Cycles and components in geometric graphs: adjacency operator approach

René Schott, Stacey Staples

Cycles and Components in Geometric Graphs: Adjacency Operator Approach

René Schott, *G. Stacey Staples†

Abstract

Nilpotent and idempotent adjacency operator methods are applied to the study of random geometric graphs in a discretized, d-dimensional unit cube $[0, 1]^d$. Cycles are enumerated, sizes of maximal connected components are computed, and closed formulas are obtained for graph circumference and girth. Expected numbers of k-cycles, expected sizes of maximal components, and expected circumference and girth are also computed by considering powers of adjacency operators.

1 Introduction

Consider n points distributed uniformly and independently in the unit cube $[0, 1]^d$. Given a fixed real number $r > 0$, connect two points by an edge if their Euclidean distance is at most r. More specifically, as described in the monograph by Penrose [11], given some probability density on $\mathbb{R}^d$, let $X_1, X_2, \ldots$ be i.i.d. d-dimensional random variables with common density f, and let $\chi_n = \{X_1, \ldots, X_n\}$. The geometric graph $G^{(d)}(\chi_n, ; r)$ is called a random geometric graph. Random geometric graphs are of particular interest as models of wireless networks [7], [8].

Asymptotic properties of random geometric graphs have been studied in a number of papers. For example, fix $d \geq 2$, and let n points be uniformly and independently distributed in $[0, 1]^d$. Letting ρ_n denote the minimum r at which the corresponding geometric graph is k-connected and letting σ_n denote the minimum r at which the graph has minimum degree k, Penrose [12] showed that $\mathbb{P}(\rho_n = \sigma_n) \to 1$ as $n \to \infty$.

In studies of the capacity of wireless networks, Gupta and Kumar have considered connectivity in the case $d = 2$. In particular, they showed that for appropriate constants c_n, if $\pi r(n)^2 = (\log n + c_n)/n$, then as $n \to \infty$, the graph is connected almost surely if $c_n \to \infty$ and is disconnected almost surely if $c_n \to -\infty$ [6].

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In contrast to previous works on random geometric graphs, the goal of the current work is to recover information about a geometric graph's cycles and connected components using methods from algebraic probability theory. In earlier works by the current authors (cf. [15], [16], [17], [21]), adjacency operators associated with finite graphs and Bernoulli random graphs were constructed using commuting elements that square to zero. Cycles were enumerated by considering powers of the resulting nilpotent operators. Similarly, by constructing adjacency operators over algebras generated by commuting idempotents, sizes of maximal connected components can be recovered by considering powers of the operators [19].

In order to apply the adjacency operator methods to random geometric graphs, it is necessary to somehow discretize the space. In the current work, vertices of d-dimensional random geometric graphs will be points in the unit cube $[0,1]^d$ having rational coordinates; i.e. vertices will be elements of the space $(\mathbb{Q}\cap[0,1])^d$. More specifically, the d-dimensional cube is partitioned into equal sub-cubes whose centers serve as the vertices of geometric graphs.

2 Partitions of $[0,1]^d$ and Notational Preliminaries

Consider first the unit d-cube $[0,1]^d$. Dividing the sides into N equal subintervals yields N^d sub-cubes. Center points of the sub d-cubes will serve as vertices of a geometric graph.

The set of vertices V is defined by

$$V = \{\left(\frac{2j_1-1}{2N},\ldots,\frac{2_{j_d}-1}{2N}\right) : 1 \le j_1,\ldots,j_d \le N\}. \tag{1}$$

The partitioned d-cube just described will be said to have *mesh* $1/N^d$.

Given any subset $U \subseteq V$, the topology of the geometric graph on vertex set U is uniquely determined by

$$v_1 \sim v_2 \Leftrightarrow 0 < \|v_1 - v_2\| \le r. \tag{2}$$

Let $\mathbb{P}$ be a probability measure on V such that elements of V are pairwise-independent, and let $\mathcal{F}$ be the σ-algebra of subsets of V. In particular, for $U \subseteq V$,

$$\mathbb{P}(U) = \prod_{v\in U}\mathbb{P}(v)\prod_{w\notin U}(1-\mathbb{P}(w)). \tag{3}$$

The resulting probability space $(V,\mathcal{F},\mathbb{P})$ then induces a probability measure on the collection of geometric graphs.

Let $\mathcal{G}$ denote the collection of geometric graphs on the partitioned d-cube with mesh $1/N^d$. The induced probability measure μ on $\mathcal{G}$ is defined by

$$\mu(G_U) = \prod_{v\in U}\mathbb{P}(v)\prod_{w\notin U}(1-\mathbb{P}(w)). \tag{4}$$

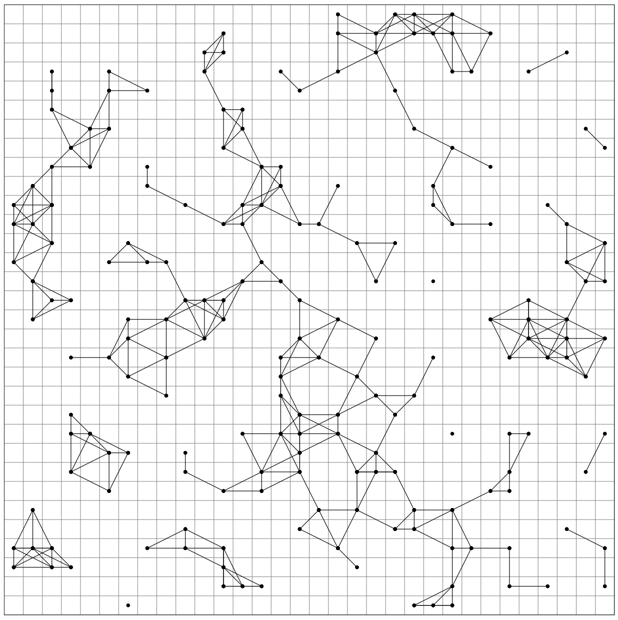


Figure 1: Two-dimensional geometric graph with radius $r = \frac{\sqrt{5}}{32}$, and vertex probability $p = 0.2$.

Example 2.1. In Figure 1, the unit square $[0,1]^2$ is partitioned into 4096 subsquares. Vertices are present with equal probability $p = 0.2$, and adjacency is determined using $r = \sqrt{5}/32$.

