Hinfinity/LPV observer for an industrial semi-active suspension

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$\mathcal{H}_\infty/LPV$ observer for an industrial semi-active suspension

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***Abstract*— In this paper, an $\mathcal{H}_\infty/LPV$ observer to be used in an automotive suspension control application is proposed. The system considered is a road disturbance affected quarter car equipped with an industrial SOBEN damper. This observer is designed in the $\mathcal{H}_\infty$ framework in order to minimize the effect of the unknown road disturbance on the estimated states. The damper studied in this paper is highly nonlinear, therefore an adaptative linear parameter varying (LPV) structure is proposed to improve the robustness of the observer. The observer presented here uses a single position sensor and is easy to implement in a real industrial application because of its simple linear structure. Some simulation results highlight the performances of this observer in realistic noise and uncertainty conditions. The estimated state variables of the quarter car model could be used for example in a state feedback control strategy to improve the comfort and roadholding level of a vehicle.**

I. INTRODUCTION

Suspension control based on quarter vehicles has been widely explored in the past few years to improve vertical movements. Active control laws have been developed [5], [7], [6], and semi-active control laws [17], [3], [8], [14]. Active suspensions provide excellent performances but are not realistic in an industrial context because of the excessive cost of the actuators and their huge energy consumption. Semi-active suspensions provide satisfying performances and can be adopted in mass-produced vehicles if the number and the cost of the sensors required by the control startegy is low, which has not always been the case in the past studies. Furthermore, many control strategies assume a full-state measurement [18], [21], or require at least two sensors as in the well-known Skyhook control strategy [17], [14]. Therefore the state estimation problem is very important if we wish to reduce the number of sensors, i.e. reduce the cost and improve the reliability of the system. Unknown input observers have been studied by many authors [11], [10], [13], [12], [20], [19], and also applied to automotive systems affected by road disturbances [9], [22]. In [22], a disturbance decoupled quarter car observer is designed using the vertical accelerations of the sprung and unsprung masses, but these measurements are very noisy and the sensors are very expensive. Therefore this observer is difficult to implement and sensible to measurement noises.

The main contribution of this paper is to build an observer that estimates the state of the vertical quarter car model using a single reliable and cheap deflection sensor. The observer is designed in the $\mathcal{H}_\infty$ framework in order to minimize the effect of the unknown road disturbance on the estimated states. The real damper considered in the application under study in this paper and described in a previous paper [2] is a SOBEN industrial damper. This system is highly nonlinear, therefore an adaptative linear parameter varying (LPV) structure is proposed to improve the robustness of the observer in front of damping nonlinearities.

This paper is organised as follows: Section II presents the system to be observed, Section III formulates the estimation problem considered in this paper, Section IV deals with the synthesis of the $\mathcal{H}_\infty/LPV$ observer and Section V gives some simulation results that emphasize the performances of the proposed observer. This paper is finally concluded in Section VI and some possible future works are proposed.

II. VEHICLE MODEL

In this section, the system to be observed is presented. This is a vertical linear quarter car model represented on Figure 1.

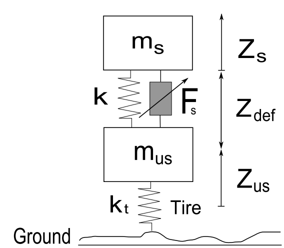

Fig. 1. Vertical quarter car vehicle

This simple vehicle model is made up of a sprung mass, a spring, a damper, an unsprung mass and a tire modelled by a spring. The parameters of this model are given in the Table I.

TABLE I
QUARTER CAR PARAMETERS AND VARIABLES

m_s, m_{us}	Sprung, unsprung mass
k, k_t	Suspension, tire stiffness
z_r	Ground vertical position
$\ddot{z}_s$, $\ddot{z}_{us}$	Sprung, unsprung mass acceleration
z_s, z_{us}	Sprung, unsprung mass position
$z_{def} = z_s - z_{us}$	Suspension deflection
F_s	Damping force

The equations of this model are given by (1).

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$$\begin{cases} m_s\ddot{z}_s = k(z_{us}-z_s) + c\cdot(\dot{z}_{us}-\dot{z}_s) \\ m_{us}\ddot{z}_{us} = k(z_s-z_{us}) + c\cdot(\dot{z}_s-\dot{z}_{us}) + k_t(z_r-z_{us}) \end{cases} \tag{1}$$

where c is a varying parameter that represents the damping rate of the suspension. This parameter depends on the nonlinearities and on the control signal of the damper as well. Therefore considering c as a varying parameter in the observer allows the estimation to take the control signal and the nonlinearities of the damper into account. The calculation of this parameter in an online application is detailed in Section IV.

This quarter car model will be used in the synthesis of the observer and can be formulated as a LPV system given by (2).

$$\begin{cases} \dot{x} = A(c)\cdot x + D\cdot v \\ y = C\cdot x \end{cases} \tag{2}$$

where c is the variable damping rate, $v = \dot{z}_r \in \mathcal{R}^d$ is the unknown road disturbance, $x = (z_{def}, \dot{z}_s, z_{us}-z_r, \dot{z}_{us})^T \in \mathcal{R}^n$ are the state variables of the quarter car model, $y \in \mathcal{R}^m$ is the deflection of the suspension given by a position sensor and $A \in \mathcal{R}^{n,n}$, $D \in \mathcal{R}^{n,d}$ and $C \in \mathcal{R}^{m,n}$ are given by

$$A(c) = \begin{pmatrix} 0 & 1 & 0 & -1 \\ -\frac{k}{m_s} & -\frac{c}{m_s} & 0 & \frac{c}{m_s} \\ 0 & 0 & 0 & 1 \\ \frac{k}{m_{us}} & \frac{c}{m_{us}} & -\frac{kt}{m_{us}} & -\frac{c}{m_{us}} \end{pmatrix},$$

$$D = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}, C = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}^T$$

Remark:

In (2), no control signal u is considered, because in a suspension control application, the control signal modifies the damping rate c, which is already considered here as a varying parameter.

