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COMBINATORIAL INTERPRETATIONS OF THE JACOBI-STIRLING NUMBERS

YOANN GELINEAU AND JIANG ZENG

ABSTRACT. The Jacobi-Stirling numbers of the first and second kinds were introduced in 2006 in the spectral theory and are polynomial refinements of the Legendre-Stirling numbers. Andrews and Littlejohn have recently given a combinatorial interpretation for the second kind of the latter numbers. Noticing that these numbers are very similar to the classical central factorial numbers, we give combinatorial interpretations for the Jacobi-Stirling numbers of both kinds, which provide a unified treatment of the combinatorial theories for the two previous sequences and also for the Stirling numbers of both kinds.

1. INTRODUCTION

It is well known that Jacobi polynomials $P_n^{(\alpha, \beta)}(t)$ satisfy the classical second-order Jacobi differential equation:

$$(1-t^2)y''(t) + (\beta - \alpha - (\alpha + \beta + 2)t)y'(t) + n(n + \alpha + \beta + 1)y(t) = 0. \quad (1.1)$$

Let $\ell_{\alpha, \beta}[y](t)$ be the Jacobi differential operator:

$$\ell_{\alpha, \beta}[y](t) = \frac{1}{(1-t)^\alpha(1+t)^\beta} \left(-(1-t)^{\alpha+1}(1+t)^{\beta+1}y'(t) \right)'$$

Then, equation (1.1) is equivalent to say that $y = P_n^{(\alpha, \beta)}(t)$ is a solution of

$$\ell_{\alpha, \beta}[y](t) = n(n + \alpha + \beta + 1)y(t).$$

In [5, Theorem 4.2], for each $n \in \mathbb{N}$, Everitt et al. gave the following expansion of the n -th composite power of $\ell_{\alpha, \beta}$:

$$(1-t)^\alpha(1+t)^\beta \ell_{\alpha, \beta}^n[y](t) = \sum_{k=0}^n (-1)^k \left(P^{(\alpha, \beta)} S_n^k (1-t)^{\alpha+k} (1+t)^{\beta+k} y^{(k)}(t) \right)^{(k)},$$

where $P^{(\alpha, \beta)} S_n^k$ are called the Jacobi-Stirling numbers of the second kind. They [5, (4.4)] also gave an explicit summation formula for $P^{(\alpha, \beta)} S_n^k$ numbers, showing that these numbers depend only on one parameter $z = \alpha + \beta + 1$. So we can define the Jacobi-Stirling numbers as the connection coefficients in the following equation:

$$x^n = \sum_{k=0}^n \text{JS}_n^k(z) \prod_{i=0}^{k-1} (x - i(z + i)), \quad (1.2)$$

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TABLE 1. The first values of $\text{JS}_n^k(z)$

$k \setminus n$	1	2	3	4	5	6
1	1	$z+1$	$(z+1)^2$	$(z+1)^3$	$(z+1)^4$	$(z+1)^5$
2		1	$5+3z$	$21+24z+7z^2$	$85+141z+79z^2+15z^3$	$341+738z+604z^2+222z^3+31z^4$
3			1	$14+6z$	$147+120z+25z^2$	$1408+1662z+664z^2+90z^3$
4				1	$30+10z$	$627+400z+65z^2$
5					1	$55+15z$
6						1

where $\text{JS}_n^k(z) = P^{(\alpha, \beta)} S_n^k$, while the Jacobi-Stirling numbers of the first kind can be defined by inverting the above equation:

$$\prod_{i=0}^{n-1} (x - i(z+i)) = \sum_{k=0}^n \text{js}_n^k(z) x^k, \quad (1.3)$$

where $\text{js}_n^k(z) = P^{(\alpha, \beta)} s_n^k$ in the notations of [5].

It follows from (1.2) and (1.3) that the Jacobi-Stirling numbers $\text{JS}_n^k(z)$ and $\text{js}_n^k(z)$ satisfy, respectively, the following recurrence relations:

$$\begin{cases} \text{JS}_0^0(z) = 1, & \text{JS}_n^k(z) = 0, & \text{if } k \notin \{1, \dots, n\}, \\ \text{JS}_n^k(z) = \text{JS}_{n-1}^{k-1}(z) + k(k+z) \text{JS}_{n-1}^k(z), & n, k \geq 1. \end{cases} \quad (1.4)$$

and

$$\begin{cases} \text{js}_0^0(z) = 1, & \text{js}_n^k(z) = 0, & \text{if } k \notin \{1, \dots, n\}, \\ \text{js}_n^k(z) = \text{js}_{n-1}^{k-1}(z) - (n-1)(n-1+z) \text{js}_{n-1}^k(z), & n, k \geq 1. \end{cases} \quad (1.5)$$

The first values of $\text{JS}_n^k(z)$ and $\text{js}_n^k(z)$ are given, respectively, in Tables 1 and 2.

As remarked in [4, 5, 1], the previous definitions are reminiscent to the well-known *Stirling numbers* of the second (resp. the first) kind $S(n, k)$ (resp. $s(n, k)$), which are defined (see [2]) by

$$x^n = \sum_{k=0}^n S(n, k) \prod_{i=0}^{k-1} (x-i), \quad \prod_{i=0}^{n-1} (x-i) = \sum_{k=0}^n s(n, k) x^k.$$

and satisfy the following recurrences:

$$S(n, k) = S(n-1, k-1) + kS(n-1, k), \quad n, k \geq 1, \quad (1.6)$$

$$s(n, k) = s(n-1, k-1) - (n-1)s(n-1, k), \quad n, k \geq 1. \quad (1.7)$$

The starting point of this paper is the observation that the *central factorial numbers* of the second (resp. the first) kind $T(n, k)$ (resp. $t(n, k)$) seem to be more appropriate for comparison. Indeed, these numbers are defined in Riordan's book [8, p. 213-217] by

$$x^n = \sum_{k=0}^n T(n, k) x \prod_{i=1}^{k-1} \left(x + \frac{k}{2} - i \right), \quad (1.8)$$

and

$$x \prod_{i=1}^{n-1} \left(x + \frac{n}{2} - i \right) = \sum_{k=0}^n t(n, k) x^k. \quad (1.9)$$

