\infty)_j, (\tilde{\xi}_j^\infty)_j)$ is a supersolution of (2.1) and is again above the subsolution $((\overline{u}_j^\infty)_j, (\overline{\xi}_j^\infty)_j)$. Therefore from the definition of $((u_j^\infty)_j, (\xi_j^\infty)_j)$, we deduce that

$$\tilde{u}_j^\infty = u_j^\infty \quad \text{and} \quad \tilde{\xi}_j^\infty = \xi_j^\infty$$

which implies that u_j^∞ and ξ_j^∞ satisfy (4.25), *i.e* the first two equalities of (4.21).

Similarly, we can consider, for $j\in\{1,\dots,n\}$

$$\hat{u}_j^\infty(\tau,y) = \left(\inf_{b\in[0,+\infty)} u_j^\infty(\tau,y+b)\right)_*$$

$$\hat{\xi}_j^\infty(\tau,y) = \left(\inf_{b\in[0,+\infty)} \xi_j^\infty(\tau,y+b)\right)_*$$

which is again supersolution above the subsolution $((\overline{u}_j^\infty)_j, (\overline{\xi}_j^\infty)_j)$. Therefore

$$\hat{u}_j^\infty = u_j^\infty \quad \text{and} \quad \hat{\xi}_j^\infty = \xi_j^\infty$$

which implies that u_j^∞ and ξ_j^∞ are non-decreasing in y, *i.e.* the third line of (4.21) is satisfied.

Let us now prove that u_j^∞ and ξ_j^∞ are non-decreasing in j. We consider, for $j\in\{1,\dots,n\}$

$$\check{u}_j^\infty(\tau,y) = \left(\inf_{k\ge 0} u_{j+k}^\infty(\tau,y)\right)_* = \left(\inf_{0\le k<n} u_{j+k}^\infty(\tau,y)\right)_*$$

$$\check{\xi}_j^\infty(\tau,y) = \left(\inf_{k\ge 0} \xi_{j+k}^\infty(\tau,y)\right)_* = \left(\inf_{0\le k<n} \xi_{j+k}^\infty(\tau,y)\right)_*.$$

The fact that this is a supersolution uses assuption (A6). Indeed, let us assume that the infimum for u_j is reached for the index k_u and that the infimum for ξ_j is reached for the index k_ξ. Then, formally, on one hand we have

$$\begin{aligned}(\check{u}_j^\infty)_\tau(\tau,y) =& \alpha_0(\xi_{j+k_u}^\infty(\tau,y) - u_{j+k_u}^\infty(\tau,y))\\ \ge& \alpha_0(\xi_{j+k_\xi}^\infty(\tau,y) - u_{j+k_u}^\infty(\tau,y))\\ \ge& \alpha_0(\check{\xi}_j^\infty(\tau,y) - \check{u}_j^\infty(\tau,y))\end{aligned}$$

where we have used the fact that $\xi_{j+k_u}^\infty(\tau,y) \ge \xi_{j+k_\xi}^\infty(\tau,y)$. On the other hand, we have

$$\begin{aligned}(\check{\xi}_j^\infty)_\tau(\tau,y) =& G_{j+k_\xi}(\tau,[u^\infty(\tau,\cdot)]_{j+k_\xi}(y), \xi_{j+k_\xi}^\infty(\tau,y), \inf_{y'}(\xi_{j+k_\xi}^\infty(\tau,y') - py') + py - \xi_{j+k_\xi}^\infty(\tau,y), (\xi_{j+k_\xi}^\infty)_y)\\ \ge& G_{j+k_\xi}(\tau,[\check{u}^\infty(\tau,\cdot)]_{j+k_\xi}(y), \check{\xi}_j^\infty(\tau,y), \inf_{y'}(\check{\xi}_j^\infty(\tau,y') - py') + py - \check{\xi}_j^\infty(\tau,y), (\check{\xi}_j^\infty)_y)\\ \ge& G_{j+k_\xi-1}(\tau,[\check{u}^\infty(\tau,\cdot)]_{j+k_\xi-1}(y), \check{\xi}_j^\infty(\tau,y), \inf_{y'}(\check{\xi}_j^\infty(\tau,y') - py') + py - \check{\xi}_j^\infty(\tau,y), (\check{\xi}_j^\infty)_y)\\ \ge& \dots\\ \ge& G_j(\tau,[\check{u}^\infty(\tau,\cdot)]_j(y), \check{\xi}_j^\infty(\tau,y), \inf_{y'}(\check{\xi}_j^\infty(\tau,y') - py') + py - \check{\xi}_j^\infty(\tau,y), (\check{\xi}_j^\infty)_y)\end{aligned}$$

