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STABILIZED DDFV SCHEMES FOR STOKES PROBLEM WITH VARIABLE VISCOSITY ON GENERAL 2D MESHES

STELLA KRELL*

Abstract. “Discrete Duality Finite Volume” schemes (DDFV for short) on general meshes are studied here for Stokes problems with variable viscosity with Dirichlet boundary conditions. The aim of this work is to analyze the wellposedness of the scheme and its convergence properties. The classical Stokes problem has already been presented with DDFV discretization in [14]. They first considered the natural extension of the DDFV scheme classically used for the Laplace problem. Unfortunately, its wellposedness is still an open problem on general meshes. To overcome this difficulty, we provide here a stabilized DDFV scheme, for Stokes problems with variable viscosity. This scheme is wellposed on general meshes and is proved to be first order convergent. Numerical examples are given, including those on locally refined non conformal meshes.

Key words. Finite-volume methods, Stokes problem, DDFV methods, variable viscosity.

1. Introduction. This paper is devoted to the finite volume approximation of the 2D steady Stokes model with variable viscosity:

$$\begin{aligned} -\operatorname{div}(2\eta(x)\mathbf{D}\mathbf{u}) + \nabla p &= \mathbf{f}, & \text{in } \Omega, \\ \operatorname{div}(\mathbf{u}) &= 0, & \text{in } \Omega, \end{aligned} \tag{1.1}$$

where $\mathbf{u} : \Omega \mapsto \mathbb{R}^2$ is the velocity, $p : \Omega \mapsto \mathbb{R}$ is the pressure and $\mathbf{D}\mathbf{u} = \frac{1}{2}(\nabla\mathbf{u} + {}^t\nabla\mathbf{u})$ is the symmetric part of the gradient of \mathbf{u} . Of course, this system will be supplemented with boundary conditions. We assume that the viscosity η simply depends on the space variable. Note that in more physically interesting models, not considered here, the viscosity depends in fact on other characteristics of the flow like density, temperature, through the coupling with other equations. Nevertheless, solving a problem like (1.1) is very often one of the steps needed to approximate the solution of those more complex models by the mean of some time splitting methods.

Since the velocity is divergence free, the momentum conservation equation can be written $-\eta\Delta\mathbf{u} + \nabla p = \mathbf{f}$ when the viscosity is a constant. Hence, in this case, after integration of the equation on each control volume, we only need to approximate the normal component of $\nabla\mathbf{u}$ on the interface between two adjacent control volumes (see, for instance, [20, 21] and [29] for methods on admissible and conformal meshes), whereas for variable viscosity, the presence of the symmetric part of the gradient $\mathbf{D}\mathbf{u}$ imposes to address the problem of the reconstruction of the full velocity gradient on the whole domain, even for admissible and conformal meshes. Moreover, even in the case where the viscosity is a constant, a possible outflow boundary condition of physical interest is to impose the normal component of the stress on the boundary. In that case, we really need to deal with the original formulation (1.1) of the problem which makes appear the stress tensor $2\eta\mathbf{D}\mathbf{u} - p\operatorname{Id}$. Hence, from a numerical point of view we need a discretization of $\mathbf{D}\mathbf{u}$ in order to treat this problem. Although we will only consider here the case of Dirichlet boundary conditions, we emphasize the fact that our framework naturally allows to take into account those outflow boundary conditions.

Different methods of gradient reconstruction have been proposed since the last ten years, one can refer for finite volume scheme to [3], [13], [18], [19] and [16, 25]. In all cases, the crucial feature is that the summation-by-parts procedure permits to reconstitute, starting from two point finite differences, the whole two dimensional discrete gradient. Many of them

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have been compared in the benchmark [24] of the FVCA5 conference, for scalar diffusion problems.

We consider here the class of finite volume schemes called “Discrete Duality Finite Volume” (DDFV for short). The DDFV schemes have been first introduced and studied in [16, 25] to approximate the solution of the Laplace equation on a large class of 2D meshes including non-conformal and distorted meshes. Those schemes require unknowns on both vertices and “centers” of control volumes. These two sets of unknowns allow to reconstitute two-dimensional discrete gradient (defined on new geometric elements called *diamonds*) and discrete divergence operators that are in duality in a discrete sense (see Theorem 3.2). The number of unknowns doubles compared to usual cell-centered finite volume schemes, but the gradient approximation becomes simple and quite efficient. In the benchmark [24], we see that the DDFV method is a competitive first order method especially as far as the accuracy of the gradient is concerned. The DDFV framework is thoroughly recalled in Section 3.

From now on, the DDFV strategy has then been applied for several linear and nonlinear problems: linear anisotropic diffusion equations in [6, 16, 25, 26]; convection-diffusion problems in [10]; div-curl problems in [15]; the nonlinear diffusion equations for Leray-Lions operators in [2, 5]. We can also mention [11] where the DDFV method is adapted to solve numerically a bidomain problem arising in biomathematics.

Concerning the DDFV discretization of the Stokes problem we are interested in here, the first results can be found in [14] where the author first considered the natural extension of the DDFV scheme classically used for the Laplace problem, that is: velocity unknowns located at both vertices and centers of control volumes and pressure unknowns at the diamond cells (those cells where the discrete gradient operator is defined). Unfortunately, the corresponding scheme is only proved to be wellposed for particular classes of meshes. Indeed, the wellposedness result relies on a uniform discrete inf-sup condition, which is still an open problem for general meshes.

To overcome this difficulty, we propose here to add to this scheme a stabilization term in the mass conservation equation. This stabilization term is inspired by the well known Brezzi-Pitkäranta scheme [8] in the finite element framework. We then prove that the DDFV stabilized scheme is wellposed for 2D general meshes. Moreover, the stabilization term plays a key role in proving error estimates. Indeed, for both finite element schemes [8] and finite volume schemes [20], the appropriate choice of the stabilization term enables to prove a stability result (see Theorem 6.1) which is the first step towards the error estimates. More precisely, we prove here a first order convergence for the velocity, for its gradient and for the pressure in the L^2 -norm provided that the exact solution satisfies usual regularity assumptions.

Note that, an alternative strategy has also been proposed in [14] to overcome the difficulties of the analysis of the non-stabilized scheme. The author proposed to formulate the Stokes problem in the vorticity-velocity-pressure form and then to approximate the velocity on the diamond cells and the pressure on both vertices and centers of primal control volumes. This approach uses the fact that $\Delta \mathbf{v}$ is equal to $\nabla \operatorname{div} \mathbf{v} - \operatorname{curl} \operatorname{curl} \mathbf{v}$, for any vector field \mathbf{v} . Thus, it seems that it can not be easily generalized to the case where the viscosity is variable.

Only the 2D case is on purpose in this article. 3D extensions of DDFV schemes have been proposed in [9, 12, 27] for linear and nonlinear anisotropic diffusion equations and the extension of the present work is under study.

Outline. This paper is organized as follows. Section 2 is dedicated to introducing basic notation. In Section 3, we recall the DDFV framework. In Section 4, we introduce the DDFV stabilized scheme for the Stokes problem (2.1) and prove its wellposedness (see Theorem 4.1). In Section 5, we present the main results of discrete functional analysis necessary for the theoretical study of the finite volume method. These results include properties of the

discrete strain rate tensor, in particular we prove a discrete Korn inequality (see Theorem 5.1). In Section 6, we study the stability properties of the approximate solution with respect to the data (see Corollary 6.1). Then we prove error estimates provided that the exact solution lies in $(H^2(\Omega))^2 \times H^1(\Omega)$ (see Theorem 7.1). Finally, in Section 8, theoretical error estimates are illustrated with numerical results. In the concluding Section 9, we discuss the extension of our study to some fully practical variants of the finite volume scheme and to even more general viscosities, for instance discontinuous viscosities, that will be addressed in future works.

2. Stokes model. We are concerned with the finite volume approximation of the Stokes equations with variable viscosity: Find $\mathbf{u} : \Omega \rightarrow \mathbb{R}^2$ and $p : \Omega \rightarrow \mathbb{R}$ such that:

$$\begin{aligned} \operatorname{div}(-2\eta(x)\mathbf{D}\mathbf{u} + p\mathbf{Id}) &= \mathbf{f}, & \text{in } \Omega, \\ \operatorname{div}(\mathbf{u}) &= 0, & \text{in } \Omega, \\ \mathbf{u} &= \mathbf{g}, & \text{on } \partial\Omega, \end{aligned} \quad \int_{\Omega} p(x)dx = 0, \quad (2.1)$$

For the sake of simplicity, we restrict the presentation to the case of Dirichlet boundary conditions and regular right-hand sides. We assume that Ω is a polygonal connected open bounded subset of \mathbb{R}^2 , \mathbf{f} is a function in $(L^2(\Omega))^2$, the viscosity η is a function in $L^\infty(\Omega)$ and \mathbf{g} is a function in $(H^{\frac{1}{2}}(\partial\Omega))^2$ which verifies the compatibility condition:

$$\int_{\partial\Omega} \mathbf{g}(s) \cdot \bar{\mathbf{n}} ds = 0. \quad (2.2)$$

Assumptions for the viscosity. The viscosity $\eta : \Omega \rightarrow \mathbb{R}$ in (2.1) is Lipschitz continuous on the whole domain $\bar{\Omega}$ and is supposed to be bounded : there exists $\underline{C}_\eta, \bar{C}_\eta > 0$ such that

$$\underline{C}_\eta \leq \eta(x) \leq \bar{C}_\eta, \quad \text{for a.e. } x \in \Omega, \quad (2.3)$$

and

$$|\eta(x) - \eta(x')| \leq C_\eta |x - x'|, \quad \forall x, x' \in \bar{\Omega}. \quad (2.4)$$

The wellposedness of the problem (2.1) is studied in [7] with a constant viscosity. In order to study convergence rates of our approximate solution, we need to assume regularity of the solution (\mathbf{u}, p) of the problem (2.1). In [1], it is show that if Ω is a convex polygon, \mathbf{g} is equal to zero and the viscosity is smooth, then the regularity of the solution is the following

$$\mathbf{u} \in (H^2(\Omega))^2 \text{ and } p \in H^1(\Omega).$$

In the sequel, $\|\cdot\|_2$ stands for the natural L^2 -norm when we consider scalar and vector functions and for the Frobenius norm when we consider matrix functions:

$$\begin{aligned} \|\xi\|_{\mathcal{F}}^2 &= (\xi : \xi), \quad \forall \xi \in \mathcal{M}_2(\mathbb{R}), \\ \|\xi\|_2^2 &= \int_{\Omega} \|\xi(x)\|_{\mathcal{F}}^2 dx, \quad \forall \xi \in L^2(\Omega, \mathcal{M}_2(\mathbb{R})), \end{aligned}$$

where $(\xi : \tilde{\xi}) = \sum_{1 \leq i, j \leq 2} \xi_{i,j} \tilde{\xi}_{i,j} = \operatorname{Trace}({}^t \xi \tilde{\xi}), \quad \forall \xi, \tilde{\xi} \in \mathcal{M}_2(\mathbb{R}).$

REMARK 2.1. The matrix norm $\|\cdot\|_{\mathcal{F}}$ satisfies the following property

$$\left\| \frac{A + {}^t A}{2} \right\|_{\mathcal{F}} \leq \|A\|_{\mathcal{F}}, \quad \forall A \in \mathcal{M}_2(\mathbb{R}).$$

3. The DDFV framework.

3.1. The meshes and notation.

The meshes. We recall here the main notation and definitions taken from [2]. A DDFV mesh \mathcal{T} is constituted by a primal mesh $\mathfrak{M} \cup \partial\mathfrak{M}$ and a dual mesh $\mathfrak{M}^* \cup \partial\mathfrak{M}^*$ (Figure 3.1).

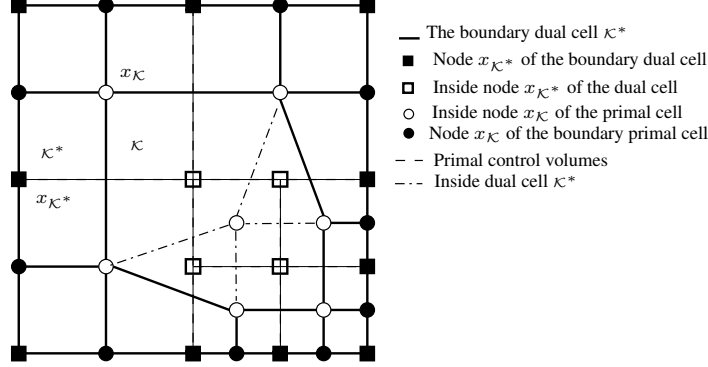


FIG. 3.1. The mesh \mathcal{T}

The primal mesh \mathfrak{M} is a set of disjoint open polygonal control volumes $\kappa \subset \Omega$ such that $\overline{\cup \kappa} = \overline{\Omega}$. We denote by $\partial\mathfrak{M}$ the set of edges of the control volumes in \mathfrak{M} included in $\partial\Omega$, that we consider as degenerate control volumes. To each control volume and degenerate control volume $\kappa \in \mathfrak{M} \cup \partial\mathfrak{M}$, we associate a point x_{κ} . For each degenerate control volume $\kappa \in \partial\mathfrak{M}$, we choose the point x_{κ} on the middle point of the control volume. This family of points is denoted by $X = \{x_{\kappa}, \kappa \in \mathfrak{M} \cup \partial\mathfrak{M}\}$.

Let X^* denote the set of the vertices of the primal control volumes in \mathfrak{M} that we split into $X^* = X_{int}^* \cup X_{ext}^*$ where $X_{int}^* \cap \partial\Omega = \emptyset$ and $X_{ext}^* \subset \partial\Omega$. With any point $x_{\kappa^*} \in X_{int}^*$ (resp. $x_{\kappa^*} \in X_{ext}^*$), we associate the polygon κ^* whose vertices are $\{x_{\kappa} \in X, \text{ such that } x_{\kappa^*} \in \overline{\kappa}, \kappa \in \mathfrak{M}\}$ (resp. $\{x_{\kappa^*}\} \cup \{x_{\kappa} \in X, \text{ such that } x_{\kappa^*} \in \overline{\kappa}, \kappa \in (\mathfrak{M} \cup \partial\mathfrak{M})\}$) sorted with respect to the clockwise order of the corresponding control volumes. This defines the set $\mathfrak{M}^* \cup \partial\mathfrak{M}^*$ of dual control volumes.

REMARK 3.1. Remark that our dual control volumes differ from the one proposed in [15] or [26]. In [15], they built κ^* by joining not only the barycenters x_{κ} associated to the elements of the primal mesh of which x_{κ^*} is a vertex but also the middle points of the edges of which x_{κ^*} is a vertex. This construction is usually called the barycentric dual mesh.

For all control volumes κ and \mathcal{L} , we assume that $\partial\kappa \cap \partial\mathcal{L}$ is either empty or a common vertex or an edge of the primal mesh denoted by $\sigma = \kappa|\mathcal{L}$. We note by \mathcal{E} the set of such edges. We also note $\sigma^* = \kappa^*|\mathcal{L}^*$ and \mathcal{E}^* for the corresponding dual definitions.

Given the primal and dual control volumes, we define the diamond cells $\mathcal{D}_{\sigma, \sigma^*}$ being the quadrangles whose diagonals are a primal edge $\sigma = \kappa|\mathcal{L} = (x_{\kappa^*}, x_{\mathcal{L}^*})$ and a corresponding dual edge $\sigma^* = \kappa^*|\mathcal{L}^* = (x_{\kappa}, x_{\mathcal{L}})$, (see Fig. 3.2). Note that the diamond cells are not necessarily convex. If $\sigma \in \mathcal{E} \cap \partial\overline{\Omega}$, the quadrangle $\mathcal{D}_{\sigma, \sigma^*}$ degenerate into a triangle. The set of the diamond cells is denoted by \mathcal{D} and we have $\overline{\Omega} = \bigcup_{\mathcal{D} \in \mathcal{D}} \overline{\mathcal{D}}$.

Notation. For any primal control volume $\kappa \in \mathfrak{M} \cap \partial\mathfrak{M}$, we note

- m_{κ} its Lebesgue measure,
- \mathcal{E}_{κ} the set of its edges (if $\kappa \in \mathfrak{M}$), or the one-element set $\{\kappa\}$ if $\kappa \in \partial\mathfrak{M}$.
- $\mathcal{D}_{\kappa} = \{\mathcal{D}_{\sigma, \sigma^*} \in \mathcal{D}, \sigma \in \mathcal{E}_{\kappa}\}$,
- d_{κ} its diameter,

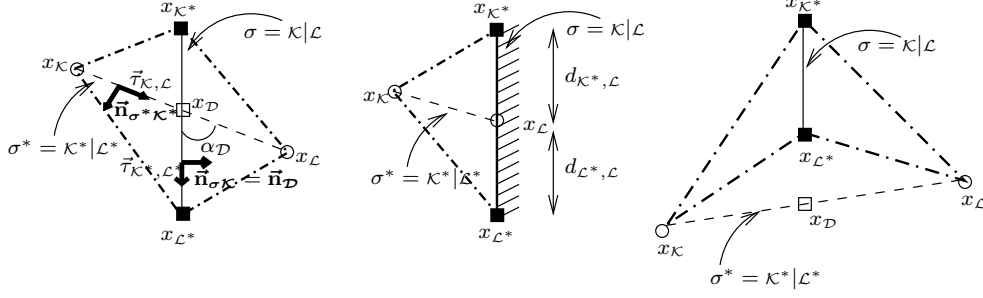


FIG. 3.2. Notations in the diamond cells. (Left) Interior cell. (Middle) Boundary cell. (Right) Non convex interior cell.

- $B_\kappa := B(x_\kappa, \rho_\kappa) \subset \Omega$ the open ball of radius $\rho_\kappa > 0$ for $\kappa \in \mathfrak{M}$, m_{B_κ} its measure. The value ρ_κ is chosen such that the inclusion is verified.
- $\sigma_\kappa = \kappa \cap \partial\Omega$ for $\kappa \in \partial\mathfrak{M}$, m_{σ_κ} its length.

We will also use corresponding dual notation: m_{κ^*} , \mathcal{E}_{κ^*} , \mathcal{D}_{κ^*} , d_{κ^*} , B_{κ^*} , $m_{B_{\kappa^*}}$, ρ_{κ^*} , σ_{κ^*} and $m_{\sigma_{\kappa^*}}$. For a diamond cell $\mathcal{D} = \mathcal{D}_{\sigma, \sigma^*}$ whose vertices are $(x_\kappa, x_{\kappa^*}, x_\mathcal{L}, x_{\mathcal{L}^*})$, we note

- $x_\mathcal{D}$ the center of the diamond cell \mathcal{D} : $x_\mathcal{D} = \sigma \cap \sigma^*$,
- $m_\mathcal{D}$ its measure,
- m_σ the length of the primal edge σ ,
- m_{σ^*} the length of the dual edge σ^* ,
- $\vec{n}_{\sigma\kappa}$ the unit vector normal to σ oriented from x_κ to $x_\mathcal{L}$,
- $\vec{n}_{\sigma^*\kappa^*}$ the unit vector normal to σ^* oriented from x_{κ^*} to $x_{\mathcal{L}^*}$,
- $\vec{\tau}_{\kappa, \mathcal{L}}$ the unit vector parallel to σ^* (oriented from x_κ to $x_\mathcal{L}$),
- $\vec{\tau}_{\kappa^*, \mathcal{L}^*}$ the unit vector parallel to σ (oriented from x_{κ^*} to $x_{\mathcal{L}^*}$),
- $\alpha_\mathcal{D}$ the angle between $\vec{\tau}_{\kappa, \mathcal{L}}$ and $\vec{\tau}_{\kappa^*, \mathcal{L}^*}$,
- $d_{\kappa^*, \mathcal{L}}$ (respectively $d_{\mathcal{L}^*, \mathcal{L}}$) the length between x_{κ^*} (respectively $x_{\mathcal{L}^*}$) and $x_\mathcal{L}$,
- $h_\mathcal{D}$ its diameter,
- \mathfrak{s} its edges (for example $\mathfrak{s} = [x_\kappa, x_{\kappa^*}]$),
- $\mathcal{E}_\mathcal{D} = \{\mathfrak{s}, \mathfrak{s} \in \partial\mathcal{D} \text{ and } \mathfrak{s} \not\subset \partial\Omega\}$ the set of edges of \mathcal{D} ,
- $\mathcal{G} = \{\mathfrak{s} \in \mathcal{E}_\mathcal{D}, \forall \mathcal{D} \in \mathcal{D}\}$ the set of inside edges of the diamond cells $\mathcal{D} \in \mathcal{D}$,
- $m_\mathfrak{s}$ the length of a diamond edge \mathfrak{s} ,
- $m_{\mathfrak{s}^*}$ the length between $x_\mathcal{D}$ and $x_{\mathcal{D}'}$ if $\mathfrak{s} = \mathcal{D}|\mathcal{D}'$,
- $\vec{n}_{\mathfrak{s}\mathcal{D}}$ the unit vector normal to $\mathfrak{s} = \mathcal{D}|\mathcal{D}'$ oriented from \mathcal{D} to \mathcal{D}' .

In a diamond cell $\mathcal{D} \in \mathcal{D}$, we have two direct orthonormal basis: $(\vec{\tau}_{\kappa^*, \mathcal{L}^*}, \vec{n}_{\sigma\kappa})$ and $(\vec{n}_{\sigma^*\kappa^*}, \vec{\tau}_{\kappa, \mathcal{L}})$. The boundary unit normal vectors are denoted by $\vec{n}^\mathcal{D} \in (\mathbb{R}^2)^\mathcal{D}$ such that $\vec{n}_\mathcal{D} = \vec{n}_{\sigma\kappa}$. We distinguish the interior diamond cells and the boundary diamond cells:

$$\mathcal{D}_{ext} = \{\mathcal{D} \in \mathcal{D}, \mathcal{D} \cap \partial\Omega \neq \emptyset\} \quad \text{and} \quad \mathcal{D}_{int} = \mathcal{D} \setminus \mathcal{D}_{ext}.$$

REMARK 3.2. In practice, during implementation, we do not construct explicitly the dual mesh but a local data structure, that contains the information on the vertices and centers of a diamond cell, we also calculate the measures of $\mathcal{D} \cap \kappa$, $\mathcal{D} \cap \kappa^*$ and the normal vectors $m_\sigma \vec{n}_{\sigma\kappa}$, $m_{\sigma^*} \vec{n}_{\sigma^*\kappa^*}$, which depend on the center $x_\mathcal{D}$ of the diamond cell \mathcal{D} . By perusing this structure, we assemble completely the matrix.

ASSUMPTION 3.1. We assume that all the diamond cells \mathcal{D} are convex.

Assumption 3.1 implies that the center $x_\mathcal{D}$ of the diamond cell \mathcal{D} (resp. the node x_{κ^*} of the dual cell κ^*) is inside \mathcal{D} (resp. κ^*). We also have for all $(\kappa^*, \mathcal{L}^*) \in \mathfrak{M}^* \cup \partial\mathfrak{M}^*$ such that $\kappa^* \neq \mathcal{L}^*$, we have $\overset{\circ}{\kappa^*} \cap \overset{\circ}{\mathcal{L}^*} = \emptyset$. It is not the case if we do not assume 3.1 (see Figure 3.3).

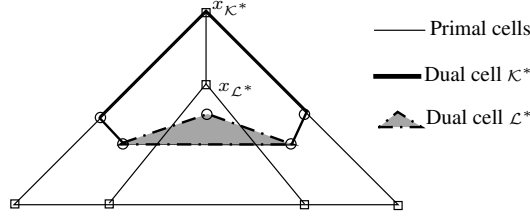


FIG. 3.3. An example where two dual cells κ^* and \mathcal{L}^* overlap: $\mathcal{L}^* \subset \kappa^*$.

REMARK 3.3. If Assumption 3.1 is not satisfied, for instance in Figure 3.4, we take the barycentric dual mesh (defined in Remark 3.1). In that case, the center $x_{\mathcal{D}}$ of the diamond cell $\mathcal{D}_{\sigma, \sigma^*}$ is defined as the barycenter of the primal edge σ .

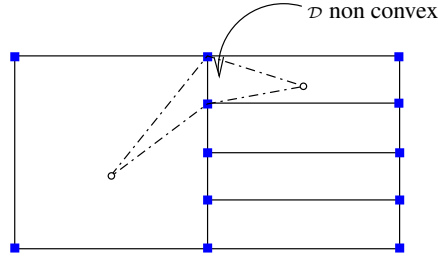


FIG. 3.4. An example where the diamond cells \mathcal{D} could be non convex.

