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On calibrated and separating sub-actions

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Abstract

We consider a one-sided transitive subshift of finite type $\sigma : \Sigma \rightarrow \Sigma$ and a Hölder observable A . In the ergodic optimization model, one is interested in properties of A -minimizing probability measures. If \bar{A} denotes the minimizing ergodic value of A , a sub-action u for A is by definition a continuous function such that $A \geq u \circ \sigma - u + \bar{A}$. We call contact locus of u with respect to A the subset of Σ where $A = u \circ \sigma - u + \bar{A}$. A calibrated sub-action u gives the possibility to construct, for any point $x \in \Sigma$, backward orbits in the contact locus of u . In the opposite direction, a separating sub-action gives the smallest contact locus of A , that we call $\Omega(A)$, the set of non-wandering points with respect to A .

We prove that, under certain conditions on $\Omega(A)$, any calibrated sub-action is of the form $u(x) = u(x_i) + h_A(x_i, x)$ for some $x_i \in \Omega(A)$, where $h_A(x, y)$ denotes the Peierls barrier of A . We also prove that separating sub-actions are generic among Hölder sub-actions. We present the proofs in the holonomic optimization model, a formalism which allows to take into account a two-sided transitive subshift of finite type $(\hat{\Sigma}, \hat{\sigma})$.

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1 Introduction

In the *ergodic optimization model* (see, for instance, [2, 3, 4, 7, 13, 14, 16]), given a continuous observable $A : X \rightarrow \mathbb{R}$, one is interested in understanding which T -invariant Borel probability measure μ of a compact metric space X minimizes the average $\int_X A d\mu$. Such measures are called *minimizing probability measures*¹.

Minimizing probability measures admit dual objects: the *sub-actions*. A sub-action $u : X \rightarrow \mathbb{R}$ associated to an observable A enables to replace A by a cohomologous observable whose ergodic minimizing value is actually the absolute minimum. To each sub-action u one associate a compact subset of X called *contact locus* which contains the support of any minimizing probability measure. A sub-action gives therefore important informations on T -invariant Borel probability measures that minimize the average of A . It is a relevant problem to investigate the existence of a particular sub-action having the smallest contact locus, that is, the smallest “trapping region” of all minimizing probability measures.

In section 2, we give a simplified version for the ergodic optimization model of the main results, namely, of the theorems 9, 10 and 11. In section 3, we recall the definition of the *holonomic optimization model* and state the main results. We give in section 4 the proof of theorem 9 and in section 5 the proof of theorem 11. We will adopt throughout the text the point of view which consists in interpreting ergodic optimization problems as questions of variational dynamics (see, for instance, [7, 11, 16]), similar to Aubry-Mather technics for Lagrangian systems. For an expository introduction to the general theory of ergodic optimization, we refer the reader to the article of O. Jenkinson (see [14]).

We still would like to point out that one of the main conjectures in the theory of ergodic optimization on compact spaces can be roughly formulated in the following way: *in any hyperbolic dynamics, a generic Hölder (or Lipschitz) observable possesses an unique minimizing probability measure, which is supported by a periodic orbit*. Concerning this problem, partial answers were already obtained, among them [3, 7, 13, 16, 18, 19]. Working with a transitive expanding dynamical system, J. Brémont (see [4]) has recently shown how such conjecture might follow from a carefull study of the contact loci of typical sub-actions with finitely many connected components.

We are in particular interested in finding *separating sub-actions*, that is sub-actions whose contact locus is the smallest one. Our main theorem states that such sub-actions are actually generic among the set of Hölder sub-actions.

¹Maximizing probabilities also appears in the literature. Obviously, replacing the observable A by $-A$, both vocabularies can be interchanged and the rephrased statements will be immediately verified. The maximizing terminology seems more convenient to study the connections with the thermodynamic formalism (see, for example, [7, 15]).

2 A simplified version of theorems 9, 10 and 11

Let (X, T) be a transitive expanding dynamical system, that is, a continuous covering several-to-one map $T : X \rightarrow X$ on a compact metric space X whose inverse branches are uniformly contracting by a factor $0 < \lambda < 1$. We denote by \mathcal{M}_T the set of T -invariant Borel probability measures. Our objective in this section is to summarize the conclusions of theorems 9, 10 and 11 in ergodic optimization theory. We first recall basic definitions from [7] (see also [14]).

Given a continuous observable $A : X \rightarrow \mathbb{R}$, we call *ergodic minimizing value* the quantity

$$\bar{A} := \min_{\mu \in \mathcal{M}_T} \int A d\mu.$$

We call *A-minimizing probability* a measure $\mu \in \mathcal{M}_T$ which realizes the above minimum.

We say that a function $u : X \rightarrow \mathbb{R}$ is a sub-action with respect to the observable A if the following inequality holds everywhere on X

$$A \geq u \circ T - u + \bar{A}.$$

Definition 1. A sub-action $u : X \rightarrow \mathbf{R}$ is said *calibrated* if

$$u(x) = \min_{T(y)=x} [u(y) + A(y) - \bar{A}].$$

Definition 2. We call *contact locus* of a sub-action u the set

$$\mathbb{M}_A(u) := (A - u \circ T + u)^{-1}(\bar{A}).$$

A point $x \in X$ is said to be *non-wandering with respect to A* if, for every $\epsilon > 0$, there exists an integer $k \geq 1$ and a point $y \in X$ such that

$$d(x, y) < \epsilon, \quad d(x, T^k(y)) < \epsilon \quad \text{and} \quad \left| \sum_{j=0}^{k-1} (A - \bar{A}) \circ T^j(y) \right| < \epsilon.$$

We denote by $\Omega(A)$ the *set of non-wandering points* with respect to the observable $A \in C^0(X)$. When the observable is Hölder, $\Omega(A)$ is a non-empty compact T -invariant set containing the support of all minimizing probability measures. Moreover,

$$\Omega(A) \subset \bigcap \left\{ \mathbb{M}_A(u) \mid u \text{ is a continuous sub-action} \right\}.$$

We are interested in finding u so that $\Omega(A) = \mathbb{M}_A(u)$.

Definition 3. A sub-action $u \in C^0(X)$ is said to be *separating (with respect to A)* if it satisfies $\mathbb{M}_A(u) = \Omega(A)$.

The main conclusion of theorem 9 can be stated in the following way. The proof of this particular case will not be given and can be adapted from the one of the general situation (see section 4).

Theorem 4. *Let (X, T) be a transitive expanding dynamical system on a compact metric space and $A : X \rightarrow \mathbb{R}$ be a θ -Hölder observable. Then there exist a θ -Hölder separating sub-action for A . Furthermore, in the θ -Hölder topology, the subset of θ -Hölder separating sub-actions is generic among all θ -Hölder sub-actions.*

Contrary to a separating sub-action, a calibrated sub-action u possesses a large contact locus in the sense $T(\mathbb{M}_A(u)) = X$. Calibrated sub-actions are built using a particular sub-action called the *Peierls barrier*. For Hölder observable A , the Peierls barrier of A , $h_A : \Omega(A) \times X \rightarrow \mathbb{R}$, is a Hölder calibrated sub-action in the second variable defined by

$$h_A(x, y) := \lim_{\epsilon \rightarrow 0} \liminf_{k \rightarrow +\infty} \inf \left\{ \sum_{j=0}^{k-1} (A - \bar{A}) \circ T^j(z) \mid \right. \\ \left. z \in X, d(z, x) < \epsilon \text{ and } d(T^k(z), y) < \epsilon \right\}.$$

The equivalent theorem to 10 may be stated in the following form.

Theorem 5. *Let (X, T) be a transitive expanding dynamical system on a compact metric space and $A : X \rightarrow \mathbb{R}$ be a Hölder observable. Then the set of calibrated sub-actions coincides with the set of functions of the form*

$$u(y) = \min_{x \in \Omega(A)} [\phi(x) + h_A(x, y)], \quad \forall y \in X,$$

where $\phi : \Omega(A) \rightarrow \mathbb{R}$ is any continuous function satisfying

$$\phi(y) - \phi(x) \leq h_A(x, y), \quad \forall x, y \in \Omega(A).$$

Moreover, u extends ϕ and is thus uniquely characterized by ϕ .

In the case $\Omega(A)$ is reduced to a finite number of disjoint irreducible components, the set of calibrated sub-actions is parametrized by a finite number of conditions. The condition $x \sim y \Leftrightarrow h_A(x, y) + h_A(y, x) = 0$ defines an equivalent relation on $\Omega(A)$. An equivalence class is called an *irreducible component* and is a closed T -invariant set. If $\Omega(A) = \sqcup_{i=1}^r C_i$ is equal to a disjoint union of irreducible components and $x_i \in C_i$ are chosen, the *sub-action constraint set* is by definition

$$\mathcal{C}_A(x_1, \dots, x_r) := \{(u_1, \dots, u_r) \in \mathbb{R}^r \mid u_j - u_i \leq h_A(x_i, x_j), \quad \forall i, j\}.$$

The analogous result to theorem 11 can be stated as follows.

Theorem 6. *Let (X, T) be a transitive expanding dynamical system on a compact metric space and $A : X \rightarrow \mathbb{R}$ be a Hölder observable. Assume that $\Omega(A) = \sqcup_{i=1}^r C_i$ is equal to a disjoint union of irreducible components.*

1. *There is a one-to-one correspondance between the sub-action constraint set and the set of calibrated sub-actions,*

$$\left\{ \begin{array}{l} (u_1, \dots, u_r) \in \mathcal{C}_A(x_1, \dots, x_r) \\ u(x) = \min_{1 \leq i \leq r} [u_i + h_A(x_i, x)] \end{array} \right\} \iff \left\{ \begin{array}{l} u \text{ is a calibrated sub-action} \\ u_i = u(x_i) \end{array} \right\}.$$

2. *Let $i_0 \in \{1, \dots, r\}$ and $u_{i_0} \in \mathbb{R}$ fixed. Define $u_i = u_{i_0} + h_A(x_{i_0}, x_i)$ for all i , then $(u_1, \dots, u_r) \in \mathcal{C}_A(x_1, \dots, x_r)$ and the unique calibrated sub-action u satisfying $u(x_i) = u_i$, for all i , is of the form*

$$u(x) := \min_{1 \leq i \leq r} [u_i + h_A(x_i, x)] = u_{i_0} + h_A(x_{i_0}, x).$$

3 Basic Concepts and Main Results

For simplicity, we will restrict the exposition of the *holonomic optimization model* to the symbolic dynamics case. Let (Σ, σ) be a one-sided transitive subshift of finite type given by a $s \times s$ irreducible transition matrix \mathbf{M} . More precisely

$$\Sigma := \left\{ \mathbf{x} \in \{1, \dots, s\}^{\mathbb{N}} \mid \mathbf{M}(x_j, x_{j+1}) = 1 \text{ for all } j \geq 0 \right\}$$

and σ is the left shift acting on Σ by $\sigma(x_0, x_1, \dots) = (x_1, x_2, \dots)$. Fix $\lambda \in (0, 1)$. We choose a particular metric on Σ defined by $d(\mathbf{x}, \bar{\mathbf{x}}) = \lambda^k$, for any $\mathbf{x}, \bar{\mathbf{x}} \in \Sigma$, $\mathbf{x} = (x_0, x_1, \dots)$, $\bar{\mathbf{x}} = (\bar{x}_0, \bar{x}_1, \dots)$ and $k = \min\{j : x_j \neq \bar{x}_j\}$.