where we have used the fact that $u^\infty_{j+k_\xi+k} \geq \check{u}^\infty_{j+k_\xi+k}$ and $\xi^\infty_{j+k_\xi}(\tau, y') \geq \check{\xi}^\infty_j(\tau, y')$ joint to the monotonicity assumption of G in the variable V_i and a for the first inequality and assumtion (A6) joint to the fact that $\check{u}^\infty_j$ is non-decreasing in j (by construction) for the other inequalities.

We then conclude that $(\check{u}^\infty_j, \check{\xi}^\infty_j)$ is again supersolution above the subsolution $((\overline{u}^\infty_j)_j, (\overline{\xi}^\infty_j)_j)$. Therefore

$$u^\infty_j = \check{u}^\infty_j \quad \text{and} \quad \xi^\infty_j = \check{\xi}^\infty_j$$

which implies that u^∞_j and ξ^∞_j are non-decreasing in j, *i.e.* the forth line of (4.21) is satisfied.

Finally, the function $((u^\infty_j - \lceil C_3 \rceil)_j, (\xi^\infty_j - \lceil C_3 \rceil)_j)$ still satisfies (4.21) and also (4.20).

Step 3: Further properties when the G_j are independent on τ
When the G_j do not depend on τ, we can apply Steps 1 and 2 with $k \in \mathbb{Z}$ in (4.26), (4.27) and (4.28) replaced with $k \in \mathbb{R}$. This implies that the hull function $((h_j)_j, (g_j)_j)$ does not depend on τ. This ends the proof of the proposition. □

Proposition 4.11 (Definition and continuity of the effective Hamiltonian).
Consider $p > 0$ and assume (A1)-(A6). *Then*

- *there exists a unique real number $\lambda \in \mathbb{R}$ such that there exists a solution $((u^\infty_j)_j, (\xi^\infty_j)_j)$ of* (4.19) *on $\mathbb{R} \times \mathbb{R}$ such that there exists $C > 0$,*

$$|h_j(0,z) - z| \leq C, \quad \text{and} \quad |g_j(0,z) - z| \leq C, \tag{4.29}$$

  where the h_j and the g_j are defined in (4.22) *and* (4.23)*; Moreover, we can choose $C = 2\lceil C_3 \rceil$ with C_3 given in* (4.6).

- *if λ is seen as a function $\overline{G}$ of p ($\lambda = \overline{G}(p)$), then this function $\overline{G} : (0, +\infty) \to \mathbb{R}$ is continuous.*

The proof is quite classical. However, we give it in Appendix for the reader's convenience. We can now prove Theorem 1.10.

Proof of Theorem 1.10. Just apply Proposition 4.11 with $G = F$. □

5 Construction of Lipschitz continuous approximate hull functions

When proving the convergence Theorem 1.5, we explained that, on the one hand, it is necessary to deal with hull functions $(h, g) = ((h_j(\tau, z))_j, (g_j(\tau, z))_j)$ that are uniformly continuous in z (uniformly in τ and j) in order to apply Evans' perturbed test function method; on the other hand, given some $p > 0$, we also know some Hamiltonian F_j, with effective Hamiltonian $\overline{F}(p)$, such that every corresponding hull function h_j is necessarily discontinuous in z (see [9]). Recall that a hull function (h, g) solves, with $\lambda = \overline{F}(p)$,

$$\begin{cases} (h_j)_\tau + \lambda (h_j)_z = \alpha_0 (g_j - h_j) \\ (g_j)_\tau + \lambda (g_j)_z = 2F_j(\tau, [h(\tau, \cdot)]_{j,m}) + \alpha_0 (h_j - g_j) \end{cases} \tag{5.1}$$

and

$$h_{j+n}(\tau, z) = h_j(\tau, z + p), \; g_{j+n}(\tau, z) = g_j(\tau, z + p)\,.$$

We overcome this difficulty as in [9] (see also [12]).

