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A NOTE ON FURSTENBERG'S FILTERING PROBLEM

RODOLPHE GARBIT

Abstract. This note gives a positive answer to an old question in elementary probability theory that arose in Furstenberg's seminal article "Disjointness in Ergodic Theory." As a consequence, Furstenberg's filtering theorem holds without any integrability assumption.

1. Disjointness and Filtering

In his seminal article [2], H. Furstenberg introduced the notion of *disjointness* of two stationary random processes. A *joining* of two stationary processes $\{X_n\}$ and $\{Y_n\}$ is a two-dimensional stationary process $\{(X'_n, Y'_n)\}$ whose marginal distributions are those of $\{X_n\}$ and $\{Y_n\}$, respectively. The processes $\{X_n\}$ and $\{Y_n\}$ are said to be *disjoint* if any joining $\{(X'_n, Y'_n)\}$ is such that $\{X'_n\}$ and $\{Y'_n\}$ are independent. Among other purposes this notion was used to study the following filtering problem: if $\{X_n\}$ and $\{Y_n\}$ are two independent stationary sequences of real-valued random variables, $\{X_n\}$ being the emitted signal and $\{Y_n\}$ the noise, is it possible to recover $\{X_n\}$ from the received signal $\{X_n + Y_n\}$? More precisely, the problem was to determine whether $\{X_n\}$ is a function of the sum process $\{X_n + Y_n\}$. In that case, the sequence $\{(X_n, Y_n)\}$ is said to admit a *perfect filter*. Furstenberg proved the following theorem:

Theorem 1.1 ([2], Theorem I.5)**.** *Let $\{X_n\}$ and $\{Y_n\}$ be two stationary sequences of integrable random variables and suppose that the two sequences are disjoint. Then $\{(X_n, Y_n)\}$ admits a perfect filter.*

Furstenberg then raised the question whether the integrability requirement was essential for the conclusion of the theorem. He noticed that the integrability stipulation may be removed if the disjointness assumption is replaced by a much stronger assumption of *double disjointness*. The idea of double disjointness was then exploited in [3] to treat a more general filtering problem where the received signal has the form $Z_n = f(X_n, Y_n)$. In [1], the authors deal with a $\mathbb{Z}^2$-variant of the filtering problem. But, as far as we know, the original question about the integrability assumption was still open. It is shown here that it can be removed from the statement of Theorem 1.1.

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2. Probabilistic Background

From a technical point of view, the necessity of the integrability assumption in Theorem 1.1 is purely a probabilistic question. In the course of the proof of the filtering theorem, Furstenberg uses the following lemma:

Lemma 2.1 ([2], Lemma I.3)**.** *Let U_1, U_2, V_1, V_2 denote four integrable random variables with each of the U_i independent of each V_j. Then $U_1 + V_1 = U_2 + V_2$ together with $\mathbb{E}[U_1] = \mathbb{E}[U_2]$ implies $U_1 = U_2$ and $V_1 = V_2$.*

Furstenberg noted that "It would be of interest to know if the integrability stipulation may be omitted, replacing the equality of the expectations of U_1 and U_2 by equality of their distributions," because a positive answer would mean that the integrability assumption in Theorem 1.1 can also be omitted. This remark leads to the following problem:

Problem 2.2. *Let U_1, U_2, V_1, V_2 be four real random variables such that:*

(1) *U_1 and U_2 have the same distribution;*
(2) *For all $i, j \in \{1, 2\}$, U_i and V_j are independent.*

Is it true that $U_1 + V_1 = U_2 + V_2$ implies $U_1 = U_2$?

In an attempt to answer this question he proposed the more general problem:

Problem 2.3. *Given random variables Z_1, Z_2, W_1, W_2 such that Z_1 has the same distribution as Z_2 and W_1 has the same distribution as W_2, is it true that $Z_1 + W_1 \leq Z_2 + W_2$ implies that $Z_1 + W_1 = Z_2 + W_2$?*

Taking $Z_1 = U_1V_1$, $Z_2 = U_1V_2$, $W_1 = U_2V_2$ and $W_2 = U_2V_1$, shows that an affirmative answer to Problem 2.3 would imply an affirmative answer to Problem 2.2. It is easy to see that the answer to Problem 2.3 is positive when the variables are integrable since the expectation of the positive variable $Z_1 + W_1 - Z_2 - W_2$ is equal to zero. However, B. Weiss gave in [4] a negative answer to Problem 2.3 by exhibiting a simple counterexample with non-integrable stable random variables for which the inequality holds without equality.

We now show that the answer to Problem 2.2 is positive.

Lemma 2.4. *Let U_1, U_2, V_1, V_2 be four real random variables such that:*

(1) *U_1 and U_2 have the same distribution;*
(2) *For all $i, j \in \{1, 2\}$, U_i and V_j are independent.*

Then $U_1 + V_1 = U_2 + V_2$ implies $U_1 = U_2$.

Proof. Fix $n \geq 1$ and let ϕ_n be the continuous function defined by

$$\phi_n(x) = \begin{cases} n & \text{if } x > n \\ x & \text{if } x \in [-n, n] \\ -n & \text{if } x < -n \end{cases}$$

Set $U_{i,n} = \phi_n(U_i)$ and $V_{j,n} = \phi_n(V_j)$ for $i, j = 1, 2$. These variables verify:

(1) $U_{1,n}$ and $U_{2,n}$ have the same distribution;
(2) For all $i, j \in \{1, 2\}$, $U_{i,n}$ and $V_{j,n}$ are independent.

We apply Furstenberg's argument to these truncated random variables. Let $H_n = (U_{1,n} - U_{2,n})(V_{2,n} - V_{1,n})$. By linearity, independence and equality of distributions, we get

$$\begin{aligned}\mathbb{E}[H_n] &= \mathbb{E}[U_{1,n}]\mathbb{E}[V_{2,n}] - \mathbb{E}[U_{1,n}]\mathbb{E}[V_{1,n}] - \mathbb{E}[U_{2,n}]\mathbb{E}[V_{2,n}] + \mathbb{E}[U_{2,n}]\mathbb{E}[V_{1,n}] \\ &= (\mathbb{E}[U_{1,n}] - \mathbb{E}[U_{2,n}])(\mathbb{E}[V_{2,n}] - \mathbb{E}[V_{1,n}]) \\ &= 0\ .\end{aligned}$$

Furthermore, since ϕ_n is a non-decreasing function and $U_1 - U_2 = V_2 - V_1$ a.s, we see that $H_n \geq 0$ a.s: if $U_1 - U_2 = V_2 - V_1 \geq 0$ then $U_{1,n} - U_{2,n} \geq 0$ and $V_{2,n} - V_{1,n} \geq 0$, hence $H_n \geq 0$; the same argument holds if $U_1 - U_2 = V_2 - V_1 \leq 0$.

Since $H_n \geq 0$ a.s. and $\mathbb{E}[H_n] = 0$, we have $H_n = 0$ a.s. Now, observe that $H_n \to (U_1 - U_2)(V_2 - V_1)$ as $n \to \infty$. Thus,

$$(U_1 - U_2)^2 = (U_1 - U_2)(V_2 - V_1) = 0 \text{ a.s.}$$

and the lemma is proven. □

As already mentioned, a consequence of this lemma is that Furstenberg's filtering theorem holds without any integrability assumption. Thus, we can formulate:

Theorem 2.5. *Let $\{X_n\}$ and $\{Y_n\}$ be two stationary sequences of random variables and suppose that the two sequences are disjoint. Then $\{(X_n, Y_n)\}$ admits a perfect filter.*

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