III. PROBLEM STATEMENT

The system to be observed is the quarter car model presented in Section II and given in (2). The full-order observer synthesized in this paper has the general structure given by (3). In a first approach, the varying parameter c is considered as a constant parameter, and the problem is formulated as a linear time invariant (LTI) problem. The LPV form of the observer is given in paragraph IV-D.

$$\begin{cases} \dot{z} = N\cdot z + L\cdot y \\ \hat{x} = z - E\cdot y \end{cases} \tag{3}$$

Where $z \in \mathcal{R}^n$ is the state variable of the observer and $\hat{x} \in \mathcal{R}^n$ the estimated state variables. $N \in \mathcal{R}^{n,n}$, $L \in \mathcal{R}^{n,m}$, $E \in \mathcal{R}^{n,m}$ are matrices to be designed. Then considering (2) and (3), the estimation error can be expressed as

$$e = x - \hat{x} = (I_n + EC)\cdot x - z \tag{4}$$

and then the dynamics of the estimation error is:

$$\begin{aligned} \dot{e} &= \dot{x} - \dot{\hat{x}} \\ \dot{e} &= Ax + Dv - Nz - Ly + EC(Ax + Dv) \end{aligned} \tag{5}$$

By using (4), (5) leads to

$$\dot{e} = Ne + (A - N(I_n+EC) - LC + ECA)x + (D + ECD)v \tag{6}$$

Let us define $K = NE + L$ and $P = I_n + EC$, then (6) turns into

$$\dot{e} = Ne + (PA - (N + KC))x + PDv \tag{7}$$

The state $\hat{x}$ is an asymptotic estimate of x for any $\hat{x}(0)$ and $x(0)$ if and only if N is Hurwitz and

$$\begin{cases} N = PA - KC \\ PD = 0 \end{cases} \tag{8}$$

The design of the observer involves the calculation of $N \in \mathcal{R}^{n,n}$, $L \in \mathcal{R}^{n,m}$, $E \in \mathcal{R}^{n,m}$ satisfying (8). A method to solve this problem is proposed in Section IV.

IV. OBSERVER DESIGN

In this section, a method is proposed to sythesize a road disturbance decoupled $\mathcal{H}_\infty/LPV$ full-order observer based on the deflection measurement. The problem formulated in Section III is solved. Some previous works on this topic have been used [12], [13].

A. Road disturbance decoupling

The first condition of (8) is equivalent to

$$z\cdot\psi = A \tag{9}$$

where $z \in \mathcal{R}^{n,(n+2m)}$ and $\psi \in \mathcal{R}^{(n+2m),n}$ are defined by

$$\begin{cases} z = \begin{pmatrix} N & K & E \end{pmatrix} \\ \psi = \begin{pmatrix} I_n \\ C \\ -CA \end{pmatrix} \end{cases} \tag{10}$$

There exist a solution z of (9) if

$$rank(\psi) = rank\begin{pmatrix} \psi \\ A \end{pmatrix} \tag{11}$$

Since condition (11) is satisfied, the solution exists and is of the form $z = \alpha + \mathbf{Y}\beta$ where

$$\begin{cases} \alpha = A\cdot\psi^+ \\ \beta = I_{n+2m} - \psi\cdot\psi^+ \end{cases} \tag{12}$$

$\mathbf{Y}$ is any matrix with appropriate dimensions and ψ^+ is any generalized inverse matrix of ψ. The matrix $\mathbf{Y}$ will be determined later. From $z = \alpha + \mathbf{Y}\beta$, (4) turns into

$$\dot{e} = N\cdot e + (I_n + EC)D\cdot v \tag{13}$$

Let us define $\tilde{N} \in \mathcal{R}^{(n+2m),n}$ and $\tilde{E} \in \mathcal{R}^{(n+2m),m}$ such that

$$\tilde{N} = \begin{pmatrix} I_n \\ 0_{m,n} \\ 0_{m,n} \end{pmatrix} \quad \tilde{E} = \begin{pmatrix} 0_{n,m} \\ 0_{m,m} \\ I_{m,m} \end{pmatrix}$$

Therefore we have

$$\begin{aligned} N &= z\cdot\tilde{N} \\ E &= z\cdot\tilde{E} \end{aligned}$$

Finally, (13) can be expressed as

$$\dot{e} = A_0 \cdot e + B_0 \cdot v \tag{14}$$

with $A_0 = (\alpha + \mathbf{Y}\beta)\tilde{N}$ and $B_0 = (I_n + (\alpha + \mathbf{Y}\beta)\tilde{E}C)D$.

The equation (14) that rules the estimation error is affected by the unknown road disturbance v. If E and $\mathbf{Y}$ can be found such that $B_0 = 0$, the disturbance decoupling is perfect. Otherwise the disturbance effect has to be minimized. Therefore the problem is to find $\mathbf{Y}$ such that A_0 is stable and the effect of v on e is minimized.

Proposition 4.1: There exist a full-order LTI observer ensuring (17) if there exist $\mathbf{X} = \mathbf{X}^T \succ 0$, $\tilde{\mathbf{Y}}$ and a scalar γ_∞ that solve the LMI (15).

$$\begin{pmatrix} Q_1 + Q_1^T & Q_2 & I_n \\ * & -\gamma_\infty I_d & \mathcal{O}_{d,n} \\ * & * & -\gamma_\infty I_n \end{pmatrix} \prec 0 \tag{15}$$

Where $\mathbf{X} = \mathbf{X}^T \succ 0$, $\tilde{\mathbf{Y}} = \mathbf{X}\mathbf{Y}$ are the decision variables and

$$\begin{cases} Q_1 = (\mathbf{X}\alpha + \tilde{\mathbf{Y}}\beta)\tilde{N} \\ Q_2 = (\mathbf{X} + (\mathbf{X}\alpha + \tilde{\mathbf{Y}}\beta)\tilde{E}C)D \end{cases} \tag{16}$$

Remark: γ_∞ has to be minimized in order to minmize the road disturbance effect on the estimated variables.