The holonomic model is a generalization of the ergodic optimization framework. The holonomic model has been introduced first by R. Mañé in an attempt to clarify Aubry-Mather theory for continuous time Lagrangian dynamics (see [6, 17]). In this model, the set of invariant minimizing probability measures is replaced by a broader class of measures called holonomic measures. In Aubry-Mather theory for discrete time Lagrangian dynamics on the n dimensional torus \mathbb{T}^n (see [12]), an holonomic probability measure $\mu(dx, dv)$ is a probability measure on $\mathbb{T}^n \times \mathbb{R}^n$ satisfying

$$\int_{\mathbb{T}^n \times \mathbb{R}^n} f(x + v) d\mu(x, v) = \int_{\mathbb{T}^n \times \mathbb{R}^n} f(x) d\mu(x, v), \quad \forall f \in C^0(\mathbb{T}^n).$$

Similarly to this example of discrete dynamics, Σ will play the role of the “space of positions” (analogous to \mathbb{T}^n in the holonomic model) and the set of inverse branches or possible pasts Σ^* will play the role of the “space of immediately anterior velocities” (analogous to \mathbb{R}^n). For a complete exposition and motivation of the holonomic optimization model, see [10, 11].

We call *dual subshift of finite type* the space

$$\Sigma^* := \left\{ \mathbf{y} \in \{1, \dots, s\}^{\mathbb{N}^*} \mid \mathbf{M}(y_{j+1}, y_j) = 1 \text{ for all } j \geq 1 \right\}.$$

We denote by $\mathbf{y} = (\dots, y_3, y_2, y_1)$ a point of Σ^* . We call *dual shift* the map $\sigma^*(\dots, y_3, y_2, y_1) := (\dots, y_3, y_2)$. The *natural extension* of (Σ, σ) will play the role of the “phase space” (analogous to $\mathbb{T}^n \times \mathbb{R}^n$) and will be identified with a subset of $\Sigma \times \Sigma^*$

$$\hat{\Sigma} := \left\{ (\mathbf{x}, \mathbf{y}) = (\dots, y_2, y_1 | x_0, x_1, \dots) \in \Sigma \times \Sigma^* \mid \right. \\ \left. \mathbf{x} = (x_0, x_1, \dots), \mathbf{y} = (\dots, y_2, y_1) \text{ and } \mathbf{M}(y_1, x_0) = 1 \right\}.$$

The analogue of the “discrete Euler-Lagrange map” is obtained by the usual left shift $\hat{\sigma}$ on the natural extension. Consider then $\tau^* : \hat{\Sigma} \rightarrow \Sigma^*$ and $\hat{\sigma} : \hat{\Sigma} \rightarrow \hat{\Sigma}$ given by

$$\tau^*(\mathbf{x}, \mathbf{y}) := \tau_{\mathbf{x}}^*(\mathbf{y}) := (\dots, y_2, y_1, x_0) \quad \text{and} \quad \hat{\sigma}(\mathbf{x}, \mathbf{y}) := (\sigma(\mathbf{x}), \tau_{\mathbf{x}}^*(\mathbf{y})).$$

Similarly, inverse branches are constructed using the map $\tau : \hat{\Sigma} \rightarrow \Sigma$,

$$\tau(\mathbf{x}, \mathbf{y}) := \tau_{\mathbf{y}}(\mathbf{x}) = (y_1, x_0, x_2, \dots) \quad \text{and} \quad \hat{\sigma}^{-1}(\mathbf{x}, \mathbf{y}) = (\tau_{\mathbf{y}}(\mathbf{x}), \sigma^*(y)).$$

Note that $\tau = \pi \circ \hat{\sigma}^{-1}$, where $\pi : \hat{\Sigma} \rightarrow \Sigma$ is the canonical projection onto the \mathbf{x} -variable.

Let $\hat{\mathcal{M}}$ be the set of probability measures over the Borel sigma-algebra of $\hat{\Sigma}$. Instead of considering the set of $\hat{\sigma}$ -invariant probability measures, we introduce the set of *holonomic probability measures*,

$$\hat{\mathcal{M}}_{\text{hol}} := \left\{ \hat{\mu} \in \hat{\mathcal{M}} \mid \int_{\hat{\Sigma}} f(\tau_{\mathbf{y}}(\mathbf{x})) d\hat{\mu}(\mathbf{x}, \mathbf{y}) = \int_{\hat{\Sigma}} f(\mathbf{x}) d\hat{\mu}(\mathbf{x}, \mathbf{y}), \quad \forall f \in C^0(\Sigma) \right\}.$$

Observe that $\hat{\mu} \in \hat{\mathcal{M}}_{\text{hol}}$ if, and only if, $\pi_*(\hat{\mu}) = \pi_*(\hat{\sigma}_*^{-1}(\hat{\mu}))$ if, and only if, $\sigma_*^{-1}(\hat{\mu})$ projects onto a σ -invariant Borel probability measure. As in section 2, we denote by \mathcal{M}_σ the set of σ -invariant Borel probability measures. The triple $(\hat{\Sigma}, \hat{\sigma}, \hat{\mathcal{M}}_{\text{hol}})$ is called the *holonomic model*. Such a formalism includes the ergodic optimization model discussed in section 2 as we will see.

Let $A \in C^\theta(\hat{\Sigma})$ be a Hölder observable. We call *holonomic minimizing value* of A

$$\bar{A} := \min \left\{ \int_{\hat{\Sigma}} A(\mathbf{x}, \mathbf{y}) d\hat{\mu}(\mathbf{x}, \mathbf{y}) \mid \hat{\mu} \in \hat{\mathcal{M}}_{\text{hol}} \right\} \\ = \min \left\{ \int_{\hat{\Sigma}} A \circ \hat{\sigma}(\mathbf{x}, \mathbf{y}) d\hat{\mu}(\mathbf{x}, \mathbf{y}) \mid \pi_*(\hat{\mu}) \in \mathcal{M}_\sigma \right\}.$$

If $A \circ \hat{\sigma} = B \circ \pi$ depends only on the \mathbf{x} -variable, $\bar{A} = \bar{B}$ as in the section 2.

The set of *minimizing (holonomic) probability measures* is denoted

$$\hat{\mathcal{M}}_{\text{hol}}(A) := \left\{ \hat{\mu} \in \hat{\mathcal{M}}_{\text{hol}} \mid \int_{\hat{\Sigma}} A(\mathbf{x}, \mathbf{y}) d\hat{\mu}(\mathbf{x}, \mathbf{y}) = \bar{A} \right\}.$$

A function $u : \Sigma \rightarrow \mathbb{R}$ is called *sub-action* with respect to A if

$$u(\mathbf{x}) - u(\tau_{\mathbf{y}}(\mathbf{x})) \leq A(\mathbf{x}, \mathbf{y}) - \bar{A}, \quad \forall (\mathbf{x}, \mathbf{y}) \in \hat{\Sigma},$$

or equivalently $A - \bar{A} \geq u \circ \pi - u \circ \pi \circ \hat{\sigma}^{-1}$. We call *contact locus* of a sub-action u the set

$$\hat{\mathbb{M}}_A(u) := (A - u \circ \pi + u \circ \pi \circ \hat{\sigma}^{-1})^{-1}(\bar{A}),$$

where the above inequality becomes an identity. If $A \circ \hat{\sigma} = B \circ \pi$, for some $B : \Sigma \rightarrow \mathbb{R}$, then $\pi \circ \hat{\sigma}^{-1}(\hat{\mathbb{M}}_A(u)) = \mathbb{M}_B(u)$.

A calibrated sub-action is a particular sub-action which possesses a large contact locus in the sense that $\pi(\hat{\mathbb{M}}_A(u)) = \Sigma$.

Definition 7. A sub-action $u : \Sigma \rightarrow \mathbb{R}$ is said to be *calibrated for A* if

$$u(\mathbf{x}) = \min_{\mathbf{y} \in \Sigma_{\mathbf{x}}^*} [u(\tau_{\mathbf{y}}(\mathbf{x})) + A(\mathbf{x}, \mathbf{y}) - \bar{A}], \quad \forall \mathbf{x} \in \Sigma,$$

where $\Sigma_{\mathbf{x}}^* := \{\mathbf{y} \in \Sigma^* \mid (\mathbf{x}, \mathbf{y}) \in \hat{\Sigma}\}$.

If $\hat{B} := A \circ \hat{\sigma}$ and $B(\mathbf{x}) := \min\{\hat{B}(\mathbf{x}, \mathbf{y}) \mid \mathbf{y} \in \Sigma_{\mathbf{x}}^*\}$, then u is a calibrated sub-action for A if, and only if, u is a calibrated sub-action for B . Indeed,

$$\begin{aligned} u(\mathbf{x}) &= \min_{\sigma(\bar{\mathbf{x}})=\mathbf{x}} \min_{\mathbf{y} \in \Sigma_{\bar{\mathbf{x}}}^*, \tau_{\mathbf{y}}(\bar{\mathbf{x}})=\mathbf{x}} [u(\bar{\mathbf{x}}) + \hat{B}(\bar{\mathbf{x}}, \sigma^*(\mathbf{y})) - \bar{A}] \\ &= \min_{\sigma(\bar{\mathbf{x}})=\mathbf{x}} [u(\bar{\mathbf{x}}) + B(\bar{\mathbf{x}}) - \bar{A}] = \min_{\sigma(\bar{\mathbf{x}})=\mathbf{x}} [u(\bar{\mathbf{x}}) + B(\bar{\mathbf{x}}) - \bar{B}]. \end{aligned}$$

(The definition of B gives $\bar{B} \leq \bar{A}$ and the calibration gives $\bar{B} \geq \bar{A}$.)