We build approximate Hamiltonians G^δ with corresponding effective Hamiltonians $\lambda^\delta = \overline{G}^\delta(p)$, and corresponding hull functions (h^δ, g^δ), such that

$$\begin{cases} (h^\delta_j, g^\delta_j) \quad \text{is Lipschitz continuous wrt } z \text{ uniformly in } \tau \text{ and } j \\ \overline{G}^\delta(p) \to \overline{F}(p) \quad \text{as} \quad \delta \to 0 \\ (h^\delta, g^\delta) \text{ is a sub-/super-solution of (5.1).} \end{cases}$$

We will show that it is enough to choose for $\delta \geq 0$

$$G_j^\delta(\tau, V, r, a, q) = 2F_j(\tau, V) + \alpha_0(V_0 - r) + \delta(a_0 + a)q^+ \tag{5.2}$$

with $a_0 \in \mathbb{R}$ (in fact, we will consider $a_0 = \pm 1$).

We have the following variant of Corollary 4.10

Proposition 5.1 (**Existence of Lipschitz continuous approximate hull functions).** *Assume* (A1)-(A3). *Given $p > 0$, $0 < \delta \leq 1$ and $a_0 \in \mathbb{R}$, then there exists a family of Lipschitz continuous functions $((h_j)_j, (g_j)_j)$ satisfying for $j \in \{1, \dots, n\}$*

$$\begin{cases} h_j(\tau, z+1) = h_j(\tau, z) + 1 \\ h_j(\tau+1, z) = h_j(\tau, z) \\ \\ 0 \leq (h_j)_z \leq 1 + \frac{2L_F}{p\delta} \end{cases} \qquad \begin{cases} g_j(\tau, z+1) = g_j(\tau, z) + 1 \\ g_j(\tau+1, z) = g_j(\tau, z) \\ \\ 0 \leq (g_j)_z \leq 1 + \frac{2L_F}{p\delta} \end{cases} \tag{5.3}$$

and there exists $\lambda \in \mathbb{R}$ such that

$$\begin{cases} \begin{cases} (h_j)_\tau + \lambda(h_j)_z = & \alpha_0(g_j - h_j) \\ (g_j)_\tau + \lambda(g_j)_z = & 2F_j(\tau, [h(\tau, \cdot)]_{j,m}) + \alpha_0(h_j - g_j) \\ & +\delta p \left\{ a_0 + \inf_{z' \in \mathbb{R}} \left(h_j(\tau, z') - z' \right) + z - h_j(\tau, z) \right\} \end{cases} (h_j)_z \\ \\ \begin{cases} h_{j+n}(\tau, z) = & h_j(\tau, z+p) \\ g_{j+n}(\tau, z) = & g_j(\tau, z+p) \end{cases} \end{cases} \tag{5.4}$$

and for all $\tau, z, z' \in \mathbb{R}$

$$|h_j(\tau, z') - z' + z - h_j(\tau, z)| \leq 1 \quad \text{and} \quad |g_j(\tau, z') - z' + z - g_j(\tau, z)| \leq 1 \,. \tag{5.5}$$

Moreover there exists a constant $C_4 > 0$ defined in (4.6) *such that*

$$|\lambda| \leq C_4 \tag{5.6}$$

and for all $(\tau, z) \in \mathbb{R} \times \mathbb{R}$,

$$\begin{aligned} |h(\tau, z) - z| &\leq C(C_4, p, \alpha_0, \delta |a_0| p) \,, \\ |g(\tau, z) - z| &\leq C(C_4, p, \alpha_0, \delta |a_0| p) \,. \end{aligned} \tag{5.7}$$

Moreover, when the F_j do not depend on τ, we can choose the hull function $((h_j)_j, (g_j)_j)$ such that it does not depend on τ either.

