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NUMERICAL SOLUTION OF A PARABOLIC SYSTEM WITH BLOW-UP OF THE SOLUTION

MARIE-NOELLE LEROUX

ABSTRACT. In this paper, the author proposes a numerical method to solve a parabolic system of two quasilinear equations of nonlinear heat conduction with sources. The solution of this system may blow up in finite time. It is proved that the numerical solution also may blow up in finite time and an estimate of this time is obtained. The convergence of the scheme is obtained for particular values of the parameters.

1. INTRODUCTION

The purpose of this paper is to study the numerical behavior of the solution of a nonlinear reaction diffusion system with nonlinear source terms.

Let Ω a smooth bounded domain in \mathbb{R}^d . We consider the system:

$$\begin{cases} u_{1t} - \Delta u_1^{\nu+1} = \alpha v_1^{\mu+1}, & x \in \Omega, t > 0 \\ v_{1t} - \Delta v_1^{\mu+1} = \alpha u_1^{\nu+1}, & x \in \Omega, t > 0 \\ u_1(x, 0) = u_{10}(x) > 0, & x \in \Omega \\ v_1(x, 0) = v_{10}(x) > 0, & x \in \Omega \\ u_1 = v_1 = 0, & x \in \partial\Omega, t > 0 \end{cases}$$

with $\nu, \mu > 0$, $\alpha \geq 0$.

Samarskii and al. [5] have studied this system and obtained the following results:

If λ_1 denotes the first eigenvalue of the Dirichlet problem: $-\Delta \rho = \lambda \rho$, $x \in \Omega$, $\rho = 0$, $x \in \partial\Omega$ and if $\alpha > \lambda_1$, the problem has no global solutions and there exists $T_0 > 0$ such that

$$\overline{\lim}_{t \rightarrow T_0^-} \left(\|u_1^{\nu+1}(t, \cdot)\|_{L^2(\Omega)}^2 + \|v_1^{\mu+1}(t, \cdot)\|_{L^2(\Omega)}^2 \right) = +\infty$$

In [3], [4], we have proposed a numerical method to solve a quasilinear parabolic equation with blow-up of the solution. The numerical solution is computed by using the function $u = u_1^{\nu+1}$; so the nonlinearity is reported on the derivative in time. This solution has the same properties as the exact solution, in particular blow-up in finite time. We generalize this method to the system of two equations.

For what follows, it is more convenient to work with a transformed equation. Let $u = u_1^{\nu+1}$, $v = v_1^{\mu+1}$, $m = \frac{1}{\nu+1}$, $p = \frac{1}{\mu+1}$; then we get $m, p \in]0, 1[$ and we suppose that $p \leq m$ (or $\mu \geq \nu$).

Then (u, v) satisfies the following system:

$$(1.1) \quad \begin{cases} mu^{m-1}u_t + Au = \alpha v \\ pv^{p-1}v_t + Av = \alpha u \\ u(x, 0) = u_{10}^{\nu+1}(x) = u_0(x), \quad x \in \Omega \\ v(x, 0) = v_{10}^{\mu+1}(x) = v_0(x), \quad x \in \Omega \end{cases}$$

where A is the operator $-\Delta$ of domain $D(A) = H_0^1(\Omega) \cap H^2(\Omega)$.

An outline of the paper is as follows: In Section 2, we study the asymptotic behavior of the solution. In Section 3, we define a numerical scheme and prove the existence of the solution of this scheme. The section 4 is devoted to the properties of the scheme, in particular, the existence of a numerical blow-up time in the case $\alpha > \lambda_1$. Finally, in Section 5, we study the particular case $p = m$ and prove the convergence of the scheme in that case for a class of initial conditions.

2. ASYMTOTIC BEHAVIOR OF THE SOLUTION

Given $u_0, v_0 \in L^\infty(\Omega)$, a couple (u, v) is a weak solution of (1.1) on $[0, T]$ if $u, v \in L^\infty((0, T) \times \Omega)$ and

$$\int_0^t \int_\Omega (u^m \phi_t - uA\phi + \alpha v\phi) dx dt = \int_\Omega u(x, t)\phi(x) dx - \int_\Omega u_0(x)\phi(x) dx$$

$$\int_0^t \int_\Omega (v^p \phi_t - vA\phi + \alpha u\phi) dx dt = \int_\Omega v(x, t)\phi(x) dx - \int_\Omega v_0(x)\phi(x) dx$$

for all $\phi \in C^2((0, T) \times \Omega) \cap C^1([0, T] \times \bar{\Omega})$, $\phi(x, t) = 0$ for $x \in \partial\Omega$.