A classification theorem for calibrated sub-actions is presented in [11]. A central concept is the set of non-wandering points with respect to A (previously defined in [7, 16] in the ergodic optimization model). We call *path of length k* a sequence $(\mathbf{z}^0, \dots, \mathbf{z}^k)$ of points of $\hat{\Sigma}$ such that

$$\mathbf{z}^i = (\mathbf{x}^i, \mathbf{y}^i) \quad \text{with} \quad \mathbf{x}^i = \tau_{\mathbf{y}^{i+1}}(\mathbf{x}^{i+1}), \quad \forall i = 0, 1, \dots, k-1,$$

that is, a sequence $(\mathbf{z}^0, \dots, \mathbf{z}^k)$ where $\mathbf{x}^i = \sigma^i(\mathbf{x}^0)$ for all $i = 0, 1, \dots, k$, $\mathbf{x}^0 = (x_0, x_1, \dots, x_{k-1}, \mathbf{x}^k)$ and

$$\begin{aligned} \mathbf{z}^0 &= (\mathbf{y}^0 | x_0, \dots, x_{k-1}, \mathbf{x}^k), \quad \mathbf{z}^1 = (\sigma^*(\mathbf{y}^1), x_0 | x_1, \dots, x_{k-1}, \mathbf{x}^k), \dots, \\ \mathbf{z}^{k-1} &= (\sigma^*(\mathbf{y}^{k-1}), x_{k-2} | x_{k-1}, \mathbf{x}^k), \quad \mathbf{z}^k = (\sigma^*(\mathbf{y}^k), x_{k-1} | \mathbf{x}^k). \end{aligned}$$

Note that the point \mathbf{y}^0 is free of any restriction except that $\mathbf{M}(y_1^0, x_0) = 1$.

Given $\epsilon > 0$ and $\mathbf{x}, \bar{\mathbf{x}} \in \Sigma$, we say that a path of length k , $(\mathbf{z}^0, \dots, \mathbf{z}^k)$, begins within ϵ of \mathbf{x} and ends within ϵ of $\bar{\mathbf{x}}$ if $d(\mathbf{x}^0, \mathbf{x}) < \epsilon$ and $d(\mathbf{x}^k, \bar{\mathbf{x}}) < \epsilon$. Denote by $\mathcal{P}_k(\mathbf{x}, \bar{\mathbf{x}}, \epsilon)$ the set of such paths. Denote by $\mathcal{P}_k(\mathbf{x})$ the set of paths of length k beginning exactly at \mathbf{x} .

A point $\mathbf{x} \in \Sigma$ will be called *non-wandering with respect to A* if, for every $\epsilon > 0$, one can find a path $(\mathbf{z}^0, \dots, \mathbf{z}^k)$ in $\mathcal{P}_k(\mathbf{x}, \mathbf{x}, \epsilon)$, with $k \geq 1$, such that

$$\left| \sum_{i=1}^k (A - \bar{A})(\mathbf{z}^i) \right| < \epsilon.$$

We will denote by $\Omega(A)$ the set of non-wandering points with respect to A . If $A \circ \hat{\sigma} = B \circ \pi$, notice that $\Omega(A) = \Omega(B)$ as in section 2.

The first two authors have proved in [11] that $\Omega(A)$ is a non-empty compact σ -invariant set and satisfies

$$\Omega(A) \subset \bigcap \left\{ \pi(\hat{\mathbb{M}}_A(u)) \mid u \text{ is a continuous sub-action} \right\}.$$

The set $\Omega(A)$ is analogous to the projected Aubry set in the continuous time Lagrangian dynamics. One could have introduced the corresponding Aubry set $\hat{\Omega}(A) \subset \hat{\Sigma}$ and proved $\pi(\hat{\Omega}(A)) = \Omega(A)$. Unfortunately, even for Hölder observable A , the graph property is not any more true: $\pi : \hat{\Omega}(A) \rightarrow \Omega(A)$ is no more bijective. A counter-example can be found in [11]. It would be interesting to find the right assumptions on $A \in C^\theta(\hat{\Sigma})$ in order to get this property.

Contrarily to a calibrated sub-action, a *separating sub-action* is a sub-action with the smallest contact locus. More precisely,

Definition 8. *A sub-action $u : \Sigma \rightarrow \mathbb{R}$ is said to be separating with respect to A if $\pi(\hat{\mathbb{M}}_A(u)) = \Omega(A)$.*

Our first result is the following one.

Theorem 9. *If A is a θ -Hölder observable then there exists a θ -Hölder separating sub-action. Moreover, in the θ -Hölder topology, the subset of θ -Hölder separating sub-actions is generic among all θ -Hölder sub-actions.*

According to the analogy with continuous time Lagrangian dynamics, sub-actions correspond to viscosity sub-solutions of the stationary Hamilton-Jacobi equation, calibrated sub-actions correspond to the weak KAM solutions introduced by A. Fathi (see [8]) and separating sub-actions correspond to special sub-solutions as described in [9].

By adapting the proof of theorem 10 in [11] and by using definition 13 of the the Peierls barrier h_A , we obtain a structure theorem for calibrated sub-actions. Such characterization corresponds to the one obtained for weak KAM solutions in Lagrangian dynamics (see [5]). The proof of the following theorem will be omitted.

Theorem 10. *Let A be a θ -Hölder observable.*

1. *If u is a continuous calibrated sub-action for A , then*

$$u(\mathbf{x}) = \min_{\bar{\mathbf{x}} \in \Omega(A)} [u(\bar{\mathbf{x}}) + h_A(\bar{\mathbf{x}}, \mathbf{x})].$$

2. *Conversely, for every continuous application $\phi : \Omega(A) \rightarrow \mathbb{R}$ satisfying*

$$\phi(\mathbf{x}) - \phi(\bar{\mathbf{x}}) \leq h_A(\bar{\mathbf{x}}, \mathbf{x}), \quad \forall \mathbf{x}, \bar{\mathbf{x}} \in \Omega(A),$$

the function $u(\mathbf{x}) := \min_{\bar{\mathbf{x}} \in \Omega(A)} [\phi(\bar{\mathbf{x}}) + h_A(\bar{\mathbf{x}}, \mathbf{x})]$ is a continuous calibrated sub-action extending ϕ on $\Omega(A)$.

In particular, this representation formula for calibrated sub-actions implies immediately that, in order to compare two such functions, we just need to compare their restrictions to $\Omega(A)$. For instance, if two calibrated sub-actions coincide for every non-wandering point with respect to A , then they are the same.

In the case the set of non-wandering points for A is reduced to a finite union of irreducible components $\Omega(A) = C_1 \cup \dots \cup C_r$, the set of calibrated sub-actions admits a simpler characterization. We first show that the condition $\mathbf{x} \sim \bar{\mathbf{x}} \Leftrightarrow h_A(\mathbf{x}, \bar{\mathbf{x}}) + h_A(\bar{\mathbf{x}}, \mathbf{x}) = 0$ defines an equivalent relation. Each one of its equivalent classes is called an irreducible component. Let $\bar{\mathbf{x}}^1 \in C_1, \dots, \bar{\mathbf{x}}^r \in C_r$ fixed. We call *sub-action constraint set* the set

$$\mathcal{C}_A(\bar{\mathbf{x}}^1, \dots, \bar{\mathbf{x}}^r) = \{(u_1, \dots, u_r) \in \mathbb{R}^r \mid u_j - u_i \leq h_A(\bar{\mathbf{x}}^i, \bar{\mathbf{x}}^j), \quad \forall i, j\}.$$

Our second result is the following one.

Theorem 11. *Let A be a Hölder observable. Assume $\Omega(A)$ is a finite union of disjoint irreducible components, namely, $\Omega(A) = \sqcup_{i=1}^r C_i$. Let $\bar{\mathbf{x}}^1 \in C_1, \dots, \bar{\mathbf{x}}^r \in C_r$ fixed.*

1. *If u is a continuous calibrated sub-action and $u_i := u(\bar{\mathbf{x}}^i)$ for every $i = 1, \dots, r$, then*

$$(u_1, \dots, u_r) \in \mathcal{C}(\bar{\mathbf{x}}^1, \dots, \bar{\mathbf{x}}^r) \quad \text{and} \quad u(\mathbf{x}) = \min_{1 \leq i \leq r} [u(\bar{\mathbf{x}}^i) + h_A(\bar{\mathbf{x}}^i, \mathbf{x})].$$

2. *If $(u_1, \dots, u_r) \in \mathcal{C}(\bar{\mathbf{x}}^1, \dots, \bar{\mathbf{x}}^r)$ and $u(\mathbf{x}) := \min_{1 \leq i \leq r} [u_i + h_A(\bar{\mathbf{x}}^i, \mathbf{x})]$, then u is a continuous calibrated sub-action satisfying $u(\bar{\mathbf{x}}^i) = u_i$ for all $i = 1, \dots, r$.*

3. *Take $i_0 \in \{1, \dots, r\}$ and (u_1, \dots, u_r) such that $u_i := u_{i_0} + h_A(\bar{\mathbf{x}}^{i_0}, \bar{\mathbf{x}}^i)$ for all $i = 1, \dots, r$. Then i_0 is unique, $(u_1, \dots, u_r) \in \mathcal{C}(\bar{\mathbf{x}}^1, \dots, \bar{\mathbf{x}}^r)$ and the unique calibrated sub-action u satisfying $u(\bar{\mathbf{x}}^i) = u_i$, for all $i = 1, \dots, r$, is of the form $u(\mathbf{x}) = u_{i_0} + h_A(\bar{\mathbf{x}}^{i_0}, \mathbf{x})$.*

The application we present here has a certain similarity to lemma 6 in [1]. We point out that the local character of viscosity solutions (as in definition 1 of [1]) is not present in our setting.

