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## Coherence and finiteness spaces

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# Finiteness spaces, graphs and “coherence”

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WORKING DRAFT

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**Abstract.** We look at a sub-collection of finiteness spaces introduced in [2] based on the notion of coherence spaces from [4]. The original idea was to generalize the notion of stable functions between coherence spaces to interpret the algebraic lambda-calculus ([6]) or even the differential lambda-calculus ([3]). An important tool for this analysis is the infinite Ramsey theorem.

**0. Introduction.** The category of coherence spaces was the first denotational model for linear logic (see [4]): the basic objects are reflexive, non oriented graphs; and we are more specifically interested by their *cliques* (complete subgraph). If  $C$  is such a graph, we write  $\mathcal{C}(C)$  for the collection of its cliques.

Coherence spaces enjoy a very rich algebraic structure where the most important operations on them are:

- taking the (reflexive closure of the) complement (written  $C_1^\perp$ );
- taking a cartesian product (written  $C_1 \otimes C_2$ );
- taking a disjoint union (written  $C_1 \oplus C_2$ ).

If one only looks at the vertices, the corresponding operations are simply the identity, the usual cartesian product “ $\times$ ” and the disjoint union “ $\oplus$ ”.

More recently, Thomas Ehrhard introduced the notion of finiteness spaces ([2]) to give a model to the differential  $\lambda$ -calculus ([3]), which can be seen as an enrichment of linear logic. The point that interests us most here is that the collection of finitary sets of a finiteness space are closed under finite sums (*i.e.* finite unions) to take into account a notion of “non-deterministic sum” of terms. (See also [6].) This is definitely not true of cliques of a coherence space...

Very briefly, a finiteness space is given by a set  $|\mathcal{F}|$  and a collection  $\mathcal{F}$  of subsets of  $|\mathcal{F}|$  such that

$$\mathcal{F}^{\perp\perp} = \mathcal{F}$$

where

$$\mathcal{D}^\perp = \{x \mid \forall y \in \mathcal{D}, \#(x \cap y) < \omega\}.$$

Constructions similar to the one above can be defined; and they are characterized by:

- the dual  $\mathcal{F}^\perp$ ;
- $\mathcal{F}_1 \oplus \mathcal{F}_2 = \{x_1 \uplus x_2 \mid x_i \in \mathcal{F}_i\}$ ;
- $\mathcal{F}_1 \otimes \mathcal{F}_2 = \{r \mid \pi_i(r) \in \mathcal{F}_i\}$ .

Here again, if one looks only at the web  $|\mathcal{F}|$  of finiteness spaces, the corresponding operations are just the identity, the usual cartesian product and the disjoint union.

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*Remarks:*

- the operations on finiteness spaces are actually defined in a way that makes it clear that they yield finiteness spaces. They are latter *proved* to be equivalent what is given above. (See [2].)
- There is another presentation of coherent spaces that closely resembles the definition of finiteness spaces: a coherent space is given by a collection  $\mathcal{C}$  of subsets of  $|C|$  which satisfy  $\mathcal{C}^{\bullet\bullet} = \mathcal{C}$ , where  $\mathcal{D}^\bullet = \{x \mid \forall y \in \mathcal{D}, \#(x \cap y) \leq 1\}$ .
- Any operator of the form  $\mathcal{X} \mapsto \mathcal{X}^\bullet = \{y \mid \forall x \in \mathcal{X} R(x, y)\}$  is contravariant and yields a *closure operator* when applied twice.

**1. From “coherence” to finiteness.** The idea is rather simple: we would like to close the collection of cliques of a coherence space under finite unions. Unfortunately (but unsurprisingly), the notion of “finite unions of cliques” is not very well behaved. We instead consider the following notion:

**Definition.** *If  $C$  is a coherent space, we call a subset of  $|C|$  finitely incoherent if it doesn’t contain infinite anticliques. We write  $\mathcal{F}(C)$  for the collection of all finitely incoherent subsets of  $C$ .*

The following follows directly from the definition:

**Lemma.**

- any finite subset of  $|C|$  is finitely incoherent;
- any clique is finitely incoherent;
- a subset of a finitely incoherent subset is finitely incoherent;
- finitely incoherent subsets are closed under finite unions.

