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Rodolphe Garbit

To cite this version:

Rodolphe Garbit. A note on Furstenberg's filtering problem. 2009. hal-00387100v1

HAL Id: hal-00387100

https://hal.science/hal-00387100v1

Preprint submitted on 24 May 2009 (v1), last revised 12 Jun 2009 (v3)

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A NOTE ON FURSTENBERG'S FILTERING PROBLEM

RODOLPHE GARBIT

Abstract. This short note gives a positive answer to an elementary question in probability theory that arose in Furstenberg's famous article "Disjointness in Ergodic Theory". As a consequence, Furstenberg's filtering theorem holds without any integrability assumption.

1. The context

In his famous article [1], Furstenberg applied his concept of disjointness to a filtering problem : if (X_n) and (Y_n) are two stationary sequences of real-valued random variables, X_n being the emitted signal and Y_n the noise, is it possible to recover X_n from the knowledge of the received signal $X_n + Y_n$? Furstenberg obtained the following result :

Theorem 1.1 ([1], Theorem I.5)**.** *Let (X_n) and (Y_n) be two stationary sequences of integrable random variables, and suppose that the two sequences are absolutely independent (i.e., that the processes they determine are disjoint). Then the σ-fields spanned by $(X_n + Y_n)$ and (X_n, Y_n) are identical, so that (X_n) is a "function" of $(X_n + Y_n)$.*

For the defintion of *disjoint processes*, we refer the reader to [1] or [2].

This Theorem is a consequence of an isomorphism lemma for two dynamical systems, and the following elementary lemma :

Lemma 1.2 ([1], Lemma I.3)**.** *Let U_1, U_2, V_1, V_2 denote four integrable random variables with each of the U_i independent of each V_j. Then $U_1+V_1 = U_2 + V_2$ together with $\mathbb{E}[U_1] = \mathbb{E}[U_2]$ implies $U_1 = U_2$ and $V_1 = V_2$.*

Furstenberg noted that "It would be of interest to know if the integrability stipulation may be omitted, replacing the equality of the expectations of U_1 and U_2 by equality of their distributions", because a positive answer would mean that the integrability assumption in Theorem 1.1 can also be omitted.

As far as we know, this elementary question remained unsolved. It is shown in the next section that the answer is positive.

2. An elementary lemma in probability theory

Lemma 2.1. *Let U_1, U_2, V_1, V_2 be four real random variables such that :*

(1) *U_1 and U_2 have the same distribution;*
(2) *For all $i, j \in \{1, 2\}$, U_i and V_j are independent.*

Date: May 22, 2009.

Then $U_1 + V_1 = U_2 + V_2$ *a.s. implies* $U_1 = U_2$ *a.s.*

Proof. We first note that the equality of the distributions of U_1 and U_2, and the independence of U_1 with V_1 and U_2 with V_2, together with $U_1 + V_1 = U_2 + V_2$ imply that V_1 and V_2 have the same distribution. This is easily seen by the mean of characteristic functions.

Fix $n \geq 1$ and let ϕ_n be the continuous function defined by

$$\phi_n(x) = \begin{cases} n & \text{if } x > n \\ x & \text{if } x \in [-n, n] \\ -n & \text{if } x < -n \end{cases}$$

Set $U_{i,n} = \phi_n(U_i)$ and $V_{j,n} = \phi_n(V_j)$. These variables are bounded and verify :

(1) $U_{1,n}$ and $U_{2,n}$ have the same distribution;
(2) $V_{1,n}$ and $V_{2,n}$ have the same distribution;
(3) For all $i, j \in \{1, 2\}$, $U_{i,n}$ and $V_{j,n}$ are independent;

Let $H_n = (U_{1,n} - U_{2,n})(V_{2,n} - V_{1,n})$. By independence and equality of distributions, we get

$$\begin{aligned} \mathbb{E}[H_n] &= \mathbb{E}[U_{1,n}V_{2,n}] - \mathbb{E}[U_{1,n}V_{1,n}] - \mathbb{E}[U_{2,n}V_{2,n}] + \mathbb{E}[U_{2,n}V_{1,n}] \\ &= \mathbb{E}[U_{1,n}]\mathbb{E}[V_{2,n}] - \mathbb{E}[U_{1,n}]\mathbb{E}[V_{1,n}] - \mathbb{E}[U_{2,n}]\mathbb{E}[V_{2,n}] + \mathbb{E}[U_{2,n}]\mathbb{E}[V_{1,n}] \\ &= 0 \ . \end{aligned}$$

Furthermore, since ϕ_n is a non-decreasing function and $U_1 - U_2 = V_2 - V_1$ a.s, we see that $H_n \geq 0$ a.s: if $U_1 - U_2 = V_2 - V_1 \geq 0$ then $U_{1,n} - U_{2,n} \geq 0$ and $V_{2,n} - V_{1,n} \geq 0$, hence $H_n \geq 0$; the same argument holds if $U_1 - U_2 = V_2 - V_1 \leq 0$.

Since $H_n \geq 0$ a.s. and $\mathbb{E}[H_n] = 0$, we get $H_n = 0$ a.s. Now, observe that $H_n \to (U_1 - U_2)(V_2 - V_1)$ as $n \to \infty$. Thus,

$$(U_1 - U_2)^2 = (U_1 - U_2)(V_2 - V_1) = 0 \text{ a.s.}$$

and the lemma is proven. □

References

[1] Furstenberg, H. (1967). Disjointness in ergodic theory, minimal sets, and a problem in diophantine approximation. *Math. Sys. Theory* **1** : 1–49.

[2] De la Rue, T. (2006). An introduction to joinings in ergodic theory. *Discrete and Continuous Dynamical Systems* **15**(1) : 121–142.

Laboratoire de Mathématiques Jean Leray, UMR CNRS 6629, Université de Nantes, BP 92208, 44322 Nantes Cedex 3, France.

E-mail address: `rodolphe.garbit@univ-nantes.fr`