Mesh regularity measurement. Set $\text{size}(\mathcal{T})$ the maximum of the diameters of the diamond cells in \mathcal{D} . To measure how flat the diamond cells can be, we note $\alpha_{\mathcal{T}}$ the unique real in $]0, \frac{\pi}{2}]$ such that $\sin(\alpha_{\mathcal{T}}) := \min_{\mathcal{D} \in \mathcal{D}} |\sin(\alpha_{\mathcal{D}})|$. We introduce a positive number $\text{reg}(\mathcal{T})$ that quantifies the regularity of a given mesh and is useful to perform the convergence analysis of finite volume schemes like in [2] and [5].

$$\begin{aligned} \mathcal{N}_{\mathcal{T}} &:= \sup_{x \in \Omega} \# \left\{ \kappa^*, x \in \widehat{\kappa^* \cup B_{\kappa^*}} \right\} + \sup_{x \in \Omega} \# \left\{ \kappa, x \in \widehat{\kappa \cup B_{\kappa}} \right\} \\ &\quad + \sup_{x \in \Omega} \# \left\{ \mathcal{D}, x \in \widehat{\mathcal{D} \cup B_{\mathcal{D}}} \right\}, \\ \text{reg}(\mathcal{T}) &:= \max \left(\frac{1}{\sin(\alpha_{\mathcal{T}})}, \mathcal{N}_{\mathcal{T}}, \max_{\mathcal{D} \in \mathcal{D}} \max_{s \in \mathcal{E}_{\mathcal{D}}} \frac{h_{\mathcal{D}}}{\min(m_s, m_{s^*})}, \max_{\substack{\kappa \in \mathfrak{M} \\ \mathcal{D} \in \mathfrak{D}_{\kappa}}} \frac{d_{\kappa}}{h_{\mathcal{D}}}, \right. \\ &\quad \left. \max_{\substack{\kappa^* \in \mathfrak{M}^* \cup \partial \mathfrak{M}^* \\ \mathcal{D} \in \mathfrak{D}_{\kappa^*}}} \frac{d_{\kappa^*}}{h_{\mathcal{D}}}, \max_{\kappa \in \mathfrak{M}} \frac{d_{\kappa}}{\rho_{\kappa}} + \frac{\rho_{\kappa}}{d_{\kappa}}, \max_{\kappa^* \in \mathfrak{M}^* \cup \partial \mathfrak{M}^*} \frac{d_{\kappa^*}}{\rho_{\kappa^*}} + \frac{\rho_{\kappa^*}}{d_{\kappa^*}} \right), \end{aligned} \quad (3.1)$$

where \widehat{E} is the convex hull of E . For instance, this number $\text{reg}(\mathcal{T})$ is involved in the following geometrical result: there exists two constants C_1 and C_2 depending on $\text{reg}(\mathcal{T})$ such that for any $\kappa \in \mathfrak{M}$, $\kappa^* \in \mathfrak{M}^* \cup \partial \mathfrak{M}^*$ and $\mathcal{D} \in \mathcal{D}$ such that $\mathcal{D} \cap \kappa \neq \emptyset$ and $\mathcal{D} \cap \kappa^* \neq \emptyset$, we have

$$C_1 m_{\kappa} \leq m_{\mathcal{D}} \leq C_2 m_{\kappa}, \quad C_1 m_{\kappa^*} \leq m_{\mathcal{D}} \leq C_2 m_{\kappa^*}.$$

We define two discrete sets $\mathbb{R}^{\mathcal{T}}$ and $(\mathbb{R}^2)^{\mathfrak{D}}$. The first one $\mathbb{R}^{\mathcal{T}}$ associates to all primal cells $\kappa \in \mathfrak{M} \cup \partial \mathfrak{M}$ a value $u_{\kappa} \in \mathbb{R}$ and to all dual cells $\kappa^* \in \mathfrak{M}^* \cup \partial \mathfrak{M}^*$ a value $u_{\kappa^*} \in \mathbb{R}$.

The second one $(\mathbb{R}^2)^{\mathfrak{D}}$ associates to all diamond cells $\mathcal{D} \in \mathfrak{D}$ a value $\phi^{\mathcal{D}} \in \mathbb{R}^2$. So that we have

$$\begin{aligned} u^{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}} &\iff u^{\mathcal{T}} = \left((u_{\kappa})_{\kappa \in (\mathfrak{M} \cup \partial\mathfrak{M})}, (u_{\kappa^*})_{\kappa^* \in (\mathfrak{M}^* \cup \partial\mathfrak{M}^*)} \right), \\ \phi^{\mathfrak{D}} \in (\mathbb{R}^2)^{\mathfrak{D}} &\iff \phi^{\mathfrak{D}} = (\phi^{\mathcal{D}})_{\mathcal{D} \in \mathfrak{D}}. \end{aligned}$$

We recall the two discrete operators defined in [2] for a scalar-value function.

DEFINITION 3.1. *We define a gradient operator of a scalar fields in $\mathbb{R}^{\mathcal{T}}$ denoted by $\nabla^{\mathfrak{D}} : u^{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}} \mapsto (\nabla^{\mathcal{D}} u^{\mathcal{T}})_{\mathcal{D} \in \mathfrak{D}} \in (\mathbb{R}^2)^{\mathfrak{D}}$, as follows:*

$$\nabla^{\mathcal{D}} u^{\mathcal{T}} = \frac{1}{2m_{\mathcal{D}}} [(u_{\mathcal{L}} - u_{\kappa})m_{\sigma}\vec{\mathbf{n}}_{\sigma\kappa} + (u_{\mathcal{L}^*} - u_{\kappa^*})m_{\sigma^*}\vec{\mathbf{n}}_{\sigma^*\kappa^*}], \quad \forall \mathcal{D} \in \mathfrak{D}.$$

Using the value of the diamond cell measure $2m_{\mathcal{D}} = \sin(\alpha_{\mathcal{D}})m_{\sigma}m_{\sigma^*}$, it can also be written as follows:

$$\nabla^{\mathcal{D}} u^{\mathcal{T}} = \frac{1}{\sin(\alpha_{\mathcal{D}})} \left[\frac{u_{\mathcal{L}} - u_{\kappa}}{m_{\sigma^*}} \vec{\mathbf{n}}_{\sigma\kappa} + \frac{u_{\mathcal{L}^*} - u_{\kappa^*}}{m_{\sigma}} \vec{\mathbf{n}}_{\sigma^*\kappa^*} \right].$$

We also define the discrete divergence of a vector field in $(\mathbb{R}^2)^{\mathfrak{D}}$.

DEFINITION 3.2. *We define a discrete divergence operator applied to vector fields in $(\mathbb{R}^2)^{\mathfrak{D}}$ denoted by $\operatorname{div}^{\mathcal{T}} : \phi^{\mathfrak{D}} = (\phi^{\mathcal{D}})_{\mathcal{D} \in \mathfrak{D}} \in (\mathbb{R}^2)^{\mathfrak{D}} \mapsto \operatorname{div}^{\mathcal{T}} \phi^{\mathfrak{D}} \in \mathbb{R}^{\mathcal{T}}$, as follows:*

$$\begin{aligned} \operatorname{div}^{\kappa} \phi^{\mathfrak{D}} &= \frac{1}{m_{\kappa}} \sum_{\sigma \in \partial\kappa} m_{\sigma} \phi^{\mathcal{D}} \cdot \vec{\mathbf{n}}_{\sigma\kappa}, \quad \forall \kappa \in \mathfrak{M}, \quad \text{and} \quad \operatorname{div}^{\kappa} \phi^{\mathfrak{D}} = 0, \quad \forall \kappa \in \partial\mathfrak{M}, \\ \operatorname{div}^{\kappa^*} \phi^{\mathfrak{D}} &= \frac{1}{m_{\kappa^*}} \sum_{\sigma^* \in \partial\kappa^*} m_{\sigma^*} \phi^{\mathcal{D}} \cdot \vec{\mathbf{n}}_{\sigma^*\kappa^*}, \quad \forall \kappa^* \in \mathfrak{M}^* \cup \partial\mathfrak{M}^*. \end{aligned}$$

In [2, 15, 16], the discrete gradient and discrete divergence for a scalar-value function are linked by a discrete Stokes formula, as follows.

THEOREM 3.1. *For all $\psi^{\mathfrak{D}} \in (\mathbb{R}^2)^{\mathfrak{D}}$ and for all $u^{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$:*

$$[\operatorname{div}^{\mathcal{T}} \psi^{\mathfrak{D}}, u^{\mathcal{T}}]_{\mathcal{T}} = - \sum_{\mathcal{D} \in \mathfrak{D}} m_{\mathcal{D}} \psi^{\mathcal{D}} \cdot \nabla^{\mathcal{D}} u^{\mathcal{T}} + \sum_{\mathcal{D}, \sigma, \sigma^* \in \mathfrak{D}_{ext}} m_{\sigma} (\psi^{\mathcal{D}} \cdot \vec{\mathbf{n}}_{\mathcal{D}}) \gamma_{\sigma}(u^{\mathcal{T}}),$$

where $[v^{\mathcal{T}}, u^{\mathcal{T}}]_{\mathcal{T}} = \frac{1}{2} \sum_{\kappa \in \mathfrak{M}} m_{\kappa} u_{\kappa} v_{\kappa} + \frac{1}{2} \sum_{\kappa^* \in \mathfrak{M}^* \cup \partial\mathfrak{M}^*} m_{\kappa^*} u_{\kappa^*} v_{\kappa^*}$, $\forall u^{\mathcal{T}}, v^{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$ and $\gamma_{\sigma}(u^{\mathcal{T}}) = \frac{d_{\kappa^*, \mathcal{L}}(u_{\kappa^*} + u_{\mathcal{L}}) + d_{\mathcal{L}^*, \mathcal{L}}(u_{\mathcal{L}^*} + u_{\mathcal{L}})}{2m_{\sigma}}$, $\forall \mathcal{D}, \sigma, \sigma^* \in \mathfrak{D}_{ext}$ and $\forall u^{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$.

3.2. Unknowns and discrete projections. The DDFV method for the Stokes problem associates to any primal control volume $\kappa \in \mathfrak{M} \cup \partial\mathfrak{M}$ an unknown value $\mathbf{u}_{\kappa} \in \mathbb{R}^2$ for the velocity, to any dual control volume $\kappa^* \in \mathfrak{M}^* \cup \partial\mathfrak{M}^*$ an unknown value $\mathbf{u}_{\kappa^*} \in \mathbb{R}^2$ for the velocity and to any diamond cell $\mathcal{D} \in \mathfrak{D}$ an unknown value $p^{\mathcal{D}} \in \mathbb{R}$ for the pressure. The approximate solution on the mesh $(\mathcal{T}, \mathfrak{D})$ is denoted by $(\mathbf{u}^{\mathcal{T}}, p^{\mathfrak{D}}) \in (\mathbb{R}^2)^{\mathcal{T}} \times \mathbb{R}^{\mathfrak{D}}$:

$$\mathbf{u}^{\mathcal{T}} = \left((\mathbf{u}_{\kappa})_{\kappa \in (\mathfrak{M} \cup \partial\mathfrak{M})}, (\mathbf{u}_{\kappa^*})_{\kappa^* \in (\mathfrak{M}^* \cup \partial\mathfrak{M}^*)} \right), \quad p^{\mathfrak{D}} = ((p^{\mathcal{D}})_{\mathcal{D} \in \mathfrak{D}}).$$

We piecewise the two constant functions $\mathbf{u}^{\mathfrak{M}} = \sum_{\kappa \in \mathfrak{M}} \mathbf{u}_{\kappa} \mathbf{1}_{\kappa}$ and $\mathbf{u}^{\mathfrak{M}^*} = \sum_{\kappa^* \in \mathfrak{M}^* \cup \partial \mathfrak{M}^*} \mathbf{u}_{\kappa^*} \mathbf{1}_{\kappa^*}$, where $\mathbf{1}_E$ is the characteristic function of any set E . As a consequence, one can define the L^2 -norm of $\mathbf{u}^{\mathcal{T}}$.

$$\|\mathbf{u}^{\mathcal{T}}\|_2^2 = \frac{1}{2} \left(\|\mathbf{u}^{\mathfrak{M}}\|_2^2 + \|\mathbf{u}^{\mathfrak{M}^*}\|_2^2 \right), \quad \forall \mathbf{u}^{\mathcal{T}} \in (\mathbb{R}^2)^{\mathcal{T}}.$$

In a same way, we associate the piecewise constant function to a discrete function $p^{\mathfrak{D}}$

$$p^{\mathfrak{D}} \sim \sum_{\mathfrak{D} \in \mathfrak{D}} p^{\mathfrak{D}} \mathbf{1}_{\mathfrak{D}}, \quad \text{and} \quad \|p^{\mathfrak{D}}\|_2^2 = \sum_{\mathfrak{D} \in \mathfrak{D}} m_{\mathfrak{D}} |p^{\mathfrak{D}}|^2, \quad \forall p^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}}.$$

We define the two projections of functions defined on Ω over the primal and dual meshes \mathcal{T} , the mean-value one and the center-value one. We begin with the mean-value projection. We call the interior mean-value projection for any integrable vector function \mathbf{v} on Ω :

$$\begin{aligned} \mathbb{P}_{\mathbf{m}}^{\mathfrak{M}} \mathbf{v} &= \left(\left(\frac{1}{m_{B_{\kappa}}} \int_{B_{\kappa}} \mathbf{v}(x) dx \right)_{\kappa \in \mathfrak{M}} \right), \\ \mathbb{P}_{\mathbf{m}}^{\mathfrak{M}^*} \mathbf{v} &= \left(\left(\frac{1}{m_{B_{\kappa^*}}} \int_{B_{\kappa^*}} \mathbf{v}(x) dx \right)_{\kappa^* \in \mathfrak{M}^*} \right). \end{aligned} \quad (3.2)$$

Then we consider the discrete mean-value boundary data, for any integrable vector function \mathbf{v} on $\bar{\Omega}$, denoted by $\mathbb{P}_{\mathbf{m}}^{\partial \Omega}$ and defined as follows:

$$\mathbb{P}_{\mathbf{m}}^{\partial \Omega} \mathbf{v} = \left(\left(\frac{1}{m_{\sigma_{\kappa}}} \int_{\sigma_{\kappa}} \mathbf{v}(x) dx \right)_{\kappa \in \partial \mathfrak{M}}, \left(\frac{1}{m_{\sigma_{\kappa^*}}} \int_{\sigma_{\kappa^*}} \mathbf{v}(x) dx \right)_{\kappa^* \in \partial \mathfrak{M}^*} \right). \quad (3.3)$$

Finally, we define the mean-value projection for any integrable vector function \mathbf{v} on $\bar{\Omega}$ with the two other projections as follows:

$$\mathbb{P}_{\mathbf{m}}^{\mathcal{T}} \mathbf{v} = (\mathbb{P}_{\mathbf{m}}^{\mathfrak{M}} \mathbf{v}, \mathbb{P}_{\mathbf{m}}^{\mathfrak{M}^*} \mathbf{v}, \mathbb{P}_{\mathbf{m}}^{\partial \Omega} \mathbf{v}). \quad (3.4)$$

In particular, the mean-value projection is well defined for any vector function \mathbf{v} lying in $(H^1(\Omega))^2$. The second one is the center-value projection for any continuous function \mathbf{v} on $\bar{\Omega}$:

$$\mathbb{P}_{\mathbf{c}}^{\mathcal{T}} \mathbf{v} = ((\mathbf{v}(x_{\kappa}))_{\kappa \in (\mathfrak{M} \cup \partial \mathfrak{M})}, (\mathbf{v}(x_{\kappa^*}))_{\kappa^* \in (\mathfrak{M}^* \cup \partial \mathfrak{M}^*)}). \quad (3.5)$$

In particular, the center-value projection is well defined for any vector function \mathbf{v} lying in $(H^2(\Omega))^2$.

We also define a mean-value projection on Ω over the diamond mesh \mathfrak{D} for any integrable function q :

$$\mathbb{P}_{\mathbf{m}}^{\mathfrak{D}} q = \left(\left(\frac{1}{m_{\mathfrak{D}}} \int_{\mathfrak{D}} q(x) dx \right)_{\mathfrak{D} \in \mathfrak{D}} \right). \quad (3.6)$$

We introduce two discrete subsets of $(\mathbb{R}^2)^{\mathcal{T}}$ needing in the following

$$\begin{aligned} \mathbb{E}_0 &= \left\{ \mathbf{v}^{\mathcal{T}} \in (\mathbb{R}^2)^{\mathcal{T}} \text{ s. t. } \mathbf{v}_{\kappa} = 0, \forall \kappa \in \partial \mathfrak{M} \text{ and } \mathbf{v}_{\kappa^*} = 0, \forall \kappa^* \in \partial \mathfrak{M}^* \right\}, \\ \mathbb{E}_{m,g} &= \left\{ \mathbf{v}^{\mathcal{T}} \in (\mathbb{R}^2)^{\mathcal{T}} \text{ s. t. } \mathbf{v}_{\kappa} = (\mathbb{P}_{\mathbf{m}}^{\partial \Omega} \mathbf{g})_{\kappa}, \forall \kappa \in \partial \mathfrak{M} \text{ and } \mathbf{v}_{\kappa^*} = (\mathbb{P}_{\mathbf{m}}^{\partial \Omega} \mathbf{g})_{\kappa^*}, \forall \kappa^* \in \partial \mathfrak{M}^* \right\}. \end{aligned}$$

We define the projection $\mathfrak{P}_{m,g}$ on the set $\mathbb{E}_{m,g}$:

$$\begin{aligned} \mathfrak{P}_{m,g} : \quad (\mathbb{R}^2)^T &\longrightarrow \mathbb{E}_{m,g} \\ \mathbf{u}^T &\longmapsto ((\mathbf{u}_\kappa)_{\kappa \in \mathfrak{M}}, (\mathbb{P}_m^{\partial\Omega} \mathbf{g})_{\kappa \in \partial\mathfrak{M}}, (\mathbf{u}_{\kappa^*})_{\kappa^* \in \mathfrak{M}^*}, (\mathbb{P}_m^{\partial\Omega} \mathbf{g})_{\kappa^* \in \partial\mathfrak{M}^*}). \end{aligned} \quad (3.7)$$

3.3. Discrete operators. In this subsection, we define the discrete operators which are needed in order to write the DDFV scheme. We begin with the discrete gradient.

DEFINITION 3.3 (Discrete gradient). *We define a consistent approximation of the gradient operator of a vector field in $(\mathbb{R}^2)^T$ denoted by $\nabla^{\mathfrak{D}} : \mathbf{u}^T \in (\mathbb{R}^2)^T \mapsto (\nabla^{\mathfrak{D}} \mathbf{u}^T)_{\mathfrak{D} \in \mathfrak{D}} \in (\mathcal{M}_2(\mathbb{R}))^{\mathfrak{D}}$, as follows:*

$$\nabla^{\mathfrak{D}} \mathbf{u}^T = \begin{pmatrix} {}^t(\nabla^{\mathfrak{D}} u_1^T) \\ {}^t(\nabla^{\mathfrak{D}} u_2^T) \end{pmatrix}, \quad \forall \mathfrak{D} \in \mathfrak{D},$$

where $\nabla^{\mathfrak{D}} u_i^T$ is defined in Definition 3.1, for $i = 1, 2$.

DEFINITION 3.4 (Discrete divergence). *We define a consistent approximation of the divergence operator applied to discrete tensor fields denoted by $\mathbf{div}^{\mathfrak{T}} : \xi^{\mathfrak{D}} = (\xi^{\mathfrak{D}})_{\mathfrak{D} \in \mathfrak{D}} \in (\mathcal{M}_2(\mathbb{R}))^{\mathfrak{D}} \mapsto \mathbf{div}^{\mathfrak{T}} \xi^{\mathfrak{D}} \in (\mathbb{R}^2)^T$, as follows:*

$$\begin{aligned} \mathbf{div}^{\kappa} \xi^{\mathfrak{D}} &= \frac{1}{m_\kappa} \sum_{\sigma \in \partial\kappa} m_\sigma \xi^{\mathfrak{D}} \bar{\mathbf{n}}_{\sigma\kappa}, \quad \forall \kappa \in \mathfrak{M}, \quad \text{and} \quad \mathbf{div}^{\kappa} \xi^{\mathfrak{D}} = 0, \quad \forall \kappa \in \partial\mathfrak{M}, \\ \mathbf{div}^{\kappa^*} \xi^{\mathfrak{D}} &= \frac{1}{m_{\kappa^*}} \sum_{\sigma^* \in \partial\kappa^*} m_{\sigma^*} \xi^{\mathfrak{D}} \bar{\mathbf{n}}_{\sigma^*\kappa^*}, \quad \forall \kappa^* \in \mathfrak{M}^* \cup \partial\mathfrak{M}^*. \end{aligned}$$

To write the DDFV scheme, we also need to denote the discrete divergence on the primal mesh and on the interior dual mesh as follows:

$$\mathbf{div}^{\mathfrak{M}} \xi^{\mathfrak{D}} = \left((\mathbf{div}^{\kappa} \xi^{\mathfrak{D}})_{\kappa \in \mathfrak{M}} \right), \quad \mathbf{div}^{\mathfrak{M}^*} \xi^{\mathfrak{D}} = \left((\mathbf{div}^{\kappa^*} \xi^{\mathfrak{D}})_{\kappa^* \in \mathfrak{M}^*} \right).$$

Also, these two operators are in *discrete duality* since they can be linked by a discrete Stokes formula Theorem 3.2. Thanks to the discrete gradient we can define a discrete strain rate tensor and a discrete divergence of a vector field in $(\mathbb{R}^2)^T$.

DEFINITION 3.5 (Discrete strain rate tensor). *Let us define a discrete strain rate tensor of a vector field in $(\mathbb{R}^2)^T$, by using the discrete gradient: $\mathbf{D}^{\mathfrak{D}} : \mathbf{u}^T \in (\mathbb{R}^2)^T \mapsto (\mathbf{D}^{\mathfrak{D}} \mathbf{u}^T)_{\mathfrak{D} \in \mathfrak{D}} \in (\mathcal{M}_2(\mathbb{R}))^{\mathfrak{D}}$, such that $\mathbf{D}^{\mathfrak{D}} \mathbf{u}^T = \frac{\nabla^{\mathfrak{D}} \mathbf{u}^T + {}^t(\nabla^{\mathfrak{D}} \mathbf{u}^T)}{2}$.*

DEFINITION 3.6. *Let us define a discrete divergence of a vector field in $(\mathbb{R}^2)^T$, by using the discrete gradient: $\mathbf{div}^{\mathfrak{D}} : \mathbf{u}^T \in (\mathbb{R}^2)^T \mapsto (\mathbf{div}^{\mathfrak{D}} \mathbf{u}^T)_{\mathfrak{D} \in \mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}}$, such that $\mathbf{div}^{\mathfrak{D}} \mathbf{u}^T = \text{Trace}(\nabla^{\mathfrak{D}} \mathbf{u}^T)$.*

REMARK 3.4. *Remark that $\mathbf{div}^{\mathfrak{D}} \mathbf{u}^T$ can be expressed in the following way:*

$$\mathbf{div}^{\mathfrak{D}} \mathbf{u}^T = \frac{1}{2m_{\mathfrak{D}}} [m_\sigma (\mathbf{u}_\sigma - \mathbf{u}_\kappa) \cdot \bar{\mathbf{n}}_{\sigma\kappa} + m_{\sigma^*} (\mathbf{u}_{\sigma^*} - \mathbf{u}_{\kappa^*}) \cdot \bar{\mathbf{n}}_{\sigma^*\kappa^*}].$$

The discrete counterpart of the equality $\int_{\mathfrak{D}} \text{div}(u)(z) dz = \int_{\mathcal{E}_{\mathfrak{D}}} (u(s), \bar{\mathbf{n}}_{\mathfrak{D}}) ds$ is the following:

$$m_{\mathfrak{D}} \mathbf{div}^{\mathfrak{D}}(\mathbf{u}^T) = \sum_{s \in \mathcal{E}_{\mathfrak{D}}} m_s \left(\frac{\mathbf{u}_{\kappa^*} + \mathbf{u}_\kappa}{2}, \bar{\mathbf{n}}_{s\mathfrak{D}} \right), \quad \forall \mathbf{u}^T \in (\mathbb{R}^2)^T. \quad (3.8)$$

Finally, we define the stabilization term as follows.

DEFINITION 3.7. *The stabilization term is a non consistent discrete approximation of $2 \text{size}(\mathcal{T})\Delta p$, denoted by $\Delta^{\mathfrak{D}} : p^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}} \mapsto \Delta^{\mathfrak{D}} p^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}}$, and defined as follows:*

$$\Delta^{\mathfrak{D}} p^{\mathfrak{D}} = \frac{1}{m_{\mathfrak{D}}} \sum_{s=\mathfrak{D} | \mathfrak{D}' \in \mathcal{E}_{\mathfrak{D}}} \frac{h_{\mathfrak{D}}^2 + h_{\mathfrak{D}'}^2}{h_{\mathfrak{D}}^2} (p^{\mathfrak{D}'} - p^{\mathfrak{D}}), \quad \forall \mathfrak{D} \in \mathfrak{D}.$$

Note that we do not need a consistent discrete form of a laplacian.

3.4. Discrete Stokes formula. We want to generalize the Theorem 3.1 for vector-value functions. First of all, we define trace operators on both $(\mathbb{R}^2)^{\mathcal{T}}$ and $(\mathbb{R}^2)^{\mathfrak{D}}$.

Set $\gamma^{\mathcal{T}} : \mathbf{u}^{\mathcal{T}} \in (\mathbb{R}^2)^{\mathcal{T}} \mapsto \gamma^{\mathcal{T}}(\mathbf{u}^{\mathcal{T}}) = (\gamma_{\sigma}(\mathbf{u}^{\mathcal{T}}))_{\sigma \in \partial \mathfrak{M}} \in (\mathbb{R}^2)^{\partial \mathfrak{M}}$, as follows:

$$\gamma_{\sigma}(\mathbf{u}^{\mathcal{T}}) = \frac{d_{\kappa^*, \mathcal{L}}(\mathbf{u}_{\kappa^*} + \mathbf{u}_{\mathcal{L}}) + d_{\mathcal{L}^*, \mathcal{L}}(\mathbf{u}_{\mathcal{L}^*} + \mathbf{u}_{\mathcal{L}})}{2m_{\sigma}}, \quad \forall \sigma \in \partial \mathfrak{M}.$$

This trace operator enables to impose the Dirichlet boundary conditions in a weak way. The second one is denoted by $\gamma^{\mathfrak{D}} : \phi^{\mathfrak{D}} \in (\mathbb{R}^2)^{\mathfrak{D}} \mapsto (\phi^{\mathfrak{D}})_{\mathfrak{D} \in \mathfrak{D}_{ext}} \in (\mathbb{R}^2)^{\mathfrak{D}_{ext}}$.