This problem admits a local solution and from the maximum principle, we get $u(t), v(t) > 0$ in Ω , $0 < t < T$.

We prove the following results:

ι) if $\alpha > \lambda_1$, the solution blows up in finite time T ; we get:

$$\lim_{t \rightarrow T_-} \int_\Omega \left(\frac{m}{m+1} u^{m+1}(t) + \frac{p}{p+1} v^{p+1}(t) \right) dx = +\infty$$

u) if $\alpha < \lambda_1$, the problem has a global solution which tends to 0 when $t \rightarrow \infty$

uu) if $\alpha = \lambda_1$, the problem has a global solution (u, v) which tends to $\theta\rho_1$ when $t \rightarrow \infty$ where ρ_1 is the first eigenfunction of A ($A\rho_1 = \lambda_1\rho_1$ and $\|\rho_1\|_{L^1(\Omega)} = 1$) and θ is a constant depending on the initial condition.

We introduce the functions Φ, Z defined on $[0, T]$ by

$$\Phi(t) = \int_{\Omega} \left(\frac{m}{m+1} u^{m+1}(t) + \frac{p}{p+1} v^{p+1}(t) \right) dx$$

$$(2.1) \quad Z(t) = \Phi(t)^{\frac{m-1}{m+1}} = \left(\int_{\Omega} \left(\frac{m}{m+1} u^{m+1}(t) + \frac{p}{p+1} v^{p+1}(t) \right) dx \right)^{\frac{m-1}{m+1}}$$

and the functional defined on $H_0^1(\Omega) \times H_0^1(\Omega)$ by

$$(2.2) \quad J(u, v) = \int_{\Omega} (|\nabla u|^2 + |\nabla v|^2 - 2\alpha uv) dx$$

Lemma 2.1. *The function Φ is convex and the function Z is concave*

Proof: We prove that the second derivative of Φ is nonnegative.

We have: $\Phi'(t) = \int_{\Omega} (mu^m u_t + pv^p v_t) dx$

By multiplying the first equation of (1.1) by u , the second by v and integrating over Ω , we get:

$$\begin{aligned} \Phi'(t) &= \int_{\Omega} (mu^m u_t + pv^p v_t) dx = - \int_{\Omega} ((Au - \alpha v)u + (Av - \alpha u)v) dx = \\ &= - \int_{\Omega} (|\nabla u|^2 + |\nabla v|^2 - 2\alpha uv) dx = -J(u, v). \end{aligned}$$

Then we deduce:

$$\begin{aligned} \Phi''(t) &= -2 \int_{\Omega} ((Au - \alpha v)u_t + (Av - \alpha u)v_t) dx \\ &= 2 \int_{\Omega} \left(\frac{1}{m} (Au - \alpha v)^2 u^{1-m} + \frac{1}{p} (Av - \alpha u)^2 v^{1-p} \right) dx \end{aligned}$$

and $\Phi''(t) \geq 0, \forall t \geq 0$. The function Φ is convex.

Besides, we have:

$$Z'(t) = \frac{m-1}{m+1} \Phi(t)^{-2/(m+1)} \Phi'(t)$$

$$\text{and } Z''(t) = \frac{1-m}{m+1} (\Phi(t))^{-2/(m+1)-1} \left(\frac{2}{m+1} (\Phi'(t))^2 - \Phi(t)\Phi''(t) \right).$$

By using Cauchy-Schwarz inequality, we get:

$$\int_{\Omega} (Au - \alpha v)u dx \leq \left(\frac{m+1}{m} \int_{\Omega} (Au - \alpha v)^2 u^{1-m} dx \right)^{1/2} \left(\int_{\Omega} \frac{m}{m+1} u^{m+1} dx \right)^{1/2}$$

and

$$\int_{\Omega} (Av - \alpha u)v dx \leq \left(\frac{p+1}{p} \int_{\Omega} (Av - \alpha u)^2 v^{1-p} dx \right)^{1/2} \left(\int_{\Omega} \frac{p}{p+1} v^{p+1} dx \right)^{1/2}$$