Note however that a finitely incoherent set needs not be a finite union of cliques: take for example the graph composed of the disjoint union of all the complete graphs  $K_n$  for  $n \geq 1$ . This graph doesn’t contain an infinite clique, but it is not a finite union of anticliques; so, its dual doesn’t contain infinite anticliques, but is not a finite union of cliques.

The next lemma is more interesting as it implies that the collection of finitely incoherent subsets forms a finiteness space in the sense of [2]:

**Lemma.** *If  $C$  is a coherence space, we have:*

$$\mathcal{C}(C)^\perp = \mathcal{F}(C^\perp)$$

*Proof:*

- ( $\subseteq$ ) let  $x$  be in  $\mathcal{C}(C)^\perp$ , and suppose, by contradiction, that  $x$  is not in  $\mathcal{F}(C^\perp)$ , *i.e.*  $x$  contains an infinite anticlique  $y$  of  $C^\perp$ . This set  $y$  is a clique in  $C$ , *i.e.*  $y \in \mathcal{C}(C)$ . Since  $x \cap y = y$  is infinite, this contradicts the hypothesis that  $x \in \mathcal{C}(C)^\perp$ .
- ( $\supseteq$ ) let  $x$  be finitely incoherent in  $C^\perp$ , *i.e.*  $x$  doesn’t contain an infinite clique of  $C$ ; let  $y$  be in  $\mathcal{C}(C)$ . Since  $x \cap y \in \mathcal{C}(C)$  and  $x \cap y$  is contained in  $x$ , it cannot be infinite. This shows that  $x \in \mathcal{C}(C)^\perp$ .

■

We thus get the expected corollary:

**Corollary.** *If  $C$  is a coherent space, then  $\mathcal{F}(C)$  is a finiteness space.*

What was slightly unexpected was the following:

**Lemma.** *If  $C$  is a coherence space, then:*

$$\mathcal{F}(C^\perp) = \mathcal{F}(C)^\perp .$$

*Proof:* because of the previous lemma, and because  $\perp$  is contravariant, we only need to show that  $\mathcal{C}(C)^\perp \subseteq \mathcal{F}(C)^\perp$ . Suppose that  $x \in \mathcal{C}(C)^\perp$ , and let  $y \in \mathcal{F}(C)$ ; we need to show that  $x \cap y$  is finite.

- Since  $x \cap y \subseteq y \in \mathcal{F}(C)$ ,  $x \cap y$  cannot contain an infinite anticlique;
- since  $x \cap y \subseteq x \in \mathcal{C}(C)^\perp$ ,  $x \cap y$  cannot contain an infinite clique.

Those two points imply, by the infinite Ramsey theorem, that  $x \cap y$  is finite. ■

The other linear connectives are similarly behaved with respect to the notion of finitely incoherent sets. We have:

**Lemma.** *If  $C_1$  and  $C_2$  are coherent spaces, then we have both*

$$\mathcal{F}(C_1 \oplus C_2) = \mathcal{F}(C_1) \oplus \mathcal{F}(C_2) ,$$

and

$$\mathcal{F}(C_1 \otimes C_2) = \mathcal{F}(C_1) \otimes \mathcal{F}(C_2)$$

where the connectives on the left are the coherent spaces ones, and the connectives on the right are the finiteness ones.

*Proof:* the  $\oplus$  part is direct; for the  $\otimes$  part, recall that  $r \in \mathcal{F}(C_1) \otimes \mathcal{F}(C_2)$  is equivalent to  $\pi_1(r) \in \mathcal{F}(C_1)$  and  $\pi_2(r) \in \mathcal{F}(C_2)$ .