Proof of Proposition 5.1. The proof follow the construction made in Proposition 4.3 and Proposition 4.9. The only difference is the proof of Proposition 4.9. Indeed, here we can use the Lipschitz continuity of $((u_j), (\xi_j))$ to get enough compacity to pass to the limit (so we do not have to use Perron's method). The Lipschitz continuity in space comes from Proposition 4.1. The Lipschitz continuity in time of the u_j follows from Lemma 4.6 and the equation satisfied by u_j while the Lipschitz continuity in time of the ξ_j is obtained in the same way, using the fact that we can bound the right hand side of the equation satisfied by ξ_j (use the control of the space oscillation of u to bound $F(t, [u(t, \cdot)]_{j,m}(x))$ as in (4.12)-(4.13) and Lemma 4.6 and Proposition 4.1 to bound the other ones).

□

We finally have

Proposition 5.2 (Sub- and super- Lipschitz continuous hull functions). *We consider* $0 < \delta \leq 1$ *and the Lipschitz continuous hull function obtained in Proposition 5.1 for* $a_0 = \pm 1$*, that we call* $((h_j^{\delta,\pm})_j, (g_j^{\delta,\pm})_j)$*, and the corresponding value* $\lambda^{\delta,\pm}$ *of the effective Hamiltonian. Then we have*

$$\begin{aligned}(h_j^{\delta,+})_\tau + \lambda^{\delta,+}(h_j^{\delta,+})_z &= \alpha_0(g_j^{\delta,+} - h_j^{\delta,+}) \\ (g_j^{\delta,+})_\tau + \lambda^{\delta,+}(g_j^{\delta,+})_z &\geq 2F_j(\tau, [h^{\delta,+}(\tau,\cdot)]_{j,m}) + \alpha_0(h_j^{\delta,+} - g_j^{\delta,+})\end{aligned}$$

and

$$\lambda \leq \lambda^{\delta,+} \to \lambda \quad as \quad \delta \to 0$$

and

$$\begin{aligned}(h_j^{\delta,-})_\tau + \lambda^{\delta,-}(h_j^{\delta,-})_z &= \alpha_0(g_j^{\delta,-} - h_j^{\delta,-}) \\ (g_j^{\delta,-})_\tau + \lambda^{\delta,-}(g_j^{\delta,-})_z &\geq 2F_j(\tau, [h^{\delta,-}(\tau,\cdot)]_{j,m}) + \alpha_0(h_j^{\delta,-} - g_j^{\delta,-})\end{aligned}$$

and

$$\lambda \geq \lambda^{\delta,-} \to \lambda \quad as \quad \delta \to 0$$

where $\lambda = \overline{F}(p)$.

Proof of Proposition 5.2. Inequalities $\pm\lambda^{\delta,\pm} \geq \pm\lambda$ follow from the comparison principle. Remark that bounds (5.6) and (5.7) on $\lambda^{\delta,\pm}$ and $h_j^{\delta,\pm}$ are uniform as δ goes to zero. Hence the convergence $\lambda^{\delta,\pm} \to \lambda$ holds true as $\delta \to 0$. Indeed, it suffices to adapt Step 2 of the proof of Proposition 4.11. □

6 Qualitative properties of the effective Hamiltonian

Proof of Theorem 1.11. We recall that we have hull functions $((h_j)_j, (g_j)_j)$ solutions of

$$\begin{cases} (h_j)_\tau + \lambda(h_j)_z = \alpha_0(g_j - h_j) \\ (g_j)_\tau + \lambda(g_j)_z = 2L + 2F(\tau, [h(\tau,\cdot)]_{j,m}(z)) + \alpha_0(h_j - g_j) \end{cases}$$

with $\lambda = \overline{F}(L,p)$.
The continuity of the map $(L,p) \mapsto \overline{F}(L,p)$ is easily proved as in step 2 of the proof of Proposition 4.11.
(i) Bound
This is a straightforward adaptation of the proof of (4.14).
(ii) Monotonicity in L
The monotonicity of the map $L \mapsto \overline{F}(L,p)$ follows from the comparison principle on

$$((u_j(\tau,y) = h_j(\tau, \lambda\tau + py))_j, (\xi_j(\tau,y) = g_j(\tau, \lambda\tau + py))_j$$

where $((h_j)_j, (g_j)_j)$ is the hull function and $\lambda = \overline{F}(L,p)$.