Therefore, if we denote the central factorial numbers of even indices by $U(n, k) = T(2n, 2k)$ and $u(n, k) = t(2n, 2k)$, then :

TABLE 2. The first values of $js_n^k(z)$

$k \setminus n$	1	2	3	4	5
1	1	$-z - 1$	$2z^2 + 6z + 4$	$-6z^3 - 36z^2 - 66z - 36$	$24z^4 + 240z^3 + 840z^2 + 1200z + 576$
2		1	$-3z - 5$	$11z^2 + 48z + 49$	$-50z^3 - 404z^2 - 1030z - 820$
3			1	$-6z - 14$	$35z^2 + 200z + 273$
4				1	$-10z - 30$
5					1

$$U(n, k) = U(n - 1, k - 1) + k^2 U(n - 1, k), \quad (1.10)$$

$$u(n, k) = u(n - 1, k - 1) - (n - 1)^2 u(n - 1, k). \quad (1.11)$$

From (1.4)-(1.11), we easily derive the following result.

Theorem 1. *Let n, k be positive integers with $n \geq k$. The Jacobi-Stirling numbers $JS_n^k(z)$ and $(-1)^{n-k} js_n^k(z)$ are polynomials in z of degree $n - k$ with positive integer coefficients. Moreover, if*

$$JS_n^k(z) = a_{n,k}^{(0)} + a_{n,k}^{(1)}z + \cdots + a_{n,k}^{(n-k)}z^{n-k}, \quad (1.12)$$

$$(-1)^{n-k} js_n^k(z) = b_{n,k}^{(0)} + b_{n,k}^{(1)}z + \cdots + b_{n,k}^{(n-k)}z^{n-k}, \quad (1.13)$$

then

$$a_{n,k}^{(n-k)} = S(n, k), \quad a_{n,k}^{(0)} = U(n, k), \quad b_{n,k}^{(n-k)} = |s(n, k)|, \quad b_{n,k}^{(0)} = |u(n, k)|.$$

Note that when $z = 1$, the Jacobi-Stirling numbers reduce to the *Legendre-Stirling numbers* of the first and the second kinds [4]:

$$LS(n, k) = JS_n^k(1), \quad ls(n, k) = js_n^k(1). \quad (1.14)$$

The integral nature of the involved coefficients in the above polynomials ask for combinatorial interpretations. Indeed, it is folklore (see [2]) that the Stirling number $S(n, k)$ (resp. $|s(n, k)|$) counts the number of partitions (resp. permutations) of $[n] := \{1, \dots, n\}$ into k blocks (resp. cycles). In 1974, in his study of Genocchi numbers, Dumont [3] discovered the first combinatorial interpretation for the central factorial number $U(n, k)$ in terms of couples of supdiagonal quasi-permutations of $[n]$ (cf. § 2). Recently, Andrews and Littlejohn [1] interpreted $JS_n^k(1)$ in terms of set partitions (cf. § 2).

Several questions arise naturally in the light of the above known results:

- First of all, what is the combinatorial refinement of Andrews and Littlejohn's model which gives the combinatorial counterpart for the coefficient $a_{n,k}^{(i)}$?
- Secondly, is there any connection between the model of Dumont and that of Andrews and Littlejohn?
- Thirdly, is there any combinatorial interpretation for the coefficient $b_{n,k}^{(i)}$ in the Jacobi-Stirling numbers of the first kind, generalizing that for the Stirling number $|s(n, k)|$?

The aim of this paper is to settle all of these questions. Additional results of the same type are also provided.

In Section 2, after introducing some necessary definitions, we give two combinatorial interpretations for the coefficient $a_{n,k}^{(i)}$ in $JS_n^k(z)$ ($0 \leq i \leq n - k$), and explicitly

construct a bijection between the two models. In Section 3, we give a combinatorial interpretation for the coefficient $b_{n,k}^{(i)}$ in $js_n^k(z)$ ($0 \leq i \leq n - k$). In Section 4, we give the combinatorial interpretation for two sequences which are multiples of the central factorial numbers of odd indices and we also establish a simple derivation of the explicit formula of Jacobi-Stirling numbers.

2. JACOBI-STIRLING NUMBERS OF THE SECOND KIND $JS_n^k(z)$

2.1. First interpretation. For any positive integer n we define

$$[\pm n]_0 := \{0, 1, -1, 2, -2, 3, -3, \dots, n, -n\}.$$

The following definition is equivalent to that given by Andrews and Littlejohn [1] in order to interpret Legendre-Stirling numbers, where 0 is added to avoid empty block and also to be consistent with the model for the Jacobi-Stirling numbers of the first kind.

Definition 1. A *signed k -partition* of $[\pm n]_0$ is a set partition of $[\pm n]_0$ with $k + 1$ non-empty blocks B_0, B_1, \dots, B_k with the following rules:

- (1) $0 \in B_0$ and $\forall i \in [n]$, $\{i, -i\} \not\subset B_0$,
- (2) $\forall j \in [k]$ and $\forall i \in [n]$, we have $\{i, -i\} \subset B_j \iff i = \min B_j \cap [n]$.

For example, the partition $\pi = \{\{2, -5\}_0, \{\pm 1, -2\}, \{\pm 3\}, \{\pm 4, 5\}\}$ is a signed 3-partition of $[\pm 5]_0$, with $\{2, -5\}_0 := \{0, 2, -5\}$ being the zero-block.