- ( $\subseteq$ ) suppose  $r$  doesn't contain an infinite anticlique; neither  $\pi_1(r)$  nor  $\pi_2(r)$  can contain an infinite anticlique, as it would imply the existence of an infinite anticlique in  $r$ .
- ( $\supseteq$ ) suppose that  $r \in \mathcal{F}(C_1) \otimes \mathcal{F}(C_2)$  contains an infinite anticlique  $r'$  of  $C_1 \otimes C_2$ . At least one of  $\pi_1(r')$  or  $\pi_2(r')$  must be infinite, otherwise,  $r'$  itself would be finite. Suppose  $\pi_1(r')$  is infinite; because  $\pi_1(r') \subseteq \pi_1(r)$ , it cannot contain an infinite anticlique. By the infinite Ramsey theorem, it thus contains an infinite clique  $x$ . For each  $a \in x$ , chose one element  $b$  inside the "fiber"  $r'(a) = \{b \mid (a, b) \in r'\}$ . Two such  $b$ 's can't be coherent as it would contradict the fact that  $r'$  is an anticlique: we have constructed an infinite anticlique in  $\pi_2(r')$ . Contradiction! ■

Note that because the same is true of finiteness spaces, the previous corollary implies in particular that

$$\mathcal{F}((C_1 \oplus C_2)^\perp) = \mathcal{F}(C_1^\perp \oplus C_2^\perp) .$$

The direct proof of this equality is also quite easy.

From all the above (and anticipating on the section about exponentials), we can conclude that

**Proposition.**  $\mathcal{F}(\_)$  can be lifted to a functor from **Coh** to **Fin**; moreover, this functor commutes with the logical connectives.

Note that this functor is faithful (but not full); and that it is not injective on objects: adding and removing any finite number of edges to a coherence space doesn't change its image via  $\mathcal{F}(\_)$ . We will see in a moment that it is not surjective on objects.

So, coherence spaces can be used to define a (non-full) subcategory of finiteness spaces, closed under the logical operations  $(\_ \perp, \_ \otimes \_)$  and  $(\_ \oplus \_)$ . We will later see that the functor  $\mathcal{F}$  commutes with the (different flavors of the) exponentials. The subcategory thus obtained contains all the examples used in [2], like for example the space of natural numbers  $\mathbf{N}$ . However, not all finiteness spaces are equal to the bi-orthogonal of a coherence space:

**Proposition.** If  $A$  is infinite countable, there are strictly more finiteness spaces on  $A$  than coherence spaces on  $A$ .

*Proof :* let  $A$  be countable; “up to isomorphism”, we can assume that  $A = \mathbf{B}^{<\omega}$ , the set of finite sequences of bits. If  $x$  is an infinite sequence of bits (a non dyadic real), write  $x^\perp$  for the set of finite approximations of  $x$ ; and if  $X$  is a set of such “real numbers”, write  $X^\perp$  for the set  $\{x^\perp \mid x \in X\}$ . We have  $X^\perp \subset \mathcal{P}(A)$  for any such  $X$ .

Suppose now that  $X \neq X'$  with, for example,  $x \in X$  but  $x \notin X'$ . Since  $x^\perp$  is infinite and  $x^\perp \in X^\perp$ , we have  $x \notin X'^{\perp\perp}$ . However since two different reals must differ on some finite approximation, we have that  $x^\perp \in X'^{\perp\perp}$ . Thus, the finiteness spaces  $(A, X^{\perp\perp})$  and  $(A, X'^{\perp\perp})$  differ.

Thus, finiteness spaces on  $A$  have the same cardinality as  $\mathcal{P}(\mathbf{R}) \simeq \mathcal{P}(\mathcal{P}(A))$ . Since the cardinality of coherence spaces on  $A$  is the same as that of  $\mathcal{P}(A \times A) \simeq \mathcal{P}(A)$ , we can conclude.

Note that the above reasoning also provides a proof that the cardinality of finiteness spaces on  $A$  modulo permutation on  $A$  is the same as that of  $\mathcal{P}(\mathcal{P}(A))$ : any equivalence class is of cardinality at most  $\#(\mathcal{P}(A))$  (because there are exactly  $\#(\mathcal{P}(A))$  permutations on  $A$ ), and since  $\kappa \times \#(\mathcal{P}(A)) = \max(\kappa, \#(\mathcal{P}(A)))$ , there must be at least  $\#(\mathcal{P}(\mathcal{P}(A)))$  such equivalence classes to cover the whole collection of finiteness spaces...