Proof: In the application considered here, the effect of the road disturbance on the estimation error has to be eliminated or minimized. Here the gains of the observer are determined by studying the stability and the $\mathcal{H}_\infty$-norm bound of the transfer e/v generating the estimation error. This problem can be solved by minimizing γ_∞ such that

$$\| \, e/v \, \|_\infty < \gamma_\infty \tag{17}$$

The Bouded Real Lemma [15] (BRL) applied to the system (14) gives the solution of (17) and leads to the bilinear matrix inequality (BMI) (18) where $\mathbf{X} = \mathbf{X}^T \succ 0$ and $\mathbf{Y}$ are the unknown matrices to be determined. Therefore the full order, stable and disturbance decoupled observer design problem consists in solving (18).

$$\begin{pmatrix} Q_1 + Q_1^T & Q_2 & I_n \\ * & -\gamma_\infty I_d & \mathcal{O}_{d,n} \\ * & * & -\gamma_\infty I_n \end{pmatrix} \prec 0 \tag{18}$$

In (18), Q_1 and Q_2 are given by

$$\begin{cases} Q_1 = \mathbf{X}A_0 = (\mathbf{X}\alpha + \mathbf{X}\mathbf{Y}\beta)\tilde{N} \\ Q_2 = \mathbf{X}B_0 = (\mathbf{X} + (\mathbf{X}\alpha + \mathbf{X}\mathbf{Y}\beta)\tilde{E}C)D \end{cases} \tag{19}$$

The matrix inequality (18) is a BMI because Q_1 and Q_2 are bilinear. Therefore the variable change $\tilde{\mathbf{Y}} = \mathbf{X}\mathbf{Y}$ is introduced to transform the BMI into a solvable LMI where Q_1 and Q_2 become

$$\begin{cases} Q_1 = (\mathbf{X}\alpha + \tilde{\mathbf{Y}}\beta)\tilde{N} \\ Q_2 = (\mathbf{X} + (\mathbf{X}\alpha + \tilde{\mathbf{Y}}\beta)\tilde{E}C)D \end{cases} \tag{20}$$

Solving (18) with (20) leads to find $\mathbf{X}$ and $\tilde{\mathbf{Y}}$. Thereafter $\mathbf{Y} = \mathbf{X}^{-1}\tilde{\mathbf{Y}}$ and then $z = \alpha + \mathbf{Y}\beta$ can be deduced using (12). N, K and E are given by z and $L = K - NE$ can be computed. Finally, the observer proposed is designed so that the first and second conditions of (8) are respected, and the third one is approached by minimizing γ_∞ subject to (17). ■

B. Filtering

In this paragraph, a weighting filter has been added to the system to focus the interesting frequency range where the disturbance effect minimization has to be done. The new estimation variable to be considered in this section is the filtered estimation variable e_f. Therefore the problem is now to minimize γ_∞ such that

$$\| \, e_f/v \, \|_\infty < \gamma_\infty \tag{21}$$

Proposition 4.2: There exist a full-order LTI observer ensuring (21) if there exist $\mathbf{X}_1 = \mathbf{X}_1^T \succ 0$, $\mathbf{X}_2 = \mathbf{X}_2^T \succ 0$, $\tilde{\mathbf{Y}}$ and a scalar γ_∞ that solve (22).

$$\begin{pmatrix} A_0^T\mathbf{X}_1 + \mathbf{X}_1 A_0 & \mathcal{O}_n & \mathbf{X}_1 B_0 & \mathcal{O}_n \\ \mathbf{X}_2 B_f & \mathbf{X}_2 A_f & \mathcal{O}_{n,d} & I_n \\ * & * & -\gamma_\infty I_d & \mathcal{O}_{n,d} \\ * & * & * & -\gamma_\infty I_n \end{pmatrix} \prec 0 \tag{22}$$

Where $\mathbf{X} = \mathbf{X}^T \succ 0$ is defined such that

$$\mathbf{X} = \begin{pmatrix} \mathbf{X}_1 & \mathcal{O}_n \\ \mathcal{O}_n & \mathbf{X}_2 \end{pmatrix} \tag{23}$$

$A_f \in \mathcal{R}^{n,n}$ and $B_f \in \mathcal{R}^{n,n}$ determine a given weighting filter.

Proof: From (14) the augmented system (24) is built using the state variable $x_a = (e, e_f)^T$ and the weighting filter: $\dot{e}_f = A_f \cdot e_f + B_f \cdot e$.

$$\begin{cases} \dot{x}_a = A_a \cdot e_a + B_a \cdot v \\ e_f = C_a \cdot e_a + D_a \cdot v \end{cases} \tag{24}$$

Where $A_a \in \mathcal{R}^{2n}$, $B_a \in \mathcal{R}^{2n,d}$, $C_a \in \mathcal{R}^{n,2n}$ and $D_a \in \mathcal{R}^{n,d}$ are given by

$$\begin{array}{ll} A_a = \begin{pmatrix} A_0 & \mathcal{O}_n \\ B_f & A_f \end{pmatrix} & B_a = \begin{pmatrix} B_0 \\ \mathcal{O}_{n,d} \end{pmatrix} \\ C_a = \begin{pmatrix} \mathcal{O}_n & I_n \end{pmatrix} & D_a = \mathcal{O}_{n,d} \end{array} \tag{25}$$

The weighting filter can be chosen as $A_f = -diag(\frac{1}{\tau}) \in \mathcal{R}^{n,n}$ and $B_f = -diag(G) \in \mathcal{R}^{n,n}$, for example with $\tau = \frac{1}{2\pi 30}$ and $G = 2$. The function $diag(x)$ refers to a diagonal matrix with the term x on the diagonal. This corresponds to a simple first order low-pass filter with a cut-off frequency equal to $30Hz$, appropriate in the case of the application considered here.