Given a collection of commuting null-square elements $\{\zeta_j\}$ in one-to-one correspondence with the vertex set V, let $\mathcal{Z}_V$ denote the associative algebra generated by $\{\zeta_j\}$ and the unit scalar $1 = \zeta_\emptyset$. In particular, $\zeta_i\,\zeta_j = \zeta_j\,\zeta_i$ when $i \neq j$ and ${\zeta_i}^2 = 0$ for each i.

For convenience, generators of $\mathcal{Z}_V$ will be labeled with elements of V. The basis of $\mathcal{Z}_V$ is then in one-to-one correspondence with the power set of V. For any subset $U \subseteq V$, define the notation $\zeta_U = \prod_{v\in U} \zeta_v$. An arbitrary element $z \in \mathcal{Z}_V$ then has canonical expansion of the form

$$z = \sum_{U\subseteq V} \alpha_U\,\zeta_U, \tag{5}$$

where $\alpha_U \in \mathbb{R}$.

Remark 2.2. The algebra $\mathcal{Z}_V$ is referred to as a *zeon* algebra in Feinsilver [4]. It is the algebra referred to as ${\mathcal{C}\ell_{|V|}}^{\mathrm{nil}}$ in Staples [22], and it is the algebra referred to as $\mathcal{N}_V$ in Schott and Staples [16].

Assuming a fixed enumeration of elements of V, a probability mapping φ is induced on the generators of $\mathcal{Z}_V$ by

$$\varphi(\zeta_{v_j}) = \mu(v_j). \tag{6}$$

Denote by $\{e_i\}$ the collection of orthonormal basis vectors of $\mathbb{R}^{|V|2^{|V|}}$. The Dirac notation $\langle e_i|$ will represent a row vector, while the conjugate transpose $|e_i\rangle$ represents a column vector. In this way,

$$\langle e_i|e_j\rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

Moreover, $|e_i\rangle\langle e_i|$ is the rank-one orthogonal projector onto the linear subspace $\mathrm{span}(e_i)$.

Fix an enumeration $f : 2^V \to \{1, \ldots, 2^{|V|}\}$ of the power set 2^V. Notation of the form $|e_U\rangle$ and $\langle e_U|$ should be understood to use the fixed enumeration of 2^V for subsets $U \subseteq V$.

Define an enumeration of $2^V \times V$ by

$$(U, \{v_j\}) \mapsto (f(U) - 1)|V| + j. \tag{8}$$

The enumeration of $2^V \times V$ is then used as a double-index for the unit basis vectors of $\mathbb{R}^{|V|2^{|V|}}$. Notation of the form $|e_{U,v_i}\rangle$ and $\langle e_{U,v_i}|$ should be viewed in this context.

For each subset of vertices $U \subseteq V$, denote the nilpotent adjacency operator of the corresponding subgraph G_U by $\Lambda_r^{(U)}$. In particular,

$$\Lambda_r^{(U)} = \sum_{\substack{v_i, v_j \in U \\ 0 < \|v_1 - v_2\| < Nr}} \zeta_{\{v_2\}} |e_{U,v_1}\rangle \langle e_{U,v_2}|. \tag{9}$$

Use the fixed enumeration of 2^V to define the *second quantization nilpotent adjacency operator* by

$$\Lambda_r = \sum_{U \in 2^V} \left(\Lambda_r^{(U)} \otimes |e_U\rangle \langle e_U| \right). \tag{10}$$

By construction, Λ_r is an operator on the $N^d 2^{N^d}$-dimensional product space ${\mathcal{Z}_V}^{|V|2^{|V|}}$. In particular, Λ_r is defined by

$$\langle e_{U,v_i}| \Lambda_r |e_{U,v_j}\rangle = \begin{cases} \zeta_{v_j} & \text{if } v_i \sim v_j \text{ in } G_U, \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

Recalling the canonical expansion $x = \sum_{U \subseteq V} x_U \, \zeta_U \in \mathcal{Z}_V$, let $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ and define the function $\psi : \mathcal{Z}_{\mathrm{V}} \to \mathbb{N}_0$ by

$$\psi(x) = \sum_{U \subseteq V} \langle x, \zeta_U \rangle = \sum_{U \subseteq V} x_u. \tag{12}$$

In other words, $\psi(x)$ is the sum of the scalar coefficients in the canonical expansion of x.

For convenience, define the notation $\vec{e}_V = e_1 + e_2 + \cdots + e_{|V|!}$, and for any $U \subseteq V$, define the U-*trace* of Λ_r by

$$\mathrm{tr}_U(\Lambda_r) = \sum_{j=1}^{|V|} \langle e_{U,j} | \, \Lambda_r \, | e_{U,j} \rangle \,. \tag{13}$$

Given a collection of commuting idempotent elements $\{\gamma_j\}$ in one-to-one correspondence with the vertex set V, let $\mathcal{I}_V$ denote the associative algebra generated by $\{\gamma_j\}$ and the unit scalar $1 = \gamma_\emptyset$. In particular, $\gamma_i \, \gamma_j = \gamma_j \, \gamma_i$ when $i \neq j$, and ${\gamma_i}^2 = \gamma_i$ for each i.

For convenience, generators of $\mathcal{I}_V$ will be labeled with elements of V. The basis of $\mathcal{I}_V$ is then in one-to-one correspondence with the power set of V. For any subset $U \subseteq V$, define the notation $\gamma_U = \prod_{v \in U} \gamma_v$. An arbitrary element $z \in \mathcal{I}_V$ then has canonical expansion of the form

$$z = \sum_{U \subseteq V} \alpha_U \, \gamma_U, \tag{14}$$

where $\alpha_U \in \mathbb{R}$.

Define the *degree* mapping $\delta : \mathcal{I}_V \to \mathbb{N}_0$ by

$$\delta\left(\sum_{U \in 2^V} \alpha_U \, \gamma_U \right) = \max_{\alpha_U \neq 0} \{|U|\}. \tag{15}$$

In other word, $\delta(z)$ is the size of the maximal multi-index in the canonical expansion of $z \in \mathcal{I}_V$.

Example 2.3. For example, given $V = \{v_1, \ldots, v_5\}$, let

$$u = \gamma_{\{v_1,v_4\}} + 2\gamma_{\{v_1,v_2,v_5\}} + 5\gamma_{\{v_2,v_3,v_4,v_5\}} \in \mathcal{I}_V.$$

Then, $\delta(u) = 4$.