Then, we define the four following inner products

$$\begin{aligned} \llbracket \mathbf{v}^{\mathcal{T}}, \mathbf{u}^{\mathcal{T}} \rrbracket_{\mathcal{T}} &= \frac{1}{2} \left(\sum_{\kappa \in \mathfrak{M}} m_{\kappa} \mathbf{u}_{\kappa} \cdot \mathbf{v}_{\kappa} + \sum_{\kappa^* \in \mathfrak{M}^* \cup \partial \mathfrak{M}^*} m_{\kappa^*} \mathbf{u}_{\kappa^*} \cdot \mathbf{v}_{\kappa^*} \right), \quad \forall \mathbf{u}^{\mathcal{T}}, \mathbf{v}^{\mathcal{T}} \in (\mathbb{R}^2)^{\mathcal{T}}, \\ (\phi^{\mathfrak{D}}, \mathbf{v}^{\mathcal{T}})_{\partial \Omega} &= \sum_{\mathfrak{D}, \sigma^* \in \mathfrak{D}_{ext}} m_{\sigma} \phi^{\mathfrak{D}} \cdot \mathbf{v}_{\sigma}, \quad \forall \phi^{\mathfrak{D}} \in (\mathbb{R}^2)^{\mathfrak{D}_{ext}}, \mathbf{v}^{\mathcal{T}} \in (\mathbb{R}^2)^{\partial \mathfrak{M}}, \\ (p^{\mathfrak{D}}, q^{\mathfrak{D}})_{\mathfrak{D}} &= \sum_{\mathfrak{D} \in \mathfrak{D}} m_{\mathfrak{D}} p^{\mathfrak{D}} q^{\mathfrak{D}}, \quad \forall p^{\mathfrak{D}}, q^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}}, \\ (\xi^{\mathfrak{D}} : \phi^{\mathfrak{D}})_{\mathfrak{D}} &= \sum_{\mathfrak{D} \in \mathfrak{D}} m_{\mathfrak{D}} (\xi^{\mathfrak{D}} : \phi^{\mathfrak{D}}), \quad \forall \xi^{\mathfrak{D}}, \phi^{\mathfrak{D}} \in (\mathcal{M}_2(\mathbb{R}))^{\mathfrak{D}}. \end{aligned}$$

The following result is the generalization of the discrete Stokes formula 3.1 to the case of vector-value functions.

THEOREM 3.2 (Discrete Stokes formula). *For all $\xi^{\mathfrak{D}} \in (\mathcal{M}_2(\mathbb{R}))^{\mathfrak{D}}$, $\mathbf{u}^{\mathcal{T}} \in (\mathbb{R}^2)^{\mathcal{T}}$:*

$$\llbracket \text{div}^{\mathcal{T}} \xi^{\mathfrak{D}}, \mathbf{u}^{\mathcal{T}} \rrbracket_{\mathcal{T}} = -(\xi^{\mathfrak{D}} : \nabla^{\mathfrak{D}} \mathbf{u}^{\mathcal{T}})_{\mathfrak{D}} + (\gamma^{\mathfrak{D}}(\xi^{\mathfrak{D}} \vec{\mathbf{n}}^{\mathfrak{D}}), \gamma^{\mathcal{T}}(\mathbf{u}^{\mathcal{T}}))_{\partial \Omega}.$$

Proof. Let $\xi^{\mathfrak{D}} \in (\mathcal{M}_2(\mathbb{R}))^{\mathfrak{D}}$ with $\xi^{\mathfrak{D}} = \begin{pmatrix} \xi_1^{\mathfrak{D}} \\ \xi_2^{\mathfrak{D}} \end{pmatrix}$, $\xi_1^{\mathfrak{D}}, \xi_2^{\mathfrak{D}} \in \mathbb{R}^2$, and $\mathbf{u}^{\mathcal{T}} = \begin{pmatrix} u^{\mathcal{T}} \\ v^{\mathcal{T}} \end{pmatrix} \in (\mathbb{R}^2)^{\mathcal{T}}$.

By definition of $(\cdot : \cdot)_{\mathfrak{D}}$, we have

$$(\xi^{\mathfrak{D}} : \nabla^{\mathfrak{D}} \mathbf{u}^{\mathcal{T}})_{\mathfrak{D}} = \sum_{\mathfrak{D} \in \mathfrak{D}} m_{\mathfrak{D}} \xi_1^{\mathfrak{D}} \cdot \nabla^{\mathfrak{D}} u^{\mathcal{T}} + \sum_{\mathfrak{D} \in \mathfrak{D}} m_{\mathfrak{D}} \xi_2^{\mathfrak{D}} \cdot \nabla^{\mathfrak{D}} v^{\mathcal{T}}.$$

We apply the discrete Stokes formula Theorem 3.1 to each component of the velocity

$$\begin{aligned} (\xi^{\mathfrak{D}} : \nabla^{\mathfrak{D}} \mathbf{u}^{\mathcal{T}})_{\mathfrak{D}} &= -[\text{div}^{\mathcal{T}} \xi_1^{\mathfrak{D}}, u^{\mathcal{T}}]_{\mathcal{T}} + \sum_{\mathfrak{D}, \sigma^* \in \mathfrak{D}_{ext}} m_{\sigma} (\xi_1^{\mathfrak{D}} \cdot \vec{\mathbf{n}}_{\mathfrak{D}}) \gamma_{\sigma}(u^{\mathcal{T}}) \\ &\quad - [\text{div}^{\mathcal{T}} \xi_2^{\mathfrak{D}}, v^{\mathcal{T}}]_{\mathcal{T}} + \sum_{\mathfrak{D}, \sigma^* \in \mathfrak{D}_{ext}} m_{\sigma} (\xi_2^{\mathfrak{D}} \cdot \vec{\mathbf{n}}_{\mathfrak{D}}) \gamma_{\sigma}(v^{\mathcal{T}}). \end{aligned}$$

Furthermore, we have as $(\xi^{\mathfrak{D}} \bar{\mathbf{n}}^{\mathfrak{D}}) \cdot \gamma_{\sigma}(\mathbf{u}^{\mathfrak{T}}) = (\xi_1^{\mathfrak{D}} \cdot \bar{\mathbf{n}}^{\mathfrak{D}}) \gamma_{\sigma}(u^{\mathfrak{T}}) + (\xi_2^{\mathfrak{D}} \cdot \bar{\mathbf{n}}^{\mathfrak{D}}) \gamma_{\sigma}(v^{\mathfrak{T}})$, we get

$$\sum_{\mathfrak{D}_{\sigma}, \sigma^* \in \mathfrak{D}_{ext}} m_{\sigma}(\xi_1^{\mathfrak{D}} \cdot \bar{\mathbf{n}}_{\mathfrak{D}}) \gamma_{\sigma}(u^{\mathfrak{T}}) + \sum_{\mathfrak{D}_{\sigma}, \sigma^* \in \mathfrak{D}_{ext}} m_{\sigma}(\xi_2^{\mathfrak{D}} \cdot \bar{\mathbf{n}}_{\mathfrak{D}}) \gamma_{\sigma}(v^{\mathfrak{T}}) = (\gamma^{\mathfrak{D}}(\xi^{\mathfrak{D}} \bar{\mathbf{n}}^{\mathfrak{D}}), \gamma^{\mathfrak{T}}(\mathbf{u}^{\mathfrak{T}}))_{\partial\Omega}.$$

Apply the first inner product $[[\cdot, \cdot]]_{\mathfrak{T}}$ and Definition 3.4, we get

$$2[[\operatorname{div}^{\mathfrak{T}} \xi^{\mathfrak{D}}, \mathbf{u}^{\mathfrak{T}}]]_{\mathfrak{T}} = \sum_{\kappa \in \mathfrak{M}} \sum_{\sigma \in \partial\kappa} m_{\sigma} \mathbf{u}_{\kappa} \cdot (\xi^{\mathfrak{D}} \bar{\mathbf{n}}_{\sigma\kappa}) + \sum_{\kappa^* \in \mathfrak{M}^* \cup \partial\mathfrak{M}^*} \sum_{\sigma^* \in \partial\kappa^*} m_{\sigma^*} \mathbf{u}_{\kappa^*} \cdot (\xi^{\mathfrak{D}} \bar{\mathbf{n}}_{\sigma^*\kappa^*}).$$

We conclude using as previous that $\mathbf{u}_{\kappa} \cdot (\xi^{\mathfrak{D}} \bar{\mathbf{n}}_{\sigma\kappa}) = (\xi_1^{\mathfrak{D}} \cdot \bar{\mathbf{n}}_{\sigma\kappa}) u_{\kappa} + (\xi_2^{\mathfrak{D}} \cdot \bar{\mathbf{n}}_{\sigma\kappa}) v_{\kappa}$ and $\mathbf{u}_{\kappa^*} \cdot (\xi^{\mathfrak{D}} \bar{\mathbf{n}}_{\sigma^*\kappa^*}) = (\xi_1^{\mathfrak{D}} \cdot \bar{\mathbf{n}}_{\sigma^*\kappa^*}) u_{\kappa^*} + (\xi_2^{\mathfrak{D}} \cdot \bar{\mathbf{n}}_{\sigma^*\kappa^*}) v_{\kappa^*}$, so that

$$[[\operatorname{div}^{\mathfrak{T}} \xi_1^{\mathfrak{D}}, u^{\mathfrak{T}}]]_{\mathfrak{T}} + [[\operatorname{div}^{\mathfrak{T}} \xi_2^{\mathfrak{D}}, v^{\mathfrak{T}}]]_{\mathfrak{T}} = [[\operatorname{div}^{\mathfrak{T}} \xi^{\mathfrak{D}}, \mathbf{u}^{\mathfrak{T}}]]_{\mathfrak{T}},$$

which concludes the proof. \square

In addition, we define a mesh dependent seminorm $|\cdot|_h$ over $\mathbb{R}^{\mathfrak{D}}$ by:

$$|p^{\mathfrak{D}}|_h^2 = \sum_{\mathfrak{s} = \mathfrak{D} | \mathfrak{D}' \in \mathfrak{S}} (h_{\mathfrak{D}}^2 + h_{\mathfrak{D}'}^2) (p^{\mathfrak{D}'} - p^{\mathfrak{D}})^2, \quad \forall p^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}}. \quad (3.9)$$

REMARK 3.5. We have, for any $p^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}}$, by reorganizing the summation over $\mathfrak{s} \in \mathfrak{S}$,

$$\begin{aligned} -(h_{\mathfrak{D}}^2 \Delta^{\mathfrak{D}} p^{\mathfrak{D}}, p^{\mathfrak{D}})_{\mathfrak{D}} &= \sum_{\mathfrak{D} \in \mathfrak{D}} p^{\mathfrak{D}} \sum_{\mathfrak{s} \in \mathcal{E}_{\mathfrak{D}}} (h_{\mathfrak{D}}^2 + h_{\mathfrak{D}'}^2) (p^{\mathfrak{D}} - p^{\mathfrak{D}'}) \\ &= \sum_{\mathfrak{s} = \mathfrak{D} | \mathfrak{D}' \in \mathfrak{S}} (h_{\mathfrak{D}}^2 + h_{\mathfrak{D}'}^2) (p^{\mathfrak{D}'} - p^{\mathfrak{D}})^2 \\ &= |p^{\mathfrak{D}}|_h^2. \end{aligned}$$

4. DDFV schemes for the Stokes equation. We note $\eta_{\mathfrak{D}} = \int_{\bar{\mathfrak{D}}} \eta(s) d\mu_{\bar{\mathfrak{D}}}(s)$, for all $\mathfrak{D} \in \mathfrak{D}$, where $\mu_{\bar{\mathfrak{D}}}$ is a probability measure on $\bar{\mathfrak{D}}$. This includes the case $\eta_{\mathfrak{D}} = \eta(x_{\mathfrak{D}})$ or $\eta_{\mathfrak{D}} = \frac{1}{m_{\mathfrak{D}}} \int_{\mathfrak{D}} \eta(x) dx$. Furthermore, we always have the following inequality

$$\underline{C}_{\eta} \leq \eta_{\mathfrak{D}} \leq \bar{C}_{\eta}, \quad \forall \mathfrak{D} \in \mathfrak{D}.$$

As claimed in introduction, we approximate the velocity on both vertices and centers of primal control volumes and the pressure on the diamond cells. We integrate the momentum conservation law of problem (2.1) on the primal mesh \mathfrak{M} and on the interior dual mesh \mathfrak{M}^* . The mass conservation equation is directly approached on the diamond mesh using the discrete operator $\operatorname{div}^{\mathfrak{D}}$ and a stabilized term inspired by the well known Brezzi-Pitkäranta scheme. We impose the Dirichlet boundary conditions on $\partial\mathfrak{M}$ and on $\partial\mathfrak{M}^*$. Finally, the integral of the pressure is imposed to be equal to zero.

The scheme for the problem (2.1) reads as follows:

$$\left\{ \begin{array}{l} \text{Find } \mathbf{u}^{\mathfrak{T}} \in \mathbb{E}_{m,g} \text{ and } p^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}} \text{ such that,} \\ \operatorname{div}^{\mathfrak{M}}(-2\eta^{\mathfrak{D}} \mathbb{D}^{\mathfrak{D}} \mathbf{u}^{\mathfrak{T}} + p^{\mathfrak{D}} \operatorname{Id}) = \mathbf{f}^{\mathfrak{M}}, \\ \operatorname{div}^{\mathfrak{M}^*}(-2\eta^{\mathfrak{D}} \mathbb{D}^{\mathfrak{D}} \mathbf{u}^{\mathfrak{T}} + p^{\mathfrak{D}} \operatorname{Id}) = \mathbf{f}^{\mathfrak{M}^*}, \\ \operatorname{div}^{\mathfrak{D}}(\mathbf{u}^{\mathfrak{T}}) - \lambda h_{\mathfrak{D}}^2 \Delta^{\mathfrak{D}} p^{\mathfrak{D}} = 0, \\ \sum_{\mathfrak{D} \in \mathfrak{D}} m_{\mathfrak{D}} p^{\mathfrak{D}} = 0, \end{array} \right. \quad (4.1)$$

with $\lambda > 0$ given, $\mathbf{f}^{\mathfrak{m}} = \mathbb{P}_m^{\mathfrak{m}} \mathbf{f}$ and $\mathbf{f}^{\mathfrak{m}*} = \mathbb{P}_m^{\mathfrak{m}*} \mathbf{f}$, where the projection is defined by (3.2).

REMARK 4.1. *With these choices and if \mathbf{g} satisfies the compatibility condition (2.2), we easily check that*
$$\sum_{\mathcal{D}_{\sigma, \sigma^*} \in \mathfrak{D}_{ext}} m_{\sigma} \gamma_{\sigma}(\mathbf{g}^{\mathcal{T}}) \cdot \vec{\mathbf{n}}_{\sigma\kappa} = 0.$$

THEOREM 4.1. *Assume that η satisfies (2.3). For all mesh \mathcal{T} as described in Section 3, the finite volume scheme (4.1) with $\lambda > 0$ admits a unique solution $(\mathbf{u}^{\mathcal{T}}, p^{\mathfrak{D}}) \in (\mathbb{R}^2)^{\mathcal{T}} \times \mathbb{R}^{\mathfrak{D}}$.*

In order to prove the existence and uniqueness of the solution of (4.1), a discrete Korn inequality is needed. Note that the discrete Korn inequality (Theorem 5.1) is proved in Subsection 5.1.

Proof. Let $N = \#((\mathbb{R}^2)^{\mathcal{T}} \times \mathbb{R}^{\mathfrak{D}})$. Scheme (4.1) is a linear system of the form: $A\mathbf{v} = \mathbf{b}$ with

$$A : \mathbb{R}^N \rightarrow V = \left\{ (\mathbf{g}^{\mathcal{T}}, \mathbf{f}^{\mathfrak{m}}, \mathbf{f}^{\mathfrak{m}*}, q^{\mathfrak{D}}, \alpha)' \in \mathbb{R}^{N+1}, \sum_{\mathcal{D} \in \mathfrak{D}} m_{\mathcal{D}} q^{\mathfrak{D}} = \sum_{\mathcal{D}_{\sigma, \sigma^*} \in \mathfrak{D}_{ext}} m_{\sigma} \gamma_{\sigma}(\mathbf{g}^{\mathcal{T}}) \cdot \vec{\mathbf{n}}_{\sigma\kappa} \right\}.$$

As $\dim V = N$, it suffices to prove that the matrix A is injective. Let us then study the kernel of the matrix A . Let $\mathbf{u}^{\mathcal{T}} \in \mathbb{E}_0$ and $p^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}}$ such that:

$$\begin{cases} \operatorname{div}^{\mathfrak{m}}(-2\eta^{\mathfrak{D}} \mathbb{D}^{\mathfrak{D}} \mathbf{u}^{\mathcal{T}} + p^{\mathfrak{D}} \operatorname{Id}) = 0, \\ \operatorname{div}^{\mathfrak{m}*}(-2\eta^{\mathfrak{D}} \mathbb{D}^{\mathfrak{D}} \mathbf{u}^{\mathcal{T}} + p^{\mathfrak{D}} \operatorname{Id}) = 0, \\ \operatorname{div}^{\mathfrak{D}}(\mathbf{u}^{\mathcal{T}}) - \lambda h_{\mathfrak{D}}^2 \Delta^{\mathfrak{D}} p^{\mathfrak{D}} = 0, \\ \sum_{\mathcal{D} \in \mathfrak{D}} m_{\mathcal{D}} p^{\mathfrak{D}} = 0, \end{cases}$$

By definition of $\llbracket \cdot, \cdot \rrbracket_{\mathcal{T}}$, we deduce that

$$\llbracket \operatorname{div}^{\mathcal{T}}(-2\eta^{\mathfrak{D}} \mathbb{D}^{\mathfrak{D}} \mathbf{u}^{\mathcal{T}} + p^{\mathfrak{D}} \operatorname{Id}), \mathbf{u}^{\mathcal{T}} \rrbracket_{\mathcal{T}} = 0.$$

Using the discrete Stokes formula Theorem 3.2, noting that $\mathbf{u}^{\mathcal{T}} \in \mathbb{E}_0$ implies that $\gamma^{\mathcal{T}}(\mathbf{u}^{\mathcal{T}}) = 0$, $\mathbb{D}^{\mathfrak{D}} \mathbf{u}^{\mathcal{T}}$ is a symmetric tensor, and substituting $\operatorname{div}^{\mathfrak{D}} \mathbf{u}^{\mathcal{T}} = \operatorname{Trace} \nabla^{\mathfrak{D}} \mathbf{u}^{\mathcal{T}} = (\operatorname{Id} : \nabla^{\mathfrak{D}} \mathbf{u}^{\mathcal{T}})$, we obtain

$$\llbracket \operatorname{div}^{\mathcal{T}}(-2\eta^{\mathfrak{D}} \mathbb{D}^{\mathfrak{D}} \mathbf{u}^{\mathcal{T}} + p^{\mathfrak{D}} \operatorname{Id}), \mathbf{u}^{\mathcal{T}} \rrbracket_{\mathcal{T}} = (2\eta^{\mathfrak{D}} \mathbb{D}^{\mathfrak{D}} \mathbf{u}^{\mathcal{T}} : \mathbb{D}^{\mathfrak{D}} \mathbf{u}^{\mathcal{T}})_{\mathfrak{D}} - (\operatorname{div}^{\mathfrak{D}} \mathbf{u}^{\mathcal{T}}, p^{\mathfrak{D}})_{\mathfrak{D}}.$$

Furthermore, the mass conservation equation and Remark 3.5 give:

$$-(\operatorname{div}^{\mathfrak{D}} \mathbf{u}^{\mathcal{T}}, p^{\mathfrak{D}})_{\mathfrak{D}} = -\lambda (h_{\mathfrak{D}}^2 \Delta^{\mathfrak{D}} p^{\mathfrak{D}}, p^{\mathfrak{D}})_{\mathfrak{D}} = \lambda |p^{\mathfrak{D}}|_h^2,$$

where $|\cdot|_h$ is the seminorm introduced in (3.9). Using the discrete Korn inequality, c.f. Theorem 5.1, and the boundness on η given in (2.3), we have:

$$\llbracket \operatorname{div}^{\mathcal{T}}(-2\eta^{\mathfrak{D}} \mathbb{D}^{\mathfrak{D}} \mathbf{u}^{\mathcal{T}} + p^{\mathfrak{D}} \operatorname{Id}), \mathbf{u}^{\mathcal{T}} \rrbracket_{\mathcal{T}} \geq \underline{C}_{\eta} \|\nabla^{\mathfrak{D}} \mathbf{u}^{\mathcal{T}}\|_2^2 + \lambda |p^{\mathfrak{D}}|_h^2.$$

We finally get

$$0 \geq \underline{C}_{\eta} \|\nabla^{\mathfrak{D}} \mathbf{u}^{\mathcal{T}}\|_2^2 + \lambda |p^{\mathfrak{D}}|_h^2,$$

which implies that

$$\|\nabla^{\mathfrak{D}} \mathbf{u}^{\mathcal{T}}\|_2^2 = 0 \quad \text{and} \quad |p^{\mathfrak{D}}|_h^2 = 0.$$

Therefore, we have that $\nabla^{\mathcal{D}} \mathbf{u}^{\mathcal{T}} = 0$ and $p^{\mathcal{D}}$ is constant for every $\mathcal{D} \in \mathcal{D}$, which implies that there exists three constants $\mathbf{c}_0 \in \mathbb{R}^2$, $\mathbf{c}_1 \in \mathbb{R}^2$ and $c_3 \in \mathbb{R}$ such that :

$$\begin{aligned} \forall \kappa \in (\mathfrak{M} \cup \partial\mathfrak{M}), \quad \mathbf{u}_{\kappa} &= \mathbf{c}_0, \\ \forall \kappa^* \in (\mathfrak{M}^* \cup \partial\mathfrak{M}^*), \quad \mathbf{u}_{\kappa^*} &= \mathbf{c}_1, \\ \forall \mathcal{D} \in \mathcal{D}, \quad p^{\mathcal{D}} &= c_3. \end{aligned}$$

Furthermore, $\mathbf{u}^{\mathcal{T}}$ belongs to \mathbb{E}_0 so that $\mathbf{c}_0 = \mathbf{c}_1 = 0$. Consequently, we have $\mathbf{u}^{\mathcal{T}} = \mathbf{0}$ and $p^{\mathcal{D}} = c_3$. And then, we use the fact that $p^{\mathcal{D}}$ verifies $\sum_{\mathcal{D} \in \mathcal{D}} m_{\mathcal{D}} p^{\mathcal{D}} = 0$ so $p^{\mathcal{D}} = 0$. \square

5. Results on discrete operators. In this section, we present some several results on the discrete operators. In Subsections 5.1, we begin with the properties of the discrete strain rate tensor in order to prove a discrete Korn inequality. In Subsection 5.2, we review the results of [2] and adapt them to the vector-valued. Then, in Subsections 5.3, 5.4 and 5.5, we focus on the properties of mean-value and center value projection operators.

5.1. Properties of discrete strain rate tensor. Korn inequality. We first have the bound of the discrete strain rate tensor by the discrete gradient.

PROPOSITION 5.1. *For all $\mathbf{u}^{\mathcal{T}} \in (\mathbb{R}^2)^{\mathcal{T}}$, we get*

$$\|\mathbb{D}^{\mathcal{D}} \mathbf{u}^{\mathcal{T}}\|_2 \leq \|\nabla^{\mathcal{D}} \mathbf{u}^{\mathcal{T}}\|_2.$$

Proof. Thanks to Remark 2.1 we have

$$\|\mathbb{D}^{\mathcal{D}} \mathbf{u}^{\mathcal{T}}\|_2^2 = \sum_{\mathcal{D} \in \mathcal{D}} m_{\mathcal{D}} \|\mathbb{D}^{\mathcal{D}} \mathbf{u}^{\mathcal{T}}\|_{\mathcal{F}}^2 \leq \sum_{\mathcal{D} \in \mathcal{D}} m_{\mathcal{D}} \|\nabla^{\mathcal{D}} \mathbf{u}^{\mathcal{T}}\|_{\mathcal{F}}^2 = \|\nabla^{\mathcal{D}} \mathbf{u}^{\mathcal{T}}\|_2^2.$$

\square

We introduce the following notation

$$\forall \mathbf{u}, \mathbf{v} \in \mathbb{R}^2, \quad \mathbf{u} \wedge \mathbf{v} = u_1 v_2 - u_2 v_1.$$

From calculus, we know that

$$\operatorname{div} \left({}^t(\nabla \mathbf{u}) \right) = \operatorname{div} (\operatorname{div}(\mathbf{u}) \operatorname{Id}) = \nabla(\operatorname{div}(\mathbf{u})).$$

The corresponding discrete property is proved in the following proposition.