Then applying the bounded real lemma to system (24) leads to the BMI (26).

$$\begin{pmatrix} A_a^T\mathbf{X} + \mathbf{X}A_a & \mathbf{X}B_a & C_a^T \\ * & -\gamma_\infty I_d & D_a^T \\ * & * & -\gamma_\infty I_n \end{pmatrix} \prec 0 \tag{26}$$

Let us define the unknown matrix $\mathbf{X} \in \mathcal{R}^{2n,2n}$ such that

$$\mathbf{X} = \begin{pmatrix} \mathbf{X}_1 & \mathcal{O}_n \\ \mathcal{O}_n & \mathbf{X}_2 \end{pmatrix} \quad (27)$$

Therefore, from (25) and (27), (26) turns into

$$\begin{pmatrix} A_0^T\mathbf{X}_1 + \mathbf{X}_1 A_0 & \mathcal{O}_n & \mathbf{X}_1 B_0 & \mathcal{O}_n \\ \mathbf{X}_2 B_f & \mathbf{X}_2 A_f & \mathcal{O}_{n,d} & I_n \\ * & * & -\gamma_\infty I_d & \mathcal{O}_{n,d} \\ * & * & * & -\gamma_\infty I_n \end{pmatrix} \prec 0 \quad (28)$$

Then using $\tilde{\mathbf{Y}} = \mathbf{X}_1\mathbf{Y}$ as a variable change, (28) can be easily transformed into a solvable LMI where the unknown matrices are $\tilde{\mathbf{Y}}$, $\mathbf{X}_1 = \mathbf{X}_1^T \succ 0$ and $\mathbf{X}_2 = \mathbf{X}_2^T \succ 0$. ■

C. Pole placement

This method ensures the stability of the observer and the minimization of the disturbance effect, but the poles of the observer may be excessively high and comprise high imaginary parts. Such poles may render the observer oscillating and sensible to measurement noises. In order to avoid such a behavior that may lead to implementation problems and bad estimation performances, a pole placement method [4] using LMI regions has been introduced. The poles of the observer can be placed in the intersection of a cone $\mathcal{D}_1$, given by the LMI region (29) and a half plane $\mathcal{D}_2$, given by (30). The cone is defined with apex at the origin and inner angle 2θ to ensure that the observer is stable and has poles with moderate imaginary parts. The half plane is delimited by a vertical straight line to ensure that the poles have real parts higher than $-p_m$.

$$\mathcal{D}_1 = \left\{ z \in \mathcal{C} : \begin{pmatrix} \sin\theta(z+\bar{z}) & \cos\theta(z-\bar{z}) \\ \cos\theta(\bar{z}-z) & \sin\theta(z+\bar{z}) \end{pmatrix} \prec 0 \right\} \quad (29)$$

$$\mathcal{D}_2 = \{ z \in \mathcal{C} : -z - \bar{z} - 2p_m \prec 0 \} \quad (30)$$

Proposition 4.3: There exist a full-order LTI observer ensuring (21) with poles in LMI regions $\mathcal{D}_1$ and $\mathcal{D}_2$ if there exist $\mathbf{X}_1 = \mathbf{X}_1^T \succ 0$, $\mathbf{X}_2 = \mathbf{X}_2^T \succ 0$, $\tilde{\mathbf{Y}}$ and a scalar γ_∞ that solve (31) and (32).

$$\begin{pmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} & \mathcal{M}_{13} \\ * & \mathcal{M}_{22} & \mathcal{M}_{23} \\ * & * & \mathcal{M}_{33} \end{pmatrix} \prec 0 \quad (31)$$

$$\begin{pmatrix} Q_1 + Q_1^T + 2p_m & \mathcal{O}_n & Q_2 & \mathcal{O}_n \\ \mathbf{X}_2 B_f & \mathbf{X}_2 A_f & \mathcal{O}_{n,d} & -I_n \\ * & * & \gamma_\infty I_d & \mathcal{O}_{n,d} \\ * & * & * & \gamma_\infty I_n \end{pmatrix} \succ 0 \quad (32)$$

Where the $\mathcal{M}_{ii}$ terms are given by

$$\begin{aligned} \mathcal{M}_{11} &= \begin{pmatrix} \sin\theta(\mathbf{X}A_a + A_a^T\mathbf{X}) & \cos\theta(\mathbf{X}A_a - A_a^T\mathbf{X}) \\ -\cos\theta(\mathbf{X}A_a - A_a^T\mathbf{X}) & \sin\theta(\mathbf{X}A_a + A_a^T\mathbf{X}) \end{pmatrix} \\ \mathcal{M}_{12} &= \begin{pmatrix} \mathbf{X}B_a & \mathcal{O}_{2n,d} \\ \mathcal{O}_{2n,d} & \mathbf{X}B_a \end{pmatrix} \\ \mathcal{M}_{13} &= \begin{pmatrix} \sin\theta(\ \mathcal{O}_n \quad I_n\ )^T & \cos\theta(\ \mathcal{O}_n \quad I_n\ )^T \\ -\cos\theta(\ \mathcal{O}_n \quad I_n\ )^T & \sin\theta(\ \mathcal{O}_n \quad I_n\ )^T \end{pmatrix} \\ \mathcal{M}_{22} &= -\gamma_\infty I_{2d} \\ \mathcal{M}_{23} &= \mathcal{O}_{2d,2n} \\ \mathcal{M}_{33} &= -\gamma_\infty I_{2n} \end{aligned} \quad (33)$$

and XA_a and XB_a are expressed as

$$XA_a = \begin{pmatrix} Q_1 + Q_1^T & \mathcal{O}_n \\ \mathbf{X}_2 B_f & \mathbf{X}_2 A_f \end{pmatrix} \quad XB_a = \begin{pmatrix} Q_2 \\ \mathcal{O}_{n,d} \end{pmatrix} \quad (34)$$