Application 12. *Let A be a Hölder observable. Consider any continuous sub-action v and a continuous calibrated sub-action u .*

1. *Then $u - v$ is constant on every irreducible component and*

$$\min_{\Sigma}(u - v) = \min_{\Omega(A)}(u - v).$$

2. *Assume $\Omega(A) = \sqcup_{i=1}^r C_i$ is a finite union of disjoint irreducible components. If $\min_{\Sigma}(u - v)$ is realized on an unique component C_{i_1} and the other components C_i , $i \neq i_1$, are not local minimum for $u - v$, then*

$$u(\mathbf{x}) = u(\bar{\mathbf{x}}_{i_1}) + h_A(\bar{\mathbf{x}}_{i_1}, \mathbf{x}), \quad \forall \mathbf{x} \in \Sigma,$$

where $\bar{\mathbf{x}}_{i_1}$ is any point in C_{i_1} .

4 Proof of theorem 9

We first recall two notions of action potential between two points: the Mañé potential and the Peierls barrier. Given $\epsilon > 0$, $\mathbf{x}, \bar{\mathbf{x}} \in \Sigma$ and $k \geq 1$, we denote

$$S_A^\epsilon(\mathbf{x}, \bar{\mathbf{x}}, k) = \inf \left\{ \sum_{i=1}^k (A - \bar{A})(\mathbf{z}^i) \mid (\mathbf{z}^0, \dots, \mathbf{z}^k) \in \mathcal{P}_k(\mathbf{x}, \bar{\mathbf{x}}, \epsilon) \right\}.$$

If $\hat{B} := A \circ \hat{\sigma}$ and $B := \min\{\hat{B}(\mathbf{x}, \mathbf{y}) \mid \mathbf{y} \in \Sigma_{\mathbf{x}}^*\}$, notice that

$$S_A^\epsilon(\mathbf{x}, \bar{\mathbf{x}}, k) = \inf \left\{ \sum_{i=0}^{k-1} (B - \bar{B}) \circ \sigma^i(\mathbf{x}^0) \mid d(\mathbf{x}^0, \mathbf{x}) < \epsilon, d(\sigma^k(\mathbf{x}^0), \bar{\mathbf{x}}) < \epsilon \right\}.$$

Definition 13. *We call Mañé potential the function $\phi_A : \Sigma \times \Sigma \rightarrow \mathbb{R} \cup \{+\infty\}$ defined by*

$$\phi_A(\mathbf{x}, \bar{\mathbf{x}}) = \lim_{\epsilon \rightarrow 0} \inf_{k \geq 1} S_A^\epsilon(\mathbf{x}, \bar{\mathbf{x}}, k).$$

We call Peierls barrier the function $h_A : \Sigma \times \Sigma \rightarrow \mathbb{R} \cup \{+\infty\}$ defined by

$$h_A(\mathbf{x}, \bar{\mathbf{x}}) = \lim_{\epsilon \rightarrow 0} \liminf_{k \rightarrow +\infty} S_A^\epsilon(\mathbf{x}, \bar{\mathbf{x}}, k).$$

Clearly, $\phi_A \leq h_A$ and both functions are lower semi-continuous. We summarize the main properties of these action potentials.

Proposition 14. *Let A be a Hölder observable.*

1. If u is a continuous sub-action then $u(\bar{\mathbf{x}}) - u(\mathbf{x}) \leq \phi_A(\mathbf{x}, \bar{\mathbf{x}})$.
2. For any points $\mathbf{x}, \bar{\mathbf{x}}, \bar{\bar{\mathbf{x}}} \in \Sigma$, $\phi_A(\mathbf{x}, \bar{\bar{\mathbf{x}}}) \leq \phi_A(\mathbf{x}, \bar{\mathbf{x}}) + \phi_A(\bar{\mathbf{x}}, \bar{\bar{\mathbf{x}}})$.
3. If $\mathbf{x} \in \Sigma$ and $0 < L < \min\{j > 0 : \sigma^j(\mathbf{x}) = \mathbf{x}\} \leq +\infty$, then

$$\phi_A(\mathbf{x}, \mathbf{x}) = \phi_A(\mathbf{x}, \sigma^L(\mathbf{x})) + \phi_A(\sigma^L(\mathbf{x}), \mathbf{x}).$$

Moreover, if $\phi_A(\mathbf{x}, \mathbf{x}) < +\infty$, then there exists a path of length L , $(\bar{\mathbf{z}}^0 = (\bar{\mathbf{x}}^0, \bar{\mathbf{y}}^0), \dots, \bar{\mathbf{z}}^L = (\bar{\mathbf{x}}^L, \bar{\mathbf{y}}^L))$, beginning at \mathbf{x} ($\bar{\mathbf{x}}^j = \sigma^j(\mathbf{x})$ for all $j = 0, \dots, L$), such that

$$\phi_A(\mathbf{x}, \sigma^L(\mathbf{x})) = \sum_{j=1}^L (A - \bar{A})(\bar{\mathbf{z}}^j).$$

4. For any points $\mathbf{x}, \bar{\mathbf{x}}, \bar{\bar{\mathbf{x}}} \in \Sigma$ and any sequence $\{\bar{\mathbf{x}}^l\}$ converging to $\bar{\mathbf{x}}$,

$$h_A(\mathbf{x}, \bar{\bar{\mathbf{x}}}) \leq \liminf_{l \rightarrow +\infty} \phi_A(\mathbf{x}, \bar{\mathbf{x}}^l) + h_A(\bar{\mathbf{x}}, \bar{\bar{\mathbf{x}}}).$$

5. If $\mathbf{x} \in \Sigma$, then $\mathbf{x} \in \Omega(A) \Leftrightarrow \phi_A(\mathbf{x}, \mathbf{x}) = 0 \Leftrightarrow h_A(\mathbf{x}, \mathbf{x}) = 0$.
6. If $\mathbf{x} \in \Omega(A)$, then $\phi_A(\mathbf{x}, \cdot) = h_A(\mathbf{x}, \cdot)$ and $h_A(\mathbf{x}, \cdot)$ is a Hölder calibrated sub-action with respect to the second variable.

This proposition shows how to construct Hölder calibrated sub-actions without the use of the Lax-Oleinik fixed point method.

In Lagrangian Aubry-Mather theory on a compact manifold M , it is well known that, for any point $x \in M$, the map $y \in M \mapsto h(x, y) \in \mathbb{R}$ defines a weak KAM solution, where $h : M \times M \rightarrow \mathbb{R}$ denotes the corresponding Peierls barrier. The analogous result for $h_A(\mathbf{x}, \cdot)$ is however false in the holonomic optimization model. Using item 3, it is not difficult to build examples where

$$\lim_{L \rightarrow +\infty} \phi_A(\mathbf{x}, \sigma^L(\mathbf{x})) = \lim_{L \rightarrow +\infty} h_A(\mathbf{x}, \sigma^L(\mathbf{x})) = +\infty,$$

which shows that $h_A(\mathbf{x}, \cdot)$ is not always a continuous function.

Proof. Items 1, 2, 5 and 6 are well known and a demonstration can be found, for instance, in [7, 11]. So let us prove items 3 and 4.

Item 3. We already know from item 2, that

$$\phi_A(\mathbf{x}, \mathbf{x}) \leq \phi_A(\mathbf{x}, \sigma^L(\mathbf{x})) + \phi_A(\sigma^L(\mathbf{x}), \mathbf{x}).$$

Define $\eta = \min\{d(\sigma^i(\mathbf{x}), \sigma^j(\mathbf{x})) : 0 \leq i < j \leq L\}$. Fix $\gamma > 0$ and take $\epsilon \in (0, \min\{\lambda, \eta/2\})$ such that $\text{Höld}(A)L\epsilon^\theta < \gamma$. Consider also $\rho \in (0, \epsilon)$

such that $d(\mathbf{x}, \bar{\mathbf{x}}) < \rho$ implies $d(\sigma^j(\mathbf{x}), \sigma^j(\bar{\mathbf{x}})) < \epsilon$ for $1 \leq j \leq L$. Take then a path $(\mathbf{z}^0, \dots, \mathbf{z}^l) \in \mathcal{P}_l(\mathbf{x}, \mathbf{x}, \rho)$ satisfying

$$\sum_{j=1}^l (A - \bar{A})(\mathbf{z}^j) < \inf_{k \geq 1} S_A^\rho(\mathbf{x}, \mathbf{x}, k) + \gamma \leq \phi_A(\mathbf{x}, \mathbf{x}) + \gamma.$$

Let $\mathbf{z}^j = (\mathbf{x}^j, \mathbf{y}^j)$ where $\mathbf{x}^j = \sigma^j(\mathbf{x}^0)$ for all $j = 0, 1, \dots, l$. Note that $l > L$. Indeed, ρ has been chosen so that, for each $j \in \{1, 2, \dots, L\}$,

$$d(\mathbf{x}^j, \mathbf{x}) = d(\sigma^j(\mathbf{x}^0), \mathbf{x}) \geq d(\sigma^j(\mathbf{x}), \mathbf{x}) - d(\sigma^j(\mathbf{x}), \sigma^j(\mathbf{x}^0)) > \eta - \epsilon > \rho.$$

Introduce a new path $(\bar{\mathbf{z}}^0, \dots, \bar{\mathbf{z}}^L) \in \mathcal{P}_L(\mathbf{x}, \mathbf{x}, \epsilon)$ given by $\bar{\mathbf{z}}^j = (\sigma^j(\mathbf{x}), \mathbf{y}^j)$, for all $j = 0, \dots, L$. The definition of ρ guarantees

$$\sum_{j=1}^L (A - \bar{A})(\bar{\mathbf{z}}^j) < \sum_{j=1}^L (A - \bar{A})(\mathbf{z}^j) + \text{Höld}_\theta(A) L \epsilon^\theta \leq \sum_{j=1}^L (A - \bar{A})(\mathbf{z}^j) + \gamma.$$