For each subset U of the collection of vertices V, denote the corresponding idempotent adjacency operator by

$$\Xi_r^{(U)} = \sum_{\substack{v_1, v_2 \in U \\ \|v_1 - v_2\| \leq r}} \gamma_{v_2} \, |e_{U,v_1}\rangle \, \langle e_{U,v_2}| \,. \tag{16}$$

Remark 2.4. By using the inequality $\|v_1 - v_2\| \leq r$ in place of $0 < \|v_1 - v_2\| \leq r$, "loops" are placed at each vertex of the graph. This allows every pair of vertices in a given component to be joined by a closed walk of length $2|V| - 1$.

The *second quantization idempotent adjacency operator* is defined by

$$\Xi_r = \sum_{U\in 2^V} \left(\Xi_r^{(U)} \otimes |e_U\rangle \langle e_U|\right). \tag{17}$$

By construction, Ξ_r is an operator on the $N^d 2^{N^d}$-dimensional product space ${\mathcal{I}_V}^{|V|2^{|V|}}$. In particular, Ξ_r is defined by

$$\langle e_{U,v_i}|\,\Xi_r\,|e_{U,v_j}\rangle = \begin{cases}\gamma_{v_j} & \text{if } v_i \sim v_j \text{ in } G_U,\\ 0 & \text{otherwise.}\end{cases} \tag{18}$$

3 Main Results

Theorem 3.1. *Let $k \geq 3$ be fixed. Let Λ_r denote the second quantization nilpotent adjacency operator. Let X_k denote the number of k-cycles in a random geometric graph in the partitioned d-cube with mesh $1/N^d$. Then,*

$$\mathbb{E}(X_k) = \frac{1}{2k}\sum_{U\in 2^V} \mu(G_U)\,\psi\left(\mathrm{tr}_U({\Lambda_r}^k)\right). \tag{19}$$

Proof. Fix vertex set $U \subseteq V$ and integer $k \geq 3$, and consider the nilpotent adjacency operator $\Lambda_r^{(U)}$. A well-known result in graph theory states that the diagonal elements of the k^{th} power of a graph's adjacency matrix correspond to the graph's closed k-walks. Similarly, a diagonal element of the k^{th} power of a graph's nilpotent adjacency operator is a sum of products of k commuting null-square generators indexed by subsets of vertices. Because each generator ζ_i squares to zero, a straightforward inductive argument shows that

$$\left\langle e_j|{\Lambda_r^{(U)}}^k|e_j\right\rangle = \sum_{\substack{k\text{-cycles } W\subseteq U\\ \text{based at } v_j}} \zeta_W \tag{20}$$

for any integer j satisfying $1 \leq j \leq |V|$. Applying the mapping ψ to this result reveals the number of distinct k-cycles based at vertex v_j in G_U, if $v_j \in U$. Moreover, each cycle appears with multiplicity 2 due to the two possible orientations of the cycle.

It follows that applying ψ to the trace of $\Lambda_r^{(U)}$ reveals all of the k-cycles contained in G_U. Each k-cycle now appears with multiplicity $2k$ due to the k choices of basepoint for each cycle. By construction, applying the U-trace of the second quantization nilpotent adjacency operator is equivalent to the trace of $\Lambda_r^{(U)}$.

Finally, summing the products of graph probabilities and numbers of k-cycles over all geometric graphs G_U yields the expected value of X_k. □

Example 3.2. Consider the partition of $[0,1]^2$ of mesh $1/9$. Assume each vertex has equal probability $p = 0.2$ of existence in the random geometric graph

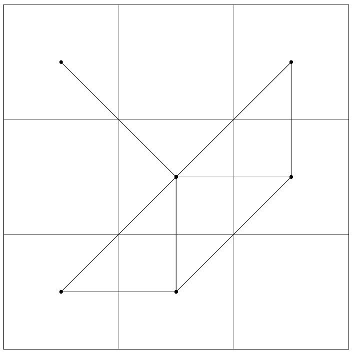


Figure 2: A geometric graph plotted with Mathematica in $[0,1]^2$ partitioned with mesh 1/9. Adjacency determined by $r = \sqrt{2}/3$.

```
In[18]:= (* Build all possible adjacency matrices in [0,1]^2 with mesh 1/9 and r=√2 /3 *)Λ0 = {{0}};
         For[m = 1, m ≤ 511, m++,
          Λm = BuildMatrix[m, 3, √2 /3];
          If[Mod[m, 50] == 0, Print[m],]]

In[43]:= (* Calculate expected number of 4-cycles in random geometric graph
            on partition of mesh 1/9 in [0,1]^2.  Assume equiprobable vertices p=.2,
          and adjacency determined by radius √2 /3. *)
          Sum[μ[gnum, 3, .2]/8 Expand[Simplify[Tr[MatPwr[Λgnum, 4]]]] /. {ξ_ → 1}, {gnum, 0, 511}]

Out[43]= 0.0464
```

Figure 3: Mathematica computation of expected number of 4-cycles.

G_U. In this case, for any $U \subseteq V$, $\mu(G_U) = (0.2)^{|U|}(0.8)^{(9-|U|)}$. Further, let the topology of G_U be determined by $r = \sqrt{2}/3$. An example of one such graph appears in Figure 2.

The expected number of 4-cycles in a random geometric graph G_U is then given by

$$\mathbb{E}(X_4) = \frac{1}{8} \sum_{U \subseteq V} (0.2)^{|U|}(0.8)^{(9-|U|)} \psi \left(\mathrm{tr}_U \left(\Lambda_{\frac{\sqrt{2}}{3}}{}^4 \right) \right).$$

In Figure 3, Mathematica computations (see Appendix for more details of Mathematica code) reveal $\mathbb{E}(X_4) = 0.0464$.

Theorem 3.3. *The probability that a random geometric graph contains exactly ℓ cycles of length $k \geq 3$ is given by*

$$\mathbb{P}(X_k = \ell) = \sum_{U \in 2^V} \left\langle e_{\psi\left(\mathrm{tr}_U(\Lambda_r{}^k)\right)}, e_{2k\ell} \right\rangle \mu(G_U). \tag{21}$$

Proof. As in the proof of Theorem 3.1, $\psi\left(\mathrm{tr}_U(\Lambda_r{}^k)\right)$ is a positive integer representing the number of k-cycles in the geometric graph G_U. Due to multiple choices of orientation and basepoint, the correction factor $2k$ must be considered.