We deduce:

$$(\Phi'(t))^2 \leq \left(\frac{m+1}{m} \int_{\Omega} (Au - \alpha v)^2 u^{1-m} dx + \frac{p+1}{p} \int_{\Omega} (Av - \alpha u)^2 v^{1-p} dx \right) \Phi(t)$$

and

$$Z''(t) \leq 2 \frac{(1-m)}{(m+1)^2} \frac{p-m}{p} \left(\int_{\Omega} \left(\frac{m}{m+1} u^{m+1} + \frac{p}{p+1} v^{p+1} \right) dx \right)^{-2/(m+1)} \int_{\Omega} (Av - \alpha u)^2 v^{1-p} dx.$$

Since $p \leq m$, the function Z'' is nonpositive, that is the function Z is concave.

In the two next propositions, we prove that in the case $\alpha > \lambda_1$, the solution (u, v) of (1.1) blows up in finite time and obtain estimates of this time for a class of initial conditions.

Proposition 2.2. *If $\alpha > \lambda_1$ and if the initial condition satisfies $J(u_0, v_0) < 0$, the solution blows up in finite time T such that*

$$(2.3) \quad T < \frac{1+m}{1-m} \frac{\int_{\Omega} \left(\frac{m}{m+1} u_0^{m+1} + \frac{p}{p+1} v_0^{p+1} \right) dx}{-\int_{\Omega} (|\nabla u_0|^2 + |\nabla v_0|^2 - 2\alpha u_0 v_0) dx}.$$

Proof: The function Z' is nonincreasing, so we get:

$$(2.4) \quad Z(t) - Z(0) \leq tZ'(0)$$

$$\text{and } Z'(0) = \frac{1-m}{1+m} (\Phi(0))^{\frac{-2}{m+1}} J(u_0, v_0).$$

The inequality (2.4) may be written as

$$(2.5) \quad \left(\int_{\Omega} \left(\frac{m}{m+1} u(t)^{m+1} + \frac{p}{p+1} v(t)^{p+1} \right) dx \right)^{\frac{m-1}{m+1}} \leq \left(\int_{\Omega} \left(\frac{m}{m+1} u_0^{m+1} + \frac{p}{p+1} v_0^{p+1} \right) dx \right)^{\frac{m-1}{m+1}} \left(1 + \frac{1-m}{m+1} t \left(\int_{\Omega} \left(\frac{m}{m+1} u_0^{m+1} + \frac{p}{p+1} v_0^{p+1} \right) dx \right)^{-1} J(u_0, v_0) \right).$$

If $\alpha > \lambda_1$, the set $S = \{(u_0, v_0) \in H_0^1(\Omega) / J(u_0, v_0) < 0\}$ is not empty. If $J(u_0, v_0) < 0$, the right member of (2.5) becomes equal to zero for a finite time and so the solution blows up in a finite time T such that

$$T \leq \frac{m+1}{1-m} \frac{\int_{\Omega} \left(\frac{m}{m+1} u_0^{m+1} + \frac{p}{p+1} v_0^{p+1} \right) dx}{-J(u_0, v_0)}.$$

Proposition 2.3. : *If $\alpha > \lambda_1$, the solution blows up in finite time T and satisfies the inequality:*

$$\left(\int_{\Omega} \left(\frac{m}{m+1} u^{m+1}(t) + \frac{p}{p+1} v^{p+1}(t) \right) dx \right)^{\frac{1}{m+1}} \leq \left(\frac{T}{T-t} \right)^{\frac{1}{1-m}} \left(\int_{\Omega} \left(\frac{m}{m+1} u_0^{m+1} + \frac{p}{p+1} v_0^{p+1} \right) dx \right)^{\frac{1}{m+1}}.$$

Proof: If $J(u_0, v_0) < 0$, from Proposition 2.3, we know that the solution blows up in finite time. If $J(u_0, v_0) \geq 0$, the result is obtained by using the same proof as Friedman-McLeod in [1].

Since the function Z is concave, it satisfies

$$Z(t) \geq \frac{s-t}{s} Z(0) + \frac{t}{s} Z(s), \quad 0 \leq t \leq s < T \quad \text{and} \quad \lim_{s \rightarrow T_-} Z(s) = 0.$$

We deduce

$$\left(\int_{\Omega} \left(\frac{m}{m+1} u^{m+1}(t) + \frac{p}{p+1} v^{p+1}(t) \right) dx \right)^{\frac{m-1}{m+1}} \geq \frac{T-t}{T} \left(\int_{\Omega} \left(\frac{m}{m+1} u_0^{m+1} + \frac{p}{p+1} v_0^{p+1} \right) dx \right)^{\frac{m-1}{m+1}},$$

that is the inequality (2.3).