Theorem 2. For any positive integers n and k , the integer $a_{n,k}^{(i)}$ ($0 \leq i \leq n - k$) is the number of signed k -partitions of $[\pm n]_0$ such that the zero-block contains i signed entries.

Proof. Let $\mathcal{A}_{n,k}^{(i)}$ be the set of signed k -partitions of $[\pm n]_0$ such that the zero-block contains i signed entries and $\tilde{a}_{n,k}^{(i)} = |\mathcal{A}_{n,k}^{(i)}|$. By convention $\tilde{a}_{0,0}^{(0)}$. Clearly $\tilde{a}_{1,1}^{(0)} = 1$ and for $\tilde{a}_{n,k}^{(i)} \neq 0$ we must have $n \geq k \geq 1$ and $0 \leq i \leq n - k$. We divide $\mathcal{A}_{n,k}^{(i)}$ into four parts:

- (i) the signed k -partitions of $[\pm n]_0$ with $\{-n, n\}$ as a block. Clearly, the number of such partitions is $\tilde{a}_{n-1,k-1}^{(i)}$.
- (ii) the signed k -partitions of $[\pm n]_0$ with n in the zero-block. We can construct such partitions by first constructing a signed k -partition of $[\pm(n-1)]_0$ with i signed entries in the zero block and then insert n into the zero block and $-n$ into one of the k other blocks; so there are $k\tilde{a}_{n-1,k}^{(i)}$ such partitions.
- (iii) the signed k -partitions of $[\pm n]_0$ with $-n$ in the zero-block. We can construct such partitions by first constructing a signed k -partition of $[\pm(n-1)]_0$ with $i - 1$ signed entries in the zero-block, and then placing n into one of the k non-empty blocks, so there are $k\tilde{a}_{n-1,k}^{(i-1)}$ possibilities.
- (iv) the signed k -partitions of $[\pm n]_0$ where neither n nor $-n$ appears in the zero-block and $\{-n, n\}$ is not a block. We can construct such partitions by first choosing a signed k -partition of $[\pm(n-1)]_0$ with i signed entries in the zero block, and then placing n and $-n$ into two different non-zero blocks, so there are $k(k-1)\tilde{a}_{n-1,k}^{(i)}$ possibilities.

Summing up we get the following equation:

$$\tilde{a}_{n,k}^{(i)} = \tilde{a}_{n-1,k-1}^{(i)} + k\tilde{a}_{n-1,k}^{(i-1)} + k^2\tilde{a}_{n-1,k}^{(i)}. \quad (2.1)$$

TABLE 3. The first values of $JS_n^k(z)$ in the basis $\{(z+1)^i\}_{i=0,\dots,n-k}$

$k \setminus n$	1	2	3	4	5
1	1	$(z+1)$	$(z+1)^2$	$(z+1)^3$	$(z+1)^4$
2		1	$2+3(z+1)$	$4+10(z+1)+7(z+1)^2$	$8+28(z+1)+34(z+1)^2+15(z+1)^3$
3			1	$8+6(z+1)$	$52+70(z+1)+25(z+1)^2$
4				1	$20+10(z+1)$
5					1

By (1.4) it is easy to see that $a_{n,k}^{(i)}$ satisfies the same recurrence and initial conditions as $\tilde{a}_{n,k}^{(i)}$, so they agree. \square

Since $LS(n, k) = \sum_{k=0}^{n-k} a_{n,k}^{(i)}$, Theorem 2 implies immediately the following result of Andrews and Littlejohn [1].

Corollary 1. The integer $LS(n, k)$ is the number of signed k -partitions of $[\pm n]_0$.

By Theorems 1 and 2, we derive that the integer $S(n, k)$ is the number of signed k -partitions of $[\pm n]_0$ such that the zero-block contains $n - k$ signed entries. By definition, in this case there is no positive entry in the zero-block. By deleting the signed entries in the remaining k blocks we recover then the following known interpretation for the Stirling number of the second kind.

Corollary 2. The integer $S(n, k)$ is the number of partitions of $[n]$ in k blocks.

For a partition $\pi = \{B_1, B_2, \dots, B_k\}$ of $[n]$ in k blocks, denote by $\min \pi$ the set of minima of blocks

$$\min \pi = \{\min(B_1), \dots, \min(B_k)\}.$$

The following partition version of Dumont's interpretation for the central factorial number of even indices can be found in [6, Chap. 3].

Corollary 3. The integer $U(n, k)$ is the number of couples (π_1, π_2) of partitions of $[n]$ in k blocks such that $\min(\pi_1) = \min(\pi_2)$.

Proof. As $U(n, k) = a_{n,k}^{(0)}$, by Theorem 2, the integer $U(n, k)$ counts the number of signed k -partitions of $[\pm n]_0$ such that the zero-block doesn't contain any signed entry. For any such a signed k -partition π , we apply the following algorithm: (i) move each positive entry j of the zero-block into the block containing $-j$ to obtain a signed k -partition $\pi' = \{\{0\}, B_1, \dots, B_k\}$, (ii) π_1 is obtained by deleting the negative entries in each block B_i of π' , and π_2 is obtained by deleting the positive entries and taking the opposite values of signed entries in each block of π' . For example, if $\pi = \{\{3\}_0, \{\pm 1, -3, 4\}, \{\pm 2, -4\}\}$ is the signed 2-partition of $[\pm 4]_0$, the corresponding couple of partitions is (π_1, π_2) with $\pi_1 = \{\{1, 3, 4\}, \{2\}\}$ and $\pi_2 = \{\{1, 3\}, \{2, 4\}\}$. \square

The following result shows that the coefficients in the expansion of the Jacobi-Stirling numbers $JS_n^k(z)$ in the basis $\{(z+1)^i\}_{i=0,\dots,n-k}$ are also interesting.