□

A Proof of Proposition 4.11

The proof is divided in two steps.
Step 1: Uniqueness of λ
Given some $p \in (0, +\infty)$, assume that there exist $\lambda_1, \lambda_2 \in \mathbb{R}$ with their corresponding hull functions $((h_j^1)_j, (g_j^1)_j), ((h_j^2)_j, (g_j^2)_j)$. Then define for $i = 1, 2$, $j \in \{1, \ldots, n\}$

$$u_j^i(\tau,y) = h_j^i(\tau, \lambda_i\tau + py) \quad \text{and} \quad \xi_j^i(\tau,y) = g_j^i(\tau, \lambda_i\tau + py)$$

which are both solutions of equation (2.1) on $[0,+\infty)\times\mathbb{R}$. By Corollary 4.10, we know that h_j and g_j satisfy (4.29). Then we have

$$u_j^1(0,y) \le u_j^2(0,y) + 2C \quad \text{and} \quad \xi_j^1(0,y) \le \xi_j^2(0,y) + 2C$$

which implies (from the comparison principle) for all $(\tau,y)\times[0,+\infty)\times\mathbb{R}$

$$u_j^1(\tau,y) \le u_j^2(\tau,y) + 2C \quad \text{and} \quad \xi_j^1(\tau,y) \le \xi_j^2(\tau,y) + 2C\,.$$

Using the fact that $h_j^i(\tau+1,z) = h_j^i(\tau,z)$ and $g_j^i(\tau+1,z) = g_j^i(\tau,z)$, we deduce that for $\tau = k \in \mathbb{N}$ and $y = 0$ we have

$$h_j^1(0,\lambda_1 k) \le h_j^2(0,\lambda_2 k) + 2C \quad \text{and} \quad g_j^1(0,\lambda_1 k) \le g_j^2(0,\lambda_2 k) + 2C$$

which implies by (4.29)

$$\lambda_1 k \le \lambda_2 k + 4C\,.$$

Because this is true for any $k \in \mathbb{N}$, we deduce that

$$\lambda_1 \le \lambda_2\,.$$

The reverse inequality is obtained exchanging $((h_j^1)_j,(g_j^1)_j)$ and $((h_j^2)_j,(g_j^2)_j)$. We finally deduce that $\lambda_1 = \lambda_2$, which proves the uniqueness of the real λ, that we call $\overline{G}(p)$.

Step 2: Continuity of the map $p \mapsto \overline{G}(p)$
Let us consider a sequence $(p_m)_m$ such that $p_m \to p > 0$. Let $\lambda_m = \overline{G}(p_m)$ and $((h_j^m)_j,(g_j^m)_j)$ be the corresponding hull functions. From Corollary 4.10, we can choose these hull functions such that for $j \in \{1,\dots,n\}$

$$|h_j^m(\tau,z) - z| \le 2\lceil C_3\rceil, \quad \text{and} \quad |g_j^m(\tau,z) - z| \le 2\lceil C_3\rceil$$

and we have

$$|\lambda_m| \le C_4$$

where we recall that C_4 is defined in (4.6). Remark that both C_3 and C_4 depends on p_m, but can be bounded for p_m in a neighbourhood of p. We deduce in particular that there exists a constant $C_5 > 0$ such that

$$|h_j^m(\tau,z) - z| \le C_5, \quad |g_j^m(\tau,z) - z| \le C_5 \quad \text{and} \quad |\lambda_m| \le C_5\,.$$

Let us consider a limit λ_∞ of $(\lambda_m)_m$, and let us define

$$\overline{h}_j = \limsup_{m\to+\infty}{}^* h_j^m, \quad \text{and} \quad \overline{g}_j = \limsup_{m\to+\infty}{}^* g_j^m\,.$$