Where

$$\begin{cases} Q_1 = (\mathbf{X}_1\alpha + \tilde{\mathbf{Y}}\beta)\tilde{N} \\ Q_2 = (\mathbf{X}_1 + (\mathbf{X}_1\alpha + \mathbf{Y}\beta)\tilde{E}C)D \end{cases} \quad (35)$$

Proof: According to the pole placement method proposed in [4] (*Theorem 3.3*), the regions (29) and (30) have been combined with the disturbance effect minimization constraint (28). Therefore we obtain respectively the two BMI (31) and (32) to be solved at the same time. For more details concerning pole placement in LMI regions intersection, see [4]. The structure of the unknown matrix $\mathbf{X}$ has been chosen according to (27). (31) and (32) are deduced from *Theorem 3.3* in [4] applied respectively to the LMI regions (29) and (30).

If XA_a and XB_a given by (43) are directly expressed with Q_1 and Q_2 given by (36), the matrices $\mathcal{M}_{1,1}$ and $\mathcal{M}_{1,2}$ in (31) and (32) contain some bilinear terms due to $\mathbf{X}_1$ and $\mathbf{Y}$.

$$\begin{cases} Q_1 = \mathbf{X}_1 A_0 = (\mathbf{X}_1\alpha + \mathbf{X}_1\mathbf{Y}\beta)\tilde{N} \\ Q_2 = \mathbf{X}_1 B_0 = (\mathbf{X}_1 + (\mathbf{X}_1\alpha + \mathbf{X}_1\mathbf{Y}\beta)\tilde{E}C)D \end{cases} \quad (36)$$

Using $\tilde{\mathbf{Y}} = \mathbf{X}_1\mathbf{Y}$ as a variable change, the bilinear form (36) becomes the linear form (35). Therefore (31) and (32) become solvable LMI where the unknown matrices are $\tilde{\mathbf{Y}}$, $\mathbf{X}_1 = \mathbf{X}_1^T \succ 0$ and $\mathbf{X}_2 = \mathbf{X}_2^T \succ 0$. ■

Therefore to summerize IV-A, IV-B and IV-C, the method to design the proposed LTI observer can be formulated as follows:

1) Choose the weighting filter A_f and B_f appropriate to the system
2) Choose $\mathcal{D}_1$ and $\mathcal{D}_2$ according to the desired poles real and imaginary parts bound
3) Solve LMI (31) and (32) to find $\mathbf{X}_1$ and $\tilde{\mathbf{Y}}$
4) Calculate $\mathbf{Y} = \mathbf{X}_1^{-1}\tilde{\mathbf{Y}}$, $z = \alpha + \mathbf{Y}\beta$ using (12)
5) Deduce N, K, E, $L = K - NE$

D. LPV observer

In suspension control application, the damper is controlled. Therefore the damping rate c is varying and depends on the control signal. In the previous paragraphs, the control signal has not been taken into account, and c was a constant. Here, c is considered as a varying parameter so that the control signal and the nonlinearities of the damper are taken into account in the observer dynamics. The observer design method proposed in IV-C will be extended to the LPV case using the LPV form of system (14), given by (37).

$$\dot{e} = A_0(c) \cdot e + B_0 \cdot v \quad (37)$$

The parameter c can be computed on-line with the available measurements. In the application considered here, the damping rate provided by the damper can be easily computed using measurements, but this part is confidential due to patented results. Another method to evaluate the varying parameter c in real-time consists in using an off-line identified damper model presented in a previous work to be published [1]. This model provides realistic damping forces. Then dividing the computed force by the deflection velocity $(\hat{\dot{z}}_s - \hat{\dot{z}}_{us})$ estimated by the proposed observer, the damping rate c can be calculated. This measured or estimated damping rate c can be used on-line as a varying parameter to schedule the $\mathcal{H}_\infty/LPV$ observer with filtering and pole placement proposed in this section.

Proposition 4.4: There exist a full-order LPV observer ensuring (21) with poles in LMI regions $\mathcal{D}_1$ and $\mathcal{D}_2$ if there exist $\mathbf{X}_1 = \mathbf{X}_1^T \succ 0$, $\mathbf{X}_2 = \mathbf{X}_2^T \succ 0$, $\tilde{\mathbf{Y}}_{c_{min}}$, $\tilde{\mathbf{Y}}_{c_{max}}$ and a scalar γ_∞ that solve the finite set of LMI (38), (39), (40) and (41).

$$\begin{pmatrix} \mathcal{M}_{11}(c_{min}) & \mathcal{M}_{12} & \mathcal{M}_{13} \\ * & \mathcal{M}_{22} & \mathcal{M}_{23} \\ * & * & \mathcal{M}_{33} \end{pmatrix} \prec 0 \tag{38}$$

$$\begin{pmatrix} \mathcal{M}_{11}(c_{max}) & \mathcal{M}_{12} & \mathcal{M}_{13} \\ * & \mathcal{M}_{22} & \mathcal{M}_{23} \\ * & * & \mathcal{M}_{33} \end{pmatrix} \prec 0 \tag{39}$$

$$\left(\begin{array}{c|c|c|c} Q_1(c_{min})+ & \mathcal{O}_n & Q_2(c_{min}) & \mathcal{O}_n \\ Q_1(c_{min})^T + 2p_m & & & \\ \hline \mathbf{X}_2 B_f & \mathbf{X}_2 A_f & \mathcal{O}_{n,d} & -I_n \\ \hline * & * & \gamma_\infty I_d & \mathcal{O}_{n,d} \\ \hline * & * & * & \gamma_\infty I_n \end{array}\right) \succ 0 \tag{40}$$

$$\left(\begin{array}{c|c|c|c} Q_1(c_{max})+ & \mathcal{O}_n & Q_2(c_{max}) & \mathcal{O}_n \\ Q_1(c_{max})^T + 2p_m & & & \\ \hline \mathbf{X}_2 B_f & \mathbf{X}_2 A_f & \mathcal{O}_{n,d} & -I_n \\ \hline * & * & \gamma_\infty I_d & \mathcal{O}_{n,d} \\ \hline * & * & * & \gamma_\infty I_n \end{array}\right) \succ 0 \tag{41}$$