Theorem 3. Let

$$JS_n^k(z) = d_{n,k}^{(0)} + d_{n,k}^{(1)}(z+1) + \dots + d_{n,k}^{(n-k)}(z+1)^{n-k}. \quad (2.2)$$

Then the coefficient $d_{n,k}^{(i)}$ is a positive integer, which counts the number of signed k -partitions of $[\pm n]_0$ such that the zero-block contains only zero and i negative values.

Proof. We derive from (1.4) that the coefficients $d_{n,k}^{(i)}$ verify the following recurrence relation:

$$d_{n,k}^{(i)} = d_{n-1,k-1}^{(i)} + kd_{n-1,k}^{(i-1)} + k(k-1)d_{n-1,k}^{(i)}. \quad (2.3)$$

As for the $a_{n,k}^{(i)}$, we can prove the result by a similar argument in proof of Theorem 2 \square

Corollary 4. The integer $J_n^k(-1) = d_{n,k}^{(0)}$ is the number of signed k -partitions of $[\pm n]_0$ with $\{0\}$ as zero-block.

Remark 1. A priori, it was not obvious that $J_n^k(-1) = \sum_{i=0}^{n-k} (-1)^i a_{n,k}^{(i)}$ was positive.

From Theorem 1 and (2.2), we derive the following relations :

$$a_{n,k}^{(i)} = \sum_{j=i}^{n-k} \binom{j}{i} d_{n,k}^{(j)}, \quad U(n,k) = \sum_{j=i}^{n-k} d_{n,k}^{(j)}, \quad \text{LS}(n,k) = \sum_{j=i}^{n-k} 2^j d_{n,k}^{(j)}. \quad (2.4)$$

We can give combinatorial interpretations for these formulas. For example, for the first one, we can split the set $\mathcal{A}_{n,k}^{(i)}$ by counting the total number j of elements in the zero-block ($1 \leq j \leq n-k$). Then to construct such an element, we first take a signed k -partition of $[\pm n]_0$ with no positive values in the zero-block, so there are $d_{n,k}^{(j)}$ possibilities, and then we choose the $j-i$ numbers that are positive among the j possibilities in the zero-block. Similar proofs can be easily described for the two other formulas.

2.2. Second interpretation. We propose now a second model for the coefficient $a_{n,k}^{(i)}$, inspired by Foata and Schützenberger [7] and Dumont [3]. Let \mathcal{S}_n be the set of permutations of $[n]$. In the rest of this paper, we identify any permutation σ in \mathcal{S}_n with its diagram $\mathcal{D}(\sigma) = \{(i, \sigma(i)) : i \in [n]\}$.

For any finite set X , we denote by $|X|$ its cardinality. If $\alpha = (i, j) \in [n] \times [n]$, we define $\text{pr}_x(\alpha) = i$ and $\text{pr}_y(\alpha) = j$ to be its x and y projections. For any subset Q of $[n] \times [n]$, we define the x and y projections by

$$\text{pr}_x(Q) = \{\text{pr}_x(\alpha) : \alpha \in Q\}, \quad \text{pr}_y(Q) = \{\text{pr}_y(\alpha) : \alpha \in Q\};$$

and the supdiagonal and subdiagonal parts by

$$Q^+ = \{(i, j) \in Q : i \leq j\}, \quad Q^- = \{(i, j) \in Q : i \geq j\}.$$

Definition 2. A *simply hooked k -quasi-permutation* of $[n]$ is a subset Q of $[n] \times [n]$ such that

- i) $Q \subset \mathcal{D}(\sigma)$ for some permutation σ of $[n]$,
- ii) $|Q| = n - k$ and $\text{pr}_x(Q^-) \cap \text{pr}_y(Q^+) = \emptyset$.

A simply hooked k -quasi-permutation Q of $[n]$ can be depicted by darkening the $n-k$ corresponding boxes of Q in the $n \times n$ square tableau. Conversely, if we define the diagonal hook $H_i := \{(i, j) : i \leq j\} \cup \{(j, i) : i \leq j\}$ ($1 \leq i \leq n$), then a black subset of the $n \times n$ square tableau represents a simply hooked quasi-permutation if there is no black box on the main diagonal and at most one black box in each row, in each column and in each diagonal hook. An example is given in Figure 1.

FIGURE 1. The diagonal hook H_4 and a simply hooked quasi-permutation of $[6]$: $Q = \{(2, 5), (4, 2), (6, 3)\}$

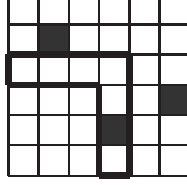
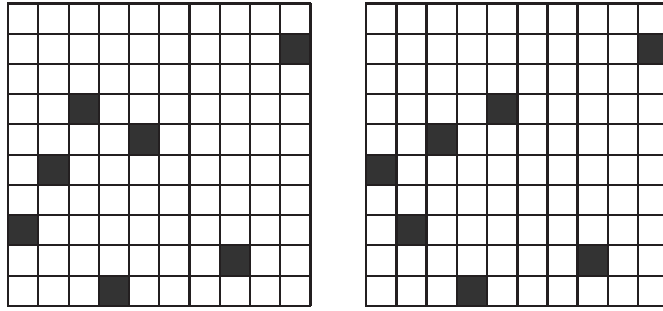


FIGURE 2. A couple of simply hooked quasi-permutations in $\mathcal{C}_{10,3}^{(3)}$



Theorem 4. The integer $a_{n,k}^{(i)}$ ($1 \leq i \leq n - k$) is the number of couples (Q_1, Q_2) of simply hooked k -quasi-permutations of $[n]$ satisfying the following conditions:

$$Q_1^- = Q_2^-, \quad |Q_1^-| = |Q_2^-| = i \quad \text{and} \quad pr_y(Q_1) = pr_y(Q_2). \quad (2.5)$$

Proof. Let $\mathcal{C}_{n,k}^{(i)}$ be the set of couples (Q_1, Q_2) of simply hooked k -quasi-permutations of $[n]$ verifying (2.5), and let $c_{n,k}^{(i)} = |\mathcal{C}_{n,k}^{(i)}|$.