By definition of the inner-product on $\mathbb{R}^{|V|2^{|V|}}$, $\langle e_i, e_j\rangle$ takes values 1 if $i = j$ and 0 if $i \neq j$. Hence,

$$\left\langle e_{\psi\left(\mathrm{tr}_U(\Lambda_r{}^k)\right)}, e_{2k\ell}\right\rangle = \begin{cases} 1 & \text{if } G_U \text{ contains exactly } \ell \ k\text{-cycles,} \\ 0 & \text{otherwise.} \end{cases} \tag{22}$$

Summing the probabilities over all geometric graphs G_U containing ℓ k-cycles yields the required result. □

Example 3.4. Recall the random geometric graphs of Example 3.2. Mathematica calculations reveal the probability that such a random graph contains exactly ℓ 5-cycles in Figure 4.

Definition 3.5. The *circumference* of a graph G is the length of the longest cycle contained in G. Circumference will be denoted by $\mathrm{Circ}(G)$.

Theorem 3.6. *Fix integer ℓ such that $3 \leq \ell \leq |V|$. Then, the geometric graph G_U has circumference ℓ if and only if*

$$\left\langle e_{\psi\left(\mathrm{tr}_U(\Lambda_r{}^\ell)\right)}, \vec{e}_V\right\rangle \prod_{k=\ell+1}^{|V|} \left\langle e_{\psi\left(\mathrm{tr}_U(\Lambda_r{}^k)\right)}, e_0\right\rangle = 1. \tag{23}$$

Proof. Recall that $\vec{e}_V = e_1 + e_2 + \cdots + e_{|V|!}$. It follows that

$$\left\langle e_{\psi\left(\mathrm{tr}_U(\Lambda_r{}^\ell)\right)}, \vec{e}_V\right\rangle = \begin{cases} 1 & \text{if } G_U \text{ contains 1 or more } \ell\text{-cycles,} \\ 0 & \text{otherwise.} \end{cases} \tag{24}$$

Moreover, $\prod_{k=\ell+1}^{|V|} \left\langle e_{\psi\left(\mathrm{tr}_U(\Lambda_r{}^k)\right)}, e_0\right\rangle = 1$ if and only if the number of k-cycles in G_U is zero for all $\ell < k \leq |V|$. In other words, ℓ is the length of the longest cycle in G_U. □

Proposition 3.7. *The expected circumference of a random geometric graph G_U is given by*

$$\mathbb{E}(\mathrm{Circ}(G_U)) = \sum_{\ell=3}^{|V|} \ell \left(\sum_{U \in 2^V} \left\langle e_{\psi\left(\mathrm{tr}_U(\Lambda_r{}^\ell)\right)}, \vec{e}_V\right\rangle \prod_{k=\ell+1}^{|V|} \left\langle e_{\psi\left(\mathrm{tr}_U(\Lambda_r{}^k)\right)}, e_0\right\rangle \mu(G_U) \right). \tag{25}$$

```
In[16]:= (* Calculate probability that G_U contains exactly ell 5-cycles *)
s_0 = μ[0, 3, .2];
For[gnum = 1, gnum ≤ 511, gnum++,
 Λ = BuildMatrix[gnum, 3, √2 / 3];
 t = 1/10 * Expand[Simplify[Tr[MatPwr[Λ, 5]]]] /. {ξ_ → 1};
 s_t = s_t + μ[gnum, 3, .2];
]

In[136]:= For[ell = 0, ell ≤ Binomial[9, 5] / 10, ell++,
  Print["The probability that G_U contains ", ell, " 5-cycles is ", s_ell, "."]]

The probability that G_U contains 0 5-cycles is 0.993772.
The probability that G_U contains 1 5-cycles is 0.00222822.
The probability that G_U contains 2 5-cycles is 0.00268698.
The probability that G_U contains 3 5-cycles is 0.000032768.
The probability that G_U contains 4 5-cycles is 0.000395264.
The probability that G_U contains 5 5-cycles is 0.000425984.
The probability that G_U contains 6 5-cycles is 0.
The probability that G_U contains 7 5-cycles is 0.000065536.
The probability that G_U contains 8 5-cycles is 0.000131072.
The probability that G_U contains 9 5-cycles is 0.
The probability that G_U contains 10 5-cycles is 0.
The probability that G_U contains 11 5-cycles is 0.000131072.
The probability that G_U contains 12 5-cycles is 0.000065536.
```

Figure 4: Probability that a random graph in $[0,1]^2$ contains ℓ 5-cycles.

Proof. This is an immediate corollary of Theorem 3.6 by noting that the probability that a random geometric graph has circumference ℓ is given by

$$\mathbb{P}\left(\mathrm{Circ}(G_U)=\ell\right)=\sum_{U\in 2^V}\left\langle e_{\psi\left(\mathrm{tr}_U(\Lambda_r{}^\ell)\right)},\vec{e}_V\right\rangle\prod_{k=\ell+1}^{|V|}\left\langle e_{\psi\left(\mathrm{tr}_U(\Lambda_r{}^k)\right)},e_0\right\rangle\mu(G_U). \tag{26}$$

□

Definition 3.8. The *girth* of a graph G is the length of the shortest cycle contained in G. Girth will be denoted by $\mathrm{Girth}(G)$.