The terms $\mathcal{M}_{1,1}$ and $\mathcal{M}_{1,2}$ in (38) are given by

$$\begin{array}{l} \mathcal{M}_{11} = \left(\begin{array}{c|c} \sin\theta(\mathbf{X}A_a(c) & \cos\theta(\mathbf{X}A_a(c) \\ +A_a(c)^T\mathbf{X}) & -A_a(c)^T\mathbf{X}) \\ \hline -\cos\theta(\mathbf{X}A_a(c) & \sin\theta(\mathbf{X}A_a(c) \\ -A_a(c)^T\mathbf{X}) & +A_a(c)^T\mathbf{X}) \end{array}\right) \\ \mathcal{M}_{12} = \begin{pmatrix} \mathbf{X}B_a(c) & \mathcal{O}_{2n,d} \\ \mathcal{O}_{2n,d} & \mathbf{X}B_a(c) \end{pmatrix} \end{array} \tag{42}$$

Where $XA_a(c)$, $XB_a(c)$, $Q_1(c_{min})$, $Q_2(c_{min})$, $Q_1(c_{max})$ and $Q_2(c_{max})$ are expressed as

$$\begin{array}{l} XA_a(c) = \begin{pmatrix} Q_1(c) + Q_1(c)^T & \mathcal{O}_n \\ \mathbf{X}_2 B_f & \mathbf{X}_2 A_f \end{pmatrix} \\ XB_a(c) = \begin{pmatrix} Q_2(c) \\ \mathcal{O}_{n,d} \end{pmatrix} \end{array} \tag{43}$$

$$\begin{cases} Q_1(c_{min}) = (\mathbf{X}_1\alpha + \tilde{\mathbf{Y}}_{\mathbf{c_{min}}}\beta)\tilde{N} \\ Q_2(c_{min}) = (\mathbf{X}_1 + (\mathbf{X}_1\alpha + \tilde{\mathbf{Y}}_{\mathbf{c_{min}}}\beta)\tilde{E}C)D \end{cases} \tag{44}$$

$$\begin{cases} Q_1(c_{max}) = (\mathbf{X}_1\alpha + \tilde{\mathbf{Y}}_{\mathbf{c_{max}}}\beta)\tilde{N} \\ Q_2(c_{max}) = (\mathbf{X}_1 + (\mathbf{X}_1\alpha + \tilde{\mathbf{Y}}_{\mathbf{c_{max}}}\beta)\tilde{E}C)D \end{cases} \tag{45}$$

The other terms $\mathcal{M}_{i,i}, i \neq 1,2$ in (39) are given by

$$\begin{array}{l} \mathcal{M}_{13} = \begin{pmatrix} \sin\theta(\ \mathcal{O}_n \quad I_n\ )^T & \cos\theta(\ \mathcal{O}_n \quad I_n\ )^T \\ -\cos\theta(\ \mathcal{O}_n \quad I_n\ )^T & \sin\theta(\ \mathcal{O}_n \quad I_n\ )^T \end{pmatrix} \\ \mathcal{M}_{22} = -\gamma_\infty I_{2d} \\ \mathcal{M}_{23} = \mathcal{O}_{2d,2n} \\ \mathcal{M}_{33} = -\gamma_\infty I_{2n} \end{array} \tag{46}$$

Proof: The Bounded Real Lemma extended to LPV systems, detailed in [15], [16], has been applied to the system (37). This system depends on the varying parameter $c \in [c_{min}, c_{max}]$, therefore an infinite set of LMI is obtained. The polytopic approach detailed in [15], [16] gives a solution to this problem. This method ensures the quadratic stability using a single Lyapunov function through the evaluation of the previous LMI at each corner of the polytope only, thereafter the infinite problem becomes finite. This polytope is defined by the extremal varying parameters $[c_{min}, c_{max}]$.

The LMI set including (38), (39), (40) and (41) is obtained applying *Theorem 3.3* in [4] respectively to the LMI regions (29) and (30) for $c = c_{min}$ and $c = c_{max}$, according to the polytopic approach.

As a single Lyapunov function has to be used, the same matrix X, chosen according to (27), has been used for the four LMI (38), (39), (40) and (41). The same variable change $\tilde{\mathbf{Y}} = \mathbf{X}_1\mathbf{Y}$ has been used to eliminate the bilinear terms. Therefore the unknown matrices to be determined are $\mathbf{X}_1$, $\mathbf{X}_2$, $\tilde{\mathbf{Y}}_{\mathbf{c_{min}}}$ and $\tilde{\mathbf{Y}}_{\mathbf{c_{max}}}$, where $\tilde{\mathbf{Y}}_{\mathbf{c_{min}}}$ and $\tilde{\mathbf{Y}}_{\mathbf{c_{max}}}$ respectively give the observer matrices at the polytope corner $c = c_{min}$ and $c = c_{max}$. ■

Then the LPV controller is a linear combination of the controllers computed at each corner. Here there is only one parameter $c \in [c_{min}, c_{max}]$, therefore the corners of the polytope are simply given by c_{min} and c_{max}. Let us define $G^{obs}_{c_{min}}$ and $G^{obs}_{c_{max}}$ the observers calculated at each corner of the polytope. Thereafter, the LPV observer is given by (47).

$$G^{obs}(c) = \frac{c_{max} - c}{c_{max} - c_{min}} \cdot G^{obs}_{c_{min}} + \frac{c - c_{min}}{c_{max} - c_{min}} \cdot G^{obs}_{c_{max}} \tag{47}$$