Proposition 2.4. *If $\alpha < \lambda_1$, the solution tends to 0 when $t \rightarrow +\infty$.*

Proof: If $\lambda_1 \geq \alpha$, we get: $J(u, v) \geq (\lambda_1 - \alpha) \int_{\Omega} (u^2 + v^2) dx \geq 0$ and $\Phi'(t) \leq 0$. Since the function Φ is convex, the function Φ' is nondecreasing and $\lim_{t \rightarrow +\infty} \Phi'(t) = l \leq 0$.

We deduce: $\Phi(t) \leq \Phi(0) + lt$, $\forall t > 0$; since $\Phi(t) \geq 0$, we obtain that $l = 0$ and

$$(2.6) \quad \lim_{t \rightarrow +\infty} J(u(t), v(t)) = 0.$$

If $\lambda_1 - \alpha > 0$, then, we get $\lim_{t \rightarrow +\infty} \int_{\Omega} (u^2 + v^2) dx = 0$ and the solution tends to 0 when $t \rightarrow +\infty$.

Proposition 2.5. *If $\lambda_1 = \alpha$, then $\lim_{t \rightarrow +\infty} u(t) = \lim_{t \rightarrow +\infty} v(t) = \theta \rho_1$ in $L^2(\Omega)$ where θ is a constant depending on the initial conditions.*

Proof: If $\lambda_1 = \alpha$, we get from (2.6) that $\lim_{t \rightarrow +\infty} J(u(t), v(t)) = 0$ and $\Phi(t)$ is bounded for $t \geq 0$.

By interpolation, we obtain

$$\int_{\Omega} u^2 dx \leq \delta \int_{\Omega} |\nabla u|^2 dx + C_1(\delta) \left(\int_{\Omega} u^{m+1} dx \right)^{2/(m+1)},$$

$$\int_{\Omega} v^2 dx \leq \delta \int_{\Omega} |\nabla v|^2 dx + C_2(\delta) \left(\int_{\Omega} v^{p+1} dx \right)^{2/(p+1)},$$

where $C_1(\delta)$ and $C_2(\delta)$ are constants depending on Ω and m and p respectively and we get:

$$\int_{\Omega} (u^2 + v^2) dx \leq \delta J(u(t), v(t)) + 2\delta\lambda_1 \int_{\Omega} uv dx + C_1(\delta) \left(\int_{\Omega} u^{m+1} dx \right)^{2/(m+1)} + C_2(\delta) \left(\int_{\Omega} v^{p+1} dx \right)^{2/(p+1)}.$$

We deduce

$$(1 - \delta\lambda_1) \int_{\Omega} (u^2 + v^2) dx \leq \delta J(u(t), v(t)) + C_1(\delta) \left(\int_{\Omega} u^{m+1} dx \right)^{2/(m+1)} + C_2(\delta) \left(\int_{\Omega} v^{p+1} dx \right)^{2/(p+1)}.$$

Since $J(u(t), v(t))$ is bounded, if δ is chosen such that $1 - \delta\lambda_1 > 0$, we deduce that $u(t)$ and $v(t)$ are uniformly bounded in $L^2(\Omega)$ and in $H_0^1(\Omega)$. So, we can extract subsequences $t_n \rightarrow +\infty$ such that $u(t_n)$ and $v(t_n)$ converge weakly in $H_0^1(\Omega)$ and strongly in $L^2(\Omega)$ to z_1 and z_2 respectively.

We have : $J(z_1, z_2) \leq \underline{\lim} J(u(t), v(t)) = 0$, that is $\int_{\Omega} (|\nabla z_1|^2 + |\nabla z_2|^2 - 2\lambda_1 z_1 z_2) dx = 0$ or

$$\int_{\Omega} (|\nabla z_1|^2 - \lambda_1 z_1^2) dx + \int_{\Omega} (|\nabla z_2|^2 - \lambda_1 z_2^2) dx + \lambda_1 \int_{\Omega} (z_1 - z_2)^2 dx = 0$$

We deduce that $z_1 = z_2 = \theta\rho_1$.