PROPOSITION 5.2. *For all $\mathbf{u}^{\mathcal{T}} = \begin{pmatrix} u^{\mathcal{T}} \\ v^{\mathcal{T}} \end{pmatrix} \in (\mathbb{R}^2)^{\mathcal{T}}$, we have*

$$\begin{aligned} \forall \kappa \in \mathfrak{M}, \quad \operatorname{div}^{\kappa} \left({}^t(\nabla^{\mathcal{D}} \mathbf{u}^{\mathcal{T}}) \right) &= \operatorname{div}^{\kappa} \left(\operatorname{div}^{\mathcal{D}}(\mathbf{u}^{\mathcal{T}}) \operatorname{Id} \right), \\ \forall \kappa^* \in \mathfrak{M}^*, \quad \operatorname{div}^{\kappa^*} \left({}^t(\nabla^{\mathcal{D}} \mathbf{u}^{\mathcal{T}}) \right) &= \operatorname{div}^{\kappa^*} \left(\operatorname{div}^{\mathcal{D}}(\mathbf{u}^{\mathcal{T}}) \operatorname{Id} \right), \\ \forall \kappa^* \in \partial\mathfrak{M}^*, \quad \operatorname{div}^{\kappa^*} \left({}^t(\nabla^{\mathcal{D}} \mathbf{u}^{\mathcal{T}}) \right) &= \operatorname{div}^{\kappa^*} \left(\operatorname{div}^{\mathcal{D}}(\mathbf{u}^{\mathcal{T}}) \operatorname{Id} \right) + \frac{1}{m_{\kappa^*}} \sum_{\mathcal{D} \in \mathcal{D}_{\kappa^*}} \begin{pmatrix} v_{\mathcal{L}} - v_{\kappa} \\ u_{\kappa} - u_{\mathcal{L}} \end{pmatrix} \\ &\quad + \frac{1}{m_{\kappa^*}} \sum_{\mathcal{D} \in \mathcal{D}_{\kappa^*} \cap \mathcal{D}_{ext}} \frac{d_{\kappa^*, \mathcal{L}}}{m_{\sigma}} \begin{pmatrix} v_{\kappa^*} - v_{\mathcal{L}^*} \\ u_{\mathcal{L}^*} - u_{\kappa^*} \end{pmatrix}. \end{aligned}$$

Proof. Let $\kappa \in \mathfrak{M}$. On any diamond $\mathcal{D} \in \mathfrak{D}$, the matrix $\left({}^t(\nabla^{\mathcal{D}} \mathbf{u}^{\mathcal{T}}) - \operatorname{div}^{\mathcal{D}} \mathbf{u}^{\mathcal{T}} \operatorname{Id}\right)$ is given by

$$\left({}^t(\nabla^{\mathcal{D}} \mathbf{u}^{\mathcal{T}}) - \operatorname{div}^{\mathcal{D}} \mathbf{u}^{\mathcal{T}} \operatorname{Id}\right) = \begin{pmatrix} -(\nabla^{\mathcal{D}} v^{\mathcal{T}})_2 & (\nabla^{\mathcal{D}} v^{\mathcal{T}})_1 \\ (\nabla^{\mathcal{D}} u^{\mathcal{T}})_2 & -(\nabla^{\mathcal{D}} u^{\mathcal{T}})_1 \end{pmatrix}.$$

So that

$$m_{\kappa} \operatorname{div}^{\kappa} \left({}^t(\nabla^{\mathcal{D}} \mathbf{u}^{\mathcal{T}}) - \operatorname{div}^{\mathcal{D}}(\mathbf{u}^{\mathcal{T}}) \operatorname{Id} \right) = \sum_{\mathcal{D} \in \mathfrak{D}_{\kappa}} m_{\sigma} \begin{pmatrix} \nabla^{\mathcal{D}} v^{\mathcal{T}} \wedge \vec{\mathbf{n}}_{\sigma \kappa} \\ -\nabla^{\mathcal{D}} u^{\mathcal{T}} \wedge \vec{\mathbf{n}}_{\sigma \kappa} \end{pmatrix}.$$

Using the Definition 3.1 and the fact that $\vec{\mathbf{n}}_{\sigma \kappa} \wedge \vec{\mathbf{n}}_{\sigma^* \kappa^*} = \sin(\alpha_{\mathcal{D}})$, we deduce

$$\nabla^{\mathcal{D}} u^{\mathcal{T}} \wedge \vec{\mathbf{n}}_{\sigma \kappa} = \frac{u_{\mathcal{L}^*} - u_{\kappa^*}}{\sin(\alpha_{\mathcal{D}}) m_{\sigma}} \vec{\mathbf{n}}_{\sigma^* \kappa^*} \wedge \vec{\mathbf{n}}_{\sigma \kappa} = \frac{u_{\kappa^*} - u_{\mathcal{L}^*}}{m_{\sigma}}.$$

We do the same with the first component and we deduce

$$m_{\kappa} \operatorname{div}^{\kappa} \left({}^t(\nabla^{\mathcal{D}} \mathbf{u}^{\mathcal{T}}) - \operatorname{div}^{\mathcal{D}}(\mathbf{u}^{\mathcal{T}}) \operatorname{Id} \right) = \sum_{\mathcal{D} \in \mathfrak{D}_{\kappa}} \begin{pmatrix} v_{\kappa^*} - v_{\mathcal{L}^*} \\ u_{\mathcal{L}^*} - u_{\kappa^*} \end{pmatrix}. \quad (5.1)$$

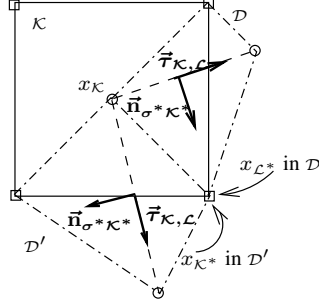


FIG. 5.1. Trigonometrical path.

Recall that in each diamond cell $\mathcal{D} \in \mathfrak{D}_{\kappa}$, the basis $(\vec{\mathbf{r}}_{\kappa, \mathcal{L}}, \vec{\mathbf{n}}_{\sigma^* \kappa^*})$ is supposed to be directly oriented. It implies that for each diamond cell $\mathcal{D} \in \mathfrak{D}_{\kappa}$ the points x_{κ^*} and $x_{\mathcal{L}^*}$ are well defined with this choice and for two diamond cells \mathcal{D} and \mathcal{D}' , such that $\overline{\mathcal{D}} \cap \overline{\mathcal{D}'}$ is a side of \mathcal{D} , the point $x_{\mathcal{L}^*}$ of \mathcal{D} coincides with the point x_{κ^*} of \mathcal{D}' . (see Figure 5.1). Thus the last sum (5.1) is equal to zero,

$$m_{\kappa} \operatorname{div}^{\kappa} \left({}^t(\nabla^{\mathcal{D}} \mathbf{u}^{\mathcal{T}}) - \operatorname{div}^{\mathcal{D}}(\mathbf{u}^{\mathcal{T}}) \operatorname{Id} \right) = 0.$$

Hence for all $\kappa \in \mathfrak{M}$, we conclude

$$\operatorname{div}^{\kappa} \left({}^t(\nabla^{\mathcal{D}} \mathbf{u}^{\mathcal{T}}) \right) = \operatorname{div}^{\kappa} \left(\operatorname{div}^{\mathcal{D}}(\mathbf{u}^{\mathcal{T}}) \operatorname{Id} \right).$$

The same result holds for all $\kappa^* \in \mathfrak{M}^*$:

$$\begin{aligned} m_{\kappa^*} \operatorname{div}^{\kappa^*} \left({}^t(\nabla^{\mathcal{D}} \mathbf{u}^{\mathcal{T}}) - \operatorname{div}^{\mathcal{D}}(\mathbf{u}^{\mathcal{T}}) \operatorname{Id} \right) &= \sum_{\mathcal{D} \in \mathfrak{D}_{\kappa^*}} m_{\sigma^*} \begin{pmatrix} \nabla^{\mathcal{D}} v^{\mathcal{T}} \wedge \vec{\mathbf{n}}_{\sigma^* \kappa^*} \\ -\nabla^{\mathcal{D}} u^{\mathcal{T}} \wedge \vec{\mathbf{n}}_{\sigma^* \kappa^*} \end{pmatrix} \\ &= \sum_{\mathcal{D} \in \mathfrak{D}_{\kappa^*}} \begin{pmatrix} v_{\mathcal{L}} - v_{\kappa} \\ u_{\kappa} - u_{\mathcal{L}} \end{pmatrix} \\ &= 0. \end{aligned}$$

The case where $\kappa^* \in \partial\mathfrak{M}^*$ is slightly different

$$\begin{aligned} m_{\kappa^*} \mathbf{div}^{\kappa^*} \left({}^t(\nabla^{\mathcal{D}} \mathbf{u}^{\mathcal{T}}) - \operatorname{div}^{\mathcal{D}}(\mathbf{u}^{\mathcal{T}}) \operatorname{Id} \right) &= \sum_{\mathcal{D} \in \mathcal{D}_{\kappa^*}} m_{\sigma^*} \left({}^t(\nabla^{\mathcal{D}} \mathbf{u}^{\mathcal{T}}) - \operatorname{div}^{\mathcal{D}} \mathbf{u}^{\mathcal{T}} \operatorname{Id} \right) \vec{\mathbf{n}}_{\sigma^* \kappa^*} \\ &+ \sum_{\mathcal{D} \in \mathcal{D}_{\kappa^*} \cap \mathcal{D}_{ext}} d_{\kappa^*, \mathcal{L}} \left({}^t(\nabla^{\mathcal{D}} \mathbf{u}^{\mathcal{T}}) - \operatorname{div}^{\mathcal{D}} \mathbf{u}^{\mathcal{T}} \operatorname{Id} \right) \vec{\mathbf{n}}_{\sigma \kappa}. \end{aligned}$$

For the first sum on $\mathcal{D} \in \mathcal{D}_{\kappa^*}$, we can do the same as for $\kappa^* \in \mathfrak{M}^*$, and the second one on $\kappa \in \mathfrak{M}$

$$m_{\kappa^*} \mathbf{div}^{\kappa^*} \left({}^t(\nabla^{\mathcal{D}} \mathbf{u}^{\mathcal{T}}) - \operatorname{div}^{\mathcal{D}}(\mathbf{u}^{\mathcal{T}}) \operatorname{Id} \right) = \sum_{\mathcal{D} \in \mathcal{D}_{\kappa^*}} \begin{pmatrix} v_{\mathcal{L}} - v_{\kappa} \\ u_{\kappa} - u_{\mathcal{L}} \end{pmatrix} + \sum_{\mathcal{D} \in \mathcal{D}_{\kappa^*} \cap \mathcal{D}_{ext}} \frac{d_{\kappa^*, \mathcal{L}}}{m_{\sigma}} \begin{pmatrix} v_{\kappa^*} - v_{\mathcal{L}^*} \\ u_{\mathcal{L}^*} - u_{\kappa^*} \end{pmatrix}.$$

□

COROLLARY 5.1. For all $\mathbf{u}^{\mathcal{T}} \in \mathbb{E}_0$, we have

$$\mathbf{div}^{\mathcal{T}} \left({}^t(\nabla^{\mathcal{D}} \mathbf{u}^{\mathcal{T}}) \right) = \mathbf{div}^{\mathcal{T}} \left(\operatorname{div}^{\mathcal{D}}(\mathbf{u}^{\mathcal{T}}) \operatorname{Id} \right). \quad (5.2)$$

Proof. Let $\mathbf{u}^{\mathcal{T}} = \begin{pmatrix} u^{\mathcal{T}} \\ v^{\mathcal{T}} \end{pmatrix} \in \mathbb{E}_0$. Thanks to Proposition 5.2, we just have to prove that

$$\forall \kappa^* \in \partial\mathfrak{M}^*, \sum_{\mathcal{D} \in \mathcal{D}_{\kappa^*}} \begin{pmatrix} v_{\mathcal{L}} - v_{\kappa} \\ u_{\kappa} - u_{\mathcal{L}} \end{pmatrix} + \sum_{\mathcal{D} \in \mathcal{D}_{\kappa^*} \cap \mathcal{D}_{ext}} \frac{d_{\kappa^*, \mathcal{L}}}{m_{\sigma}} \begin{pmatrix} v_{\kappa^*} - v_{\mathcal{L}^*} \\ u_{\mathcal{L}^*} - u_{\kappa^*} \end{pmatrix} = 0.$$

Let $\kappa^* \in \partial\mathfrak{M}^*$. Thanks to the boundary condition $\mathbf{u}^{\mathcal{T}} \in \mathbb{E}_0$, for all $\mathcal{D} \in \mathcal{D}_{\kappa^*} \cap \mathcal{D}_{ext}$, we have $v_{\kappa^*} = v_{\mathcal{L}^*} = u_{\mathcal{L}^*} = u_{\kappa^*} = 0$. It implies that

$$\sum_{\mathcal{D} \in \mathcal{D}_{\kappa^*} \cap \mathcal{D}_{ext}} \frac{d_{\kappa^*, \mathcal{L}}}{m_{\sigma}} \begin{pmatrix} v_{\kappa^*} - v_{\mathcal{L}^*} \\ u_{\mathcal{L}^*} - u_{\kappa^*} \end{pmatrix} = 0.$$

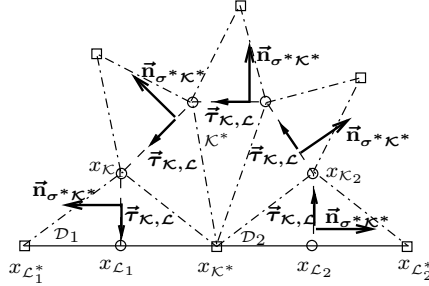


FIG. 5.2. Trigonometrical path.

Using the notation of 5.2, the sum over the diamond cells $\mathcal{D} \in \mathcal{D}_{\kappa^*}$ is equal to

$$\sum_{\mathcal{D} \in \mathcal{D}_{\kappa^*}} \begin{pmatrix} v_{\mathcal{L}} - v_{\kappa} \\ u_{\kappa} - u_{\mathcal{L}} \end{pmatrix} = \begin{pmatrix} v_{\mathcal{L}_1} - v_{\mathcal{L}_2} \\ u_{\mathcal{L}_2} - u_{\mathcal{L}_1} \end{pmatrix}.$$

But we have $\mathbf{u}^{\mathcal{T}} \in \mathbb{E}_0$, it implies that $v_{\mathcal{L}_1} = v_{\mathcal{L}_2} = u_{\mathcal{L}_2} = u_{\mathcal{L}_1} = 0$. We deduce

$$\sum_{\mathcal{D} \in \mathcal{D}_{\kappa^*}} \begin{pmatrix} v_{\mathcal{L}} - v_{\kappa} \\ u_{\kappa} - u_{\mathcal{L}} \end{pmatrix} + \sum_{\mathcal{D} \in \mathcal{D}_{\kappa^*} \cap \mathcal{D}_{ext}} \frac{d_{\kappa^*, \mathcal{L}}}{m_{\sigma}} \begin{pmatrix} v_{\kappa^*} - v_{\mathcal{L}^*} \\ u_{\mathcal{L}^*} - u_{\kappa^*} \end{pmatrix} = 0.$$

It concludes the proof. \square

From Corollary 5.2, we deduce the following proposition.

PROPOSITION 5.3. *For all $\mathbf{u}^\tau \in \mathbb{E}_0$, we have*

$$\left({}^t(\nabla^{\mathfrak{D}} \mathbf{u}^\tau) : \nabla^{\mathfrak{D}} \mathbf{u}^\tau \right)_{\mathfrak{D}} = \|\operatorname{div}^{\mathfrak{D}}(\mathbf{u}^\tau)\|_2^2 \geq 0.$$

Proof. Using the Stokes formula Theorem 3.2, the fact that $\mathbf{u}^\tau \in \mathbb{E}_0$ and then Corollary 5.2, we have

$$\left({}^t(\nabla^{\mathfrak{D}} \mathbf{u}^\tau) : \nabla^{\mathfrak{D}} \mathbf{u}^\tau \right)_{\mathfrak{D}} = -\llbracket \operatorname{div}^\tau \left({}^t(\nabla^{\mathfrak{D}} \mathbf{u}^\tau) \right), \mathbf{u}^\tau \rrbracket_{\mathcal{T}} = -\llbracket \operatorname{div}^\tau (\operatorname{div}^{\mathfrak{D}}(\mathbf{u}^\tau) \operatorname{Id}), \mathbf{u}^\tau \rrbracket_{\mathcal{T}}.$$

Using once more the Stokes formula Theorem 3.2 for $\mathbf{u}^\tau \in \mathbb{E}_0$ and substituting $\operatorname{div}^{\mathfrak{D}} \mathbf{u}^\tau = \operatorname{Trace} \nabla^{\mathfrak{D}} \mathbf{u}^\tau = (\operatorname{Id} : \nabla^{\mathfrak{D}} \mathbf{u}^\tau)$, we obtain

$$\left({}^t(\nabla^{\mathfrak{D}} \mathbf{u}^\tau) : \nabla^{\mathfrak{D}} \mathbf{u}^\tau \right)_{\mathfrak{D}} = (\operatorname{div}^{\mathfrak{D}}(\mathbf{u}^\tau) \operatorname{Id} : \nabla^{\mathfrak{D}} \mathbf{u}^\tau)_{\mathfrak{D}} = \|\operatorname{div}^{\mathfrak{D}}(\mathbf{u}^\tau)\|_2^2 \geq 0.$$

\square

We are now able to prove a discrete Korn inequality:

THEOREM 5.1 (Discrete Korn inequality). *For all $\mathbf{u}^\tau \in \mathbb{E}_0$, we have*

$$\|\nabla^{\mathfrak{D}} \mathbf{u}^\tau\|_2 \leq \sqrt{2} \|\mathbb{D}^{\mathfrak{D}} \mathbf{u}^\tau\|_2$$

Proof. This is just a consequence of the following equality and of Proposition 5.3.

$$\|\mathbb{D}^{\mathfrak{D}} \mathbf{u}^\tau\|_2^2 = \frac{1}{2} \|\nabla^{\mathfrak{D}} \mathbf{u}^\tau\|_2^2 + \frac{1}{2} \left({}^t(\nabla^{\mathfrak{D}} \mathbf{u}^\tau) : \nabla^{\mathfrak{D}} \mathbf{u}^\tau \right)_{\mathfrak{D}}.$$

\square

5.2. Technical lemmas. Poincaré inequality. We will need two technical results and the discrete Poincaré inequality whose proofs can be found in the literature. The first one is [2, Lemma 3.4] (see also [18, Lemma 6.3]).

LEMMA 5.1. *There exists a constant $C > 0$ such that for any bounded polygonal set $\mathcal{P} \subset \mathbb{R}^2$ with positive measure, any segment $\sigma \subset \mathbb{R}^2$ and any $v \in H^1(\mathbb{R}^2)$, we have*

$$|v_{\mathcal{P}} - v_\sigma|^2 \leq \frac{1}{m_\sigma m_{\mathcal{P}}} \int_\sigma \int_{\mathcal{P}} |v(x) - v(y)|^2 dx dy \leq C \frac{\operatorname{diam}(\widehat{\mathcal{P}}_\sigma)^3}{m_\sigma m_{\mathcal{P}}} \int_{\widehat{\mathcal{P}}_\sigma} |\nabla v(z)|^2 dz,$$

where $v_{\mathcal{P}}$ denotes the mean value of v on \mathcal{P} , v_σ the mean value of v on the segment σ , and $\widehat{\mathcal{P}}_\sigma$ is the convex hull of $\mathcal{P} \cup \sigma$.

The second one is the vector-valued version of the first Lemma 8.1 in the Appendix [17].

LEMMA 5.2. *Let K be a non empty open polygonal convex set in \mathbb{R}^2 such that, for some $\alpha > 0$, there exists a ball of radius $\alpha \operatorname{diam}(K)$ contained in K . Let E be an affine hyperplane of \mathbb{R}^2 and σ be a non-empty open subset of E contained in $\partial K \cap E$. Then there exists a constant $C > 0$, depending only on α , for any $\mathbf{v} \in (H^1(K))^2$:*

$$\left| \frac{1}{m_\sigma} \int_\sigma \mathbf{v}(s) ds \right|^2 \leq \frac{C \operatorname{diam}(K)}{m_\sigma} \int_K \|\nabla \mathbf{v}(s)\|_{\mathcal{F}}^2 ds + \frac{C}{\operatorname{diam}(K) m_\sigma} \int_K |\mathbf{v}(s)|^2 ds.$$

Let us finally state the discrete version of the Poincaré inequality which the vector-valued version of [2, Lemma 3.3].

THEOREM 5.2 (Discrete Poincaré inequality). *Let \mathcal{T} be a mesh of Ω . There exists a constant $C > 0$, depending only on the diameter of Ω and $\text{reg}(\mathcal{T})$, such that for any $\mathbf{g} \in (H^{\frac{1}{2}}(\partial\Omega))^2$ and any $\mathbf{u}^\tau \in \mathbb{E}_{m,g}$, we have*

$$\|\mathbf{u}^\tau\|_2 \leq C \left(\|\nabla^\mathfrak{D} \mathbf{u}^\tau\|_2 + \|\mathbf{g}\|_{H^{\frac{1}{2}}(\partial\Omega)} \right).$$

In the following Lemma, we bound the discrete seminorm $|\cdot|_h$, defined by (3.9), by the L^2 -norm $\|\cdot\|_2$.

LEMMA 5.3. *Let \mathcal{T} be a mesh of Ω . There exists $C > 0$ depending only on $\text{reg}(\mathcal{T})$, such that for any $p^\mathfrak{D} \in \mathbb{R}^\mathfrak{D}$, we have*

$$|p^\mathfrak{D}|_h \leq C \|p^\mathfrak{D}\|_2.$$

Proof. Using the definition (3.9) of the discrete seminorm $|\cdot|_h$, we have

$$\begin{aligned} |p^\mathfrak{D}|_h^2 &= \sum_{s=\mathfrak{D}|\mathfrak{D}' \in \mathfrak{G}} (h_{\mathfrak{D}}^2 + h_{\mathfrak{D}'}^2) (p^{\mathfrak{D}'} - p^\mathfrak{D})^2 \\ &\leq 2 \sum_{s=\mathfrak{D}|\mathfrak{D}' \in \mathfrak{G}} (h_{\mathfrak{D}}^2 + h_{\mathfrak{D}'}^2) ((p^{\mathfrak{D}'})^2 + (p^\mathfrak{D})^2). \end{aligned}$$

Reordering the summation over the set of diamond cells, we get

$$|p^\mathfrak{D}|_h^2 \leq 2 \sum_{\mathfrak{D} \in \mathfrak{D}} m_{\mathfrak{D}} (p^\mathfrak{D})^2 \left(\frac{1}{m_{\mathfrak{D}}} \sum_{s \in \mathcal{E}_{\mathfrak{D}}} (h_{\mathfrak{D}}^2 + h_{\mathfrak{D}'}^2) \right).$$

We conclude, using Definition (3.1) and the fact that $\#(\mathcal{E}_{\mathfrak{D}}) \leq 4$. \square

5.3. Properties of the mean-value and center-value projection operators. In this subsection, we give some Lemmas on projection operators. We only prove results which can not be deduced from their scalar counterpart (given in [2]) by working component per component. First of all, we need to recall briefly the definitions and the main properties of the Sobolev spaces defined on $\partial\Omega$. A complete study of these topics can be found, for instance, in [23]. The domain Ω is supposed to be polygonal and we note $\Gamma_1, \dots, \Gamma_k$ the sides of Ω . We define the following space:

$$\tilde{H}^{\frac{3}{2}}(\partial\Omega) = \left\{ g \in H^1(\partial\Omega) \text{ s. t. } \sum_{i=1}^k \int_{\Gamma_i} \int_{\Gamma_i} \left| \frac{\nabla g(x) - \nabla g(y)}{|x-y|^{\frac{1}{2}}} \right|^2 \frac{dx dy}{|x-y|} < +\infty \right\}, \quad (5.3)$$

with the following norm

$$\|g\|_{\tilde{H}^{\frac{3}{2}}(\partial\Omega)}^2 = \sum_{i=1}^k \int_{\Gamma_i} \int_{\Gamma_i} \left| \frac{\nabla g(x) - \nabla g(y)}{|x-y|^{\frac{1}{2}}} \right|^2 \frac{dx dy}{|x-y|}.$$

We recall that the trace operator γ is continuous from $H^2(\Omega)$ onto a finite codimensional subset of $\tilde{H}^{\frac{3}{2}}(\partial\Omega)$.

The following properties of the center-value projection operator, defined by (3.5), are used in the estimate of the consistency error of our finite volume scheme.