The method to design the LPV observer can be summerized as follows:

1) Choose the appropriate weighting filter A_f and B_f appropriate to the system
2) Choose $\mathcal{D}_1$ and $\mathcal{D}_2$ according to the desired poles real and imaginary parts bound
3) Solve LMI (38), (39), (40) and (41)
4) Deduce $\mathbf{X}_1$, $\mathbf{X}_2$, $\tilde{\mathbf{Y}}_{\mathbf{c_{min}}}$ and $\tilde{\mathbf{Y}}_{\mathbf{c_{max}}}$
5) Calculate $\mathbf{Y}_{\mathbf{c_{min}}} = \mathbf{X}_1^{-1}\tilde{\mathbf{Y}}_{\mathbf{c_{min}}}$, $\mathbf{Y}_{\mathbf{c_{max}}} = \mathbf{X}_1^{-1}\tilde{\mathbf{Y}}_{\mathbf{c_{max}}}$
6) Deduce $z_{c_{min}} = \alpha + \mathbf{Y}_{\mathbf{c_{min}}}\beta$ using (12), $N_{c_{min}}$, $K_{c_{min}}$, $E_{c_{min}}$ and then $L_{c_{min}} = K_{c_{min}} - N_{c_{min}}E_{c_{min}}$
7) Deduce $z_{c_{max}} = \alpha + \mathbf{Y}_{\mathbf{c_{max}}}\beta$ using (12), $N_{c_{max}}$, $K_{c_{max}}$, $E_{c_{max}}$ and then $L_{c_{max}} = K_{c_{max}} - N_{c_{max}}E_{c_{max}}$
8) Calculate the scheduling rule (47)

V. RESULTS

In this section, numerical results are given and different simulation results are presented to evaluate the observer performances in different conditions.

A. Numerical results

In this paragraph, the numerical values of the calculated LTI observer are given. The chosen filter is described by the diagonal structure proposed in paragraph IV-B where $\omega_f = 2\pi \cdot 20$ and $Gf = \omega_f$. The LMI regions 29 and 30 are respectively determined by $\theta = \frac{\pi}{4}$ and $p_m = 200$. The minimal γ_∞ obtained solving the LMI of the LTI is $\gamma_\infty = 2.8$. The poles $Poles$ of the observer are given bellow.

$$Poles = \begin{pmatrix} -194 + 183i \\ -194 + 183i \\ -124 \\ 0.0001 \end{pmatrix}$$

$$N = \begin{pmatrix} -124.5 & 0 & 0 & 0 \\ -0.0017 & -0.0755 & 0 & 0.0755 \\ 0.0414 & -11.9 & 0 & 12.9 \\ -184.0 & 389.4 & -5546.7 & -389.4 \end{pmatrix}$$

$$K = \begin{pmatrix} 124.5 \\ -93.6 \\ -0.04 \\ 788.5 \end{pmatrix}, E = \begin{pmatrix} -1.0 \\ 9.4 \\ -11.9 \\ 309.4 \end{pmatrix}, L = \begin{pmatrix} -0.0002 \\ -116 \\ -3888 \\ 51449 \end{pmatrix}$$

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 9.4483 & 1.0000 & 0 & 0 \\ -11.9291 & 0 & 1.0000 & 0 \\ 309.4406 & 0 & 0 & 1.0000 \end{pmatrix}$$

$$PA - (N + KC) =$$

$$10^{-13} \cdot \begin{pmatrix} 0.853 & 0.009 & 0 & 0.001 \\ -0.568 & -0.044 & 0 & -0.004 \\ 0.016 & 0.178 & 0 & 0.036 \\ 5.684 & -3.979 & 0 & -1.137 \end{pmatrix}$$

The numerical values of the observers $G^{obs}_{c_{min}}$ and $G^{obs}_{c_{min}}$ obtained in the LPV case are not given but they have similar structures.

B. Simulation results

Four simulation cases described in Table II have been tested. On each figure, the estimated state variables ($\dot{z}_s$, $z_{us} - z_r$, $\dot{z}_{us}$) are compared to the state variables of a reference quarter car model. In *case 1, 2*, the reference quarter car is linear (1), whereas in *case 3, 4*, the linear damper has been replaced by the identified nonlinear model given in [1]. The Mean Square Error ($MSE(x) = \frac{1}{n}\sum_{i=1}^{n}(x_i - \hat{x}_i)^2$) has been calculated for each state variable and is given in Table III.

TABLE II

SIMULATIONS: NOISE AND UNCERTAINTY CONDITIONS

Case	Observer	Simulation conditions
1	LTI	Without noise, uncertainties, nonlinearities
2	LTI	With noise, without uncertainties, nonlinearities
3	LTI	With noise, uncertainties, nonlinearities
4	LPV	With noise, uncertainties, nonlinearities

- *Case 1*: Figure 2 shows the simulation results when no measurement noise has been applied, and when the reference model is also the model used in the synthesis. These results are very satisfying but ideal.
- *Case 2*: Figure 3 shows the results obtained with the same observer and the same reference model, but a random white measurement noise has been added to the measurement in order to test the sensibility of the observer in a real noisy context. The amplitude of the noise has been chosen according to the noise level produced by the sensor used by SOBEN for this application. This information is given by the sensor manufacturer. The results show that the noise is not amplified, it is reduced for the state variables $\dot{z}_s$, $\dot{z}_{us}$, and satisfying for $z_{us} - z_r$.
- *Case 3*: This case has been simulated with the same system and the same noise, but the linear damper has been replaced by a nonlinear one [1]. Furthermore, the parameters of the quarter car model, given in Table I have been modified. An uncertainty of 30% has been introduced so that the model used in the synthesis is linear and uncertain. The results presented on Figure 4 show that the estimation performances are damaged by the nonlinearities and uncertainties.
- *Case 4*: In this case, the same noise and uncertainties have been applied, but the LPV observer with varying c has been used. The robustness is improved thanks to this method, because the real damper is highly nonlinear.