■

It is slightly interesting to note that the same reasoning doesn't apply to higher cardinalities since  $\#(A^{<\omega}) = \#(A)$  if  $A$  is uncountable.

In a sense, coherence spaces allow to define a collection of “simple” finiteness spaces. An informal argument regarding this “simplicity” can be found in the following remark: the logical complexity of the formula expressing  $x \in A^{\perp\perp}$  for a lower-closed collection of subsets  $x$ . In the general case (see [2]), we have

$$x \in A^{\perp\perp} \quad \Leftrightarrow \quad \forall y \subseteq x, \#(y) = \infty \quad \exists z \subseteq y, \#(z) < \infty \quad z \in A$$

In other words, logically speaking, this is a  $\Pi_2^1$  formula (second order quantifiers are  $\forall\exists$ ). For the particular case when  $A$  is the set of cliques of a coherent spaces, we obtain

$$x \in A^{\perp\perp} \Leftrightarrow \forall y \subseteq x, \#(y) = \infty \exists a, b \in y \{a, b\} \in A$$

which is only a  $\Pi_1^1$  formula.

**2. Exponentials.** Traditionally, the exponential on coherent spaces is defined using *finite subsets*, while the exponential on finiteness spaces (or on any other web-based model of linear logic) is defined using *finite multisets*. Since the original idea was to lift the notion of *stable function*, we will use here the first variant.

**Definition.** *If  $F$  is a finiteness space, then we define  $!F$  as follows:*

$$!F = \{\mathcal{P}_f(x) \mid x \in F\}^{\perp\perp}$$

Note that the usual problem arises: this operation is functorial, but isn't a comonad: dereliction isn't natural. It is only a near comonad ([5]) and the co-Kleisli construction is still available... (See also [1] for an even weaker notion.)\*

As opposed to what happens with the usual exponential on finiteness spaces, this operation applied to trivial (finite) spaces yields trivial spaces. We thus need infinite spaces to start with...

Just like in [2], we have

**Lemma.** *If  $F$  is a finiteness space, we have*

$$V \in !F \Leftrightarrow \bigcup V \in F$$

*Proof:* just follow the nice proof of lemma 4 from [2]... ■

On the coherence spaces side, the exponential is the usual one: if  $C$  is a coherence space,  $!C$  is the following coherence space:

- $!C = \mathcal{C}(|C|) \cap \mathcal{P}_f(|C|)$ : the collection of finite cliques of  $C$ ;
- $x$  and  $y$  are coherent iff  $x \cup y \in \mathcal{C}(C)$ .

With that in mind, it is now easy to show

**Lemma.** *For any coherence space  $C$ , we have*

$$\mathcal{F}(!C) = !\mathcal{F}(C) .$$

*Proof:* we have

$$\begin{aligned} V \in \mathcal{F}(!C) &\Leftrightarrow V \text{ doesn't contain an infinite anticlique in } !C \\ &\Leftrightarrow \bigcup V \text{ doesn't contain an infinite anticlique in } C \\ &\Leftrightarrow \bigcup V \in \mathcal{F}(C) \\ &\Leftrightarrow V \in !\mathcal{F}(C) \end{aligned}$$
■

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\* One slight problem is that we need to quotient morphisms (relations) by an appropriate equivalence relation.

*Remark:* the same holds if one uses the multiset variant of the exponentials...

**3. Weakly stable functions.** The notion of *stable function* has several equivalent characterizations, but the one that interests us here is the following: write  $\mathcal{T}(f, x, b)$  for the collection  $\{x_0 \mid x_0 \subseteq_f x, a \in f(x_0)\}$ .