For example, the couple (Q_1, Q_2) with

$$\begin{aligned} Q_1 &= \{(1, 3), (2, 5), (3, 7), (4, 1), (5, 6), (8, 2), (10, 9)\}, \\ Q_2 &= \{(1, 5), (2, 3), (3, 6), (4, 1), (5, 7), (8, 2), (10, 9)\}, \end{aligned} \quad (2.6)$$

is an element of $\mathcal{C}_{10,3}^{(3)}$. A graphical representation is given in Figure 2.

We divide the set $\mathcal{C}_{n,k}^{(i)}$ into three parts:

- the couples (Q_1, Q_2) such that the n -th rows and n -th columns of Q_1 and Q_2 are empty. Clearly, there are $c_{n-1,k-1}^{(i)}$ such elements.
- the couples (Q_1, Q_2) such that the n -th columns of Q_1 and Q_2 are not empty. We can first construct a couple (Q'_1, Q'_2) of $\mathcal{C}_{n-1,k}^{(i-1)}$ and then choose a box in the same position of the n -th column of both simply hooked quasi-permutations, there are $n - 1 - (n - k - 1) = k$ positions available. So there are $kc_{n-1,k}^{(i-1)}$ such elements.
- the couples (Q_1, Q_2) such that the n -th rows of Q_1 and Q_2 are not empty. We can first construct a couple (Q'_1, Q'_2) of $\mathcal{C}_{n-1,k}^{(i)}$ and then add a black box in the top of both simply hooked quasi-permutations, the box can be

2.3. The link between the two models. We introduce a third interpretation which permits to make the connection easier between the two previous models. Let $\Pi_{n,k}$ be the set of partitions of $[n]$ in k non-empty blocks.

Definition 4. Let $\mathcal{B}_{n,k}^{(i)}$ be the set of triples (π_1, π_2, π_3) in $\Pi_{n,k+i} \times \Pi_{n,k+i} \times \Pi_{n,n-i}$ such that:

- i) $\min(\pi_1) = \min(\pi_2)$ and $\text{Sing}(\pi_1) = \text{Sing}(\pi_2)$,
- ii) $\min(\pi_1) \cup \text{Sing}(\pi_3) = \text{Sing}(\pi_1) \cup \min(\pi_3) = [n]$,

where $\text{Sing}(\pi)$ denotes the set of singletons in π .

We will need the following result.

Lemma 5. For $(\pi_1, \pi_2, \pi_3) \in \mathcal{B}_{n,k}^{(i)}$, we have:

- i) $|\min(\pi_1) \cap \min(\pi_3)| = k$,
- ii) $|\text{Sing}(\pi_1) \setminus \min(\pi_3)| = i$,
- iii) $|\text{Sing}(\pi_3) \setminus \min(\pi_1)| = n - k - i$.

Proof. By definition, we have $|\min(\pi_1)| = k + i$ and $|\min(\pi_3)| = n - i$. Since $\min(\pi_1) \cup \min(\pi_3) = [n]$, by sieve formula, we deduce

$$|\min(\pi_1) \cap \min(\pi_3)| = |\min(\pi_1)| + |\min(\pi_3)| - |\min(\pi_1) \cup \min(\pi_3)| = k,$$

and

$$|\text{Sing}(\pi_1) \setminus \min(\pi_3)| = |\text{Sing}(\pi_1)| - |\text{Sing}(\pi_1) \cap \min(\pi_3)| = n - |\min(\pi_3)| = i.$$

In the same way, we obtain iii). \square

Theorem 6. There is a bijection between $\mathcal{A}_{n,k}^{(i)}$ and $\mathcal{B}_{n,k}^{(i)}$.

Proof. Let $\pi = \{B_0, B_1, \dots, B_k\}$ be a signed k -partition in $\mathcal{A}_{n,k}^{(i)}$. We construct the triple (π_1, π_2, π_3) of partitions by the following algorithm.

- π_1, π_2 :
- Let $\pi' = \{B'_0, B'_1, \dots, B'_k\}$ be the partition obtained by exchanging all j and $-j$ in π if $j \in B_0$ (resp. $j \in [n]$).
 - Let $\pi'' = \{B''_0, B''_1, \dots, B''_k\}$ be the partition obtained by removing all the negative values in π' .
 - Define π_1 (resp. π_2) to be the partition obtained by splitting the i positive elements in B''_0 into i singletons and deleting 0 in π'' .

The resulting partitions π_1 and π_2 are clearly elements of $\Pi_{n,k+i}$ and satisfy $\min(\pi_1) = \min(\pi_2)$ and $\text{Sing}(\pi_1) = \text{Sing}(\pi_2)$.

- π_3 :
- For all $p \in [n] \setminus \min \pi$ such that $B_0 \cap \{\pm p\} = \emptyset$, move p into the zero-block and obtain the partition $\pi' = \{B'_0, B'_1, \dots, B'_k\}$. So there are $n - k - i$ positive entries in the new B'_0 .
 - Let $\pi'' = \{B''_0, B''_1, \dots, B''_k\}$ be the partition obtained by removing all the negative values in π' .
 - Define π_3 to be the partition obtained by splitting the $n - k - i$ positive elements in B''_0 into i singletons and deleting 0 in π'' .