Theorem 3.9. *Fix integer ℓ such that $3\le\ell\le|V|$. Then, the geometric graph G_U has girth ℓ if and only if*

$$\left\langle e_{\psi\left(\mathrm{tr}_U(\Lambda_r{}^\ell)\right)},\vec{e}_V\right\rangle\prod_{k=3}^{\ell-1}\left\langle e_{\psi\left(\mathrm{tr}_U(\Lambda_r{}^k)\right)},e_0\right\rangle=1. \tag{27}$$

Proof. The proof is virtually identical to that of Theorem 3.6. □

Proposition 3.10. *The expected girth of a random geometric graph G_U is given by*

$$\mathbb{E}(\mathrm{Girth}(G_U))=\sum_{\ell=3}^{|V|}\ell\left(\sum_{U\in 2^V}\left\langle e_{\psi\left(\mathrm{tr}_U(\Lambda_r{}^\ell)\right)},\vec{e}_V\right\rangle\prod_{k=3}^{\ell-1}\left\langle e_{\psi\left(\mathrm{tr}_U(\Lambda_r{}^k)\right)},e_0\right\rangle\mu(G_U)\right). \tag{28}$$

Proof. This is a corollary of Theorem 3.9 by noting that the probability that a random geometric graph G_U has girth ℓ is given by

$$\mathbb{P}\left(\mathrm{Girth}(G_U)=\ell\right)=\sum_{U\in 2^V}\left\langle e_{\psi\left(\mathrm{tr}_U(\Lambda_r{}^\ell)\right)},\vec{e}_V\right\rangle\prod_{k=3}^{\ell-1}\left\langle e_{\psi\left(\mathrm{tr}_U(\Lambda_r{}^k)\right)},e_0\right\rangle\mu(G_U). \tag{29}$$

□

Example 3.11. The expected circumference and expected girth of a random graph in the partitioned square $[0,1]^2$ with mesh $1/9$ are computed with Mathematica in Figure 5. As in Example 3.2, $p=0.2$, and $r=\sqrt{2}/3$.

Theorem 3.12. *The size of the largest component $C_{\max}$ in G_U is given by*

$$|C_{\max}|=\delta\left(\mathrm{tr}_U\left({\Xi_r}^{2|V|-1}\right)\right). \tag{30}$$

Proof. Similar to the proof of Theorem 3.1, diagonal entries of the k^{th} power of the idempotent adjacency operator $\Xi_r^{(U)}$ are sums of products of k commuting idempotents γ_j corresponding to closed k-walks in G_U. Because ${\gamma_j}^2=\gamma_j$ for each j, the maximum degree of such a product is k.

```
(* Compute expected circumference *)
For[gnum = 1, gnum ≤ 511, gnum++,
  T_gnum =
    Table[If[(Expand[Simplify[Tr[MatPwr[Λ_gnum, ell]]]] /. {ξ_ → 1}) ≠ 0, ell, 0], {ell, 3, 9}];];
Print["Expected circumference = ", Sum[Max[T_gnum] * μ[gnum, 3, .2], {gnum, 1, 511}]];
Print["Expected girth = ",
  Sum[If[Norm[T_gnum] ≠ 0, (Min[DeleteCases[T_gnum, 0]] * μ[gnum, 3, .2]), 0], {gnum, 1, 511}]];

Expected circumference = 0.288265

Expected girth = 0.260005
```

Figure 5: Mathematica computation of expected circumference and girth.

Because G_U is undirected and contains at most $|V|$ vertices, the maximum length of any simple path joining two vertices u, v is $|V|$. A closed walk from $u \to v \to u$ has length at most $2|V| - 1$. By including loops at each vertex in the definition of Ξ_r, a closed walk of length $2|V| - 1$ exists from u to each vertex v in the same component. Hence, considering diagonal elements of the $(2|V| - 1)^{\text{th}}$ power of $\Xi_r^{(U)}$ is sufficient to determine the size of the connected component C containing vertex v_j:

$$|C| = \delta\left(\langle e_j | \, {\Xi_r}^{(U)^{2|V|-1}} \, | e_j \rangle\right). \tag{31}$$

The size of the maximal component in G_U is the maximum taken over all vertices in G_U, given by

$$|C_{\max}| = \delta\left(\mathrm{tr}_U\left({\Xi_r}^{(U)^{2|V|-1}}\right)\right). \tag{32}$$

Observing the equivalence

$$\langle e_{U,j} | \Xi_r | e_{U,j} \rangle = \left\langle e_j | \Xi_r^{(U)} | e_j \right\rangle \tag{33}$$

completes the proof. □

Example 3.13. The size of the maximal component in a geometric graph is computed with Mathematica in Figure 6.

Corollary 3.14. *The graph G_U on vertices $U \in 2^V$ is* connected *if and only if for every j such that $v_j \in U$,*

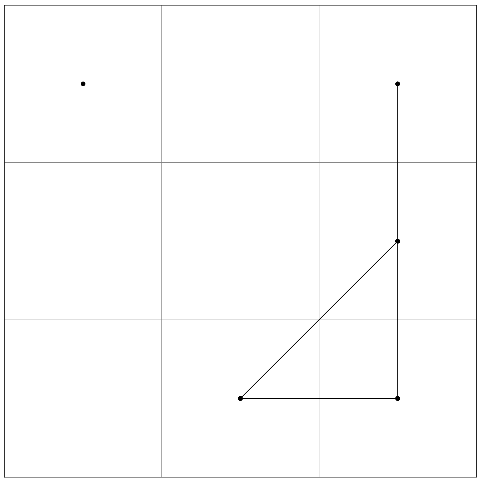
$$\delta\left(\langle e_{U,j} | \, {\Xi_r}^{2|V|-1} \, | e_{U,j} \rangle\right) = |U|. \tag{34}$$

Theorem 3.15. *The expected size of a maximal component in a random geometric graph is given by*

$$\mathbb{E}(|C_{\max}|) = \sum_{U \in 2^V} \mu(G_U) \delta\left(\mathrm{tr}_U\left({\Xi_r}^{2|V|-1}\right)\right). \tag{35}$$

```
In[84]:= (* Get size of component up to n *)
         δ[S_, n_] := Max[Table[If[(DegreeKPart[S, k] /. {ϒ_ → 1}) ≠ 0, k, 0], {k, 1, n}]]

In[79]:= GeoPlot2D[1679, 3, √2 / 3]
```

Out[79]=



```
In[81]:= Ξ = BuildIdemMatrix[1679, 3, √2 / 3]

In[86]:= Print["Size of largest component: ", δ[Expand[Simplify[Tr[MatPwr[Ξ, 9]]]], 9]]

Size of largest component: 4
```

Figure 6: Mathematica computation of maximal component size.