**Definition.** A function  $f : \mathcal{C}(C_1) \rightarrow \mathcal{C}(C_2)$  is stable if for all  $x \in \mathcal{C}(C_1)$  and  $b \in f(x)$  the collection  $\mathcal{T}(f, x, a)$  is of the form  $x_0^\uparrow$ .

$$\text{where } x_0^\uparrow = \{y_0 \in \mathcal{T}(f, x, b) \mid x_0 \subseteq y_0\}$$

This definition is equivalent to

- $\text{Tr}(f) = \{(x_0, b) \mid x_0 \in \mathcal{P}_f(C_1), a \in f(x_0), \text{ and } x_0 \text{ is minimal w.r.t. inclusion}\}$  is a clique in  $(!C_1)^\perp \mathfrak{A} C_2$
- $f$  is Scott-continuous and commutes with coherent finite intersections

We generalize the notion of stability in the following manner:

**Definition.** A function  $f : \mathcal{F}(C_1) \rightarrow \mathcal{F}(C_2)$  is weakly stable if for all  $x \in \mathcal{C}(C_1)$  and  $b \in f(x)$ , the collection  $\mathcal{T}(f, x, b)$  is of the form  $\{x_1, x_2, \dots, x_n\}^\uparrow$ .

$$\text{where } \{x_1, x_2, \dots, x_n\}^\uparrow = \{y_0 \in \mathcal{T}(f, x, b) \mid \exists i, x_i \subseteq y_0\}$$

**Lemma.**

- If  $f$  is stable, then  $f$  is weakly stable;
- if  $f$  is weakly stable, then  $f$  is Cantor continuous.

Perhaps unsurprisingly, we have

**Proposition.** If  $f$  is a function from  $\mathcal{P}(|C_1|)$  to  $\mathcal{P}(|C_2|)$ , we have

$$f \text{ is weakly stable} \iff \text{Tr}(f) \in \mathcal{F}((!C_1)^\perp \mathfrak{A} C_2)$$

*Proof:*

( $\Rightarrow$ ) let  $f$  be a weakly stable function; we need to show that  $\text{Tr}(f) \in \mathcal{C}(!C_1 \otimes C_2^\perp)^\perp$ .

Suppose it is not the case, *i.e.* that  $\text{Tr}(f)$  contains an infinite clique in  $!C_1 \otimes C_2^\perp$ ; let this clique be  $\{(x_i, b_i) \mid i \in I\}$ . Since  $x = \bigcup_{i \in I} x_i$  is a clique,  $f(x) \in \mathcal{F}(C_2)$  and so, it cannot contain an infinite anticlique. This implies that there is only a finite number of different  $b_i$ . This in turn implies that there must be at least one  $b_{i_0}$  s.t.  $\{x_i \mid (x_i, b_{i_0}) \in \text{Tr}(f)\}$  is infinite, which is impossible...

( $\Leftarrow$ ) We first need to show that  $f$  takes  $\mathcal{F}(C_1)$  to  $\mathcal{F}(C_2)$ . Suppose that  $x \in \mathcal{F}(C_1)$ ; we have  $f(x) = \{b \mid \exists x_0 \subseteq_f x, (x_0, b) \in \text{Tr}(f)\}$ . Suppose that this set contains an infinite anticlique  $\{b_i \mid i \in I\}$ , and let  $\{x_i\}$  be the corresponding  $x_0$ 's. Because  $\bigcup_i x_i \subseteq x$ , it cannot contain an infinite anticlique; and since it cannot be finite, it must contain an infinite clique by the Ramsey theorem. This implies that  $\{x_i \mid i \in I\}$  contains an infinite clique in  $!C_1$ , so that  $\{(x_i, b_i) \mid i \in I\}$  is an infinite clique in  $!C_1 \otimes C_2^\perp$ .

Moreover, we have

$$\mathcal{T}(f, x, b) = \{x_0 \mid x_0 \subseteq x, (x_0, b) \in \text{Tr}(f)\}^\uparrow$$

and since such a set is necessarily finite, we can conclude. ■

Note that by construction, we have:

**Proposition.** *The category of coherent spaces with weakly stable functions is cartesian closed.*

TO CHECK: do we have a deriving transformation (cf. Blute, Cockett and Seely)

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