This family of functions $((\overline{h}_j)_j,(\overline{g}_j)_j)$ is such that the family

$$((\overline{u}_j(\tau,y))_j,(\overline{\xi}_j(\tau,y))_j) = ((\overline{h}_j(\tau,\lambda_\infty\tau+py))_j,(\overline{g}_j(\tau,\lambda_\infty\tau+py))_j)$$

is a subsolution of (4.19) on $\mathbb{R}\times\mathbb{R}$. On the other hand, if $((h_j)_j,(g_j)_j)$ denotes the hull function associated with p and $\lambda = \overline{G}(p)$, then

$$((u_j(\tau,y))_j,(\xi_j(\tau,y))_j) = ((h_j(\tau,\lambda\tau+py))_j,(g_j(\tau,\lambda\tau+py))_j)$$

is a solution of (4.19) on $\mathbb{R}\times\mathbb{R}$. Finally, as in Step 1, we conclude that

$$\lambda_\infty \le \lambda\,.$$

Similarly, considering

$$\underline{h}_j = \liminf_{m\to+\infty}{}_* h_j^m \quad \text{and} \quad \underline{g}_j = \liminf_{m\to+\infty}{}_* g_j^m$$

we can show that

$$\lambda_\infty \ge \lambda\,.$$

Therefore $\lambda_\infty = \lambda$ and this proves that $\overline{G}(p_m) \to \overline{G}(p)$; the continuity of the map $p \mapsto \overline{G}(p)$ follows and this ends the proof of the proposition.

B An alternative proof of Proposition 4.1

In this section, we give an alternative proof of Proposition 4.1. We adapt here the method we used in [9] and we provide complementary details.

B.1 Explanation of the estimate of Proposition 4.1

In this section, we formally explain how we derive the estimate obtained in Proposition 4.1.
We can adapt the corresponding proof from [9]. For all $\eta \geq 0$, we consider the following Cauchy problem

$$\text{(B.1)} \quad \left\{ \begin{array}{l} \left\{ \begin{array}{ll} (u_j)_\tau & = \alpha_0(\xi_j - u_j) \\ (\xi_j)_\tau & = G_j^\delta(\tau, [u(\tau,\cdot)]_{j,m}, \xi_j(\tau,y), \inf_{y' \in \mathbb{R}} (\xi_j(\tau,y') - py') + py - \xi_j(\tau,y), (\xi_j)_y) + \eta(\xi_j)_{yy} \end{array} \right. \\ \\ \left\{ \begin{array}{ll} u_{j+n}(\tau,y) & = u_j(\tau,y+1) \\ \xi_{j+n}(\tau,y) & = \xi_j(\tau,y+1) \end{array} \right. \\ \\ \left\{ \begin{array}{ll} u_j(0,y) & = p\left(y + \frac{j}{n}\right) \\ \xi_j(0,y) & = p\left(y + \frac{j}{n}\right) \end{array} \right. \end{array} \right.$$

where G_j^δ is given by

$$G_j^\delta(\tau, V, r, a, q) = 2F_j(\tau, V) + \alpha_0(V_0 - r) + \delta(a_0 + a)q$$

(remark that this is not exactly the function given by (5.2)). It is convenient to introduce the modified Hamiltonian

$$\tilde{F}_i(\tau, V_{-m}, \dots, V_m) = 2F_i(\tau, V_{-m}, \dots, V_m) + \alpha_0 V_0$$

so that

$$G_j^\delta(\tau, V_{-m}, \dots, V_m, r, a, q) = \tilde{F}_j(\tau, V_{-m}, \dots, V_m) - \alpha_0 r + \delta(a_0 + a)q\,.$$