LEMMA 5.4. *Let \mathcal{T} be a mesh of Ω . There exists a constant $C > 0$, depending only on $\text{reg}(\mathcal{T})$, such that for any function \mathbf{v} in $(H^2(\Omega))^2$, we have*

$$\|\nabla \mathbf{v} - \nabla^\mathfrak{D} \mathbb{P}_c^\tau \mathbf{v}\|_2 \leq C \text{size}(\mathcal{T}) \|\nabla \mathbf{v}\|_{H^1}.$$

COROLLARY 5.2. *Let \mathcal{T} be a mesh of Ω . There exists a constant $C > 0$, depending only on $\text{reg}(\mathcal{T})$, such that for any function \mathbf{v} in $(H^2(\Omega))^2$, we have*

$$\|\nabla^{\mathcal{D}} \mathbb{P}_c^{\mathcal{T}} \mathbf{v}\|_2 \leq C \|\nabla \mathbf{v}\|_{H^1}.$$

Proof. Let $\mathbf{v} \in (H^2(\Omega))^2$. Using the triangle inequality and Lemma 5.4, we have

$$\|\nabla^{\mathcal{D}} \mathbb{P}_c^{\mathcal{T}} \mathbf{v}\|_2 \leq \|\nabla^{\mathcal{D}} \mathbb{P}_c^{\mathcal{T}} \mathbf{v} - \nabla \mathbf{v}\|_2 + \|\nabla \mathbf{v}\|_2 \leq (C \text{size}(\mathcal{T}) + 1) \|\nabla \mathbf{v}\|_{H^1}.$$

□

COROLLARY 5.3. *Let \mathcal{T} be a mesh of Ω . There exists a constant $C > 0$, depending only on $\text{reg}(\mathcal{T})$, such that for any function \mathbf{v} in $(H^2(\Omega))^2$ which satisfies $\text{div} \mathbf{v} = 0$, we have*

$$\|\text{div}^{\mathcal{D}} \mathbb{P}_c^{\mathcal{T}} \mathbf{v}\|_2 \leq C \text{size}(\mathcal{T}) \|\nabla \mathbf{v}\|_{H^1}.$$

Proof. Let $\mathbf{v} \in (H^2(\Omega))^2$ which satisfies $\text{div} \mathbf{v} = 0$. Let $\mathcal{D} \in \mathfrak{D}$; using the fact that $\text{div}^{\mathcal{D}}(\mathbb{P}_c^{\mathcal{T}} \mathbf{v}) = \text{Trace}(\nabla^{\mathcal{D}} \mathbb{P}_c^{\mathcal{T}} \mathbf{v})$ and $\text{div} \mathbf{v} = \text{Trace}(\nabla \mathbf{v}) = 0$, we have

$$\text{div}^{\mathcal{D}}(\mathbb{P}_c^{\mathcal{T}} \mathbf{v}) = \text{Trace}(\nabla^{\mathcal{D}} \mathbb{P}_c^{\mathcal{T}} \mathbf{v} - \nabla \mathbf{v}(x)), \quad \forall x \in \mathcal{D}.$$

This gives, by applying Lemma 5.4:

$$\|\text{div}^{\mathcal{D}}(\mathbb{P}_c^{\mathcal{T}} \mathbf{v})\|_2 \leq \|\nabla^{\mathcal{D}} \mathbb{P}_c^{\mathcal{T}} \mathbf{v} - \nabla \mathbf{v}\|_2 \leq C \text{size}(\mathcal{T}) \|\nabla \mathbf{v}\|_{H^1}.$$

□

We will need to evaluate the contribution in the error of the two different projections $\mathbb{P}_c^{\mathcal{T}} \mathbf{v}$, $\mathfrak{P}_{m,g} \mathbb{P}_c^{\mathcal{T}} \mathbf{v}$, defined by (3.5) and (3.7), where $\mathbf{g} = \gamma(\mathbf{v})$. Note that this two projections only differ on the boundary cells.

LEMMA 5.5. *Let \mathcal{T} be a mesh of Ω . There exists a constant $C > 0$, depending only on $\text{reg}(\mathcal{T})$, such that for any function \mathbf{v} in $(H^2(\Omega))^2$, whose trace is denoted by $\mathbf{g} = \gamma(\mathbf{v})$, we have*

$$\|\nabla^{\mathcal{D}} \mathbb{P}_c^{\mathcal{T}} \mathbf{v} - \nabla^{\mathcal{D}} \mathfrak{P}_{m,g} \mathbb{P}_c^{\mathcal{T}} \mathbf{v}\|_2 \leq C \text{size}(\mathcal{T}) \|\mathbf{g}\|_{(\tilde{H}^{\frac{3}{2}}(\partial\Omega))^2} \leq C \text{size}(\mathcal{T}) \|\mathbf{v}\|_{H^2}.$$

Next lemma gives the main properties of the mean-value projection, defined by (3.4), of a H^1 functions.

LEMMA 5.6. *Let \mathcal{T} be a mesh of Ω . There exists $C > 0$ depending only on $\text{reg}(\mathcal{T})$, such that:*

$$\begin{aligned} \|\nabla^{\mathcal{D}} \mathbb{P}_m^{\mathcal{T}} \mathbf{v}\|_2 &\leq C \|\nabla \mathbf{v}\|_2, & \forall \mathbf{v} \in (H^1(\Omega))^2, \\ \|\mathbf{v} - \mathbb{P}_m^{\mathcal{T}} \mathbf{v}\|_2 &\leq C \text{size}(\mathcal{T}) \|\nabla \mathbf{v}\|_2, & \forall \mathbf{v} \in (H^1(\Omega))^2. \end{aligned}$$

We give below the main properties of the center-value projection, defined by (3.5), onto the set of discrete functions in our framework.

LEMMA 5.7. *Let \mathcal{T} be a mesh of Ω . There exists $C > 0$ depending only on $\text{reg}(\mathcal{T})$, such that:*

$$\begin{aligned} \|\mathbf{v} - \mathbb{P}_c^{\mathcal{T}} \mathbf{v}\|_2 &\leq C \text{size}(\mathcal{T}) \|\nabla \mathbf{v}\|_{H^1}, & \forall \mathbf{v} \in (H^2(\Omega))^2, \\ \|\mathbf{v} - \mathfrak{P}_{m,g} \mathbb{P}_c^{\mathcal{T}} \mathbf{v}\|_2 &\leq C \text{size}(\mathcal{T}) \|\nabla \mathbf{v}\|_{H^1}, & \forall \mathbf{v} \in (H^2(\Omega))^2, \end{aligned}$$

where $\mathbf{g} = \gamma(\mathbf{v})$.

Proof. We only prove the second inequality. The definition of the projection $\mathfrak{P}_{m,g}$ implies that $\mathfrak{P}_{m,g}\mathbb{P}_c^\tau \mathbf{v}$ and $\mathbb{P}_c^\tau \mathbf{v}$ only differ on the boundary $\partial\mathfrak{M}$ and $\partial\mathfrak{M}^*$ whereas $\mathfrak{P}_{m,g}\mathbb{P}_c^\tau \mathbf{v}$ and $\mathbb{P}_m^\tau \mathbf{v}$ coincide on the boundary $\partial\mathfrak{M}$ and $\partial\mathfrak{M}^*$. So we get

$$\begin{aligned} \|\mathbf{v} - \mathfrak{P}_{m,g}\mathbb{P}_c^\tau \mathbf{v}\|_2^2 &= \frac{1}{2} \sum_{\kappa \in \mathfrak{M}} \int_{\kappa} |\mathbf{v}(x) - \mathbf{v}(x_\kappa)|^2 dx + \frac{1}{2} \sum_{\kappa^* \in \mathfrak{M}^*} \int_{\kappa^*} |\mathbf{v}(x) - \mathbf{v}(x_{\kappa^*})|^2 dx \\ &\quad + \frac{1}{2} \sum_{\kappa^* \in \partial\mathfrak{M}^*} \int_{\kappa^*} \left| \mathbf{v}(x) - \frac{1}{m_{\sigma_{\kappa^*}}} \int_{\sigma_{\kappa^*}} \mathbf{v}(z) dz \right|^2 dx. \end{aligned}$$

We deduce that

$$\|\mathbf{v} - \mathfrak{P}_{m,g}\mathbb{P}_c^\tau \mathbf{v}\|_2^2 \leq \|\mathbf{v} - \mathbb{P}_c^\tau \mathbf{v}\|_2^2 + \|\mathbf{v} - \mathbb{P}_m^\tau \mathbf{v}\|_2^2.$$

Lemma 5.6 and the first inequality conclude the proof. \square

5.4. Properties of mean-value projection operator on the diamond mesh. The error estimates analysis of our scheme involves some discrete approximation of test functions lying in $H^1(\Omega)$ for the pressure. Since the pressure is defined on the diamond mesh, the natural projection (as these test functions may not be continuous) is the mean-value projection \mathbb{P}_m^D , defined by (3.6), on the diamond mesh. We give below a property of such a projection onto the set of discrete functions in our framework.

LEMMA 5.8. *Let \mathcal{T} be a mesh of Ω . There exists $C > 0$ depending only on $\text{reg}(\mathcal{T})$, such that for any function p in $H^1(\Omega)$, we have*

$$\sum_{\mathfrak{s}=\mathcal{D}|\mathcal{D}' \in \mathfrak{G}} (\mathbb{P}_m^{\mathcal{D}'} p - \mathbb{P}_m^{\mathcal{D}} p)^2 \leq C \|\nabla p\|_2^2.$$

Proof. Let $p \in H^1(\Omega)$. We note for simplicity $p^{\mathcal{D}} = \mathbb{P}_m^{\mathcal{D}} p$ for any $\mathcal{D} \in \mathfrak{D}$. Introducing the value $p^{\mathfrak{s}} := \frac{1}{m_{\mathfrak{s}}} \int_{\mathfrak{s}} p(y) dy$, for any $\mathfrak{s} \in \mathfrak{G}$ (see Figure 5.3), we have the following equality

$$\sum_{\mathfrak{s}=\mathcal{D}|\mathcal{D}' \in \mathfrak{G}} (p^{\mathcal{D}'} - p^{\mathcal{D}})^2 = \sum_{\mathfrak{s}=\mathcal{D}|\mathcal{D}' \in \mathfrak{G}} m_{\mathfrak{s}^*}^2 \left| \frac{(p^{\mathcal{D}'} - p^{\mathfrak{s}}) - (p^{\mathcal{D}} - p^{\mathfrak{s}})}{m_{\mathfrak{s}^*}} \right|^2.$$

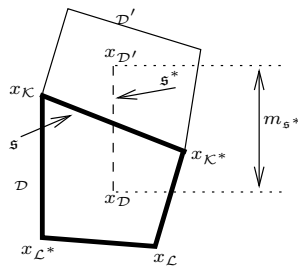


FIG. 5.3. Recall of notation

Then by using Young's inequality, we obtain

$$\sum_{\mathfrak{s}=\mathcal{D}|\mathcal{D}' \in \mathfrak{G}} (p^{\mathcal{D}'} - p^{\mathcal{D}})^2 \leq 2 \sum_{\mathfrak{s}=\mathcal{D}|\mathcal{D}' \in \mathfrak{G}} m_{\mathfrak{s}^*}^2 \left| \frac{p^{\mathcal{D}'} - p^{\mathfrak{s}}}{m_{\mathfrak{s}^*}} \right|^2 + 2 \sum_{\mathfrak{s}=\mathcal{D}|\mathcal{D}' \in \mathfrak{G}} m_{\mathfrak{s}^*}^2 \left| \frac{p^{\mathcal{D}} - p^{\mathfrak{s}}}{m_{\mathfrak{s}^*}} \right|^2.$$

Lemma 5.1 applied on a edge \mathfrak{s} and the diamond cell \mathcal{D} , leads to

$$\left| \frac{p^{\mathcal{D}} - p^{\mathfrak{s}}}{m_{\mathfrak{s}^*}} \right|^2 \leq C \frac{(h_{\mathcal{D}})^3}{m_{\mathfrak{s}^*}^2 m_{\mathfrak{s}} m_{\mathcal{D}}} \int_{\mathcal{D}} |\nabla p(z)|^2 dz.$$

Multiplying by $m_{\mathfrak{s}^*}^2$, we deduce

$$m_{\mathfrak{s}^*}^2 \left| \frac{p^{\mathcal{D}} - p^{\mathfrak{s}}}{m_{\mathfrak{s}^*}} \right|^2 \leq C \frac{(h_{\mathcal{D}})^3}{m_{\mathfrak{s}} m_{\mathcal{D}}} \int_{\mathcal{D}} |\nabla p(z)|^2 dz.$$

As $\frac{(h_{\mathcal{D}})^3}{m_{\mathfrak{s}} m_{\mathcal{D}}} \leq C(\text{reg}(T))$, (see (3.1)), we obtain

$$m_{\mathfrak{s}^*}^2 \left| \frac{p^{\mathcal{D}} - p^{\mathfrak{s}}}{m_{\mathfrak{s}^*}} \right|^2 \leq C \int_{\mathcal{D}} |\nabla p(z)|^2 dz.$$

One gets

$$\begin{aligned} \sum_{\mathfrak{s}=\mathcal{D}|\mathcal{D}' \in \mathfrak{S}} (p^{\mathcal{D}'} - p^{\mathcal{D}})^2 &\leq C \sum_{\mathfrak{s}=\mathcal{D}|\mathcal{D}' \in \mathfrak{S}} \left(\int_{\mathcal{D}} |\nabla p(z)|^2 dz + \int_{\mathcal{D}'} |\nabla p(z)|^2 dz \right) \\ &\leq 2C \sum_{\mathcal{D} \in \mathfrak{D}} \int_{\mathcal{D}} |\nabla p(z)|^2 dz \\ &\leq C \mathcal{N}_T \int_{\Omega} |\nabla p(z)|^2 dz, \end{aligned}$$

and the claim is proved. \square

PROPOSITION 5.4. *For any mesh \mathcal{T} on Ω , there exists a constant $C > 0$, depending only on $\text{reg}(T)$, such that for any function p in $H^1(\Omega)$, we have*

$$\|\mathbb{P}_m^{\mathfrak{D}} p - p\|_2 \leq C \text{size}(\mathcal{T}) \|\nabla p\|_2.$$

Proof. Let $\mathcal{D} \in \mathfrak{D}$. We apply the Cauchy-Schwarz inequality

$$\int_{\mathcal{D}} |\mathbb{P}_m^{\mathcal{D}} p - p(x)|^2 dx \leq \frac{1}{m_{\mathcal{D}}} \int_{\mathcal{D}} \int_{\mathcal{D}} |p(z) - p(x)|^2 dz dx.$$

Introducing the value $\frac{1}{m_{\mathfrak{s}}} \int_{\mathfrak{s}} p(y) dy$ for $\mathfrak{s} \in \mathcal{E}_{\mathcal{D}}$, we get

$$\begin{aligned} \int_{\mathcal{D}} |\mathbb{P}_m^{\mathcal{D}} p - p(x)|^2 dx &\leq \frac{2}{m_{\mathcal{D}}} \int_{\mathcal{D}} \int_{\mathcal{D}} \left| \frac{1}{m_{\mathfrak{s}}} \int_{\mathfrak{s}} (p(z) - p(y)) dy \right|^2 dz dx \\ &\quad + \frac{2}{m_{\mathcal{D}}} \int_{\mathcal{D}} \int_{\mathcal{D}} \left| \frac{1}{m_{\mathfrak{s}}} \int_{\mathfrak{s}} (p(y) - p(x)) dy \right|^2 dz dx. \end{aligned}$$

Applying once more the Cauchy-Schwarz inequality, we have

$$\int_{\mathcal{D}} |\mathbb{P}_m^{\mathcal{D}} p - p(x)|^2 dx \leq \frac{4}{m_{\mathfrak{s}}} \int_{\mathcal{D}} \int_{\mathfrak{s}} |p(z) - p(y)|^2 dy dz.$$

We apply Lemma 5.1 on a edge \mathfrak{s} and a diamond cell \mathcal{D} . There exists a constant $C > 0$, such that:

$$\frac{1}{m_{\mathfrak{s}}} \int_{\mathcal{D}} \int_{\mathfrak{s}} |p(y) - p(x)|^2 dy dx \leq C \frac{h_{\mathcal{D}}^3}{m_{\mathfrak{s}}} \int_{\mathcal{D}} |\nabla p(z)|^2 dz.$$

The definition (3.1) gives that $\frac{h_{\mathcal{D}}}{m_{\mathfrak{s}}} \leq C$. Finally, we obtain

$$\int_{\mathcal{D}} |\mathbb{P}_m^{\mathcal{D}} p - p(x)|^2 dx \leq \text{size}(\mathcal{T})^2 C \int_{\mathcal{D}} |\nabla p(z)|^2 dz$$

which concludes the proof. \square

We can immediately deduce that the following property of the mean-value projection on the diamond mesh \mathcal{D} holds.

COROLLARY 5.4. *For any mesh \mathcal{T} on Ω , there exists a constant $C > 0$, depending only on $\text{reg}(\mathcal{T})$, such that for any function p in $H^1(\Omega)$, we have*

$$\|\mathbb{P}_m^{\mathcal{D}} p\|_2 \leq C \|p\|_{H^1}.$$

Proof. The triangle inequality and Proposition 5.4 give

$$\|\mathbb{P}_m^{\mathcal{D}} p\|_2 \leq \|\mathbb{P}_m^{\mathcal{D}} p - p\|_2 + \|p\|_2 \leq C \text{size}(\mathcal{T}) \|\nabla p\|_2 + \|p\|_2,$$

which concludes the proof. \square

5.5. Properties of the discrete operator $\text{div}^{\mathcal{D}}$. In order to prove the stability of our finite volume scheme, we will need the following estimate.

PROPOSITION 5.5. *Let \mathcal{T} be a mesh of Ω . There exists $C > 0$ depending only on $\text{reg}(\mathcal{T})$, such that for any function \mathbf{v} in $(H^1(\Omega))^2$ and any $p^{\mathcal{D}} \in \mathbb{R}^{\mathcal{D}}$, we have*

$$\sum_{\mathcal{D} \in \mathcal{D}} \int_{\mathcal{D}} p^{\mathcal{D}} (\text{div}^{\mathcal{D}}(\mathbf{v}^{\mathcal{T}}) - \text{div}(\mathbf{v})) dz \leq C |p^{\mathcal{D}}|_h \|\mathbf{v}\|_{H^1}.$$

where $\mathbf{v}^{\mathcal{T}} = \mathbb{P}_m^{\mathcal{T}} \mathbf{v}$ is the mean-value projection of \mathbf{v} , defined by (3.4), on the mesh \mathcal{T} .

Proof. Let $\mathcal{D} \in \mathcal{D}$, by using the equality (3.8) and the Stokes formula Theorem 3.2, we have:

$$\int_{\mathcal{D}} (\text{div}^{\mathcal{D}}(\mathbf{v}^{\mathcal{T}}) - \text{div}(\mathbf{v}(z))) dz = \sum_{\mathfrak{s} \in \mathcal{E}_{\mathcal{D}}} m_{\mathfrak{s}} \frac{1}{m_{\mathfrak{s}}} \int_{\mathfrak{s}} \left(\frac{\mathbf{v}_{\mathfrak{K}} + \mathbf{v}_{\mathfrak{K}^*}}{2} - \mathbf{v}(z), \vec{\mathbf{n}}_{\mathfrak{s}\mathcal{D}} \right) dz.$$

We note $\mathbf{R}_{\text{div}}^{\mathfrak{s}}(\mathbf{v}) = \frac{1}{m_{\mathfrak{s}}} \int_{\mathfrak{s}} \left(\frac{\mathbf{v}_{\mathfrak{K}} + \mathbf{v}_{\mathfrak{K}^*}}{2} - \mathbf{v}(z) \right) dz$. First we multiply by $p^{\mathcal{D}}$ and we sum over the diamond cells $\mathcal{D} \in \mathcal{D}$. Then by reordering the summation over $\mathfrak{s} \in \mathcal{S}$, we get

$$\begin{aligned} \sum_{\mathcal{D} \in \mathcal{D}} \int_{\mathcal{D}} p^{\mathcal{D}} (\text{div}^{\mathcal{D}}(\mathbf{v}^{\mathcal{T}}) - \text{div}(\mathbf{v}(z))) dz &= \sum_{\mathcal{D} \in \mathcal{D}} p^{\mathcal{D}} \sum_{\mathfrak{s} \in \mathcal{E}_{\mathcal{D}}} m_{\mathfrak{s}} (\mathbf{R}_{\text{div}}^{\mathfrak{s}}(\mathbf{v}), \vec{\mathbf{n}}_{\mathfrak{s}\mathcal{D}}) \\ &= \sum_{\mathfrak{s} = \mathcal{D} | \mathcal{D}' \in \mathcal{S}} m_{\mathfrak{s}} (\mathbf{R}_{\text{div}}^{\mathfrak{s}}(\mathbf{v}), \vec{\mathbf{n}}_{\mathfrak{s}\mathcal{D}}) (p^{\mathcal{D}} - p^{\mathcal{D}'}). \end{aligned}$$

We apply the Cauchy-Schwarz inequality

$$\sum_{\mathcal{D} \in \mathcal{D}} \int_{\mathcal{D}} p^{\mathcal{D}} (\text{div}^{\mathcal{D}}(\mathbf{v}^{\mathcal{T}}) - \text{div}(\mathbf{v}(z))) dz \leq |p^{\mathcal{D}}|_h \left(\sum_{\mathfrak{s} = \mathcal{D} | \mathcal{D}' \in \mathcal{S}} \frac{m_{\mathfrak{s}}^2}{h_{\mathcal{D}}^2 + h_{\mathcal{D}'}^2} |\mathbf{R}_{\text{div}}^{\mathfrak{s}}(\mathbf{v})|^2 \right)^{\frac{1}{2}}.$$

In order to conclude, it remains to prove that

$$\left(\sum_{\mathfrak{s} = \mathcal{D} | \mathcal{D}' \in \mathcal{S}} \frac{m_{\mathfrak{s}}^2}{h_{\mathcal{D}}^2 + h_{\mathcal{D}'}^2} |\mathbf{R}_{\text{div}}^{\mathfrak{s}}(\mathbf{v})|^2 \right)^{\frac{1}{2}} \leq C \|\mathbf{v}\|_{H^1},$$

which is done in the following Lemma. \square

LEMMA 5.9. *Let \mathcal{T} be a mesh of Ω . There exists $C > 0$ depending only on $\text{reg}(\mathcal{T})$, such that for any function \mathbf{v} in $(H^1(\Omega))^2$, we have*

$$\sum_{\mathfrak{s}=\mathcal{D}|\mathcal{D}'\in\mathfrak{S}} \frac{m_{\mathfrak{s}}^2}{h_{\mathcal{D}}^2 + h_{\mathcal{D}'}^2} |\mathbf{R}_{\text{div}}^{\mathfrak{s}}(\mathbf{v})|^2 \leq C \|\mathbf{v}\|_{H^1}^2,$$

where $\mathbf{v}^{\mathcal{T}} = \mathbb{P}_m^{\mathcal{T}} \mathbf{v}$ and $\mathbf{R}_{\text{div}}^{\mathfrak{s}}(\mathbf{v}) = \frac{1}{m_{\mathfrak{s}}} \int_{\mathfrak{s}} \left(\frac{\mathbf{v}_{\mathcal{K}} + \mathbf{v}_{\mathcal{K}^*}}{2} - \mathbf{v}(z) \right) dz$.