By multiplying the two equations of (1.1) by ρ_1 and integrating over Ω , we get:

$$\frac{d}{dt} \left(\int_{\Omega} (u^m + v^p) \rho_1 dx \right) = 0 \text{ and then}$$

$$\int_{\Omega} (u^m(t) + v^p(t)) \rho_1 dx = \int_{\Omega} (u_0^m + v_0^p) \rho_1 dx, \text{ for all } t > 0.$$

Hence we obtain

$$\int_{\Omega} (\theta^m \rho_1^{m+1} + \theta^p \rho_1^{p+1}) dx = \int_{\Omega} (u_0^m + v_0^p) \rho_1 dx.$$

and there exists a unique positive value of θ satisfying this equation; we deduce the proposition

3. DEFINITION OF A NUMERICAL SCHEME

The classical Euler scheme cannot blow up in a finite time, so we generalize here to a system the numerical scheme used in [3].

The first equation of (1.1) may be written:

$$-\frac{m}{1-p} u^{m-p+1} \frac{d}{dt} (u^{p-1}) + Au = \alpha v$$

and the second:

$$-\frac{p}{1-p}v\frac{d}{dt}(v^{p-1}) + Av = \alpha u.$$

So, we discretize the two derivatives in time in the same manner:

If (u_n, v_n) is the approximate solution at the time level $t_n = n\Delta t$, (where Δt is the time step), the approximate solution at the time level t_{n+1} is solution of the system:

$$(3.1) \quad \frac{m}{1-p}u_{n+1}u_n^{m-p}(u_n^{p-1} - u_{n+1}^{p-1}) + \Delta t Au_{n+1} = \alpha \Delta t v_{n+1},$$

$$(3.2) \quad \frac{p}{1-p}v_{n+1}(v_n^{p-1} - v_{n+1}^{p-1}) + \Delta t Av_{n+1} = \alpha \Delta t u_{n+1}.$$

We prove first that the system (3.1), (3.2) has a unique positive solution if u_n, v_n are positive in Ω . We need several lemmas. For what follows, we denote: $\|v\|_r = \|v\|_{L^r(\Omega)}$.

Lemma 3.1. *If the functions u_n and v_n are positive on Ω , continuous in $\bar{\Omega}$ and satisfy the condition:*

$$(3.3) \quad \|u_n\|_\infty^{1-m} \|v_n\|_\infty^{1-p} < \frac{mp}{\alpha^2(1-p)^2\Delta t^2}$$

then the system (3.1), (3.2) has a positive solution $u_{n+1}, v_{n+1} \in C^2(\bar{\Omega})$

Proof: Consider the functional defined on $H_0^1(\Omega) \times H_0^1(\Omega)$ by:

$$(3.4) \quad J_n(u, v) = \int_\Omega (|\nabla u|^2 + |\nabla v|^2 - 2\alpha uv)dx + \frac{1}{(1-p)\Delta t} \int_\Omega (mu_n^{m-1}u^2 + pv_n^{p-1}v^2)dx$$

and let us denote $K = \left\{ (u, v) \in (H_0^1(\Omega))^2 / \int_\Omega (mu_n^{m-p}|u|^{p+1} + p|v|^{p+1})dx = 1 \right\}$.

Since (u_n, v_n) satisfies (3.3), we get : $mu_n^{m-1}u^2 + pv_n^{p-1}v^2 - 2\alpha(1-p)\Delta tuv \geq 0$ and $J_n(u, v) \geq 0$, $J_n(|u|, |v|) \leq J_n(u, v)$, $\forall u, v \in H_0^1(\Omega)$.

Now, we consider the problem:

$$(3.5) \quad \min_{(u,v) \in K} J_n(u, v)$$

From the preceding remark, the solution of (3.5), if it exists, will be positive.

We denote $\phi(u, v) = \int_\Omega (mu_n^{m-p}u^{p+1} + pv^{p+1})dx$.