Notice that $(\mathbf{z}^L, \dots, \mathbf{z}^l) \in \mathcal{P}_{l-L}(\sigma^L(\mathbf{x}), \mathbf{x}, \epsilon)$. So we obtain finally

$$\begin{aligned} & \inf_{k \geq 1} S_A^\epsilon(\mathbf{x}, \sigma^L(\mathbf{x}), k) + \inf_{k \geq 1} S_A^\epsilon(\sigma^L(\mathbf{x}), \mathbf{x}, k) \\ & \leq \sum_{j=1}^L (A - \bar{A})(\bar{\mathbf{z}}^j) + \sum_{j=L+1}^l (A - \bar{A})(\mathbf{z}^j) \\ & \leq \sum_{j=1}^L (A - \bar{A})(\mathbf{z}^j) + \sum_{j=L+1}^l (A - \bar{A})(\mathbf{z}^j) + \gamma \\ & \leq \inf_{k \geq 1} S_A^\rho(\mathbf{x}, \mathbf{x}, k) + 2\gamma \leq \phi_A(\mathbf{x}, \mathbf{x}) + 2\gamma. \end{aligned}$$

By letting ϵ goes to 0 and γ to 0, we get

$$\phi_A(\mathbf{x}, \sigma^L(\mathbf{x})) + \phi_A(\sigma^L(\mathbf{x}), \mathbf{x}) \leq \phi_A(\mathbf{x}, \mathbf{x}).$$

The first part of item 3 is proved. To prove the second part, the previous computation shows that, for any sufficiently small ϵ , there exists a path $(\bar{\mathbf{z}}_\epsilon^0, \dots, \bar{\mathbf{z}}_\epsilon^L) \in \mathcal{P}_L(\mathbf{x})$ such that

$$\sum_{j=1}^L (A - \bar{A})(\bar{\mathbf{z}}_\epsilon^j) + \inf_{k \geq 1} S_A^\epsilon(\sigma^L(\mathbf{x}), \mathbf{x}, k) \leq \phi_A(\mathbf{x}, \mathbf{x}) + 2\gamma.$$

By taking accumulation points of $\bar{\mathbf{z}}_\epsilon^j$ when $\epsilon \rightarrow 0$, we obtain, for any γ , a path $(\bar{\mathbf{z}}^0, \dots, \bar{\mathbf{z}}^L)$ such that

$$\phi_A(\mathbf{x}, \sigma^L(\mathbf{x})) \leq \sum_{j=1}^L (A - \bar{A})(\bar{\mathbf{z}}^j) \leq \phi_A(\mathbf{x}, \mathbf{x}) - \phi_A(\sigma^L(\mathbf{x}), \mathbf{x}) + 2\gamma$$

The result follows from item 2 and by taking once more accumulation points of $\bar{\mathbf{z}}^j$ when $\gamma \rightarrow 0$.

Item 4. Since ϕ_A is lower semi-continuous, the statement is equivalent to

$$h_A(\mathbf{x}, \bar{\mathbf{x}}) \leq \phi_A(\mathbf{x}, \bar{\mathbf{x}}) + h_A(\bar{\mathbf{x}}, \bar{\mathbf{x}}), \quad \forall \mathbf{x}, \bar{\mathbf{x}}, \bar{\bar{\mathbf{x}}} \in \Sigma.$$

Fix $\gamma > 0$ and $\epsilon \in (0, \lambda/2)$ such that $\text{Höld}(A)(2\epsilon)^\theta/(1 - \lambda^\theta) < \gamma$. There exists a path $(\mathbf{z}^0, \dots, \mathbf{z}^k) \in \mathcal{P}_k(\mathbf{x}, \bar{\mathbf{x}}, \epsilon)$ such that

$$\sum_{j=1}^k (A - \bar{A})(\mathbf{z}^j) < \inf_{n \geq 1} S_A^\epsilon(\mathbf{x}, \bar{\mathbf{x}}, n) + \gamma.$$

For any $N \geq 1$, there exists a path $(\bar{\mathbf{z}}^0, \dots, \bar{\mathbf{z}}^l) \in \mathcal{P}_l(\bar{\mathbf{x}}, \bar{\bar{\mathbf{x}}}, \epsilon)$ of length $l \geq N$ such that

$$\sum_{j=1}^l (A - \bar{A})(\bar{\mathbf{z}}^j) < \inf_{n \geq N} S_A^\epsilon(\bar{\mathbf{x}}, \bar{\bar{\mathbf{x}}}, n) + \gamma.$$

We define a path $(\bar{\bar{\mathbf{z}}}^0, \dots, \bar{\bar{\mathbf{z}}}^{k+l}) \in \mathcal{P}_{k+l}(\mathbf{x}, \bar{\mathbf{x}}, 3\epsilon)$ in the following way

$$\begin{aligned} \bar{\bar{\mathbf{z}}}^j &= \bar{\mathbf{z}}^{j-k}, \quad \forall j = k+1, \dots, k+l, & \bar{\bar{\mathbf{z}}}^j &= (\bar{\mathbf{x}}^j, \bar{\bar{\mathbf{y}}}^j), \quad \forall j = 0, \dots, k, \\ \bar{\bar{\mathbf{y}}}^j &= \mathbf{y}^j, \quad \forall j = 0, \dots, k, & \bar{\bar{\mathbf{x}}}^k &= \bar{\mathbf{x}}^0, \quad \bar{\bar{\mathbf{x}}}^{j-1} = \tau_{\mathbf{y}^j}(\bar{\mathbf{x}}^j), \quad \forall j = 1, \dots, k. \end{aligned}$$

We notice that $d(\bar{\bar{\mathbf{x}}}^j, \mathbf{x}^j) \leq \lambda^{k-j} d(\bar{\mathbf{x}}^k, \mathbf{x}^k)$, for all $j = 0, \dots, k$. Since

$$d(\bar{\mathbf{x}}^k, \mathbf{x}^k) = d(\bar{\mathbf{x}}^0, \mathbf{x}^k) \leq d(\bar{\mathbf{x}}^0, \bar{\mathbf{x}}) + d(\bar{\mathbf{x}}, \mathbf{x}^k) < 2\epsilon,$$

we obtain $d(\bar{\bar{\mathbf{x}}}^0, \mathbf{x}) \leq \lambda^k 2\epsilon + \epsilon < 3\epsilon$. Hence, it follows that

$$\begin{aligned} \inf_{n \geq N} S_A^{3\epsilon}(\mathbf{x}, \bar{\mathbf{x}}, n) &\leq \sum_{j=1}^{k+l} (A - \bar{A})(\bar{\bar{\mathbf{z}}}^j) \\ &\leq \sum_{j=1}^l (A - \bar{A})(\bar{\mathbf{z}}^j) + \sum_{j=1}^k (A - \bar{A})(\mathbf{z}^j) + \frac{(2\epsilon)^\theta}{1 - \lambda^\theta} \text{Höld}_\theta(A) \\ &\leq \inf_{n \geq 1} S_A^\epsilon(\mathbf{x}, \bar{\mathbf{x}}, n) + \inf_{n \geq N} S_A^\epsilon(\bar{\mathbf{x}}, \bar{\bar{\mathbf{x}}}, n) + 3\gamma \\ &\leq \phi_A(\mathbf{x}, \bar{\mathbf{x}}) + \inf_{n \geq N} S_A^\epsilon(\bar{\mathbf{x}}, \bar{\bar{\mathbf{x}}}, n) + 3\gamma. \end{aligned}$$

By taking first $N \rightarrow +\infty$, then $\epsilon \rightarrow 0$ and $\gamma \rightarrow 0$, we get

$$h_A(\mathbf{x}, \bar{\mathbf{x}}) \leq \phi_A(\mathbf{x}, \bar{\mathbf{x}}) + h_A(\bar{\mathbf{x}}, \bar{\bar{\mathbf{x}}}).$$

□

Other properties of the Mañé potential and the Peierls barrier can be derived from the previous proposition. For instance, item 4 gives us the following inequality

$$h_A(\mathbf{x}, \bar{\mathbf{x}}) \leq h_A(\mathbf{x}, \bar{\mathbf{x}}) + h_A(\bar{\mathbf{x}}, \bar{\mathbf{x}}), \quad \forall \mathbf{x}, \bar{\mathbf{x}}, \bar{\bar{\mathbf{x}}} \in \Sigma.$$

We now begin the proof of theorem 9. It follows immediately from the next lemma.

Lemma 15. *Let $D \subset \Sigma$ be an open set containing $\Omega(A)$. Denote by \mathcal{D}_A the subset of Hölder sub-actions u such that $\pi(\mathbb{M}_A(u)) \subset D$. Then, for the Hölder topology, \mathcal{D}_A is an open dense subset of the Hölder sub-actions.*

Indeed, if one considers, for each positive integer j , the open set $D_j = \{\mathbf{x} \in \Sigma \mid d(\mathbf{x}, \Omega(A)) < 1/j\}$ and the corresponding open dense subset of Hölder sub-actions $\mathcal{D}_{A,j}$, then the set of Hölder separating sub-actions contains the countable intersection $\bigcap_{j>0} \mathcal{D}_{A,j}$.

Proof of lemma 15. We only discuss the denseness of \mathcal{D}_A .

Part 1. Let v be any Hölder sub-action for A . We will show that, for every $\mathbf{x} \notin D$, there exists a Hölder sub-action $v_{\mathbf{x}}$ as close as we want to v in the Hölder topology with a projected contact locus disjoint from \mathbf{x} , that is, $\mathbf{x} \notin \pi(\mathbb{M}_A(v_{\mathbf{x}}))$ or

$$v_{\mathbf{x}}(\mathbf{x}) - v_{\mathbf{x}}(\tau_{\mathbf{y}}(\mathbf{x})) < A(\mathbf{x}, \mathbf{y}) - \bar{A}, \quad \forall \mathbf{y} \in \Sigma_{\mathbf{x}}^*.$$

Let $\mathbf{x} \notin D$. We discuss two cases.