Therefore these results show that the observer has satisfying performances in a realistic noisy and uncertain context.

TABLE III

SIMULATIONS: MEAN SQUARE ERRORS

State variable/Case	1	2	3	4
$MSE(\dot{z}_s)$	0.23	0.32	0.55	0.33
$MSE(z_{us} - z_r)$	0.31	0.33	0.43	0.35
$MSE(\dot{z}_{us})$	0.29	0.34	0.35	0.3

VI. CONCLUSIONS AND FUTURE WORKS

In this paper, a method to synthesize an observer for a suspension control application has been presented. This observer is based on a reliable and cheap sensor providing the damper deflection measurement. The estimation is decoupled from the unknown road disturbance through an $\mathcal{H}_\infty$ minimization, some ponderation filters are introduced to focus the accuracy of the observer on the interesting frequency range, and a varying parameter is introduced to improve the robustness of the observer when the damping rate is varying. The synthesis method proposed here also includes a pole placement in LMI regions to avoid inadapted dynamics that may preclude the implementation and damage the estimation accuracy in the real embedded application. Finally some simulations have been run in realistic conditions and emphasize the observer performance when then measurement is affected by a noise, and the model is uncertain. Future works will

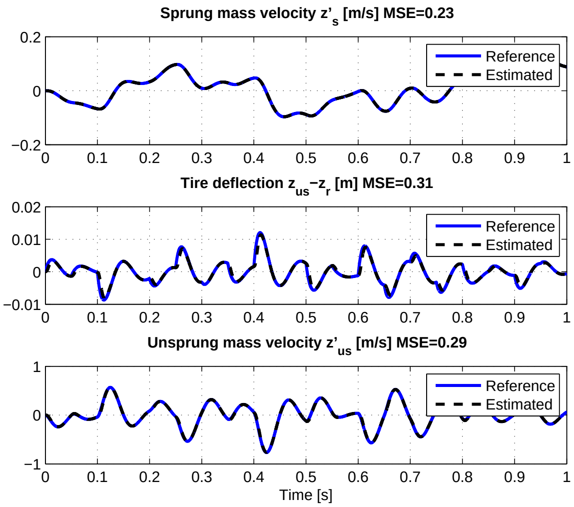

Fig. 2. Quarter car state variable estimation: *Case 1*

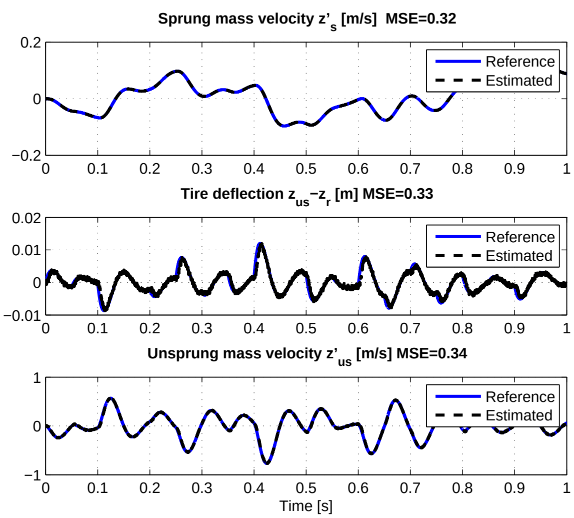

Fig. 3. Quarter car state variable estimation: *Case 2*

consist in designing the same kind of observer for a full car. Then this observer will be used with a static state feedback controller and implemented by SOBEN on a testing car in the near future. The objective is design a global attitude control strategy using the four suspensions. A reduced-order observer version of this observer could also be developped.

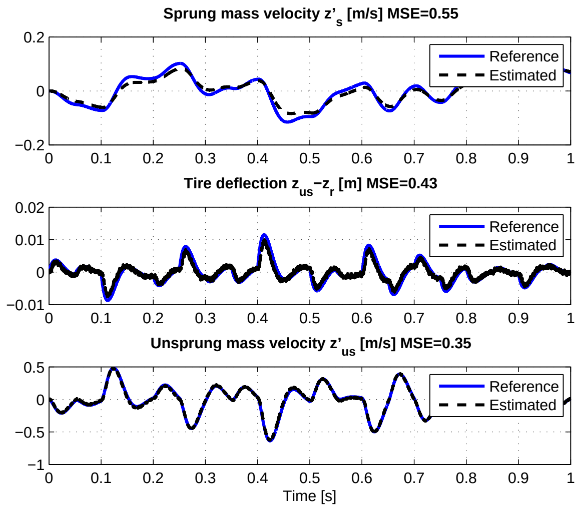

Fig. 4. Quarter car state variable estimation: *Case 3*

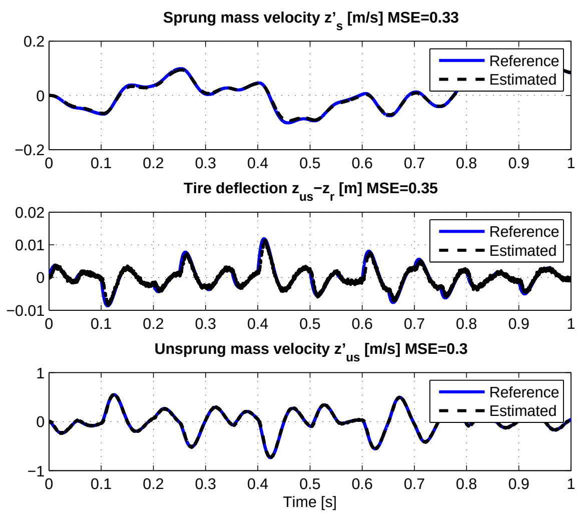

Fig. 5. Quarter car state variable estimation: *Case 4*

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