If (\hat{u}, \hat{v}) is a solution of problem (3.5), there exists $\lambda \in \mathbb{R}$ such that:

$$\begin{cases} \frac{\partial J_n(\hat{u}, \hat{v})}{\partial u} - \lambda \frac{\partial \phi}{\partial u}(\hat{u}, \hat{v}) = 0 \\ \frac{\partial J_n(\hat{u}, \hat{v})}{\partial v} - \lambda \frac{\partial \phi}{\partial v}(\hat{u}, \hat{v}) = 0 \\ \phi(\hat{u}, \hat{v}) = 1 \end{cases}$$

Then, we get for any $\psi \in H_0^1(\Omega)$,

$$\int_{\Omega} (\nabla \hat{u} \nabla \psi - \alpha \hat{v} \psi) dx + \frac{m}{(1-p)\Delta t} \int_{\Omega} u_n^{m-1} \hat{u} \psi dx = \lambda m(p+1) \int_{\Omega} u_n^{m-p} \hat{u}^p \psi dx,$$

$$\int_{\Omega} (\nabla \hat{v} \nabla \psi - \alpha \hat{u} \psi) dx + \frac{p}{(1-p)\Delta t} \int_{\Omega} v_n^{p-1} \hat{v} \psi dx = \lambda p(p+1) \int_{\Omega} \hat{v}^p \psi dx$$

and

$$\int_{\Omega} (m u_n^{p-1} \hat{u}^{p+1} + p \hat{v}^{p+1}) dx = 1.$$

Hence, (\hat{u}, \hat{v}) satisfies the equalities:

$$A\hat{u} - \alpha \hat{v} + \frac{m}{(1-p)\Delta t} u_n^{m-1} \hat{u} = \lambda m(p+1) u_n^{m-p} \hat{u}^p,$$

$$A\hat{v} - \alpha \hat{u} + \frac{p}{(1-p)\Delta t} v_n^{p-1} \hat{v} = \lambda p(p+1) \hat{v}^p$$

and $(\hat{u}, \hat{v}) \in C^2(\Omega)$. Besides $J_n(\hat{u}, \hat{v}) = \lambda(p+1)$.

A solution of (3.1), (3.2) is then defined by $u_{n+1} = \gamma \hat{u}$, $v_{n+1} = \gamma \hat{v}$; we get

$$A u_{n+1} - \alpha v_{n+1} + \frac{m}{(1-p)\Delta t} u_n^{m-1} u_{n+1} = \lambda \gamma^{1-p} m(p+1) u_n^{m-p} u_{n+1}^p,$$

$$A v_{n+1} - \alpha u_{n+1} + \frac{p}{(1-p)\Delta t} v_n^{p-1} v_{n+1} = \lambda \gamma^{1-p} p(p+1) v_{n+1}^p$$

and

$$(3.6) \quad \gamma = \left(\frac{1}{(1-p)\Delta t J_n(\hat{u}, \hat{v})} \right)^{1/(1-p)}.$$

Hence, the numerical scheme admits at least one positive solution.

Before proving the uniqueness of the solution, we first prove the existence of bounded supersolutions and of maximal solutions.

Lemma 3.2. *If the hypotheses of the lemma (3.1) are satisfied, the system (3.1), (3.2) admits a constant supersolution.*

Proof: Let $(C_n^1, C_n^2) \in \mathbb{R}_+^2$; (C_n^1, C_n^2) will be a supersolution of the system (3.1), (3.2) if these constants satisfy the inequalities:

$$(3.7) \quad -\frac{m}{(1-p)\Delta t} (C_n^1)^p u_n^{m-p} + \frac{m}{(1-p)\Delta t} C_n^1 u_n^{m-1} \geq \alpha C_n^2,$$

$$(3.8) \quad -\frac{p}{(1-p)\Delta t}(C_n^2)^p + \frac{p}{(1-p)\Delta t}C_n^2v_n^{p-1} \geq \alpha C_n^1.$$

We note $x = \frac{C_n^2}{C_n^1}$

The first inequality may be written:

$$\frac{m}{(1-p)\Delta t}u_n^{m-1} - \alpha x \geq \frac{m}{(1-p)\Delta t}(C_n^1)^{p-1}u_n^{m-p},$$

hence this inequality may be satisfied only for $x < \frac{m}{(1-p)\alpha\Delta t} \|u_n\|_\infty^{m-1}$.

The second inequality may be written:

$$\frac{p}{(1-p)\Delta t}v_n^{p-1} - \frac{\alpha}{x} \geq \frac{p}{(1-p)\Delta t}(C_n^2)^{p-1}$$

and may be satisfied only if $\frac{1}{x} < \frac{p}{(1-p)\alpha\Delta t} \|v_n\|_\infty^{p-1}$.