Case a. We assume there exists an integer $k \geq 0$ such that, for every path of length k beginning at \mathbf{x} , $(\mathbf{z}^0 = (\mathbf{x}, \mathbf{y}^0), \dots, \mathbf{z}^k = (\sigma^k(\mathbf{x}), \mathbf{y}^k)) \in \mathcal{P}_k(\mathbf{x})$, the terminal point $\mathbf{z}^k \notin \mathbb{M}_A(v)$. If $k = 0$, we choose $v_{\mathbf{x}} = v$. Assume now $k \geq 1$. Let

$$B := A - \bar{A} - v \circ \pi + v \circ \pi \circ \hat{\sigma}^{-1} \geq 0$$

be the associated normalized observable ($B \geq 0$ and $\bar{B} = 0$). We recall that $\tau_{\mathbf{y}^j}(\sigma^j(\mathbf{x})) = \sigma^{j-1}(\mathbf{x})$, for all $j = 1, \dots, k$. So by hypothesis

$$B(\mathbf{z}^k) = B(\sigma^k(\mathbf{x}), \mathbf{y}^k) > 0, \quad \forall \mathbf{y}^k \in \Sigma_{\sigma^k(\mathbf{x})}^* \text{ s.t. } \sigma^{k-1}(\mathbf{x}) = \tau_{\mathbf{y}^k}(\sigma^k(\mathbf{x})).$$

Notice first that, if $(\bar{\mathbf{z}}^0, \dots, \bar{\mathbf{z}}^k)$ is a path of length k ,

$$\begin{aligned} B(\bar{\mathbf{z}}^0) &= \sum_{j=0}^{k-1} B(\bar{\mathbf{z}}^j) - \sum_{j=1}^k B(\bar{\mathbf{z}}^j) + B(\bar{\mathbf{z}}^k) \\ &\geq \sum_{j=0}^{k-1} \gamma B(\bar{\mathbf{z}}^j) - \sum_{j=1}^k \gamma B(\bar{\mathbf{z}}^j) + \gamma B(\bar{\mathbf{z}}^k) \geq \sum_{j=0}^{k-1} \gamma B(\bar{\mathbf{z}}^j) - \sum_{j=1}^k \gamma B(\bar{\mathbf{z}}^j) \end{aligned}$$

for any constant $\gamma \in (0, 1)$. Let $w_k : \Sigma \rightarrow \mathbb{R}$ be the function given by

$$w_k(\bar{\mathbf{x}}) := \min \left\{ \sum_{j=1}^k B(\bar{\mathbf{z}}^j) \mid (\bar{\mathbf{z}}^0, \dots, \bar{\mathbf{z}}^k) \in \mathcal{P}_k(\bar{\mathbf{x}}) \right\}, \quad \forall \bar{\mathbf{x}} \in \Sigma.$$

Since $C(\bar{\mathbf{x}}) := \min\{B \circ \hat{\sigma}(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \mid \bar{\mathbf{y}} \in \Sigma_{\bar{\mathbf{x}}}^*\}$ is Hölder, $w_k = \sum_{j=0}^{k-1} C \circ \sigma^j$ is also Hölder. We first prove that $-\gamma w_k$ is a sub-action. Let $\bar{\mathbf{x}} \in \Sigma$ and $\bar{\mathbf{y}} \in \Sigma_{\bar{\mathbf{x}}}^*$. There exists a path of length k , $(\bar{\mathbf{z}}^0, \dots, \bar{\mathbf{z}}^k)$, beginning at $\bar{\mathbf{x}}$ and realizing the minimum

$$w_k(\bar{\mathbf{z}}) = \sum_{j=1}^k B(\bar{\mathbf{z}}^j).$$

Notice the only constraint on $\bar{\mathbf{y}}^0$ is $\bar{\mathbf{y}}^0 \in \Sigma_{\bar{\mathbf{x}}}^*$, besides $\bar{\mathbf{y}}^0$ does not appear in the previous sum. Choose $\bar{\mathbf{y}}^0 = \bar{\mathbf{y}}$, $\bar{\mathbf{y}}^{-1} \in \Sigma_{\bar{\mathbf{x}}^{-1}}^*$ and call $\bar{\mathbf{x}}^{-1} = \tau_{\bar{\mathbf{y}}}(\bar{\mathbf{x}})$. Then $(\bar{\mathbf{z}}^{-1}, \bar{\mathbf{z}}^0, \dots, \bar{\mathbf{z}}^{k-1})$ is a path of length k beginning at $\tau_{\bar{\mathbf{y}}}(\bar{\mathbf{x}})$ and

$$B(\bar{\mathbf{z}}) = B(\bar{\mathbf{z}}^0) \geq \gamma \sum_{j=0}^{k-1} B(\bar{\mathbf{z}}^j) - \gamma \sum_{j=1}^k B(\bar{\mathbf{z}}^j) \geq \gamma w_k(\tau_{\bar{\mathbf{y}}}(\bar{\mathbf{x}})) - \gamma w_k(\bar{\mathbf{x}}).$$

Moreover, the same computation for \mathbf{z} instead of $\bar{\mathbf{z}}$ shows

$$B(\mathbf{z}) - \gamma w_k(\tau_{\mathbf{y}}(\mathbf{x})) + \gamma w_k(\mathbf{x}) \geq \gamma B(\mathbf{z}^k) > 0,$$

for any $\mathbf{y} \in \Sigma_{\mathbf{x}}^*$. We have proved that $\mathbf{x} \notin \pi(\mathbb{M}_B(-\gamma w_k)) = \pi(\mathbb{M}_A(v - \gamma w_k))$.

Since γ can be taken as small as we want, we have shown the existence of a Hölder sub-action $v_{\mathbf{x}} = v - \gamma w_k$ close to v in the Hölder topology satisfying $\mathbf{x} \notin \pi(\mathbb{M}_A(v_{\mathbf{x}}))$.

Case b. We suppose that, for every integer $k \geq 0$, one can find a path of length k , $(\mathbf{z}^0, \dots, \mathbf{z}^k)$, beginning at \mathbf{x} , such that $\mathbf{z}^k \in \mathbb{M}_A(v)$, or equivalently $B(\mathbf{z}^k) = 0$ with B as before. In other words, for any $k \geq 1$, there exists $\mathbf{y}^k \in \Sigma_{\mathbf{x}^k}^* \cap (\sigma^*)^{-1}(\Sigma_{\mathbf{x}^{k-1}}^*)$ such that $B(\mathbf{x}^k, \mathbf{y}^k) = 0$, where $\mathbf{x}^k = \sigma^k(\mathbf{x})$. Define $\bar{\mathbf{z}}^k = (\mathbf{x}^k, \mathbf{y}^k)$ for all $k \geq 1$. Notice that $(\bar{\mathbf{z}}^0, \dots, \bar{\mathbf{z}}^k)$ is again a path of any length, beginning at \mathbf{x} , which satisfies $B(\bar{\mathbf{z}}^k) = 0$ for all $k \geq 1$.

Let $\bar{\mathbf{x}} \in \Omega(A) = \Omega(B)$ be any limit point of $(\mathbf{x}^k)_k$ chosen once for all. Let $w := h_B(\bar{\mathbf{x}}, \cdot)$ be the Hölder sub-action for B given by the corresponding Peierls barrier. We remark that $\phi_B(\mathbf{x}, \sigma^k(\mathbf{x})) = 0$ for all $k \geq 1$ and that

$$w(\mathbf{x}) = h_B(\bar{\mathbf{x}}, \mathbf{x}) = \liminf_{k \rightarrow +\infty} \phi_B(\mathbf{x}, \sigma^k(\mathbf{x})) + h_B(\bar{\mathbf{x}}, \mathbf{x}) \geq h_B(\mathbf{x}, \mathbf{x}) > 0.$$

Let $\gamma \in (0, 1)$ be any real number as close to 0 as we want. We claim that \mathbf{x} satisfies again the first case, namely, there exists $k \geq 1$ such that, for any path of length k , $(\mathbf{z}^0 = (\mathbf{x}^0, \mathbf{y}^0), \dots, \mathbf{z}^k = (\mathbf{x}^k, \mathbf{y}^k))$, beginning at \mathbf{x} , one has

$$B(\mathbf{z}^k) - \gamma h_B(\bar{\mathbf{x}}, \mathbf{x}^k) + \gamma h_B(\bar{\mathbf{x}}, \mathbf{x}^{k-1}) > 0.$$

(Notice that γw is again a sub-action for B since B is non-negative.) Indeed, by contradiction we would have

$$0 \leq B(\mathbf{z}^k) = \gamma h_B(\bar{\mathbf{x}}, \mathbf{x}^k) - \gamma h_B(\bar{\mathbf{x}}, \mathbf{x}^{k-1}), \quad \forall k \geq 1.$$

On the one hand, we would obtain $0 < h_B(\bar{\mathbf{x}}, \mathbf{x}) \leq h_B(\bar{\mathbf{x}}, \mathbf{x}^k)$ for all $k \geq 1$. On the other hand, by taking a subsequence of $\mathbf{x}^k = \{\sigma^k(\mathbf{x})\}$ converging to $\bar{\mathbf{x}}$, $h_B(\bar{\mathbf{x}}, \mathbf{x}^k)$ would converge to $h_B(\bar{\mathbf{x}}, \bar{\mathbf{x}}) = 0$, since $\bar{\mathbf{x}} \in \Omega(B)$. We have thus obtained a contradiction. Hence, case (a) implies that there exists a sub-action $v_{\mathbf{x}}$, close to v in the Hölder topology, satisfying $\mathbf{x} \notin \pi(\mathbb{M}_A(v_{\mathbf{x}}))$.

Part 2. We have just proved that, for any $\mathbf{x} \notin D$, there exists a sub-action $v_{\mathbf{x}}$ close to v and a ball $B(\mathbf{x}, \epsilon_{\mathbf{x}})$ of radius $\epsilon_{\mathbf{x}} > 0$ centered at \mathbf{x} such that

$$\forall \bar{\mathbf{x}} \in B(\mathbf{x}, \epsilon_{\mathbf{x}}), \quad \bar{\mathbf{x}} \notin \pi(\mathbb{M}_A(v_{\mathbf{x}})).$$

We can extract from the family of these balls $\{B(\mathbf{x}, \epsilon_{\mathbf{x}})\}_{\mathbf{x}}$ a finite family indexed by $\{\mathbf{x}^j\}_{1 \leq j \leq K}$ which is still a covering of the compact set $\Sigma \setminus D$. Let

$$u = \frac{1}{K} \sum_{j=1}^K v_{\mathbf{x}^j}.$$

Then it is easy to check that u is a Hölder sub-action for A satisfying $\pi(\mathbb{M}_A(u)) \subset D$, namely, $u \in \mathcal{D}_A$. Since each sub-action $v_{\mathbf{x}}$ can be taken as close as we want to v in the Hölder topology, the same is true for u . \square

5 Proof of theorem 11

It was proved in [11] that the projection of the support of a minimizing probability measure $\hat{\mu}$ is included into the A -non-wandering set $\Omega(A)$ when such projection is ergodic. If $\pi_*\hat{\mu}$ is ergodic, $\pi(\text{supp}(\hat{\mu}))$ may be seen as an irreducible component in the sense that any two points can be joined by an ϵ -closed trajectory. We introduce here a more general notion of irreducibility.