Proof. This is a corollary of Theorem 3.12 □

Theorem 3.16. *The probability that a random geometric graph is connected and contains no k-cycles for all $k \leq k_0$ is given by*

$$\sum_{U \in 2^V} \mu(G_U) \left\langle e_{\delta\left(\mathrm{tr}_U(\Xi_r{}^{2|V|-1})\right)}, e_{|U|} \right\rangle \left\langle e_{\sum_{k=3}^{k_0} \psi\left(\mathrm{tr}_U(\Lambda_r{}^k)\right)}, e_0 \right\rangle. \tag{36}$$

Proof. Let $U \subseteq V$ be fixed. By Corollary 3.14, the geometric graph G_U is connected if and only if for any j such that $v_j \in U$,

$$\delta\left(\langle e_{U,j} | \, \Xi_r{}^{2|V|-1} \, | e_{U,j} \rangle\right) = |U|. \tag{37}$$

Moreover, G_U contains no k-cycles if and only if

$$\psi\left(\mathrm{tr}_U(\Lambda_r{}^k)\right) = 0. \tag{38}$$

Summing over k from 3 to k_0 completes the proof. □

The following corollary deals with spanning trees, i.e. cycle-free connected graphs.

Corollary 3.17. *The probability that the geometric graph G_U is a spanning tree is given by*

$$\sum_{U \in 2^V} \mu(G_U) \left\langle e_{\delta\left(\mathrm{tr}_U(\Xi_r{}^{2|V|-1})\right)}, e_{|U|} \right\rangle \left\langle e_{\sum_{k=3}^{|U|} \psi\left(\mathrm{tr}_U(\Lambda_r{}^k)\right)}, e_0 \right\rangle. \tag{39}$$

One final goal is to enumerate the connected components in a geometric graph G_U. To this end, define the mapping $\eta : \mathcal{I}_V \to \mathcal{I}_V$ by

$$\eta\left(\sum_{U \in V} \alpha_U \, \gamma_U\right) = \sum_{|U| = \delta(u)} \alpha_U \, \gamma_U. \tag{40}$$

Define the function $\rho : \mathcal{I}_V \to \mathbb{N}_0$ by

$$\rho(u) = \min_{\mathcal{U} \ni u} \{\dim(\mathcal{U})\}. \tag{41}$$

In other words, $\rho(u)$ is the dimension of the smallest linear subspace of $\mathcal{I}_{\mathrm{V}}$ containing u. Now in a manner similar to the enumeration of cycles, it is possible to enumerate components.

Theorem 3.18. *Let Ξ_r denote the second quantization nilpotent adjacency operator. Let X denote the number of connected components in a random geometric graph in the partitioned d-cube with mesh $1/N^d$. Then,*

$$\mathbb{E}(X) = \sum_{U \in 2^V} \mu(G_U) \, \rho\left(\sum_{v_j \in U} \eta\left(\langle e_{U,j} | \, \Xi_r{}^{2|V|-1} \, | e_{U,j} \rangle\right)\right). \tag{42}$$

Proof. For vertex set $U \subseteq V$, letting C_j denote the collection of vertices in the connected component containing vertex v_j in geometric graph G_U, one finds

$$\eta\left(\langle e_{U,j} | \, \Xi_r{}^{2|V|-1} \, | e_{U,j} \rangle\right) = \alpha_j \, \gamma_{C_j}, \tag{43}$$

for some integer α_j. In other words, η "sieves out" the blades of $\mathcal{I}_V$ corresponding to the connected component C_j. While the multi-index is unique, the vertices can occur in many permutations when computing powers of Ξ_r, hence the constant α_j.

Accumulating all such terms by summing over $1 \leq j \leq |V|$ yields an element of $\mathcal{I}_V$ corresponding to the collection of connected components in G_U. The number of distinct multi-indices is the number of components, and this corresponds to the dimension of the linear space spanned by the blades. Hence, application of ρ to the sum completes the proof.

□

Theorem 3.19. *Let X denote the number of components in a random geometric graph G_U. The probability that a random geometric graph contains exactly ℓ components is given by*

$$\mathbb{P}(X = \ell) = \sum_{U \in 2^V} \left\langle e_{\rho \ \sum_{v_j \in U} \eta\left(\langle e_{U,j} | \Xi_r{}^{2|V|-1} | e_{U,j} \rangle\right)}, e_\ell \right\rangle \mu(G_U). \tag{44}$$

Proof. As in the proof of Theorem 3.18, $\rho\left(\sum_{v_j \in U} \eta\left(\langle e_{U,j} | \, \Xi_r{}^{2|V|-1} \, | e_{U,j} \rangle\right)\right)$ is a nonnegative integer representing the number of components in the geometric graph G_U.

By definition of the inner-product on $\mathbb{R}^{|V|2^{|V|}}$, $\langle e_i, e_j \rangle$ takes values 1 if $i = j$ and 0 if $i \neq j$. Hence,

$$\left\langle e_{\rho \ \sum_{v_j \in U} \eta\left(\langle e_{U,j} | \Xi_r{}^{2|V|-1} | e_{U,j} \rangle\right)}, e_\ell \right\rangle = \begin{cases} 1 & G_U \text{ contains exactly } \ell \text{ components}, \\ 0 & \text{otherwise.} \end{cases} \tag{45}$$

Summing the probabilities over all geometric graphs G_U containing ℓ components yields the required result. □

Example 3.20. In Figure 7, Mathematica is used to enumerate the connected components in the randomly-generated geometric graph of Figure 6.

4 Time Complexity and Clifford Algebras

For geometric graphs in the partitioned d-cube of mesh $1/N^d$, computing the k^{th} power of operators $\Lambda_r^{(U)}$ and $\Xi_r^{(U)}$ is of time complexity $O(|V| \log k)$ in terms of algebra products computed [18]. In this context, enumerating cycles, computing the size of a maximal component, and computing the circumference and girth of a fixed geometric graph G_U is of polynomial time complexity.

```
In[73]:= (* Return maximal degree terms up to degree n *)
         η[S_, n_] := DegreeKPart[S, δ[S, n]]

In[79]:= (* Generate table of components *)
         m = Simplify[MatPwr[Ξ, 9]];
         comptab = Table[η[Expand[Simplify[m[[j]][[j]]]], 5], {j, 1, 5}];
         s = Expand[Sum[comptab[[j]], {j, 1, Length[comptab]}]];
         Print["The number of components is ρ(", s, ") = ", Length[s], "."]

The number of components is ρ(ϒ{1} + 33 080. ϒ{2,3,4,5}) = 2.
```

Figure 7: Counting the components in a geometric graph.