So a necessary condition to obtain inequalities (3.7), (3.8) is that :

$$\frac{(1-p)\alpha\Delta t}{p} \|v_n\|_\infty^{1-p} < \frac{m}{(1-p)\alpha\Delta t} \|u_n\|_\infty^{m-1}, \text{ that is the condition (3.3)}$$

$$\text{and } \frac{(1-p)\alpha\Delta t}{p} \|v_n\|_\infty^{1-p} < x < \frac{m}{(1-p)\alpha\Delta t} \|u_n\|_\infty^{m-1}.$$

If we choose $C_n^1 = \frac{\|u_n\|_\infty}{\left(1 - \frac{\alpha(1-p)}{m}\Delta t x \|u_n\|_\infty^{1-m}\right)^{1/(1-p)}}$, then we get :

$$C_n^2 = \frac{x \|u_n\|_\infty}{\left(1 - \frac{\alpha(1-p)}{m}\Delta t x \|u_n\|_\infty^{1-m}\right)^{1/(1-p)}}.$$

It remains to prove that x may be chosen in the interval $\left[\frac{(1-p)\alpha\Delta t}{p} \|v_n\|_\infty^{1-p}, \frac{m}{(1-p)\alpha\Delta t} \|u_n\|_\infty^{1-m}\right]$

such that C_n^2 satisfies : $(C_n^2)^{p-1} \leq v_n^{p-1} - \frac{\alpha}{x} \frac{1-p}{p} \Delta t$.

In order to obtain this inequality, the parameter x must satisfy:

$$(3.9) \quad \frac{\|v_n\|_\infty}{\left(1 - \alpha \frac{1-p}{p} \frac{\Delta t}{x} \|v_n\|_\infty^{1-p}\right)^{1/(1-p)}} \leq \frac{x \|u_n\|_\infty}{\left(1 - \alpha \frac{1-p}{m} \Delta t x \|u_n\|_\infty^{1-m}\right)^{1/(1-p)}}$$

Let us denote $a = \frac{1-p}{p} \Delta t \alpha \|v_n\|_\infty^{1-p}$, $b = \frac{m}{(1-p)\alpha\Delta t} \|u_n\|_\infty^{m-1}$;

we have : $a < x < b$ and the condition (3.9) may be written:

$$\frac{p}{(1-p)\alpha\Delta t} ab^{(1-p)/(1-m)} \left(1 - \frac{x}{b}\right) \leq \left(\frac{m}{(1-p)\alpha\Delta t}\right)^{(1-p)/(1-m)} \left(1 - \frac{a}{x}\right) x^{1-p}.$$

If we define the function f by

$$f(x) = \frac{p}{(1-p)\alpha\Delta t} ab^{(m-p)/(1-m)} x^p (x-b) + \left(\frac{m}{(1-p)\alpha\Delta t} \right)^{(1-p)/(1-m)} (x-a)$$

the condition (3.9) becomes: $f(x) \geq 0$.

The function f satisfies $f(a) < 0$ and $f(b) > 0$; so there exists $x_0 \in]a, b[$ such that $f(x_0) = 0$ and the couple

$$\left(C_n^1 = \frac{\|u_n\|_\infty}{\left(1 - \alpha \frac{(1-p)}{m} \Delta t x_0 \|u_n\|_\infty^{1-m}\right)^{1/(1-p)}}, C_n^2 = \frac{\|v_n\|_\infty}{\left(1 - \alpha \frac{1-p}{p} \frac{\Delta t}{x_0} \|v_n\|_\infty^{1-p}\right)^{1/(1-p)}} \right)$$

is a supersolution of the system (3.1), (3.2).

Lemma 3.3. *System (3.1), (3.2) has a maximal solution (\bar{u}, \bar{v}) and any solution (u, v) satisfies: $0 \leq u \leq \bar{u}$, $0 \leq v \leq \bar{v}$.*

Proof: We use the same method as Keller in [2]. We consider the sequences defined by: $u_{n+1,0} = C_n^1$, $v_{n+1,0} = C_n^2$,

$$A u_{n+1,j+1} + \frac{m}{(1-p)\Delta t} u_n^{m-1} u_{n+1,j+1} = \alpha v_{n+1,j} + \frac{m}{(1-p)\Delta t} u_{n+1,j}^p u_n^{m-p},$$

$$A v_{n+1,j+1} + \frac{p}{(1-p)\Delta t} v_n^{p-1} v_{n+1,j+1} = \alpha u_{n+1,j} + \frac{p}{(1-p)\Delta t} v_{n+1,j}^p.$$