Proof. Let $\mathbf{v} \in (H^1(\Omega))^2$. We note $\mathbf{v} = \begin{pmatrix} v^1 \\ v^2 \end{pmatrix}$ and the value $\mathbf{v}_{\mathfrak{s}} := \frac{1}{m_{\mathfrak{s}}} \int_{\mathfrak{s}} \mathbf{v}(y) dy$, for any $\mathfrak{s} \in \mathfrak{S}$. We prove the result for any component of \mathbf{v} , and then for the function \mathbf{v} . For $i = 1, 2$. We apply Lemma 5.1 on a edge \mathfrak{s} and the open ball $B_{\mathcal{K}}$:

$$\begin{aligned} |v_{\mathcal{K}}^i - v_{\mathfrak{s}}^i|^2 &= \left| \frac{1}{m_{\mathfrak{s}} m_{B_{\mathcal{K}}}} \int_{B_{\mathcal{K}}} \int_{\mathfrak{s}} (v^i(z) - v^i(x)) dz dx \right|^2 \\ &\leq \frac{1}{m_{\mathfrak{s}} m_{B_{\mathcal{K}}}} \int_{B_{\mathcal{K}}} \int_{\mathfrak{s}} |v^i(z) - v^i(x)|^2 dz dx \\ &\leq C \frac{(\text{diam}(\widehat{B_{\mathcal{K}} \cup \mathfrak{s}}))^3}{m_{\mathfrak{s}} m_{B_{\mathcal{K}}}} \int_{\widehat{B_{\mathcal{K}} \cup \mathfrak{s}}} |\nabla v^i(z)|^2 dz. \end{aligned}$$

Thanks to (3.1) we have $\frac{m_{\mathfrak{s}}^2 (\text{diam}(\widehat{B_{\mathcal{K}} \cup \mathfrak{s}}))^3}{m_{\mathfrak{s}} m_{B_{\mathcal{K}}} (h_{\mathcal{D}}^2 + h_{\mathcal{D}'}^2)} \leq C$. It implies that

$$\frac{m_{\mathfrak{s}}^2}{h_{\mathcal{D}}^2 + h_{\mathcal{D}'}^2} |v_{\mathcal{K}}^i - v_{\mathfrak{s}}^i|^2 \leq C \int_{\widehat{B_{\mathcal{K}} \cup \mathfrak{s}}} |\nabla v^i(z)|^2 dz.$$

We deduce that

$$\begin{aligned} \frac{m_{\mathfrak{s}}^2}{h_{\mathcal{D}}^2 + h_{\mathcal{D}'}^2} |\mathbf{R}_{\text{div}}^{\mathfrak{s}}(v^i)|^2 &\leq \frac{m_{\mathfrak{s}}^2}{h_{\mathcal{D}}^2 + h_{\mathcal{D}'}^2} |v_{\mathcal{K}}^i - v_{\mathfrak{s}}^i|^2 + \frac{m_{\mathfrak{s}}^2}{h_{\mathcal{D}}^2 + h_{\mathcal{D}'}^2} |v_{\mathcal{K}^*}^i - v_{\mathfrak{s}}^i|^2 \\ &\leq C \int_{\widehat{B_{\mathcal{K}} \cup \mathfrak{s}}} |\nabla v^i(z)|^2 dz. \end{aligned}$$

Thus, we obtain for $i = 1, 2$

$$\sum_{\mathfrak{s}=\mathcal{D}|\mathcal{D}'\in\mathfrak{S}} \frac{m_{\mathfrak{s}}^2}{h_{\mathcal{D}}^2 + h_{\mathcal{D}'}^2} |\mathbf{R}_{\text{div}}^{\mathfrak{s}}(v^i)|^2 \leq C \int_{\Omega} |\nabla v^i(z)|^2 dz.$$

Suming over i gives the result. \square

6. Stability of the scheme. We used in Section 4 the stabilization to prove the well-posedness of our scheme. In this section, we prove the uniform stability of the DDFV scheme thanks to the stabilization term. We note in the following, for all $\mathbf{u}^{\mathcal{T}}, \tilde{\mathbf{u}}^{\mathcal{T}}$ and $p^{\mathcal{D}}, \tilde{p}^{\mathcal{D}}$ be two elements of respectively $(\mathbb{R}^2)^{\mathcal{T}}$ and $\mathbb{R}^{\mathcal{D}}$, with $\lambda > 0$,

$$B(\mathbf{u}^{\mathcal{T}}, p^{\mathcal{D}}; \tilde{\mathbf{u}}^{\mathcal{T}}, \tilde{p}^{\mathcal{D}}) = \llbracket \text{div}^{\mathcal{T}}(-2\eta^{\mathcal{D}} \mathbf{D}^{\mathcal{D}} \mathbf{u}^{\mathcal{T}} + p^{\mathcal{D}} \text{Id}), \tilde{\mathbf{u}}^{\mathcal{T}} \rrbracket_{\mathcal{T}} + (\text{div}^{\mathcal{D}}(\mathbf{u}^{\mathcal{T}}) - \lambda h_{\mathcal{D}}^2 \Delta^{\mathcal{D}} p^{\mathcal{D}}, \tilde{p}^{\mathcal{D}})_{\mathcal{D}}. \quad (6.1)$$

THEOREM 6.1 (Stability of the scheme). *Assume that η satisfies (2.3). Then there exists $C_1 > 0$ and $C_2 > 0$, depending only on the diameter of Ω , λ , \underline{C}_{η} , \bar{C}_{η} and $\text{reg}(\mathcal{T})$, such that*

for each pair $(\mathbf{u}^\tau, p^\mathfrak{D}) \in \mathbb{E}_0 \times \mathbb{R}^\mathfrak{D}$ such that $\sum_{\mathfrak{D} \in \mathfrak{D}} m_\mathfrak{D} p^\mathfrak{D} = 0$, there exists $\tilde{\mathbf{u}}^\tau \in \mathbb{E}_0$ and $\tilde{p}^\mathfrak{D} \in \mathbb{R}^\mathfrak{D}$:

$$\|\nabla^\mathfrak{D} \tilde{\mathbf{u}}^\tau\|_2 + \|\tilde{p}^\mathfrak{D}\|_2 \leq C_1 (\|\nabla^\mathfrak{D} \mathbf{u}^\tau\|_2 + \|p^\mathfrak{D}\|_2), \quad (6.2)$$

and

$$\|\nabla^\mathfrak{D} \mathbf{u}^\tau\|_2^2 + \|p^\mathfrak{D}\|_2^2 \leq C_2 B(\mathbf{u}^\tau, p^\mathfrak{D}; \tilde{\mathbf{u}}^\tau, \tilde{p}^\mathfrak{D}). \quad (6.3)$$

Proof. Let $(\mathbf{u}^\tau, p^\mathfrak{D}) \in \mathbb{E}_0 \times \mathbb{R}^\mathfrak{D}$ such that $\sum_{\mathfrak{D} \in \mathfrak{D}} m_\mathfrak{D} p^\mathfrak{D} = 0$. The proof of this Theorem is obtained by building explicitly $(\tilde{\mathbf{u}}^\tau, \tilde{p}^\mathfrak{D}) \in \mathbb{E}_0 \times \mathbb{R}^\mathfrak{D}$ such that the relations (6.2) and (6.3) hold.

Step 1. We apply the discrete Stokes formula Theorem 3.2, by using (6.1) and the fact that $\gamma^\tau(\mathbf{u}^\tau) = 0$, we have

$$\begin{aligned} B(\mathbf{u}^\tau, p^\mathfrak{D}; \mathbf{u}^\tau, p^\mathfrak{D}) &= (2\eta^\mathfrak{D} \mathbb{D}^\mathfrak{D} \mathbf{u}^\tau : \nabla^\mathfrak{D} \mathbf{u}^\tau)_\mathfrak{D} - \lambda(h_\mathfrak{D}^2 \Delta^\mathfrak{D} p^\mathfrak{D}, p^\mathfrak{D})_\mathfrak{D} \\ &= (2\eta^\mathfrak{D} \mathbb{D}^\mathfrak{D} \mathbf{u}^\tau : \mathbb{D}^\mathfrak{D} \mathbf{u}^\tau)_\mathfrak{D} - \lambda(h_\mathfrak{D}^2 \Delta^\mathfrak{D} p^\mathfrak{D}, p^\mathfrak{D})_\mathfrak{D}. \end{aligned}$$

Thanks to the inequality (2.3) and Remark 3.5, we obtain

$$B(\mathbf{u}^\tau, p^\mathfrak{D}; \mathbf{u}^\tau, p^\mathfrak{D}) \geq \underline{C}_\eta \|\mathbb{D}^\mathfrak{D} \mathbf{u}^\tau\|_2^2 + \lambda |p^\mathfrak{D}|_h^2.$$

Finally we use the discrete Korn inequality Theorem 5.1 in order to get

$$B(\mathbf{u}^\tau, p^\mathfrak{D}; \mathbf{u}^\tau, p^\mathfrak{D}) \geq \underline{C}_\eta \|\nabla^\mathfrak{D} \mathbf{u}^\tau\|_2^2 + \lambda |p^\mathfrak{D}|_h^2. \quad (6.4)$$

Note that the above estimate on the pressure is mesh dependent. Recall that the seminorm $|\cdot|_h$ is itself mesh dependent. That is why we could not bound uniformly the L^2 -norm of the pressure by the seminorm $|\cdot|_h$.

Step 2. We make use of the following result (see [22, Corollary 2.4] or [4, Lemma III.1.17]): since $p^\mathfrak{D} = \sum_{\mathfrak{D} \in \mathfrak{D}} p^\mathfrak{D} \mathbf{1}_\mathfrak{D} \in L^2(\Omega)$ and its integral over Ω is zero, there exists a constant $C > 0$ depending only on Ω , and $\mathbf{v} \in (H_0^1(\Omega))^2$ such that $\operatorname{div}(\mathbf{v}) = -p^\mathfrak{D}$ and

$$\|\mathbf{v}\|_{H^1} \leq C \|p^\mathfrak{D}\|_2. \quad (6.5)$$

We denote the mean-value projection $\mathbb{P}_m^\tau \mathbf{v}$ of the function \mathbf{v} on \mathcal{T} by \mathbf{v}^τ , so we have $\mathbf{v}_\kappa = (\mathbb{P}_m^\tau \mathbf{v})_\kappa$ for any $\kappa \in \mathfrak{M} \cup \partial \mathfrak{M}$ and $\mathbf{v}_{\kappa^*} = (\mathbb{P}_m^\tau \mathbf{v})_{\kappa^*}$ for any $\kappa^* \in \mathfrak{M}^* \cup \partial \mathfrak{M}^*$. In particular, we have $\mathbf{v}^\tau \in \mathbb{E}_0$. Using Lemma 5.6, we obtain

$$\|\nabla^\mathfrak{D} \mathbf{v}^\tau\|_2 \leq C \|\mathbf{v}\|_{H^1} \leq C \|p^\mathfrak{D}\|_2. \quad (6.6)$$

The discrete Stokes formula Theorem 3.2 implies

$$B(\mathbf{u}^\tau, p^\mathfrak{D}; \mathbf{v}^\tau, 0) = 2(\eta^\mathfrak{D} \mathbb{D}^\mathfrak{D} \mathbf{u}^\tau : \nabla^\mathfrak{D} \mathbf{v}^\tau)_\mathfrak{D} - (p^\mathfrak{D}, \operatorname{div}^\mathfrak{D}(\mathbf{v}^\tau))_\mathfrak{D}.$$

Using the Cauchy-Schwarz inequality, we deduce

$$\begin{aligned} B(\mathbf{u}^\tau, p^\mathfrak{D}; \mathbf{v}^\tau, 0) &\geq -\bar{C}_\eta \|\mathbb{D}^\mathfrak{D} \mathbf{u}^\tau\|_2 \|\nabla^\mathfrak{D} \mathbf{v}^\tau\|_2 - \sum_{\mathfrak{D} \in \mathfrak{D}} \int_{\mathfrak{D}} p^\mathfrak{D} \operatorname{div}(\mathbf{v}(z)) dz \\ &\quad - \sum_{\mathfrak{D} \in \mathfrak{D}} \int_{\mathfrak{D}} p^\mathfrak{D} (\operatorname{div}^\mathfrak{D}(\mathbf{v}^\tau) - \operatorname{div}(\mathbf{v}(z))) dz. \end{aligned}$$

Since we have $\operatorname{div}(\mathbf{v}) = -p^\mathfrak{D}$, the inequality (6.6) and Proposition 5.1 give

$$\begin{aligned} B(\mathbf{u}^\mathfrak{T}, p^\mathfrak{D}; \mathbf{v}^\mathfrak{T}, 0) &\geq -C \|\nabla^\mathfrak{D} \mathbf{u}^\mathfrak{T}\|_2 \|p^\mathfrak{D}\|_2 + \|p^\mathfrak{D}\|_2^2 \\ &\quad - \sum_{\mathfrak{D} \in \mathfrak{D}} \int_{\mathfrak{D}} p^\mathfrak{D} (\operatorname{div}^\mathfrak{D}(\mathbf{v}^\mathfrak{T}) - \operatorname{div}(\mathbf{v}(z))) \, dz. \end{aligned}$$

Thanks to Proposition 5.5 and to estimate (6.5) we have

$$\sum_{\mathfrak{D} \in \mathfrak{D}} \int_{\mathfrak{D}} p^\mathfrak{D} (\operatorname{div}^\mathfrak{D}(\mathbf{v}^\mathfrak{T}) - \operatorname{div}(\mathbf{v}(z))) \, dz \leq C' |p^\mathfrak{D}|_h \|\mathbf{v}\|_{H^1} \leq C |p^\mathfrak{D}|_h \|p^\mathfrak{D}\|_2.$$

We deduce that

$$B(\mathbf{u}^\mathfrak{T}, p^\mathfrak{D}; \mathbf{v}^\mathfrak{T}, 0) \geq \|p^\mathfrak{D}\|_2^2 - C \|p^\mathfrak{D}\|_2 \|\nabla^\mathfrak{D} \mathbf{u}^\mathfrak{T}\|_2 - C |p^\mathfrak{D}|_h \|p^\mathfrak{D}\|_2.$$

Using Young's inequality, we obtain the existence of three constants $C_1, C_2, C_3 > 0$, depending only on $\Omega, \underline{C}_\eta$ and $\operatorname{reg}(T)$, such that

$$B(\mathbf{u}^\mathfrak{T}, p^\mathfrak{D}; \mathbf{v}^\mathfrak{T}, 0) \geq C_1 \|p^\mathfrak{D}\|_2^2 - C_2 \|\nabla^\mathfrak{D} \mathbf{u}^\mathfrak{T}\|_2^2 - C_3 |p^\mathfrak{D}|_h^2. \quad (6.7)$$

Step 3. By bilinearity of B , the inequalities (6.4) and (6.7) give for each positive number $\xi > 0$:

$$B(\mathbf{u}^\mathfrak{T}, p^\mathfrak{D}; \mathbf{u}^\mathfrak{T} + \xi \mathbf{v}^\mathfrak{T}, p^\mathfrak{D}) \geq (\underline{C}_\eta - \xi C_2) \|\nabla^\mathfrak{D} \mathbf{u}^\mathfrak{T}\|_2^2 + \xi C_1 \|p^\mathfrak{D}\|_2^2 + (\lambda - \xi C_3) |p^\mathfrak{D}|_h^2.$$

Choosing a value of $\xi > 0$ small enough (depending only on $\underline{C}_\eta, C_2, \lambda$ and C_3), this inequality yields an estimate of the form (6.3). As the relation (6.2) is clearly verified by the pair $\tilde{\mathbf{u}}^\mathfrak{T} = \mathbf{u}^\mathfrak{T} + \xi \mathbf{v}^\mathfrak{T}$ and $\tilde{p}^\mathfrak{D} = p^\mathfrak{D}$, this concludes the proof. \square

An immediate consequence of this stability inequality is the continuous dependence of the DDFV solution with respect to the data.

COROLLARY 6.1. *Assume that η satisfies (2.3). There exists $C > 0$, depending only on the diameter of $\Omega, \lambda, \underline{C}_\eta, \tilde{C}_\eta$ and $\operatorname{reg}(T)$, such that the couple $(\mathbf{u}^\mathfrak{T}, p^\mathfrak{D}) \in \mathbb{E}_0 \times \mathbb{R}^\mathfrak{D}$, solution of the scheme (4.1) with $\mathbf{g} = 0$, satisfies:*

$$\|\nabla^\mathfrak{D} \mathbf{u}^\mathfrak{T}\|_2^2 + \|p^\mathfrak{D}\|_2^2 \leq C \|\mathbf{f}^\mathfrak{T}\|_2^2.$$

Proof. Let $(\mathbf{u}^\mathfrak{T}, p^\mathfrak{D}) \in \mathbb{E}_0 \times \mathbb{R}^\mathfrak{D}$ solution of the scheme (4.1):

$$\begin{cases} \operatorname{div}^\mathfrak{m}(-2\eta^\mathfrak{D} \mathbf{D}^\mathfrak{D} \mathbf{u}^\mathfrak{T} + p^\mathfrak{D} \operatorname{Id}) = \mathbf{f}^\mathfrak{m}, \\ \operatorname{div}^\mathfrak{m}(-2\eta^\mathfrak{D} \mathbf{D}^\mathfrak{D} \mathbf{u}^\mathfrak{T} + p^\mathfrak{D} \operatorname{Id}) = \mathbf{f}^\mathfrak{m}, \\ \operatorname{div}^\mathfrak{D}(\mathbf{u}^\mathfrak{T}) - \lambda h_\mathfrak{D}^2 \Delta^\mathfrak{D} p^\mathfrak{D} = 0, \\ \sum_{\mathfrak{D} \in \mathfrak{D}} m_\mathfrak{D} p^\mathfrak{D} = 0, \end{cases}$$

Thanks to Theorem 6.1, there exists $\tilde{\mathbf{u}}^\mathfrak{T} \in \mathbb{E}_0, \tilde{p}^\mathfrak{D} \in \mathbb{R}^\mathfrak{D}$:

$$\|\nabla^\mathfrak{D} \mathbf{u}^\mathfrak{T}\|_2^2 + \|p^\mathfrak{D}\|_2^2 \leq C_2 B(\mathbf{u}^\mathfrak{T}, p^\mathfrak{D}; \tilde{\mathbf{u}}^\mathfrak{T}, \tilde{p}^\mathfrak{D}).$$

By definition (6.1) of B , using the Cauchy-Schwarz inequality and the discrete Poincaré inequality (Theorem 5.2), we get

$$B(\mathbf{u}^\mathfrak{T}, p^\mathfrak{D}; \tilde{\mathbf{u}}^\mathfrak{T}, \tilde{p}^\mathfrak{D}) \leq C \|\mathbf{f}^\mathfrak{T}\|_2 \|\nabla^\mathfrak{D} \tilde{\mathbf{u}}^\mathfrak{T}\|_2.$$

Using (6.2) and (6.3), the claim is proved. \square

7. Error estimates.

7.1. Statement of the result and sketch of proof. We conclude by providing an error estimate in case where the exact solution of the problem (2.1) lies in the space $(H^2(\Omega))^2 \times H^1(\Omega)$ and the viscosity is smooth enough. Our main result is the following

THEOREM 7.1. *Assume that η satisfies (2.3) and (2.4). Let $\mathbf{f} \in (L^2(\Omega))^2$, $\mathbf{g} \in (\tilde{H}^{\frac{3}{2}}(\partial\Omega))^2$ and assume that the solution (\mathbf{u}, p) of the problem (2.1) belongs to $(H^2(\Omega))^2 \times H^1(\Omega)$. Let $(\mathbf{u}^\tau, p^\mathfrak{D}) \in (\mathbb{R}^2)^\tau \times \mathbb{R}^\mathfrak{D}$ be the solution of the scheme (4.1). There exists a constant $C > 0$ depending only on $\text{reg}(T)$, λ , C_η , \underline{C}_η , \bar{C}_η , $\|\mathbf{u}\|_{H^2}$ and $\|p\|_{H^1}$, such that:*

$$\|\mathbf{u} - \mathbf{u}^\tau\|_2 + \|\nabla \mathbf{u} - \nabla^\mathfrak{D} \mathbf{u}^\tau\|_2 \leq C \text{size}(T),$$

and

$$\|p - p^\mathfrak{D}\|_2 \leq C \text{size}(T).$$

As usual for the error analysis of finite volume methods, the consistency error which has to be studied is the error on the numerical fluxes across each of the edges and dual edges in the mesh. Therefore consistency errors are naturally defined on the diamond cells.

DEFINITION 7.1. *We define the consistency errors on \mathfrak{D} for any $\mathbf{v} \in (H^2(\Omega))^2$ and for any $p \in H^1(\Omega)$*

$$\begin{aligned} \mathbf{R}_\mathfrak{D}^\mathbf{v}(z) &= \eta(z)D\mathbf{v}(z) - \eta_\mathfrak{D}D^\mathfrak{D}\mathfrak{P}_{m,\gamma(\mathbf{v})}\mathbb{P}_c^\tau \mathbf{v}, & \text{if } z \in \mathfrak{D}, \\ \mathbf{R}_\mathfrak{D}^p(z) &= \mathbb{P}_m^\mathfrak{D}p - p(z), & \text{if } z \in \mathfrak{D}. \end{aligned}$$

Recall that the edges σ and σ^* are the diagonals of $\mathfrak{D} = \mathfrak{D}_{\sigma,\sigma^*}$. Let us introduce the following consistency errors on the numerical fluxes, for $i \in \{\mathbf{v}, p\}$:

$$\begin{aligned} \mathbf{R}_{\sigma,\kappa}^i &= -\mathbf{R}_{\sigma,\mathcal{L}}^i = \frac{1}{m_\sigma} \int_\sigma \mathbf{R}_\mathfrak{D}^i(s) \bar{\mathbf{n}}_{\sigma\kappa} ds, \\ \mathbf{R}_{\sigma^*,\kappa^*}^i &= -\mathbf{R}_{\sigma^*,\mathcal{L}^*}^i = \frac{1}{m_{\sigma^*}} \int_{\sigma^*} \mathbf{R}_\mathfrak{D}^i(s) \bar{\mathbf{n}}_{\sigma^*\kappa^*} ds, \\ \mathbf{R}_\sigma^i &= |\mathbf{R}_{\sigma,\kappa}^i| = |\mathbf{R}_{\sigma,\mathcal{L}}^i|, \\ \mathbf{R}_{\sigma^*}^i &= |\mathbf{R}_{\sigma^*,\kappa^*}^i| = |\mathbf{R}_{\sigma^*,\mathcal{L}^*}^i|. \end{aligned}$$

We note the L^2 -norm of the consistency error as follows, for $i = \mathbf{v}, p$:

$$\|\mathbf{R}_\sigma^i\|_2^2 = \sum_{\mathfrak{D}_{\sigma,\sigma^*} \in \mathfrak{D}} m_\mathfrak{D} |\mathbf{R}_\sigma^i|^2, \quad \|\mathbf{R}_{\sigma^*}^i\|_2^2 = \sum_{\mathfrak{D}_{\sigma,\sigma^*} \in \mathfrak{D}} m_\mathfrak{D} |\mathbf{R}_{\sigma^*}^i|^2.$$

Step 1. We denote by $\mathbf{e}^\tau = \mathfrak{P}_{m,g}\mathbb{P}_c^\tau \mathbf{u} - \mathbf{u}^\tau$ the discrete error on the velocity and $e^\mathfrak{D} = \mathbb{P}_m^\mathfrak{D}p - p^\mathfrak{D}$ the discrete error on the pressure. Recall that $\mathbf{g} = \gamma(\mathbf{u})$. Then, we have $\forall \kappa \in \mathfrak{M}$

$$\begin{aligned} \text{div}^\kappa(-2\eta^\mathfrak{D}D^\mathfrak{D}\mathbf{u}^\tau + p^\mathfrak{D}\text{Id}) &= \mathbf{f}_\kappa, \\ -\frac{1}{m_\kappa} \int_\kappa \text{div}(2\eta(x)D\mathbf{u}(x))dx + \frac{1}{m_\kappa} \int_\kappa \nabla p(x)dx &= \mathbf{f}_\kappa. \end{aligned}$$

Therefore, we deduce

$$\begin{aligned} m_\kappa \text{div}^\kappa(-2\eta^\mathfrak{D}D^\mathfrak{D}\mathbf{e}^\tau + e^\mathfrak{D}\text{Id}) &= m_\kappa \text{div}^\kappa(-2\eta^\mathfrak{D}D^\mathfrak{D}\mathfrak{P}_{m,g}\mathbb{P}_c^\tau \mathbf{u} + \mathbb{P}_m^\mathfrak{D}p\text{Id}) \\ &\quad + \int_\kappa \text{div}(2\eta(x)D\mathbf{u}(x))dx - \int_\kappa \nabla p(x)dx. \end{aligned}$$

Using Definition 7.1, it becomes

$$m_{\mathcal{K}} \operatorname{div}^{\mathcal{K}}(-2\eta^{\mathfrak{D}} D^{\mathfrak{D}} \mathbf{e}^{\mathcal{T}} + e^{\mathfrak{D}} \operatorname{Id}) = 2 \sum_{\mathfrak{D} \in \mathfrak{D}_{\mathcal{K}}} m_{\sigma} \mathbf{R}_{\sigma, \mathcal{K}}^{\mathbf{u}} + \sum_{\mathfrak{D} \in \mathfrak{D}_{\mathcal{K}}} m_{\sigma} \mathbf{R}_{\sigma, \mathcal{K}}^p.$$

In the same way, we have $\forall \mathcal{K}^* \in \mathfrak{M}^*$

$$m_{\mathcal{K}^*} (\operatorname{div}^{\mathcal{K}^*}(-2\eta^{\mathfrak{D}} D^{\mathfrak{D}} \mathbf{e}^{\mathcal{T}} + e^{\mathfrak{D}} \operatorname{Id})) = 2 \sum_{\mathfrak{D} \in \mathfrak{D}_{\mathcal{K}^*}} m_{\sigma^*} \mathbf{R}_{\sigma^*, \mathcal{K}^*}^{\mathbf{u}} + \sum_{\mathfrak{D} \in \mathfrak{D}_{\mathcal{K}^*}} m_{\sigma^*} \mathbf{R}_{\sigma^*, \mathcal{K}^*}^p.$$

Finally, the couple $(\mathbf{e}^{\mathcal{T}}, e^{\mathfrak{D}}) \in \mathbb{E}_0 \times \mathbb{R}^{\mathfrak{D}}$ satisfies:

$$\left\{ \begin{array}{l} \operatorname{div}^{\mathfrak{M}}(-2\eta^{\mathfrak{D}} D^{\mathfrak{D}} \mathbf{e}^{\mathcal{T}} + e^{\mathfrak{D}} \operatorname{Id}) = \mathbf{R}_{\mathfrak{M}}, \\ \operatorname{div}^{\mathfrak{M}^*}(-2\eta^{\mathfrak{D}} D^{\mathfrak{D}} \mathbf{e}^{\mathcal{T}} + e^{\mathfrak{D}} \operatorname{Id}) = \mathbf{R}_{\mathfrak{M}^*}, \\ \operatorname{div}^{\mathfrak{D}}(\mathbf{e}^{\mathcal{T}}) - \lambda h_{\mathfrak{D}}^2 \Delta^{\mathfrak{D}} e^{\mathfrak{D}} = R_{\mathfrak{D}}, \\ \sum_{\mathfrak{D} \in \mathfrak{D}} m_{\mathfrak{D}} e^{\mathfrak{D}} = 0, \end{array} \right. \quad (7.1)$$

where

$$\begin{aligned} \mathbf{R}_{\mathcal{K}} &= \frac{2}{m_{\mathcal{K}}} \sum_{\mathfrak{D} \in \mathfrak{D}_{\mathcal{K}}} m_{\sigma} \mathbf{R}_{\sigma, \mathcal{K}}^{\mathbf{u}} + \frac{1}{m_{\mathcal{K}}} \sum_{\mathfrak{D} \in \mathfrak{D}_{\mathcal{K}}} m_{\sigma} \mathbf{R}_{\sigma, \mathcal{K}}^p, \quad \forall \mathcal{K} \in \mathfrak{M}, \\ \mathbf{R}_{\mathcal{K}^*} &= \frac{2}{m_{\mathcal{K}^*}} \sum_{\mathfrak{D} \in \mathfrak{D}_{\mathcal{K}^*}} m_{\sigma^*} \mathbf{R}_{\sigma^*, \mathcal{K}^*}^{\mathbf{u}} + \frac{1}{m_{\mathcal{K}^*}} \sum_{\mathfrak{D} \in \mathfrak{D}_{\mathcal{K}^*}} m_{\sigma^*} \mathbf{R}_{\sigma^*, \mathcal{K}^*}^p, \quad \forall \mathcal{K}^* \in \mathfrak{M}^* \\ R_{\mathfrak{D}} &= \operatorname{div}^{\mathfrak{D}}(\mathfrak{P}_{m, g} \mathbb{P}_{\mathfrak{C}}^{\mathcal{T}} \mathbf{u}) - \lambda h_{\mathfrak{D}}^2 \Delta^{\mathfrak{D}} \mathbb{P}_{m, p}^{\mathfrak{D}}, \quad \forall \mathfrak{D} \in \mathfrak{D}. \end{aligned}$$