The resulting partition π_3 is an element of $\Pi_{n,n-i}$.

For any $p \in [n] \setminus \min(\pi_1)$, if $p \notin B_0$ then $B_0 \cap \{\pm p\} \neq \emptyset$, by definition p will be moved in the zero-block, otherwise p is already in the zero-block. Thus, the elements that are not in $\min(\pi_1)$ become singletons in π_3 . Hence $\min(\pi_1) \cup \text{Sing}(\pi_3) = [n]$. Similarly we have $\text{Sing}(\pi_1) \cup \min(\pi_3) = [n]$.

For example, for the signed 3-partition of $[\pm 10]_0$:

$$\pi = \{\{-4, 6, 7, -8, -10\}_0, \{\pm 1, 3, 4, -5, -7\}, \{\pm 2, -3, 5, -6, 8\}, \{\pm 9, 10\}\}, \quad (2.8)$$

the corresponding triple is $(\pi_1, \pi_2, \pi_3) \in \Pi_{10,6} \times \Pi_{10,6} \times \Pi_{10,7}$ with :

$$\begin{aligned} \pi_1 &= \{\{1, 3, 7\}, \{2, 5, 6\}, \{4\}, \{8\}, \{9\}, \{10\}\}, \\ \pi_2 &= \{\{1, 5, 7\}, \{2, 3, 6\}, \{4\}, \{8\}, \{9\}, \{10\}\}, \\ \pi_3 &= \{\{1, 4\}, \{2, 8\}, \{3\}, \{5\}, \{6\}, \{7\}, \{9, 10\}\}. \end{aligned} \quad (2.9)$$

Conversely, for any $(\pi_1, \pi_2, \pi_3) \in \mathcal{B}_{n,k}^{(i)}$, we construct $\pi = \{B_0, B_1, \dots, B_k\} \in \mathcal{A}_{n,k}^{(i)}$ with the following procedure:

- Use the k elements of $\min(\pi_1) \cap \min(\pi_3)$, say p_1, \dots, p_k and 0 to create $k+1$ blocks:

$$B_0 = \{\dots\}_0, B_1 = \{\pm p_1, \dots\}, \dots, B_k = \{\pm p_k \dots\}, \quad (2.10)$$

where “...” means that the blocks are not completed. For instance, for the triple (π_1, π_2, π_3) in (2.9), we create three blocks: $\{0, \dots\}$, $\{\pm 1, \dots\}$ and $\{\pm 2, \dots\}$.

- For each element x_j of $[n] \setminus \min(\pi_3)$ ($1 \leq j \leq i$), suppose that x_j appears in a non-singleton block C_j of π_3 . Then put $-x_j$ into the zero-block B_0 and x_j into the block in (2.10) that contains $\min(C_j)$. Note that we must show that $\min(C_j) \in \min(\pi_1) \cap \min(\pi_3)$ to warrant the existence of such a block in (2.10). Indeed, if $\min(C_j) \notin \min(\pi_1)$, then, by Definition 4, we would have $\min(C_j) \in \text{Sing}(\pi_3)$. For the current example, we place the number 4 in the block that contains 1.
- For each element y_j of $[n] \setminus \min(\pi_2)$ ($1 \leq j \leq n - k - i$), suppose that y_j appears in a non-singleton block D_j (resp. E_j) of π_2 (resp. π_1). Then put $-p_j$ into the block in (2.10) that contains $\min(D_j)$ and put p_j into the block (2.10) that contains $\min(E_j)$ if this block doesn't contains $-p_j$, into the zero-block B_0 otherwise. For the current example, we place the number -3 in the block that contains 2. and 5 in the block that contains 2, and 6 in the zero-block because the block that contains 2 already has -6 .

□

Since φ described in Remark 3 maps each partition to a subdiagonal quasi-permutation, for every triple (π_1, π_2, π_3) of partitions satisfying the conditions of Theorem 6, we can associate a triple $(P_1, P_2, P_3) = (\varphi(\pi_1), \varphi(\pi_2), \varphi(\pi_3))$ of subdiagonal quasi-permutations. If \overline{P}_i denotes the supdiagonal quasi-permutation obtained from P_i exchanging the x and y coordinates, then $(Q_1, Q_2) = (\overline{P}_1 \cup P_3, \overline{P}_2 \cup P_3)$ is a couple of simply hooked quasi-permutations satisfying the conditions of Theorem 4. Thus, we obtain a bijection between the signed k -partitions and the couples of simply hooked quasi-permutations.

For example, for the signed 3-partition π in (2.8), the corresponding couple of simply hooked quasi-permutations (Q_1, Q_2) is then given by (2.6) (cf. Figure 2).

3. JACOBI-STIRLING NUMBERS OF THE FIRST KIND $js_n^k(z)$

For a permutation σ of $[n]_0 := [n] \cup \{0\}$ (resp. $[n]$) and for $j \in [n]_0$ (resp. $[n]$), denote by $\text{Orb}_\sigma(j) = \{\sigma^\ell(j) : \ell \geq 1\}$ the orbit of j and $\min(\sigma)$ the set of its cyclic

minima, i.e.,

$$\min(\sigma) = \{j \in [n] : j = \min(\text{Orb}_\sigma(j) \cap [n])\}.$$

Definition 5. Given a word $w = w(1) \dots w(\ell)$ on the finite alphabet $[n]$, a letter $w(j)$ is a *record* of w if $w(k) > w(j)$ for every $k \in \{1, \dots, j-1\}$. We define $\text{rec}(w)$ to be the number of records of w and $\text{rec}_0(w) = \text{rec}(w) - 1$.

For example, if $w = 574862319$, then the records are 5, 4, 2, 1. Hence $\text{rec}(w) = 4$.