Hence, the Lipschitz constant of $\tilde{F}_j(\tau, V)$ with respect to V is $\tilde{K}_1 = 2L_F + \alpha_0$.
Case A: $\eta > 0$ and $F_j \in C^1$ For $\eta > 0$, it is possible to show that there exists a unique solution $((u_j)_j, (\xi_j)_j)$ of (B.1) in $(C^{2+\alpha,1+\alpha})^{2n}$ for any $\alpha \in (0,1)$. We will give the main idea of this existence result in the next subsection.
Step 1: bound from below on the gradient
Then, if we define $\zeta_j = (\xi_j)_y$ and $v_j = (u_j)_y$, we can derive the previous equation in order to get the following one

$$\text{(B.2)} \quad \left\{ \begin{array}{rl} (v_j)_\tau = & \alpha_0(\zeta_j - v_j) \\ (\zeta_j)_\tau - \eta(\zeta_j)_{yy} = & (\tilde{F}_j)'_V(\tau, [u(\tau,\cdot)]_{j,m}(y)) \cdot [v(\tau,\cdot)]_{j,m}(y) - \alpha_0\zeta_j - \delta(\zeta_j - p)\zeta_j \\ & +\delta\left(a_0 + \inf_{y' \in \mathbb{R}} (\xi_j(\tau,y') - py') + py - \xi_j(\tau,y)\right)(\zeta_j)_y \\ \\ v_{j+n}(\tau,y) = & v_j(\tau,y+1) \\ \zeta_{j+n}(\tau,y) = & \zeta_j(\tau,y+1) \\ \\ v_j(0,y) = & \zeta_j(0,y) = p\,. \end{array} \right.$$

Let us now define

$$\underline{m}_v(\tau) = \inf_{j \in \{1,\dots,n\}} \inf_{y \in \mathbb{R}} v_j(\tau,y) \quad \text{and} \quad \underline{m}_\zeta(\tau) = \inf_{j \in \{1,\dots,n\}} \inf_{y \in \mathbb{R}} \zeta_j(\tau,y)\,.$$

Then we have in the viscosity sense:

$$\left\{ \begin{array}{l} (\underline{m}_v)_\tau \geq \alpha_0(\underline{m}_\zeta - \underline{m}_v) \\ (\underline{m}_\zeta)_\tau \geq \tilde{L}_F \min(0, \underline{m}_v) - \alpha_0\underline{m}_\zeta - \delta(\underline{m}_\zeta - p)\underline{m}_\zeta \\ \underline{m}_v(0) = \underline{m}_\zeta(0) = p > 0 \end{array} \right.$$

where we have used the monotonicity assumptions (A2) and (A3) to get the term $\tilde{L}_F \min(0, \underline{m}_v)$ with $\tilde{L}_F = 2L_F + \alpha_0$. The fact that $(0,0)$ is a subsolution of this monotone system of ODEs implies that, for $j \in \{1, \dots, n\}$,

$$v_j \geq \underline{m}_v \geq 0 \quad \text{and} \quad \zeta_j \geq \underline{m}_\zeta \geq 0\,.$$

In particular, we see that (u, ξ) is a solution of (B.1) with G_j^δ given by (5.2).

Step 2: bound from above on the gradient
Similarly we define

$$\overline{m}_\zeta(\tau) = \sup_{j\in\{1,\dots,n\}} \sup_{y\in\mathbb{R}} \zeta_j(\tau, y) \quad \text{and} \quad \overline{m}_v(\tau) = \sup_{j\in\{1,\dots,n\}} \sup_{y\in\mathbb{R}} v_j(\tau, y)\,.$$

Then we have in the viscosity sense

$$\begin{cases} (\overline{m}_v)_\tau \leq \alpha_0(\overline{m}_\zeta - \overline{m}_v) \\ (\overline{m}_\zeta)_\tau \leq (2L_F)\overline{m}_v + \alpha_0(\overline{m}_v - \overline{m}_\zeta) - \delta(\overline{m}_\zeta - p)\overline{m}_\zeta \\ \overline{m}_v(0) = \overline{m}_\zeta(0) = p > 0 \end{cases}$$

where we have used Step 1 to ensure that $v_j \geq \underline{m}_v \geq 0$ for $j \in \{1, \dots, n\}$. The constant function $(p + (2L_F)\delta^{-1})$ (for both components) is a supersolution of the previous monotone system of ODEs. Hence, the proof is complete in Case A.

Case B: $\eta = 0$ and F general
We can use an approximation argument as in [9]. This ends the proof of the proposition.