Remark that we have $\sum_{\mathfrak{D} \in \mathfrak{D}} m_{\mathfrak{D}} R_{\mathfrak{D}} = 0$. Theorem 6.1 implies that there exists $\tilde{\mathbf{e}}^{\mathcal{T}} \in \mathbb{E}_0$, $\tilde{e}^{\mathfrak{D}} \in \mathbb{R}^{\mathfrak{D}}$ and $C > 0$ such that :

$$\|\nabla^{\mathfrak{D}} \tilde{\mathbf{e}}^{\mathcal{T}}\|_2 + \|\tilde{e}^{\mathfrak{D}}\|_2 \leq C (\|\nabla^{\mathfrak{D}} \mathbf{e}^{\mathcal{T}}\|_2 + \|e^{\mathfrak{D}}\|_2), \quad (7.2)$$

and

$$\|\nabla^{\mathfrak{D}} \mathbf{e}^{\mathcal{T}}\|_2^2 + \|e^{\mathfrak{D}}\|_2^2 \leq CB(\mathbf{e}^{\mathcal{T}}, e^{\mathfrak{D}}; \tilde{\mathbf{e}}^{\mathcal{T}}, \tilde{e}^{\mathfrak{D}}). \quad (7.3)$$

Thanks to Definition (6.1) of B ans to (7.1), we have $B(\mathbf{e}^{\mathcal{T}}, e^{\mathfrak{D}}; \tilde{\mathbf{e}}^{\mathcal{T}}, \tilde{e}^{\mathfrak{D}}) = \llbracket \mathbf{R}^{\mathcal{T}}, \tilde{\mathbf{e}}^{\mathcal{T}} \rrbracket_{\mathcal{T}} + (R^{\mathfrak{D}}, \tilde{e}^{\mathfrak{D}})_{\mathfrak{D}}$. We note $I := \llbracket \mathbf{R}^{\mathcal{T}}, \tilde{\mathbf{e}}^{\mathcal{T}} \rrbracket_{\mathcal{T}}$ and $T := (R^{\mathfrak{D}}, \tilde{e}^{\mathfrak{D}})_{\mathfrak{D}}$. Using the fact that $\tilde{\mathbf{e}}_{\mathcal{K}^*} = 0$ for any $\mathcal{K}^* \in \partial \mathfrak{M}^*$ and the definition of I , we have

$$\begin{aligned} I &= \frac{1}{2} \sum_{\mathcal{K} \in \mathfrak{M}} \sum_{\mathfrak{D} \in \mathfrak{D}_{\mathcal{K}}} m_{\sigma} (\mathbf{R}_{\sigma, \mathcal{K}}^p + 2\mathbf{R}_{\sigma, \mathcal{K}}^{\mathbf{u}}, \tilde{\mathbf{e}}_{\mathcal{K}}) \\ &\quad + \frac{1}{2} \sum_{\mathcal{K}^* \in \mathfrak{M}^* \cup \partial \mathfrak{M}^*} \sum_{\mathfrak{D} \in \mathfrak{D}_{\mathcal{K}^*}} m_{\sigma^*} (\mathbf{R}_{\sigma^*, \mathcal{K}^*}^p + 2\mathbf{R}_{\sigma^*, \mathcal{K}^*}^{\mathbf{u}}, \tilde{\mathbf{e}}_{\mathcal{K}^*}). \end{aligned}$$

Reordering the summation over the set of diamond cells and using the fact that $\mathbf{R}_{\sigma, \mathcal{K}}^i = -\mathbf{R}_{\sigma^*, \mathcal{K}^*}^i$ for $i = \mathbf{u}, p$, we deduce

$$\begin{aligned} I &= \frac{1}{2} \sum_{\mathfrak{D}_{\sigma, \sigma^*} \in \mathfrak{D}} m_{\sigma} (\mathbf{R}_{\sigma, \mathcal{K}}^p + 2\mathbf{R}_{\sigma, \mathcal{K}}^{\mathbf{u}}, \tilde{\mathbf{e}}_{\mathcal{K}} - \tilde{\mathbf{e}}_{\mathcal{L}}) \\ &\quad + \frac{1}{2} \sum_{\mathfrak{D}_{\sigma, \sigma^*} \in \mathfrak{D}} m_{\sigma^*} (\mathbf{R}_{\sigma^*, \mathcal{K}^*}^p + 2\mathbf{R}_{\sigma^*, \mathcal{K}^*}^{\mathbf{u}}, \tilde{\mathbf{e}}_{\mathcal{K}^*} - \tilde{\mathbf{e}}_{\mathcal{L}^*}). \end{aligned}$$

Using Definition 3.3 of the discrete gradient and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} I &= -\sum_{\mathcal{D}_{\sigma,\sigma^*} \in \mathfrak{D}} \frac{m_{\mathcal{D}}}{\sin(\alpha_{\mathcal{D}})} [(\mathbf{R}_{\sigma,\kappa}^p + 2\mathbf{R}_{\sigma,\kappa}^u, \nabla^{\mathcal{D}} \tilde{\mathbf{e}}^{\mathcal{T}} \tilde{\boldsymbol{\tau}}_{\kappa,\mathcal{L}}) - (\mathbf{R}_{\sigma^*,\kappa^*}^p + 2\mathbf{R}_{\sigma^*,\kappa^*}^u, \nabla^{\mathcal{D}} \tilde{\mathbf{e}}^{\mathcal{T}} \tilde{\boldsymbol{\tau}}_{\kappa^*,\mathcal{L}^*})] \\ &\leq \|\nabla^{\mathcal{D}} \tilde{\mathbf{e}}^{\mathcal{T}}\|_2 \frac{C}{\sin(\alpha_{\mathcal{T}})} [\|\mathbf{R}_{\sigma}^u\|_2 + \|\mathbf{R}_{\sigma^*}^u\|_2 + \|\mathbf{R}_{\sigma}^p\|_2 + \|\mathbf{R}_{\sigma^*}^p\|_2]. \end{aligned}$$

Furthermore, Corollary 5.3 and Lemma 5.5 imply

$$\|\operatorname{div}^{\mathcal{D}}(\mathfrak{P}_{m,g} \mathbb{P}_{\mathbf{c}}^{\mathcal{T}} \mathbf{u})\|_2 \leq C \operatorname{size}(\mathcal{T}) \|\mathbf{u}\|_{H^2}.$$

Applying the Cauchy-Schwarz inequality to $(\operatorname{div}^{\mathcal{D}}(\mathfrak{P}_{m,g} \mathbb{P}_{\mathbf{c}}^{\mathcal{T}} \mathbf{u}), \tilde{e}^{\mathcal{D}})_{\mathfrak{D}}$, we get

$$(\operatorname{div}^{\mathcal{D}}(\mathfrak{P}_{m,g} \mathbb{P}_{\mathbf{c}}^{\mathcal{T}} \mathbf{u}), \tilde{e}^{\mathcal{D}})_{\mathfrak{D}} \leq C \operatorname{size}(\mathcal{T}) \|\mathbf{u}\|_{H^2} \|\tilde{e}^{\mathcal{D}}\|_2. \quad (7.4)$$

Reordering the summation over $\mathfrak{s} \in \mathfrak{S}$ in the term $T_1 := -(\lambda h_{\mathfrak{D}}^2 \Delta^{\mathcal{D}} \mathbb{P}_m^{\mathcal{D}} p, \tilde{e}^{\mathcal{D}})_{\mathfrak{D}}$, we have

$$T_1 = -\lambda \sum_{\mathcal{D} \in \mathfrak{D}} m_{\mathcal{D}} \tilde{e}^{\mathcal{D}} h_{\mathcal{D}}^2 \Delta^{\mathcal{D}} \mathbb{P}_m^{\mathcal{D}} p = \lambda \sum_{\mathfrak{s}=\mathcal{D} | \mathcal{D}' \in \mathfrak{S}} (h_{\mathcal{D}}^2 + h_{\mathcal{D}'}^2) (\mathbb{P}_m^{\mathcal{D}'} p - \mathbb{P}_m^{\mathcal{D}} p) (\tilde{e}^{\mathcal{D}'} - \tilde{e}^{\mathcal{D}}).$$

The Cauchy-Schwarz inequality implies:

$$\begin{aligned} |T_1| &\leq \lambda \left(\sum_{\mathfrak{s}=\mathcal{D} | \mathcal{D}' \in \mathfrak{S}} (h_{\mathcal{D}}^2 + h_{\mathcal{D}'}^2) (\mathbb{P}_m^{\mathcal{D}'} p - \mathbb{P}_m^{\mathcal{D}} p)^2 \right)^{\frac{1}{2}} \left(\sum_{\mathfrak{s}=\mathcal{D} | \mathcal{D}' \in \mathfrak{S}} (h_{\mathcal{D}}^2 + h_{\mathcal{D}'}^2) (\tilde{e}^{\mathcal{D}'} - \tilde{e}^{\mathcal{D}})^2 \right)^{\frac{1}{2}} \\ &\leq 2 \operatorname{size}(\mathcal{T}) \lambda |\tilde{e}^{\mathcal{D}}|_h \left(\sum_{\mathfrak{s}=\mathcal{D} | \mathcal{D}' \in \mathfrak{S}} (\mathbb{P}_m^{\mathcal{D}'} p - \mathbb{P}_m^{\mathcal{D}} p)^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Using Lemmas 5.3 and 5.8, we obtain

$$|T_1| \leq C \operatorname{size}(\mathcal{T}) \|\tilde{e}^{\mathcal{D}}\|_2 \|\nabla p\|_2. \quad (7.5)$$

Remark that $T = (\operatorname{div}^{\mathcal{D}}(\mathfrak{P}_{m,g} \mathbb{P}_{\mathbf{c}}^{\mathcal{T}} \mathbf{u}), \tilde{e}^{\mathcal{D}})_{\mathfrak{D}} + T_1$, thanks to (7.4) and (7.5), we deduce

$$|T| \leq C \operatorname{size}(\mathcal{T}) \|\tilde{e}^{\mathcal{D}}\|_2 (\|\mathbf{u}\|_{H^2} + \|\nabla p\|_2).$$

To sum up, (7.3) becomes

$$\begin{aligned} \|\nabla^{\mathcal{D}} \mathbf{e}^{\mathcal{T}}\|_2^2 + \|e^{\mathcal{D}}\|_2^2 &\leq \frac{C}{\sin(\alpha_{\mathcal{T}})} \|\nabla^{\mathcal{D}} \tilde{\mathbf{e}}^{\mathcal{T}}\|_2 [\|\mathbf{R}_{\sigma}^u\|_2 + \|\mathbf{R}_{\sigma^*}^u\|_2 + \|\mathbf{R}_{\sigma}^p\|_2 + \|\mathbf{R}_{\sigma^*}^p\|_2] \\ &\quad + C \operatorname{size}(\mathcal{T}) \|\tilde{e}^{\mathcal{D}}\|_2 (\|\mathbf{u}\|_{H^2} + \|\nabla p\|_2). \end{aligned}$$

Finally, using (7.2), we deduce

$$\begin{aligned} \|\nabla^{\mathcal{D}} \mathbf{e}^{\mathcal{T}}\|_2^2 + \|e^{\mathcal{D}}\|_2^2 &\leq C (\|\mathbf{R}_{\sigma}^u\|_2 + \|\mathbf{R}_{\sigma^*}^u\|_2 + \|\mathbf{R}_{\sigma}^p\|_2 + \|\mathbf{R}_{\sigma^*}^p\|_2) (\|\nabla^{\mathcal{D}} \mathbf{e}^{\mathcal{T}}\|_2 + \|e^{\mathcal{D}}\|_2) \\ &\quad + C \operatorname{size}(\mathcal{T}) (\|\mathbf{u}\|_{H^2} + \|\nabla p\|_2) (\|\nabla^{\mathcal{D}} \mathbf{e}^{\mathcal{T}}\|_2 + \|e^{\mathcal{D}}\|_2). \end{aligned} \quad (7.6)$$

It remains to estimate the consistency errors.

7.2. Analysis of the consistency error.

7.2.1. Estimate of $\mathbf{R}_\sigma^{\mathbf{v}}$. The consistency error $\mathbf{R}_\sigma^{\mathbf{v}}$ can be split into four different contributions $\mathbf{R}_\sigma^{\mathbf{v},\eta}$, $\mathbf{R}_\sigma^{\mathbf{v},D\mathbf{v}}$, $\mathbf{R}_\sigma^{\mathbf{v},z}$ and $\mathbf{R}_\sigma^{\mathbf{v},bd}$ coming, respectively, from the errors due to the approximation with respect to the space variable of the flux $\eta(\cdot)D\mathbf{v}(\cdot)$, to the approximation of the gradient, to the approximation of the viscosity and to the discretization of the boundary data:

$$\mathbf{R}_\sigma^{\mathbf{v}}(z) = \mathbf{R}_\sigma^{\mathbf{v},\eta}(z) + \mathbf{R}_\sigma^{\mathbf{v},D\mathbf{v}} + \mathbf{R}_\sigma^{\mathbf{v},z} + \mathbf{R}_\sigma^{\mathbf{v},bd}, \quad (7.7)$$

where, for $z \in \mathcal{D}$,

$$\begin{aligned} \mathbf{R}_\sigma^{\mathbf{v},\eta}(z) &= \eta(z)D\mathbf{v}(z) - \frac{1}{m_{\mathcal{D}}} \int_{\mathcal{D}} \eta(x)D\mathbf{v}(x)dx, \\ \mathbf{R}_\sigma^{\mathbf{v},D\mathbf{v}} &= \frac{1}{m_{\mathcal{D}}} \int_{\mathcal{D}} \eta(x)(D\mathbf{v}(x) - D^{\mathcal{D}}\mathbb{P}_{\mathbf{c}}^{\mathcal{T}}\mathbf{v})dx, \\ \mathbf{R}_\sigma^{\mathbf{v},z} &= \left(\frac{1}{m_{\mathcal{D}}} \int_{\mathcal{D}} \eta(x)dx - \eta_{\mathcal{D}} \right) D^{\mathcal{D}}\mathbb{P}_{\mathbf{c}}^{\mathcal{T}}\mathbf{v}, \\ \mathbf{R}_\sigma^{\mathbf{v},bd} &= \eta_{\mathcal{D}}D^{\mathcal{D}}\mathbb{P}_{\mathbf{c}}^{\mathcal{T}}\mathbf{v} - \eta_{\mathcal{D}}D^{\mathcal{D}}\mathfrak{P}_{m,\gamma(\mathbf{v})}\mathbb{P}_{\mathbf{c}}^{\mathcal{T}}\mathbf{v}. \end{aligned}$$

In order to control $\mathbf{R}_\sigma^{\mathbf{v}}$ and $\mathbf{R}_{\sigma^*}^{\mathbf{v}}$, let us estimate separately the different terms in the right hand side of (7.7).

PROPOSITION 7.1 (Error due to the discrete gradient). *Assume that η satisfies (2.3). For any mesh \mathcal{T} on Ω , there exists a constant $C > 0$, depending only on $\text{reg}(\mathcal{T})$ and \bar{C}_η , such that for any function \mathbf{v} in $(H^2(\Omega))^2$, we have*

$$\|\mathbf{R}_\sigma^{\mathbf{v},D\mathbf{v}}\|_2 \leq C \text{size}(\mathcal{T}) \|\nabla\mathbf{v}\|_{H^1}.$$

Proof. Using the inequality (2.3) we get

$$\|\mathbf{R}_\sigma^{\mathbf{v},D\mathbf{v}}\|_2^2 \leq \bar{C}_\eta^2 \|D\mathbf{v} - D^{\mathcal{D}}\mathbb{P}_{\mathbf{c}}^{\mathcal{T}}\mathbf{v}\|_2^2.$$

Remark 2.1 implies that

$$\|D\mathbf{v} - D^{\mathcal{D}}\mathbb{P}_{\mathbf{c}}^{\mathcal{T}}\mathbf{v}\|_2 \leq \|\nabla\mathbf{v} - \nabla^{\mathcal{D}}\mathbb{P}_{\mathbf{c}}^{\mathcal{T}}\mathbf{v}\|_2,$$

and then, applying Lemma 5.4, we get the result. \square

PROPOSITION 7.2 (Error due to the viscosity variation). *Assume that η satisfies (2.4). For any mesh \mathcal{T} on Ω , there exists a constant $C > 0$, depending only on $\text{reg}(\mathcal{T})$ and C_η , such that for any function \mathbf{v} in $(H^2(\Omega))^2$, we have*

$$\|\mathbf{R}_\sigma^{\mathbf{v},z}\|_2 \leq C \text{size}(\mathcal{T}) \|\nabla\mathbf{v}\|_{H^1}.$$

Proof. The Jensen inequality gives

$$\|\mathbf{R}_\sigma^{\mathbf{v},z}\|_2^2 \leq \sum_{\mathcal{D} \in \mathcal{D}} m_{\mathcal{D}} \frac{1}{m_{\mathcal{D}}} \left(\int_{\mathcal{D}} \int_{\bar{\mathcal{D}}} |\eta(x) - \eta(z)|^2 d\mu_{\bar{\mathcal{D}}}(z) dx \right) \|D^{\mathcal{D}}\mathbb{P}_{\mathbf{c}}^{\mathcal{T}}\mathbf{v}\|_{\mathcal{F}}^2.$$

Thanks to (2.4) and Proposition 5.1 we have

$$\|\mathbf{R}_\sigma^{\mathbf{v},z}\|_2^2 \leq C_\eta^2 \sum_{\mathcal{D} \in \mathcal{D}} \left(\int_{\mathcal{D}} \int_{\bar{\mathcal{D}}} |x - z|^2 d\mu_{\bar{\mathcal{D}}}(z) dx \right) \|D^{\mathcal{D}}\mathbb{P}_{\mathbf{c}}^{\mathcal{T}}\mathbf{v}\|_{\mathcal{F}}^2 \leq C \text{size}(\mathcal{T})^2 \|\nabla^{\mathcal{D}}\mathbb{P}_{\mathbf{c}}^{\mathcal{T}}\mathbf{v}\|_2^2.$$

Finally, Corollary 5.2 gives the result. \square

PROPOSITION 7.3 (Error due to the boundary data). *Assume that η satisfies (2.3). For any mesh \mathcal{T} on Ω , there exists a constant $C > 0$, depending only on $\text{reg}(\mathcal{T})$ and \bar{C}_η , such that for any function \mathbf{v} in $(H^2(\Omega))^2$, whose trace is denoted by $\mathbf{g} = \gamma(\mathbf{v})$, we have*

$$\|\mathbf{R}_{\mathcal{D}}^{\mathbf{v},bd}\|_2 \leq C \text{size}(\mathcal{T}) \|\mathbf{g}\|_{(\tilde{H}^{\frac{3}{2}}(\partial\Omega))^2} \leq C \text{size}(\mathcal{T}) \|\mathbf{v}\|_{H^2}.$$

Proof. Inequality (2.3) and Proposition 5.1 imply

$$\|\mathbf{R}_{\mathcal{D}}^{\mathbf{v},bd}\|_2^2 \leq \bar{C}_\eta \|\mathbf{D}^{\mathcal{D}}(\mathbb{P}_{\mathcal{C}}^{\mathcal{T}}\mathbf{v} - \mathfrak{P}_{m,\mathbf{g}}\mathbb{P}_{\mathcal{C}}^{\mathcal{T}}\mathbf{v})\|_2^2 \leq C \|\nabla^{\mathcal{D}}(\mathbb{P}_{\mathcal{C}}^{\mathcal{T}}\mathbf{v} - \mathfrak{P}_{m,\mathbf{g}}\mathbb{P}_{\mathcal{C}}^{\mathcal{T}}\mathbf{v})\|_2^2,$$

and finally, Lemma 5.5 gives the result. \square

PROPOSITION 7.4 (Error due to the approximate flux). *Assume that η satisfies (2.3) and (2.4). For any mesh \mathcal{T} on Ω , there exists a constant $C > 0$, depending only on $\text{reg}(\mathcal{T})$, C_η and \bar{C}_η , such that for any function \mathbf{v} in $(H^2(\Omega))^2$, we have*

$$\sum_{\mathcal{D}, \sigma, \sigma^* \in \mathcal{D}} m_{\mathcal{D}} \left[\frac{1}{m_\sigma} \int_\sigma \|\mathbf{R}_{\mathcal{D}}^{\mathbf{v},\eta}(s)\|_{\mathcal{F}}^2 ds + \frac{1}{m_{\sigma^*}} \int_{\sigma^*} \|\mathbf{R}_{\mathcal{D}}^{\mathbf{v},\eta}(s)\|_{\mathcal{F}}^2 ds \right] \leq C \text{size}(\mathcal{T})^2 \|\nabla \mathbf{v}\|_{H^1}^2.$$

Proof. We apply the Jensen inequality

$$\|\mathbf{R}_{\mathcal{D}}^{\mathbf{v},\eta}(s)\|_{\mathcal{F}}^2 \leq \frac{1}{m_{\mathcal{D}}} \int_{\mathcal{D}} \|\eta(s)\mathbf{D}\mathbf{v}(s) - \eta(x)\mathbf{D}\mathbf{v}(x)\|_{\mathcal{F}}^2 dx.$$

Thus, we get

$$\begin{aligned} \|\mathbf{R}_{\mathcal{D}}^{\mathbf{v},\eta}(s)\|_{\mathcal{F}}^2 &\leq \frac{2}{m_{\mathcal{D}}} \int_{\mathcal{D}} |\eta(s) - \eta(x)|^2 \|\mathbf{D}\mathbf{v}(x)\|_{\mathcal{F}}^2 dx \\ &\quad + \frac{2}{m_{\mathcal{D}}} \int_{\mathcal{D}} |\eta(s)|^2 \|\mathbf{D}\mathbf{v}(s) - \mathbf{D}\mathbf{v}(x)\|_{\mathcal{F}}^2 dx. \end{aligned}$$

Inequalities (2.3) and (2.4) imply

$$\begin{aligned} \|\mathbf{R}_{\mathcal{D}}^{\mathbf{v},\eta}(s)\|_{\mathcal{F}}^2 &\leq \text{size}(\mathcal{T})^2 \frac{2C_\eta^2}{m_{\mathcal{D}}} \int_{\mathcal{D}} \|\mathbf{D}\mathbf{v}(x)\|_{\mathcal{F}}^2 dx \\ &\quad + \frac{2\bar{C}_\eta^2}{m_{\mathcal{D}}} \int_{\mathcal{D}} \|\mathbf{D}\mathbf{v}(s) - \mathbf{D}\mathbf{v}(x)\|_{\mathcal{F}}^2 dx. \end{aligned}$$

Suming over the diamond cells, for the first integral, we get

$$\begin{aligned} \sum_{\mathcal{D} \in \mathcal{D}} m_{\mathcal{D}} \text{size}(\mathcal{T})^2 \frac{2C_\eta^2}{m_{\mathcal{D}}} \int_{\mathcal{D}} \|\mathbf{D}\mathbf{v}(x)\|_{\mathcal{F}}^2 dx &= 2C_\eta^2 \text{size}(\mathcal{T})^2 \|\mathbf{D}\mathbf{v}\|_2^2 \\ &\leq 2C_\eta^2 \text{size}(\mathcal{T})^2 \|\nabla \mathbf{v}\|_{H^1}^2. \end{aligned}$$

We integrate the second one on σ and apply Lemma 5.1

$$\frac{1}{m_{\mathcal{D}} m_\sigma} \int_\sigma \int_{\mathcal{D}} \|\mathbf{D}\mathbf{v}(s) - \mathbf{D}\mathbf{v}(x)\|_{\mathcal{F}}^2 dx ds \leq C \frac{h_{\mathcal{D}}^3}{m_\sigma m_{\mathcal{D}}} \int_{\mathcal{D}} |\nabla(\mathbf{D}\mathbf{v}(y))|^2 dy.$$

Note that we use here an extension of Lemma 5.1 to the matrix framework. It follows

$$\sum_{\mathcal{D}, \sigma, \sigma^* \in \mathcal{D}} m_{\mathcal{D}} \frac{1}{m_{\mathcal{D}} m_\sigma} \int_\sigma \int_{\mathcal{D}} \|\mathbf{D}\mathbf{v}(s) - \mathbf{D}\mathbf{v}(x)\|_{\mathcal{F}}^2 dx ds \leq C \text{size}(\mathcal{T})^2 \|\nabla \mathbf{v}\|_{H^1}^2.$$

As a result, we obtain

$$\sum_{\mathcal{D}_{\sigma, \sigma^*} \in \mathcal{D}} m_{\mathcal{D}} \frac{1}{m_{\sigma}} \int_{\sigma} \|\mathbf{R}_{\mathcal{D}}^{\mathbf{v}, \eta}(s)\|_{\mathcal{F}}^2 ds \leq C \text{size}(\mathcal{T})^2 \|\nabla \mathbf{v}\|_{H^1}^2.$$

We use the same computation for the similar term on the dual edge σ^* . \square

Now, we can control $\mathbf{R}_{\sigma}^{\mathbf{v}}$ and $\mathbf{R}_{\sigma^*}^{\mathbf{v}}$, as follows