Theorem 7. The integer $b_{n,k}^{(i)}$ is the number of couples (σ, τ) such that σ (resp. τ) is a permutation of $[n]_0$ (resp. $[n]$) with k cycles, satisfying

- i) $1 \in \text{Orb}_\sigma(0)$,
- ii) $\min \sigma = \min \tau$,
- iii) $\text{rec}_0(w) = i$, where $w = \sigma(0)\sigma^2(0) \dots \sigma^l(0)$ with $\sigma^{l+1}(0) = 0$.

Proof. Let $\mathcal{E}_{n,k}^{(i)}$ be the set of couples (σ, τ) satisfying the conditions of Theorem 7 and $e_{n,k}^{(i)} = |\mathcal{E}_{n,k}^{(i)}|$. We divide $\mathcal{E}_{n,k}^{(i)}$ into three parts:

- (i) the couples (σ, τ) such that $\sigma^{-1}(n) = n$. Then n forms a cycle in both σ and τ and there are clearly $e_{n-1,k-1}^{(i)}$ possibilities.
- (ii) the couples (σ, τ) such that $\sigma^{-1}(n) = 0$. We can construct such couples by first choosing a couple (σ', τ') in $\mathcal{E}_{n-1,k}^{(i-1)}$ and then inserting n in σ' as image of 0 (resp. in τ'). Clearly, there are $(n-1)e_{n-1,k}^{(i-1)}$ possibilities.
- (iii) the couples (σ, τ) such that $\sigma^{-1}(n) \notin \{0, n\}$. We can construct such couples by first choosing a couple (σ', τ') in $\mathcal{E}_{n-1,k}^{(i)}$ and then inserting n in σ' (resp. in τ'). Clearly, there are $(n-1)^2 e_{n-1,k}^{(i)}$ possibilities.

Summing up, we get the following equation:

$$e_{n,k}^{(i)} = e_{n-1,k-1}^{(i)} + (n-1)e_{n-1,k}^{(i-1)} + (n-1)^2 e_{n-1,k}^{(i)}. \quad (3.1)$$

By (1.5), it is easy to see that the coefficients $b_{n,k}^{(i)}$ satisfy the same recurrence. \square

We show now how to derive from Theorems 1 and 7 the combinatorial interpretations for the numbers $|\text{ls}(n, k)|$, $|s(n, k)|$ and $|u(n, k)|$.

Corollary 8. The integer $|\text{ls}(n, k)|$ is the number of couples (σ, τ) such that σ (resp. τ) is a permutation of $[n]_0$ (resp. $[n]$) with k cycles, satisfying $1 \in \text{Orb}_\sigma(0)$ and $\min \sigma = \min \tau$.

Corollary 9. The integer $|s(n, k)|$ is the number of permutations of $[n]$ with k cycles.

Proof. By Theorem 7, the integer $|s(n, k)|$ is the number of couples (σ, τ) in $\mathcal{E}_{n,k}^{(n-k)}$. Since σ and τ both have k cycles with same cyclic minima, the permutation σ is completely determined by τ because $\text{Orb}_\sigma(1)$ is the only non singleton cycle, of cardinality $n - k + 2$, so the $n - k$ elements different from 0 and 1 are exactly the elements of $[n] \setminus \min \tau$ arranged in decreasing order in the word $w = \sigma(0)\sigma^2(0) \dots 1$ with $\sigma(1) = 0$. \square

The following result is the analogue interpretation to Corollary 3 for the central factorial numbers of the first kind. This analogy is comparable with that of Stirling numbers of the first kind $|s(n, k)|$ versus the Stirling numbers of the second kind $|S(n, k)|$.

TABLE 4. The first values of $V(n, k)$ and $|v(n, k)|$

$k \setminus n$	0	1	2	3	4	5	$k \setminus n$	0	1	2	3	4	5
0	1	1	1	1	1	1	0	1	1	9	225	11025	893025
1		1	10	91	820	7381	1		1	10	259	12916	1057221
2			1	35	966	24970	2			1	35	1974	172810
3				1	84	5082	3				1	84	8778
4					1	165	4					1	165
5						1	5						1

Corollary 10. The integer $|u(n, k)|$ is the number of couples $(\sigma, \tau) \in \mathcal{S}_n^2$ with k cycles, such that $\min(\sigma) = \min(\tau)$.

Indeed, the integer $|u(n, k)|$ is the number of couples (σ, τ) in $\mathcal{E}_{n,k}^{(0)}$. Theorem 7 implies that $\sigma^{-1}(1) = 0$. The result follows then by deleting the zero in σ .

Remark 5. By the substitution $i \rightarrow n+1-i$, we can derive that the number $|u(n, k)|$ is also the number of couples (σ, τ) in \mathcal{S}_n^2 with k cycles, such that $\max(\sigma) = \max(\tau)$, where $\max(\sigma)$ is the set of cyclic maxima of σ , i.e.,

$$\max(\sigma) = \{j \in [n] : j = \max(\text{Orb}_\sigma(j))\}.$$

4. FURTHER RESULTS

4.1. Central factorial numbers of odd indices. For all $n, k \geq 0$, set

$$V(n, k) = 4^{n-k}T(2n+1, 2k+1), \quad v(n, k) = 4^{n-k}t(2n+1, 2k+1).$$

Note that these numbers are also integers (see Table 4). By definition, we have the following recurrence relations :

$$V(n, k) = V(n-1, k-1) + (2k+1)^2V(n-1, k), \quad (4.1)$$

$$v(n, k) = v(n-1, k-1) - (2n-1)^2v(n-1, k). \quad (4.2)$$

The natural question is to find a combinatorial interpretation for these numbers. We can easily find it from combinatorial theory of generating functions.