While this in not a natural measure of computational complexity in classical computing, recent progress has been made toward a geometric computing architecture based on Clifford algebras (cf. [5], [13]). This is especially relevant to the current work because the algebras $\mathcal{N}_V$ and $\mathcal{I}_V$ can both be constructed within Clifford algebras of appropriate signature. Clifford algebras, also known as *geometric algebras*, have recently been applied to computer vision [9], [14] and automated geometric theorem proving [10].

Clifford algebras are also commonly applied to quantum physics. The Clifford algebra $\mathcal{C}\ell_{|V|,|V|}$ in which $\mathcal{Z}_V$ can be constructed is itself isomorphic to the $|V|$-particle fermion algebra familiar to quantum probabilists [2], [3], [23]. In fact, the adjacency operators themselves can be considered quantum random variables [20]. The relationship between geometric computing and quantum computing has also been the subject of recent work by Aerts and Czachor [1].

References

[1] D. Aerts, M. Czachor, Cartoon computation: quantum-like computing without quantum mechanics, J. Phys. A: Math. Theor., **40** F259-F263, (2007).

[2] D. Applebaum, Fermion stochastic calculus in Dirac-Fock space, *J. Phys. A.*, **28** 257-270, (1995).

[3] D. Applebaum, R. Hudson, Fermion Itô's formula and stochastic evolutions, *Commun. Math. Phys.*, **96** 473-96, (1984).

[4] P. Feinsilver, Zeon algebra, Fock space, and Markov chains, arXiv:0711.1680v1 [math-ph].

[5] A. Gentile, S. Segreto, F. Sorbello, G. Vassallo, S. Vitabile, V. Vullo, CliffoSor, an innovative FPGA-based architecture for geometric algebra, Proceedings of the International Conference on Engineering of Reconfigurable Systems and Algorithms, (ERSA 2005) 211-217, (2005).

[6] P. Gupta, P.R. Kumar, Critical power for asymptotic connectivity in wireless networks, *Stochastic analysis, control, optimization and applications: A volume in honor of W.H. Fleming*, Birkhaüser, Boston, 1998.

[7] P. Gupta, P.R. Kumar, The capacity of wireless networks, *IEEE Transactions on Information Theory*, **IT-46** 388-404, (2000).

[8] P. Gupta, P.R.Kumar, Internets in the sky: The capacity of three dimensional wireless networks, *Communications in Information and Systems*, **1** 33-50, (2001).

[9] J. Lasenby, W. J. Fitzgerald, C. J. L. Doran, A.N. Lasenby, New geometric methods for computer vision, *Int. J. Comp. Vision*, **36** 191-213, (1998).

[10] H. Li, *Clifford algebra approaches to automated geometry theorem proving*, Mathematics Mechanization and Applications, X.-S. Gao and D. Wang (eds.), Academic Press, London, 2000, pp. 205-230.

[11] M. Penrose, *Random geometric graphs*, Oxford Studies in Probability, vol. 5, Oxford University Press, 2003.

[12] M. Penrose, On k-connectivity for a geometric random graph, *Random Structures and Algorithms*, **15** 145-164, (1999).

[13] C. Perwass, C. Gebken, G. Sommer, Implementation of a Clifford algebra coprocessor design on a field-programmable gate array, Clifford Algebras - Applications to Mathematics, Physics, and Engineering, Series: Progress in Mathematical Physics, **34**, R. Ablamowicz (Ed.), 2004.

[14] C. Perwass, C. Gebken, G. Sommer, Estimation of geometric entities and operators from uncertain data, *Pattern Recognition*, Lecture Notes in Computer Science 3663, 459-467, (2005).

[15] R. Schott, G.S. Staples, Random walks in Clifford algebras of arbitrary signature as walks on directed hypercubes, *Markov Processes and Related Fields*, to appear.

[16] R. Schott, G.S. Staples, Nilpotent adjacency matrices and random graphs, *Ars Combinatorica*, To appear.

[17] G.S. Staples, Clifford-algebraic random walks on the hypercube, *Advances in Applied Clifford Algebras*, **15** 213-232, (2005).

[18] R. Schott, G.S. Staples, Reductions in computational complexity using Clifford algebras, Preprint, `http://www.siue.edu/~sstaple/NPtoP.pdf`, (2006).

[19] R. Schott, G.S. Staples, Connected components and evolution of random graphs: an algebraic approach, *Prépublications de l'Institut Elie Cartan*, No. 2007/27, (2007).

[20] R. Schott, G.S. Staples, Nilpotent adjacency matrices, random graphs, and quantum random variables, *Prépublications de l'Institut Elie Cartan*, No. 2007/8, (2007).

[21] G.S. Staples, Norms and generating functions in Clifford algebras, *Advances in Applied Clifford Algebras*, to appear.

[22] G.S. Staples, Graph-theoretic approach to stochastic integrals with Clifford algebras, *J. Theor. Prob.*, **20** 257-274, (2007).

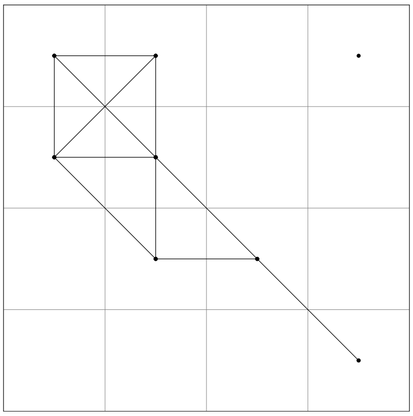
[23] W. von Waldenfels, An algebraic central limit theorem in the anti-commuting case, *Z. Wahrsch. verw. Gebiete*, **42** 135-140, (1978).