B.2 Proof of the existence of a regular solution of (B.1)

We just give the main idea.

It can be useful to remark that u_{j+l} can be rewritten as follows: for all $l \in \{-m, \dots, m\}$,

$$u_{j+l}(\tau, y) = p \cdot (y + (j+l)/n)e^{-\alpha_0\tau} + \alpha_0 \int_0^\tau e^{\alpha_0(s-\tau)}\xi_{j+l}(s, y)ds\,. \tag{B.3}$$

We set $v_j(\tau, y) = \xi_j(\tau, y) - py$. Then $(v_j)_j$ is a solution of

$$\begin{cases} (v_j)_t - \eta(v_j)_{yy} = \overline{F}_j(t, [v(\tau,\cdot) + p\cdot]_{j,m}(y)) + \delta\left(1 + \inf_{y'}(v(\tau, y')) - v(\tau, y)\right)(v_y + p) \\ v_{j+n}(\tau, y) = v_j(\tau, y+1) + p \\ v_j(0, y) = p(\frac{j}{n}) \end{cases} \tag{B.4}$$

where $\overline{F}_j[\tau, [\xi(\tau,\cdot)]_{j,m}(y)] = 2F_j(\tau, [u(\tau,\cdot)]_{j,m}(y)) + \alpha_0 u_j(\tau, y) - \xi_j(\tau, y)$ with u given by (B.3) as a function of the time integral of ξ. Since we attempt to get $\xi_j(\tau, y + \frac{1}{p}) = \xi_j(\tau, y) + 1$, we will look for functions v_j which are periodic of period $\frac{1}{p}$. The basic idea is to use a fixed point argument. First, we "regularize" the right hand side of (B.4) by considering for some given $K > 0$

$$\mathcal{F}_{K,j}(\tau, v) = T_K^0(\overline{F}_j(\tau, [v(\tau,\cdot) + p\cdot]_{j,m}(y))) + \delta\left(1 + T_K^1(\inf_{y'}(v(\tau, y')) - v(\tau, y))\right)(T_K^3(v_y + p))$$

where $T_K^i \in C_b^\infty$ are truncature functions. In particular, $\mathcal{F}_{K,j}(\tau,\cdot) \in W^{1,\infty}$ uniformly in $\tau \in [0, +\infty)$ and so for all $q > 1$, there exists a solution $w = (w_j)_j = A(v) \in W^{2,1;q}([0,T] \times [0, \frac{1}{p}))$ of

$$(w_j)_t - \eta(w_j)_{yy} = \mathcal{F}_{K,j}(v)$$

Now, we want to show that the operator A is a contraction. Let $v_1, v_2 \in W^{2,1;q}([0,T]\times[0,\frac{1}{p}))$. Standard parabolic estimates show that

$$\begin{aligned}
&|A_j(v_1) - A_j(v_2)|_{W^{2,1;q}([0,T]\times[0,\frac{1}{p}))}\\
\leq& C|\mathcal{F}_{K,j}(\tau, v_1) - \mathcal{F}_{K,j}(\tau, v_2)|_{L^q([0,T]\times[0,\frac{1}{p}))}\\
\leq& C\left(|v_2 - v_1|_{L^q([0,T]\times[0,\frac{1}{p}))} + |\inf(v_2) - v_2 - (\inf(v_1) - v_1)|_{L^q([0,T]\times[0,\frac{1}{p}))} + |(v_2 - v_1)_y|_{L^q([0,T]\times[0,\frac{1}{p}))}\right)\\
\leq& CT^\beta |v_2 - v_1|_{W^{2,1;q}([0,T]\times[0,\frac{1}{p}))}
\end{aligned}$$

for some $\beta > 0$ (see [17, 11]).

Sobolev embedding and parabolic regularity theory in Holder's spaces implies the existence for T small enough of a solution $w_j \in C^{2+\alpha,\frac{2+\alpha}{2}}$.

While we have smooth solutions below the truncature, we can applied the arguments of Subsection B.1 and get estimates on the gradient of the solution which ensures that the solution is indeed below the truncature. Finally, a posteriori, the truncature can be completely removed.

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