Theorem 8. *The integer $V(n, k)$ is the number of partitions of $[2n+1]$ into $2k+1$ blocks of odd cardinality.*

Proof. This follows from the known generating function (see [8, p. 214]):

$$\sum_{n,k \geq 0} V(n, k)t^k \frac{x^n}{n!} = \sinh(t \sinh(x)),$$

and the classical combinatorial theory of generating functions (see [7, Chp. 3] and [9, Chp. 5]). \square

To interpret the integer $|v(n, k)|$, we need to introduce the following definition.

Definition 6. A (n, k) -*Riordan complex* is a $(2k+1)$ -tuple

$$((B_1, \sigma_1, \tau_1), \dots, (B_{2k+1}, \sigma_{2k+1}, \tau_{2k+1}))$$

such that

- i) $\{B_1, \dots, B_{2k+1}\}$ is a partition of $[2n+1]$ into blocks B_i of odd cardinality;
- ii) σ_i and τ_i ($1 \leq i \leq 2k+1$) are fixed point free involutions on $B_i \setminus \max(B_i)$.

Theorem 9. *The integer $|v(n, k)|$ is the number of (n, k) -Riordan complexes.*

Proof. It is known that (see [8, p. 214]):

$$\sum_{n, k \geq 0} |v(n, k)| t^k \frac{x^n}{n!} = \sinh(t \arcsin(x)),$$

and

$$\arcsin(x) = \sum_{n \geq 0} ((2n-1)!!)^2 \frac{x^{2n+1}}{(2n+1)!},$$

where $(2n-1)!! = (2n-1)(2n-3)\cdots 3 \cdot 1$. Since $(2n-1)!!$ is the number of involutions without fixed points on $[2n]$ (see [2]), the integer $((2n-1)!!)^2$ is the number of couples of involutions without fixed points on $[2n+1] \setminus \{2n+1\}$.

Define the numbers $J(n, m)$ by:

$$\exp \left(t \sum_{n \geq 1} ((2n-1)!!)^2 \frac{x^{2n+1}}{(2n+1)!} \right) = \sum_{n, m \geq 0} J(2n+1, m) \frac{t^m}{m!} \frac{x^{2n+1}}{(2n+1)!}.$$

Then, by the theory of exponential generating functions (see [7, Chp. 3] and [9, Chp. 5]), the coefficient $J(2n+1, m)$ is the number of m -tuples

$$(B_1, \sigma_1, \tau_1), \dots, (B_m, \sigma_m, \tau_m);$$

where $\{B_1, \dots, B_m\}$ is a partition of $[2n+1]$ with $|B_i|$ odd ($1 \leq i \leq m$), and σ_i and τ_i are involutions without fixed points on $B_i \setminus \max(B_i)$. As $\sinh(x) = (e^x - e^{-x})/2$, we have $|v(n, m)| = J(2n+1, 2k+1)$ if $m = 2k+1$, and $|v(n, m)| = 0$ if m is even. \square

Remark 6. It is interesting to note that a proof of the two above theorems by using (4.1) or (4.2) is not obvious.

Example 1. There are ten $(2, 1)$ -Riordan complexes. Since the numbers n and k are small, the involved involutions are identical transpositions.

$$\begin{aligned} & \{1\}, \{(2, 3), 4\}, \{5\}, \{1\}, \{2\}, \{(3, 4), 5\}, \\ & \{(1, 2), 3\}, \{4\}, \{5\}, \{(1, 2), 5\}, \{3\}, \{4\}, \\ & \{(1, 3), 4\}, \{2\}, \{5\}, \{(1, 3), 5\}, \{2\}, \{4\}, \\ & \{(1, 2), 4\}, \{3\}, \{5\}, \{(1, 4), 5\}, \{2\}, \{3\}, \\ & \{1\}, \{(2, 3), 5\}, \{4\}, \{1\}, \{(2, 4), 5\}, \{3\}, \end{aligned}$$

where $\{1\}, \{(2, 3), 4\}, \{5\}$ means that $\pi = \{\{1\}, \{2, 3, 4\}, \{5\}\}$, and $\sigma = \tau = 13245$.

4.2. Generating functions. In [5], the authors made a long calculation to derive an explicit formula for the Jacobi-Stirling numbers. Actually, we can derive an explicit formula for the Jacobi-Stirling numbers straightforwardly from the Newton interpolation formula:

$$x^n = \sum_{j=0}^n \left(\sum_{r=0}^j \frac{x_r^n}{\prod_{k \neq i} (x_r - x_k)} \right) \prod_{i=0}^{j-1} (x - x_i). \quad (4.3)$$

Indeed, making the substitutions $x \rightarrow m(z + m)$ and $x_i \rightarrow i(z + i)$ in (4.3), we obtain

$$(m(m + z))^n = \sum_{j=0}^n \text{JS}_n^j(z)(m - j + 1)_j(z + m)_j, \quad (4.4)$$

where

$$\text{JS}_n^j(z) = \sum_{r=0}^j (-1)^r \frac{[r(r + z)]^n}{r!(j - r)!(z + r)_r(z + 2r + 1)_{j-r}}, \quad (4.5)$$

and $(z)_n = z(z + 1) \dots (z + n - 1)$.

Remark 7. If we substitute x by $m(m + z) + k$, we obtain [5, Theorem 4.1].

From the recurrence (1.4), we derive:

$$\sum_{n \geq k} \text{JS}_n^k(z)x^n = \frac{x}{1 - k(k + z)} \sum_{n \geq k-1} \text{JS}_n^{k-1}(z)x^n; \quad (4.6)$$

therefore,

$$\sum_{n \geq k} \text{JS}_n^k(z)x^n = \frac{x^k}{(1 - (z + 1)x)(1 - 2(z + 2)x) \dots (1 - k(z + k)x)}. \quad (4.7)$$

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