COROLLARY 7.1. *Assume that η satisfies (2.3) and (2.4). For any mesh \mathcal{T} on Ω , there exists a constant $C > 0$, depending only on $\text{reg}(\mathcal{T})$, C_{η} and \bar{C}_{η} , such that for any function \mathbf{v} in $(H^2(\Omega))^2$, we have*

$$\|\mathbf{R}_{\sigma}^{\mathbf{v}}\|_2 + \|\mathbf{R}_{\sigma^*}^{\mathbf{v}}\|_2 \leq C \text{size}(\mathcal{T}) \|\nabla \mathbf{v}\|_{H^1}.$$

Proof. Thanks to (7.7), we get

$$\sum_{\mathcal{D}_{\sigma, \sigma^*} \in \mathcal{D}} m_{\mathcal{D}} |\mathbf{R}_{\sigma}^{\mathbf{v}}|^2 \leq \sum_{\mathcal{D}_{\sigma, \sigma^*} \in \mathcal{D}} m_{\mathcal{D}} \frac{1}{m_{\sigma}} \int_{\sigma} \|\mathbf{R}_{\mathcal{D}}^{\mathbf{v}, \eta}(s)\|_{\mathcal{F}}^2 ds + \|\mathbf{R}_{\mathcal{D}}^{\mathbf{v}, \text{Dv}}\|_2^2 + \|\mathbf{R}_{\mathcal{D}}^{\mathbf{v}, z}\|_2^2 + \|\mathbf{R}_{\mathcal{D}}^{\mathbf{v}, \text{bd}}\|_2^2.$$

We conclude using Propositions 7.1, 7.2, 7.3 and 7.4. We proceed in the same way for the estimate of $\mathbf{R}_{\sigma^*}^{\mathbf{v}}$. \square

7.2.2. Estimate of \mathbf{R}_{σ}^p . Now, we can control \mathbf{R}_{σ}^p and $\mathbf{R}_{\sigma^*}^p$, as follows

COROLLARY 7.2. *For any mesh \mathcal{T} on Ω , there exists a constant $C > 0$, depending only on $\text{reg}(\mathcal{T})$, such that for any function p in $H^1(\Omega)$, we have*

$$\|\mathbf{R}_{\sigma}^p\|_2 + \|\mathbf{R}_{\sigma^*}^p\|_2 \leq C \text{size}(\mathcal{T}) \|\nabla p\|_2.$$

Proof. We use Lemma 5.2 on a edge σ and a half diamond cell \mathcal{D}_1 such that $\mathcal{D}_{\sigma, \sigma^*} = \mathcal{D}_1 \cup \mathcal{D}_2$ with $\mathcal{D}_1 \cap \mathcal{D}_2 = \sigma$ for $\mathbf{R}_{\mathcal{D}}^p(s) \bar{\mathbf{n}}_{\sigma \kappa}$: for $i = 1, 2$

$$\begin{aligned} |\mathbf{R}_{\sigma}^p|^2 &= \left| \frac{1}{m_{\sigma}} \int_{\sigma} \mathbf{R}_{\mathcal{D}}^p(s) \bar{\mathbf{n}}_{\sigma \kappa} ds \right|^2 \\ &\leq \frac{Ch_{\mathcal{D}_i}}{m_{\sigma}} \int_{\mathcal{D}_i} \|\nabla(\mathbf{R}_{\mathcal{D}}^p(z) \bar{\mathbf{n}}_{\sigma \kappa})\|_{\mathcal{F}}^2 dz + \frac{C}{h_{\mathcal{D}_i} m_{\sigma}} \int_{\mathcal{D}_i} |(\mathbf{R}_{\mathcal{D}}^p(z) \bar{\mathbf{n}}_{\sigma \kappa})|^2 dz \\ &\leq \frac{Ch_{\mathcal{D}_i}}{m_{\sigma}} \int_{\mathcal{D}} |\nabla \mathbf{R}_{\mathcal{D}}^p(z)|^2 dz + \frac{C}{h_{\mathcal{D}_i} m_{\sigma}} \int_{\mathcal{D}} |\mathbf{R}_{\mathcal{D}}^p(z)|^2 dz. \end{aligned}$$

Thanks to (3.1) we have $\frac{m_{\mathcal{D}} h_{\mathcal{D}_i}}{m_{\sigma}} \leq \frac{1}{2} \text{size}(\mathcal{T})^2$ and $\frac{m_{\mathcal{D}}}{h_{\mathcal{D}_i} m_{\sigma}} \leq C$. We deduce

$$\sum_{\mathcal{D}_{\sigma, \sigma^*} \in \mathcal{D}} m_{\mathcal{D}} |\mathbf{R}_{\sigma}^p|^2 \leq C \text{size}(\mathcal{T})^2 \sum_{\mathcal{D} \in \mathcal{D}} \int_{\mathcal{D}} |\nabla \mathbf{R}_{\mathcal{D}}^p(z)|^2 dz + C \sum_{\mathcal{D} \in \mathcal{D}} \int_{\mathcal{D}} |\mathbf{R}_{\mathcal{D}}^p(z)|^2 dz.$$

Finally, Proposition 5.4 gives

$$\sum_{\mathcal{D}_{\sigma, \sigma^*} \in \mathcal{D}} m_{\mathcal{D}} |\mathbf{R}_{\sigma}^p|^2 \leq C \text{size}(\mathcal{T})^2 \|\nabla p\|_2^2.$$

We proceed in the same way for the estimate of $\mathbf{R}_{\sigma^*}^p$. \square

7.3. Proof of Theorem 7.1. We may now collect all the previous results in order to conclude the proof of Theorem 7.1, that we started in Subsection 7.1.

Proof.

Having denoted by $\mathbf{e}^\tau = \mathfrak{P}_{m,g} \mathbb{P}_c^\tau \mathbf{u} - \mathbf{u}^\tau$ the error on the velocity and $e^\mathfrak{D} = \mathbb{P}_m^\mathfrak{D} p - p^\mathfrak{D}$ the error on the pressure, we have obtained the inequality (7.6)

$$\begin{aligned} \|\nabla^\mathfrak{D} \mathbf{e}^\tau\|_2^2 + \|e^\mathfrak{D}\|_2^2 &\leq C(\|\mathbf{R}_\sigma^\mathbf{u}\|_2 + \|\mathbf{R}_{\sigma^*}^\mathbf{u}\|_2 + \|\mathbf{R}_\sigma^p\|_2 + \|\mathbf{R}_{\sigma^*}^p\|_2)(\|\nabla^\mathfrak{D} \mathbf{e}^\tau\|_2 + \|e^\mathfrak{D}\|_2) \\ &\quad + C \text{size}(\mathcal{T}) (\|\mathbf{u}\|_{H^2} + \|\nabla p\|_2) (\|\nabla^\mathfrak{D} \mathbf{e}^\tau\|_2 + \|e^\mathfrak{D}\|_2). \end{aligned}$$

Corollaries 7.1 and 7.2 imply

$$\|\nabla^\mathfrak{D} \mathbf{e}^\tau\|_2 \leq C \text{size}(\mathcal{T}) \quad \text{and} \quad \|e^\mathfrak{D}\|_2 \leq C \text{size}(\mathcal{T}). \quad (7.8)$$

Estimate of $\|\mathbf{u} - \mathbf{u}^\tau\|_2$. We have

$$\|\mathbf{u} - \mathbf{u}^\tau\|_2 \leq \|\mathbf{u} - \mathfrak{P}_{m,g} \mathbb{P}_c^\tau \mathbf{u}\|_2 + \|\mathfrak{P}_{m,g} \mathbb{P}_c^\tau \mathbf{u} - \mathbf{u}^\tau\|_2.$$

Lemma 5.7 and the discrete Poincaré inequality (Theorem 5.2) imply

$$\|\mathbf{u} - \mathbf{u}^\tau\|_2 \leq C \text{size}(\mathcal{T}) \|\nabla \mathbf{u}\|_{H^1} + C \|\nabla^\mathfrak{D} \mathfrak{P}_{m,g} \mathbb{P}_c^\tau \mathbf{u} - \nabla^\mathfrak{D} \mathbf{u}^\tau\|_2.$$

Finally, (7.8) gives the estimate of $\|\mathbf{u} - \mathbf{u}^\tau\|_2$.

Estimate of $\|\nabla \mathbf{u} - \nabla^\mathfrak{D} \mathbf{u}^\tau\|_2$. We have

$$\begin{aligned} \|\nabla \mathbf{u} - \nabla^\mathfrak{D} \mathbf{u}^\tau\|_2 &\leq \|\nabla \mathbf{u} - \nabla^\mathfrak{D} \mathbb{P}_c^\tau \mathbf{u}\|_2 + \|\nabla^\mathfrak{D} \mathbb{P}_c^\tau \mathbf{u} - \nabla^\mathfrak{D} \mathfrak{P}_{m,g} \mathbb{P}_c^\tau \mathbf{u}\|_2 \\ &\quad + \|\nabla^\mathfrak{D} \mathfrak{P}_{m,g} \mathbb{P}_c^\tau \mathbf{u} - \nabla^\mathfrak{D} \mathbf{u}^\tau\|_2. \end{aligned}$$

Finally, Lemmas 5.4, 5.5 and (7.8) imply the estimate of $\|\nabla \mathbf{u} - \nabla^\mathfrak{D} \mathbf{u}^\tau\|_2$.

Estimate of $\|p - p^\mathfrak{D}\|_2$. Using (7.8), we obtain

$$\|\mathbb{P}_m^\mathfrak{D} p - p^\mathfrak{D}\|_2 \leq C \text{size}(\mathcal{T}).$$

We conclude thanks to Proposition 5.4 that the estimate of $\|p - p^\mathfrak{D}\|_2$. \square

8. Numerical results. We show here some numerical results obtained on a rectangular domain $\Omega =]0, 1[^2$. Error estimates are given on four different tests with a stabilization coefficient forced to be $\lambda = 10^{-3}$.

In the first one, the exact solution is the Green-Taylor vortex on a quadrangle and triangle mesh (see Figure 8.1(a)). The second one is a polynomial function on a non-conformal square mesh (see Figure 8.2(a)). The third and fourth tests are performed using a discontinuous viscosity function.

The exact solution (\mathbf{u}, p) and the viscosity η being chosen, we define the source term \mathbf{f} and the boundary data \mathbf{g} in such a way that (2.1) is satisfied. In Figures 8.1, 8.2, 8.3 and 8.4, we compare the L^2 -norm of the error obtained with the DDFV scheme, for the pressure $\frac{\|\mathbb{P}_c^\mathfrak{D} p - p^\mathfrak{D}\|_2}{\|\mathbb{P}_c^\mathfrak{D} p\|_2}$, for the velocity gradient $\frac{\|\nabla^\mathfrak{D} \mathbb{P}_c^\tau \mathbf{u} - \nabla^\mathfrak{D} \mathbf{u}^\tau\|_2}{\|\nabla^\mathfrak{D} \mathbb{P}_c^\tau \mathbf{u}\|_2}$ and for the velocity $\frac{\|\mathbb{P}_c^\tau \mathbf{u} - \mathbf{u}^\tau\|_2}{\|\mathbb{P}_c^\tau \mathbf{u}\|_2}$ respectively, as functions of the size of the mesh $\text{size}(\mathcal{T})$ in a logarithmic scale. In the numerical tests, we have smooth pressure, thus we use the center-value projection on $\overline{\Omega}$:

$$\mathbb{P}_c^\mathfrak{D} p = ((p(x_D))_{D \in \mathfrak{D}}).$$

8.1. Test 1 (Green-Taylor vortex) - Constant viscosity. Let us consider the following exact solution:

$$\mathbf{u}(x, y) = \begin{pmatrix} \frac{1}{2} \sin(2\pi x) \cos(2\pi y) \\ -\frac{1}{2} \cos(2\pi x) \sin(2\pi y) \end{pmatrix}, \quad p(x, y) = \frac{1}{8} \cos(4\pi x) \sin(4\pi y),$$

and the viscosity is

$$\eta(x, y) = 1.$$

The mesh considered is a quadrangle mesh (see Figure 8.1(a)).

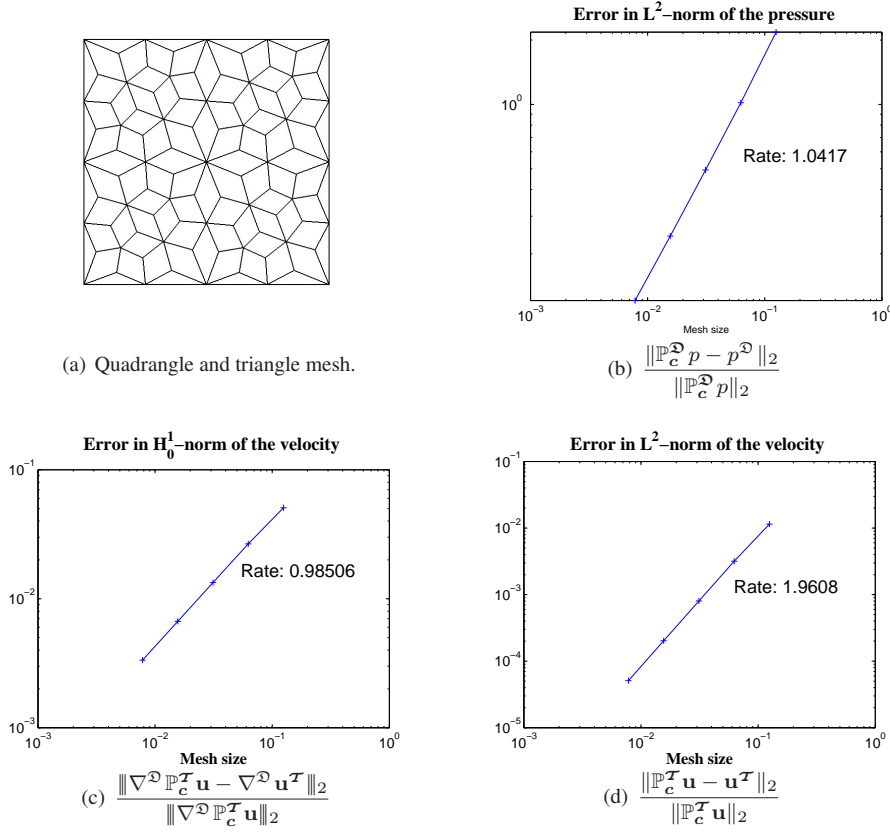


FIG. 8.1. Test 1, Green-Taylor vortex, on a quadrangle and triangle mesh.

For smooth solution and viscosity, as predicted by Theorem 7.1, we observe a first order convergence for the L^2 -norm of the velocity gradient and of the pressure, which seems to be optimal. We obtain a second order convergence for the L^2 -norm of the velocity. This superconvergence of the L^2 -norm is classical for finite volume method, however its proof still remains an open problem.

8.2. Test 2 (polynomial function) - Non constant viscosity. The exact solution on the second test are the following polynomial functions:

$$\mathbf{u}(x, y) = \begin{pmatrix} 1000x^2(1-x)^2 2y(1-y)(1-2y) \\ -1000y^2(1-y)^2 2x(1-x)(1-2x) \end{pmatrix}, \quad p(x, y) = x^2 + y^2 - \frac{2}{3},$$

and the viscosity is

$$\eta(x, y) = 2x + y + 1.$$

We use the non conformal square mesh, arbitrarily locally refined on the left bottom corner, as shown on Figure 8.2(a).

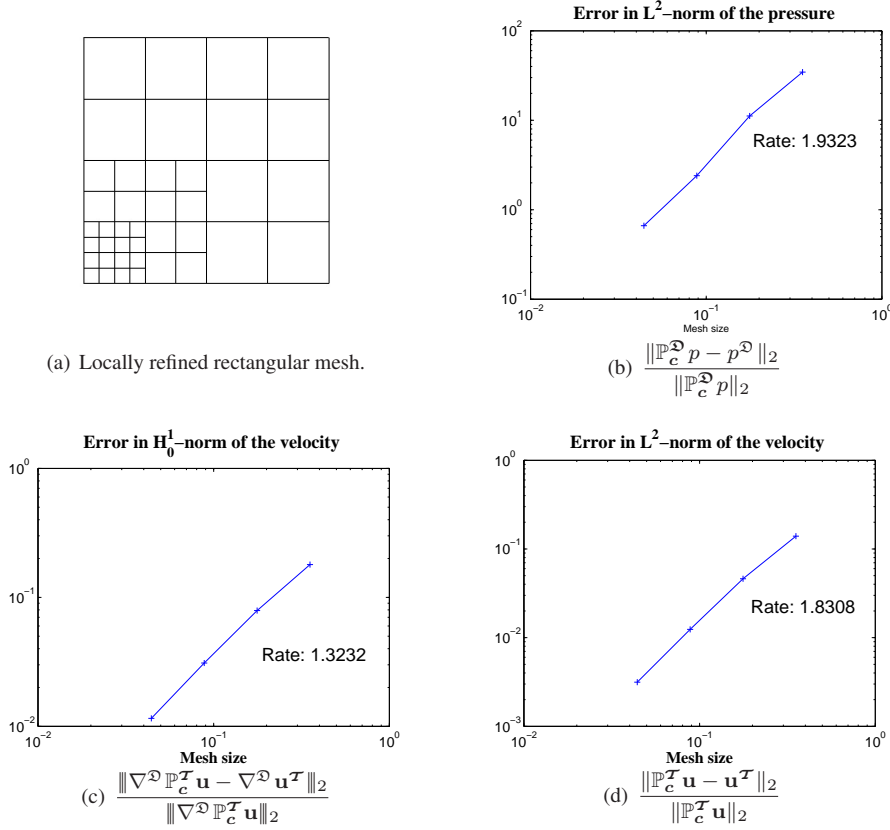


FIG. 8.2. Test 2, polynomial function, on a non conformal rectangular mesh.

Note that the convergence rates obtained in this numerical test are greater than the theoretical one given in Theorem 7.1 This is related to some uniformity of the mesh away the refinement area. For the velocity, its gradient and the pressure, we numerically obtain convergence rates equal to 1.9, 1.3 and 2 respectively for the DDFV scheme. Furthermore, let us emphasize that the convergence rate is not sensitive to the presence of non conformal control volumes.

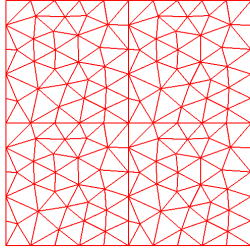
8.3. Test 3 - Discontinuous viscosity. The exact solution on the third test are the following functions:

$$\mathbf{u}(x, y) = \begin{pmatrix} -\pi y \\ \sin(\pi(x - 0.5)) \end{pmatrix}, \quad p(x, y) = 2.5(x - y).$$

Here, we consider a discontinuous viscosity:

$$\eta(x, y) = \begin{cases} 1 & \text{for } x > 0.5 \\ 10^{-4} & \text{else.} \end{cases}$$

We use the triangle mesh, shown on Figure 8.3(a).



(a) Triangle mesh.

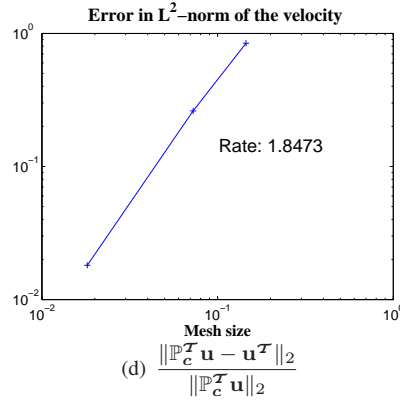
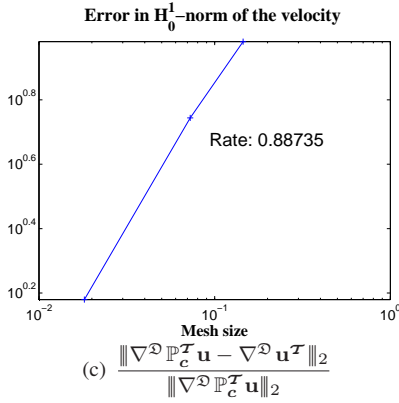
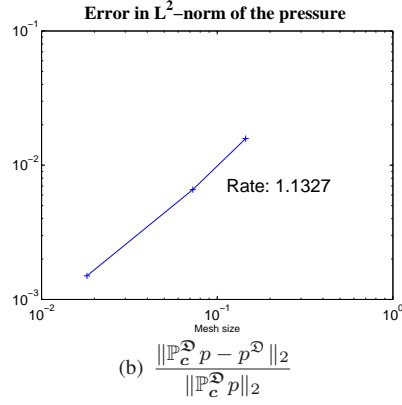


FIG. 8.3. Test 3, discontinuous viscosity on a triangle mesh.

With this viscosity, the assumptions of Theorem 7.1 are not satisfied. Nevertheless, the numerical test shows that we have a first order convergence for velocity in H_0^1 -norm and for the pressure in L^2 -norm. It seems to come from the fact that Du is equal to zero across the discontinuity of the viscosity ($x = 0.5$) so that the jump of viscosity does not affect the consistency properties of the numerical fluxes at the interface. We will see in the next test case that it is not always the case.

8.4. Test 4 - Discontinuous viscosity. Let us consider the following exact solution:

$$\mathbf{u}(x, y) = \begin{pmatrix} \begin{cases} y^2 - 0.5y & \text{for } y > 0.5 \\ 10^4(y^2 - 0.5y) & \text{else.} \end{cases} \\ 0 \end{pmatrix}, \quad p(x, y) = 2x - 1,$$

and the discontinuous viscosity:

$$\eta(x, y) = \begin{cases} 1 & \text{for } y > 0.5 \\ 10^{-4} & \text{else.} \end{cases}$$

We use the non conformal quadrangle mesh, locally refined where the viscosity is discontinuous, shown on Figure 8.4(a).

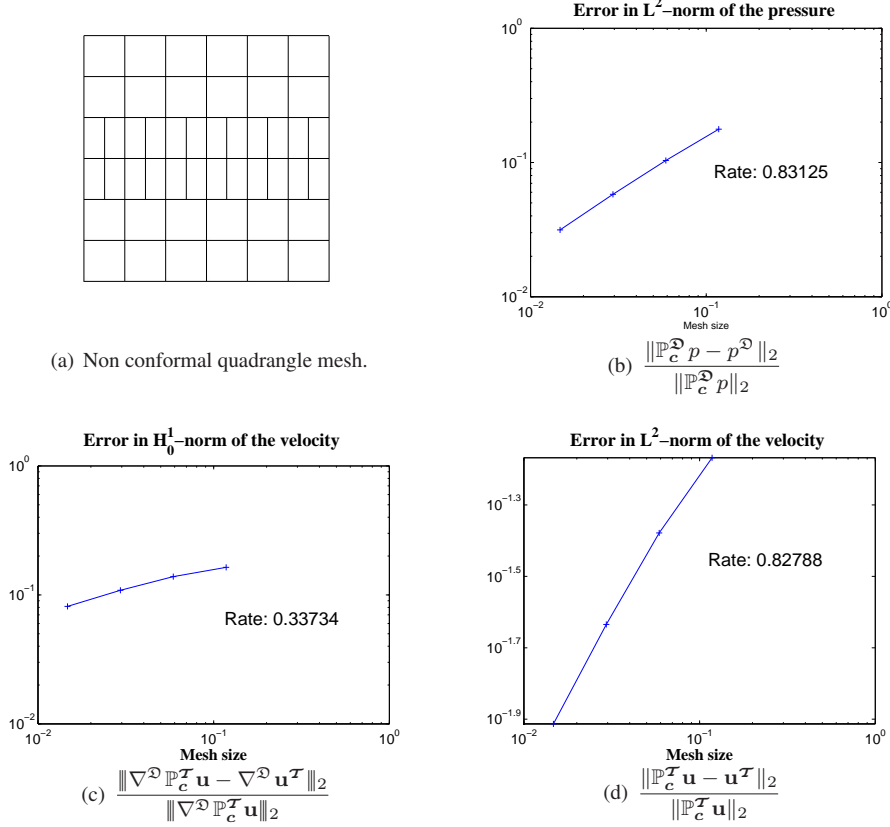


FIG. 8.4. Test 4, discontinuous viscosity on a non conformal quadrangle mesh.

Once more here the assumptions on the viscosity of Theorem 7.1 are not satisfied and the symmetric part of the gradient $D\mathbf{u}$ is discontinuous on the interface $y = 0.5$. We observe that the scheme is still convergent but we have lost the first order convergence for the DDFV scheme, as expected.

9. Conclusion. In this article, we propose stabilized DDFV schemes with Dirichlet boundary conditions for the Stokes problem with variable viscosity. The DDFV scheme is wellposed and we prove a first order convergence of it in the L^2 -norm for the velocity gradient, as well as for the velocity and for the pressure. These results are proven in the case where the viscosity is smooth on the whole domain Ω . When the viscosity is no more smooth, we still observe the convergence of the DDFV scheme. Nevertheless, the DDFV scheme is no more first order convergent.

In practice, the viscosity may present discontinuities across some interface, in multi-phase flows, for instance. Such a viscosity function is no more Lipschitz continuous on the

whole domain Ω . For anisotropic problems, in presence of discontinuities, a suitable modification of the discrete gradient allows the authors of [5] to recover the first order convergence. This approach can be adapted to the present work. We proved in [28] that the modified DDFV scheme presents a better consistency of the fluxes at near the places where discontinuities of the viscosity occur. Finally, the modified stabilized DDFV scheme is proved to be well-posed and first order convergent on 2D general meshes, even for discontinuous viscosity.

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