Definition-Proposition 16. *Let A be a Hölder observable. We say that two points $\mathbf{x}, \bar{\mathbf{x}}$ of $\Omega(A)$ are equivalent and write $\mathbf{x} \sim \bar{\mathbf{x}}$ if*

$$h_A(\mathbf{x}, \bar{\mathbf{x}}) + h_A(\bar{\mathbf{x}}, \mathbf{x}) = 0.$$

Then \sim is an equivalent relation. Its equivalent classes are called irreducible components.

Proof. It is obvious that \sim is reflexive ($h_A(\mathbf{x}, \mathbf{x}) = 0 \Leftrightarrow \mathbf{x} \in \Omega(A)$) and symmetric. Let u be a continuous sub-action and $B := A - \bar{A} - u \circ \pi + u \circ \tau$ be the associated normalized observable. Then the definition of the Peierls barrier implies easily

$$h_B(\mathbf{x}, \bar{\mathbf{x}}) = h_A(\mathbf{x}, \bar{\mathbf{x}}) - u(\bar{\mathbf{x}}) + u(\mathbf{x}), \quad \forall \mathbf{x}, \bar{\mathbf{x}} \in \Sigma.$$

Since $h_B(\mathbf{x}, \bar{\mathbf{x}}) \geq 0$, we see that $\mathbf{x} \sim \bar{\mathbf{x}} \Leftrightarrow h_B(\mathbf{x}, \bar{\mathbf{x}}) = 0$ and $h_B(\bar{\mathbf{x}}, \mathbf{x}) = 0$.

To show the transitivity property, it is enough to prove

$$\mathbf{x} \sim \bar{\mathbf{x}} \text{ and } \bar{\mathbf{x}} \sim \bar{\bar{\mathbf{x}}} \implies h_B(\mathbf{x}, \bar{\bar{\mathbf{x}}}) = 0.$$

But proposition 14 guarantees

$$0 \leq h_B(\mathbf{x}, \bar{\bar{\mathbf{x}}}) \leq h_B(\mathbf{x}, \bar{\mathbf{x}}) + h_B(\bar{\mathbf{x}}, \bar{\bar{\mathbf{x}}}) = 0.$$

The transitivity property is proved. \square

Proposition 17. *The irreducible components are closed and σ -invariant.*

Proof. Part 1. Let $\mathbf{x} \in \Omega(A)$. Consider $\{\bar{\mathbf{x}}_\epsilon\}_\epsilon$ a sequence of points of $\Omega(A)$ equivalent to \mathbf{x} and within ϵ of $\bar{\mathbf{x}} \in \Omega(A)$. Then on the one hand, $h_A(\mathbf{x}, \bar{\mathbf{x}}) + h_A(\bar{\mathbf{x}}, \mathbf{x}) \geq h_A(\mathbf{x}, \mathbf{x}) = 0$, and on the other hand,

$$h_A(\mathbf{x}, \bar{\mathbf{x}}_\epsilon) + h_A(\bar{\mathbf{x}}, \mathbf{x}) \leq h_A(\mathbf{x}, \bar{\mathbf{x}}_\epsilon) + h_A(\bar{\mathbf{x}}, \bar{\mathbf{x}}_\epsilon) + h_A(\bar{\mathbf{x}}_\epsilon, \mathbf{x}) = h_A(\bar{\mathbf{x}}, \bar{\mathbf{x}}_\epsilon).$$

By continuity of $h_A(\mathbf{x}, \cdot)$ and $h_A(\bar{\mathbf{x}}, \cdot)$ with respect to the second variable, the previous inequality gives $h_A(\mathbf{x}, \bar{\mathbf{x}}) + h_A(\bar{\mathbf{x}}, \mathbf{x}) \leq 0$. Therefore $\bar{\mathbf{x}} \sim \mathbf{x}$ and the class containing \mathbf{x} is closed.

Part 2. Let $\mathbf{x} \in \Omega(A)$. Either $\sigma(\mathbf{x}) = \mathbf{x}$ and obviously $\sigma(\mathbf{x}) \sim \mathbf{x}$ or $\sigma(\mathbf{x}) \neq \mathbf{x}$ and item 3 of proposition 14 shows $\phi_A(\mathbf{x}, \sigma(\mathbf{x})) + \phi_A(\sigma(\mathbf{x}), \mathbf{x}) = \phi_A(\mathbf{x}, \mathbf{x}) = 0$. The fact that $h_A(\mathbf{x}, \cdot) = \phi_A(\mathbf{x}, \cdot)$ whenever $\mathbf{x} \in \Omega(A)$ proves that an irreducible component is σ -invariant. \square

We assume from now on that $\Omega(A)$ is equal to a disjoint union of irreducible components, $\Omega(A) = C_1 \sqcup \dots \sqcup C_r$. The following proposition shows that the Peierls barrier normalized by a separating sub-action could play the role of a quantized set of levels of energy.

Proposition 18. *Let A be a Hölder observable and assume that $\Omega(A) = \sqcup_{i=1}^r C_i$ is equal to a finite union of irreducible components.*

1. *If u is a continuous sub-action, then*

$$(\mathbf{x}^i, \mathbf{x}^j) \mapsto h_A(\mathbf{x}^i, \mathbf{x}^j) - u(\mathbf{x}^j) + u(\mathbf{x}^i) \text{ is constant on } C_i \times C_j.$$

2. *If u is a continuous separating sub-action, then*

$$h_A(\mathbf{x}^i, \mathbf{x}^j) > u(\mathbf{x}^j) - u(\mathbf{x}^i), \quad \forall \mathbf{x}^i, \mathbf{x}^j \in C_i \times C_j, \quad \forall i \neq j.$$

Proof. We first normalize A by taking $B = A - \bar{A} - u \circ \pi + u \circ \tau$ so that $B \geq 0$ and $\bar{B} = 0$.

Part 1. Let $(\bar{\mathbf{x}}^i, \bar{\mathbf{x}}^j) \in C_i \times C_j$. Then $h_B(\mathbf{x}^i, \bar{\mathbf{x}}^i) = h_B(\mathbf{x}^j, \bar{\mathbf{x}}^j) = 0$ and

$$h_B(\bar{\mathbf{x}}^i, \bar{\mathbf{x}}^j) \leq h_B(\bar{\mathbf{x}}^i, \mathbf{x}^i) + h_B(\mathbf{x}^i, \mathbf{x}^j) + h_B(\mathbf{x}^j, \bar{\mathbf{x}}^j) \leq h_B(\mathbf{x}^i, \mathbf{x}^j).$$

Conversely $h_B(\mathbf{x}^i, \mathbf{x}^j) \leq h_B(\bar{\mathbf{x}}^i, \bar{\mathbf{x}}^j)$ and we have proved that $h_B(\cdot, \cdot)$ is constant on $C_i \times C_j$.

Part 2. Let $\{U_i^\eta\}_{\eta>0}$ be a basis of neighborhoods of C_i . Since $\sigma(C_i) \subset C_i$ is disjoint from each C_j , $j \neq i$, there exists $\eta > 0$ small enough such that $\sigma(U_i^\eta)$ is disjoint from $\cup_{j \neq i} U_j^\eta$. Let $i \neq j$ and $\mathbf{x} \in C_i$, $\bar{\mathbf{x}} \in C_j$. For $\epsilon > 0$ sufficiently small, the ball of radius ϵ centered at \mathbf{x} is included in U_i^η . Let $(\mathbf{z}^0 = (\mathbf{x}^0, \mathbf{y}^0), \dots, \mathbf{z}^k = (\mathbf{x}^k, \mathbf{y}^k))$ be a path of length k within ϵ of \mathbf{x} and $\bar{\mathbf{x}}$, more precisely, satisfying $d(\mathbf{x}^0, \mathbf{x}) < \epsilon$ and $d(\mathbf{x}^k, \bar{\mathbf{x}}) < \epsilon$. Let $p \geq 1$ be the first time $\sigma^p(\mathbf{x}) \notin U_i^\eta$. Then $\sigma^{p-1}(\mathbf{x}) \in U_i^\eta$ and $\sigma^p(\mathbf{x}) \in \sigma(U_i^\eta) \setminus U_i^\eta$. By the choice of η , $\sigma^p(\mathbf{x}) \notin \cup_{j=1}^r U_j^\eta =: \mathcal{U} \supset \Omega(A)$. Since $\Omega(A) = \pi(\mathbb{M}_A(u))$, let $\hat{\mathcal{U}} := \pi^{-1}(\mathcal{U})$, then $\mathbf{z}^p \notin \hat{\mathcal{U}}$ and

$$\sum_{l=1}^k B(\mathbf{z}^l) \geq B(\mathbf{z}^p) \geq \min_{\hat{\Sigma} \setminus \hat{\mathcal{U}}} B =: m > 0.$$

We have proved that $h_B(\mathbf{x}, \bar{\mathbf{x}}) \geq m > 0$. □

We are now in a position to prove our second result.

Proof of theorem 11. We fixed once for all $\bar{\mathbf{x}}^i \in C_i$.

Part 1. We know from theorem 10 that a continuous calibrated sub-action satisfies $u(\mathbf{x}) = \min_{\bar{\mathbf{x}} \in \Omega(A)} [u(\bar{\mathbf{x}}) + h_A(\bar{\mathbf{x}}, \mathbf{x})]$. If $\bar{\mathbf{x}} \in C_i$, then $\bar{\mathbf{x}} \sim \bar{\mathbf{x}}^i$ and $h_A(\bar{\mathbf{x}}^i, \bar{\mathbf{x}}) + h_A(\bar{\mathbf{x}}, \bar{\mathbf{x}}^i) = 0$. Then

$$\begin{aligned} u(\bar{\mathbf{x}}^i) + h_A(\bar{\mathbf{x}}^i, \mathbf{x}) &\leq u(\bar{\mathbf{x}}^i) + h_A(\bar{\mathbf{x}}^i, \bar{\mathbf{x}}) + h_A(\bar{\mathbf{x}}, \mathbf{x}) \\ &= u(\bar{\mathbf{x}}^i) - h_A(\bar{\mathbf{x}}, \bar{\mathbf{x}}^i) + h_A(\bar{\mathbf{x}}, \mathbf{x}) \leq u(\bar{\mathbf{x}}) + h_A(\bar{\mathbf{x}}, \mathbf{x}). \end{aligned}$$

We have proved that $u(\mathbf{x}) = \min_{1 \leq i \leq r} [u(\bar{\mathbf{x}}^i) + h_A(\bar{\mathbf{x}}^i, \mathbf{x})]$. The fact that $(u(\bar{\mathbf{x}}^1), \dots, u(\bar{\mathbf{x}}^r)) \in \mathcal{C}_A(\bar{\mathbf{x}}^1, \dots, \bar{\mathbf{x}}^r)$ comes from items 1 and 6 of proposition 14.