Appendix 1: Mathematica Procedures

```
(* Overload Times operator to handle null-squares ζ_{j} and idempotents γ_{j} *)
Unprotect[Times]; ClearAttributes[Times, Orderless]; ζ_{a_} ζ_{b_} := If[Length[a ⋂ b] > 0, 0, ζ_{a⋃b}];
γ_{a_} γ_{b_} := γ_{a⋃b};
Protect[Times];
Unprotect[Power];
(x_ /; ! FreeQ[x, ζ_{a_}])^n_Integer := Module[{y, f}, y = Expand[x]; Switch[EvenQ[n],
   True, If[n == 0, Return[1], Composition[Expand][Distribute[f[y, y]] /. f → Times]^(n/2)],
   False, If[n == 1, Return[x], Composition[Expand][Distribute[f[y, y]] /. f → Times]^((n-1)/2) x]]];
(x_ /; ! FreeQ[x, γ_{a_}])^n_Integer := Module[{y, f}, y = Expand[x]; Switch[EvenQ[n],
   True, If[n == 0, Return[1], Composition[Expand][Distribute[f[y, y]] /. f → Times]^(n/2)],
   False, If[n == 1, Return[x], Composition[Expand][Distribute[f[y, y]] /. f → Times]^((n-1)/2) x]]];
Protect[Power]; Unprotect[Expand];
Expand[x_ /; ! FreeQ[x, ζ__]] := DeleteCases[Distribute[x, Plus, Times], 0.` ζ__];
Expand[x_ /; ! FreeQ[x, γ__]] := DeleteCases[Distribute[x, Plus, Times], 0.` γ__]; Protect[Expand];

(* Return terms of particular degree *)
DegreeKPart[x_, k_Integer] :=
  DeleteCases[DeleteCases[If[k == 0, Expand[x] /. {ζ__ → 0, γ__ → 0},
     Expand[x - (x /. {ζ_{Table[_,{k}]} → 0, γ_{Table[_,{k}]} → 0})]], 0.` ], 0.`];

(* Matrix powers *)
MatPwr[A_, m_] := If[m == 1, Return[A], Return[Expand[A.Simplify[MatPwr[A, m - 1]]]]];

(* Plot G_U in [0,1]^2 (mesh 1/n^2) where U is a collection of vertices  *)
PlotGU[U_, n_, r_] := (

  edgs = {};
  For[k = 1, k ≤ Length[U], k ++,
   edgs = Append[edgs, {U[[k]]}];
   For[j = k + 1, j ≤ Length[U], j ++,
    edgs = If[Norm[U[[k]] - U[[j]]] ≤ r, Append[edgs, {U[[k]], U[[j]]}], edgs]]];
  ListLinePlot[edgs, GridLines → {Table[k, {k, 0, 1, 1/n}], Table[k, {k, 0, 1, 1/n}]},
   Axes → False, AspectRatio → 1, PlotRange → {{0, 1}, {0, 1}}, Frame → True,
   FrameTicks → None, Mesh → Full, PlotStyle → RGBColor[0, 0, 0] ]

 )

(* Convert binary string to coordinates of subsquares in [0,1]^2 with mesh 1/n^2
 B is an integer *)
Bin2Coord[B_, n_] := (

  d = PadLeft[IntegerDigits[B, 2], n^2];
  pnts = Table[{(2 j - 1)/(2 n), (2 k - 1)/(2 n)}, {j, 1, n}, {k, 1, n}];
  Y = Table[d[[j]] pnts[[IntegerPart[(j - 1) / n] + 1]][[Mod[j, n] + 1]], {j, 1, n^2}];
  Return[DeleteCases[Y, {0, 0}]];
 )

(* Plot 2-D Geometric graph with vertex set specified by binary expansion of integer B,
in partition of mesh 1/n^2 with topology determined by radius r*)
GeoPlot2D[B_, n_, r_] := PlotGU[Bin2Coord[B, n], n, r]
```

```
In[18]:= (* Example *)
gnum = 28 291;
GeoPlot2D[gnum, 4, √2 / 3]
```

Out[19]=



```
In[20]:= (* Build nilpotent adjacency matrix of G_U in [0,1]^2 (mesh 1/n^2) *)
BuildMatrix[B_, n_, r_] := (
  d = PadLeft[IntegerDigits[B, 2], n^2];
  pnts = Table[{(2 j - 1)/(2 n), (2 k - 1)/(2 n)}, {j, 1, n}, {k, 1, n}];
  Y = Table[d[[j]] pnts[[IntegerPart[(j - 1) / n] + 1]][[Mod[j, n] + 1]], {j, 1, n^2}];
  U = DeleteCases[Y, {0, 0}];
  edgs = {};
  For[k = 1, k ≤ Length[U], k++,
   edgs = Append[edgs, {U[[k]]}];
   For[j = k + 1, j ≤ Length[U], j++,
    edgs = If[Norm[U[[k]] - U[[j]]] ≤ r, Append[edgs, {U[[k]], U[[j]]}], edgs]]];
  (* Build the adjacency matrix *)
  A = Table[If[MemberQ[edgs, {U[[i]], U[[j]]}] || MemberQ[edgs, {U[[j]], U[[i]]}], 1, 0],
    {i, Length[U]}, {j, Length[U]}];
  Return[A.DiagonalMatrix[Table[ζ_{i}, {i, 1, Length[A]}]]];
 )

In[21]:= (* Build idempotent adjacency matrix of G_U in [0,1]^2 (mesh 1/n^2) *)
BuildIdemMatrix[B_, n_, r_] := (
  d = PadLeft[IntegerDigits[B, 2], n^2];
  pnts = Table[{(2 j - 1)/(2 n), (2 k - 1)/(2 n)}, {j, 1, n}, {k, 1, n}];
  Y = Table[d[[j]] pnts[[IntegerPart[(j - 1) / n] + 1]][[Mod[j, n] + 1]], {j, 1, n^2}];
  U = DeleteCases[Y, {0, 0}];
  edgs = {};
  For[k = 1, k ≤ Length[U], k++,
   edgs = Append[edgs, {U[[k]]}];
   For[j = k + 1, j ≤ Length[U], j++,
    edgs = If[Norm[U[[k]] - U[[j]]] ≤ r, Append[edgs, {U[[k]], U[[j]]}], edgs]]];
  (* Build the adjacency matrix *)
  A = Table[If[MemberQ[edgs, {U[[i]], U[[j]]}] || MemberQ[edgs, {U[[j]], U[[i]]}], 1, 0],
    {i, Length[U]}, {j, Length[U]}];
  Return[(A + IdentityMatrix[Length[A]]).DiagonalMatrix[Table[γ_{i}, {i, 1, Length[A]}]]];
 )

In[22]:= (* Random Geom. graph with equiprobable vertices p has probability measure μ *)
μ[B_, n_, p_] := p^Length[Bin2Coord[B,n]] (1 - p)^(n^2-Length[Bin2Coord[B,n]])
```