Part 2. Let $(u_1, \dots, u_r) \in \mathcal{C}_A(\bar{\mathbf{x}}^1, \dots, \bar{\mathbf{x}}^r)$ and define $\phi : \Omega(A) \rightarrow \mathbb{R}$ by $\phi(\mathbf{x}) := u_i + h_A(\bar{\mathbf{x}}^i, \mathbf{x})$ for all $\mathbf{x} \in C_i$. We notice that ϕ is continuous and show that $\phi(\bar{\mathbf{x}}) - \phi(\mathbf{x}) \leq h_A(\mathbf{x}, \bar{\mathbf{x}})$ for all $\mathbf{x}, \bar{\mathbf{x}} \in \Omega(A)$. Indeed, if $\mathbf{x} \in C_i$ and $\bar{\mathbf{x}} \in C_j$, then

$$\begin{aligned} \phi(\bar{\mathbf{x}}) - \phi(\mathbf{x}) &= (u_j - u_i) + h_A(\bar{\mathbf{x}}^j, \bar{\mathbf{x}}) - h_A(\bar{\mathbf{x}}^i, \mathbf{x}) \\ &\leq h_A(\bar{\mathbf{x}}^i, \bar{\mathbf{x}}^j) + h_A(\bar{\mathbf{x}}^j, \bar{\mathbf{x}}) - h_A(\bar{\mathbf{x}}^i, \mathbf{x}) \\ &= h_A(\bar{\mathbf{x}}^i, \bar{\mathbf{x}}^j) - h_A(\bar{\mathbf{x}}, \bar{\mathbf{x}}^j) - h_A(\bar{\mathbf{x}}^i, \mathbf{x}) \\ &\leq h_A(\bar{\mathbf{x}}^i, \bar{\mathbf{x}}) - h_A(\bar{\mathbf{x}}^i, \mathbf{x}) \leq h_A(\mathbf{x}, \bar{\mathbf{x}}). \end{aligned}$$

(The last but one inequality uses item 1 of proposition 14 and the fact that $h_A(\bar{\mathbf{x}}^i, \cdot)$ is a sub-action.) The function $u(\mathbf{x}) := \min_{\bar{\mathbf{x}} \in \Omega(A)} [\phi(\bar{\mathbf{x}}) + h_A(\bar{\mathbf{x}}, \mathbf{x})]$ is by proposition 10 a continuous calibrated sub-action which extends ϕ on

$\Omega(A)$. In particular, $u(\bar{\mathbf{x}}^i) = \phi(\bar{\mathbf{x}}^i) = u_i$ and, thanks to part 1, u coincides with $\min_{1 \leq i \leq r} [u_i + h_A(\bar{\mathbf{x}}^i, \cdot)]$.

Part 3. Let $i_0 \in \{1, \dots, r\}$. If (u_1, \dots, u_r) satisfies $u_i = u_{i_0} + h_A(\bar{\mathbf{x}}^{i_0}, \bar{\mathbf{x}}^i)$, then i_0 is unique. Otherwise there would exist $i_1 \neq i_0$ such that $u_i = u_{i_1} + h_A(\bar{\mathbf{x}}^{i_1}, \bar{\mathbf{x}}^i)$. Thus

$$u_{i_1} = u_{i_0} + h_A(\bar{\mathbf{x}}^{i_0}, \bar{\mathbf{x}}^{i_1}) \quad \text{and} \quad u_{i_0} = u_{i_1} + h_A(\bar{\mathbf{x}}^{i_1}, \bar{\mathbf{x}}^{i_0}).$$

We would obtain $h_A(\bar{\mathbf{x}}^{i_0}, \bar{\mathbf{x}}^{i_1}) + h_A(\bar{\mathbf{x}}^{i_1}, \bar{\mathbf{x}}^{i_0}) = 0$ contradicting $\bar{\mathbf{x}}^{i_0} \not\sim \bar{\mathbf{x}}^{i_1}$. The fact that $(u_1, \dots, u_r) \in \mathcal{C}_A(\bar{\mathbf{x}}^1, \dots, \bar{\mathbf{x}}^r)$ comes from

$$u_j - u_i = h_A(\bar{\mathbf{x}}^{i_0}, \bar{\mathbf{x}}^j) - h_A(\bar{\mathbf{x}}^{i_0}, \bar{\mathbf{x}}^i) \leq h_A(\bar{\mathbf{x}}^i, \bar{\mathbf{x}}^j).$$

The end of part 3 follows since $u(\mathbf{x}) := u_{i_0} + h_A(\bar{\mathbf{x}}^{i_0}, \mathbf{x})$ already defines a calibrated sub-action satisfying $u(\bar{\mathbf{x}}^i) = u_i$ for all i . \square

The proof of Application 12 is elementary.

Proof of application 12. Define $B := A - v \circ \pi + v \circ \tau - \bar{A}$, then the null function is a sub-action of B and $v - u$ is a sub-action calibrated to B . Moreover, $h_B(\mathbf{x}, \bar{\mathbf{x}}) = h_A(\mathbf{x}, \bar{\mathbf{x}}) - v(\bar{\mathbf{x}}) + v(\mathbf{x})$ and $\Omega(A) = \Omega(B)$. It is therefore enough to assume A normalized ($A \geq 0$ and $\bar{A} = 0$) and $v = 0$.

Part 1. If $\mathbf{x} \sim \bar{\mathbf{x}}$ are two points of $\Omega(A)$, then $h_A(\mathbf{x}, \bar{\mathbf{x}}) = 0$ and $h_A(\bar{\mathbf{x}}, \mathbf{x}) = 0$. Thanks to items 1 and 6 of proposition 14, we obtain $u(\mathbf{x}) = u(\bar{\mathbf{x}})$. If \mathbf{x} is any point of Σ , by the calibration of u , one can construct an inverse path $\{\mathbf{z}^{-i}\}_{i \geq 0}$ of $\hat{\Sigma}$, with $\pi(\mathbf{z}^0) = \mathbf{x}$, such that $u(\mathbf{x}^{-i}) - u(\mathbf{x}^{-i-1}) = A(\mathbf{z}^{-i})$, $\mathbf{x}^{-i} = \pi(\mathbf{z}^{-i})$, for all i . Let $\bar{\mathbf{x}}$ be an accumulation point of $\{\mathbf{x}^{-i}\}_{i \geq 0}$. Then $\bar{\mathbf{x}} \in \Omega(A)$ and, since $A \geq 0$, the sequence $\{u(\mathbf{x}^{-i})\}_{i \geq 0}$ is decreasing. In particular, $u(\mathbf{x}) \geq u(\bar{\mathbf{x}})$ establishes $\min_{\Sigma} u = \min_{\Omega(A)} u$.

Part 2. Let u_i be the value of u on C_i . Assume we have ordered these values as $u_{i_1} \leq u_{i_2} \leq \dots \leq u_{i_r}$. Let $\bar{\mathbf{x}}^i \in C_i$ fixed. It suffices to prove $u(\bar{\mathbf{x}}^{i_k}) = u(\bar{\mathbf{x}}^{i_1}) + h_A(\bar{\mathbf{x}}^{i_1}, \bar{\mathbf{x}}^{i_k})$ for all $k = 1, \dots, r$. It is true for $k = 1$. Since $C_{i_{k+1}}$ is not a minimum local of u , one can find a sequence of points $\{\mathbf{x}_\epsilon\}_{\epsilon > 0}$ within ϵ of $C_{i_{k+1}}$ such that $u(\mathbf{x}_\epsilon) < u(\bar{\mathbf{x}}^{i_{k+1}})$. From part 1 of theorem 11, there exists an index j such that $u(\mathbf{x}_\epsilon) = u(\bar{\mathbf{x}}^j) + h_A(\bar{\mathbf{x}}^j, \mathbf{x}_\epsilon)$. Since $h_A \geq 0$, $u_j = u(\bar{\mathbf{x}}^j) \leq u(\mathbf{x}_\epsilon) < u_{i_{k+1}}$. So j has to be one of indexes i_1, \dots, i_k . By induction, $u(\bar{\mathbf{x}}^j) = u(\bar{\mathbf{x}}^{i_1}) + h_A(\bar{\mathbf{x}}^{i_1}, \bar{\mathbf{x}}^j)$ and

$$u(\mathbf{x}_\epsilon) = u(\bar{\mathbf{x}}^{i_1}) + h_A(\bar{\mathbf{x}}^{i_1}, \bar{\mathbf{x}}^j) + h_A(\bar{\mathbf{x}}^j, \mathbf{x}_\epsilon).$$

On the one hand, $h_A(\bar{\mathbf{x}}^{i_1}, \bar{\mathbf{x}}^j) + h_A(\bar{\mathbf{x}}^j, \mathbf{x}_\epsilon) \geq h_A(\bar{\mathbf{x}}^{i_1}, \mathbf{x}_\epsilon)$ implies

$$u(\mathbf{x}_\epsilon) \geq u(\bar{\mathbf{x}}^{i_1}) + h_A(\bar{\mathbf{x}}^{i_1}, \mathbf{x}_\epsilon).$$

On the other hand, as u is a sub-action, we obtain the reverse inequality and finally

$$u(\mathbf{x}_\epsilon) = u(\bar{\mathbf{x}}^{i_1}) + h_A(\bar{\mathbf{x}}^{i_1}, \mathbf{x}_\epsilon).$$

Letting ϵ go to 0, \mathbf{x}_ϵ accumulates to $C_{i_{k+1}}$ and

$$u(\bar{\mathbf{x}}^{i_{k+1}}) = u(\bar{\mathbf{x}}^{i_1}) + h_A(\bar{\mathbf{x}}^{i_1}, \bar{\mathbf{x}}^{i_{k+1}}).$$

□

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