We get:

$$A(u_{n+1,1} - u_{n+1,0}) + \frac{m}{(1-p)\Delta t} u_n^{m-1} (u_{n+1,1} - u_{n+1,0}) = \alpha C_n^2 + \frac{m}{(1-p)\Delta t} u_n^{m-p} ((C_n^1)^p - u_n^{p-1} C_n^1).$$

The second member of this equality is negative; we deduce from the maximum principle that: $u_{n+1,1} \leq u_{n+1,0}$. In the same manner, we get: $v_{n+1,1} \leq v_{n+1,0}$. We prove recurrently that the sequences $(u_{n+1,j})_{j \geq 0}$ and $(v_{n+1,j})_{j \geq 0}$ are decreasing; in fact, we have:

$$A(u_{n+1,j+1} - u_{n+1,j}) + \frac{m}{(1-p)\Delta t} u_n^{m-1} (u_{n+1,j+1} - u_{n+1,j}) = \alpha(v_{n+1,j} - v_{n+1,j-1}) + \frac{m}{(1-p)\Delta t} u_n^{m-p} (u_{n+1,j}^p - u_{n+1,j-1}^p)$$

and the second member is negative from the recurrence hypothesis.

We deduce: $u_{n+1,j+1} \leq u_{n+1,j}$. Similarly, we get: $v_{n+1,j+1} \leq v_{n+1,j}$.

Since the two sequences $(u_{n+1,j})_{j \geq 0}$ and $(v_{n+1,j})_{j \geq 0}$ are nonnegative, they converge to \bar{u} and \bar{v} and taking the limit when $j \rightarrow +\infty$, we obtain:

$$A\bar{u} + \frac{m}{(1-p)\Delta t} u_n^{m-1} \bar{u} = \alpha \bar{v} + \frac{m}{(1-p)\Delta t} u_n^{m-p} \bar{u}^p$$

$$A\bar{v} + \frac{p}{(1-p)\Delta t} v_n^{p-1} \bar{v} = \alpha \bar{u} + \frac{p}{(1-p)\Delta t} \bar{v}^p.$$

So, the functions \bar{u} and \bar{v} are solutions of the system (3.1), (3.2).

It remains to prove that any solution (u, v) satisfies: $0 \leq u \leq \bar{u}$, $0 \leq v \leq \bar{v}$.

Let (u, v) a solution of system (3.1), (3.2), we have: $0 \leq u \leq C_n^1$, $0 \leq v \leq C_n^2$; we have the equalities:

$$A(u - u_{n+1,j+1}) + \frac{m}{(1-p)\Delta t} u_n^{m-1} (u - u_{n+1,j+1}) = \alpha(v - v_{n+1,j}) + \frac{m}{(1-p)\Delta t} u_n^{m-p} (u^p - u_{n+1,j}^p)$$

$$A(v - v_{n+1,j+1}) + \frac{p}{(1-p)\Delta t} v_n^{p-1} (v - v_{n+1,j+1}) = \alpha(u - u_{n+1,j}) + \frac{p}{(1-p)\Delta t} (v^p - v_{n+1,j}^p).$$

For $j = 0$, the second member of these inequalities is negative; then we get $u \leq u_{n+1,1}$, $v \leq v_{n+1,1}$ and recurrently, we obtain $u \leq u_{n+1,j}$, $v \leq v_{n+1,j}$ for any $j \geq 0$. It results: $u \leq \bar{u}$, $v \leq \bar{v}$.

Theorem 3.4. *If the functions u_n and v_n are positive, continuous in $\bar{\Omega}$ and satisfy the condition (3.3), then system (3.1), (3.3) has a unique positive solution.*

Proof: From the previous lemmas, we know that the system admits at least one positive solution and that any solution (u, v) satisfies $0 \leq u \leq \bar{u}$, $0 \leq v \leq \bar{v}$.

We get:

$$\int_{\Omega} (Au \bar{u} - A\bar{u} u) dx = 0 = \frac{m}{(1-p)\Delta t} \int_{\Omega} u_n^{m-p} u \bar{u} (u^{p-1} - \bar{u}^{p-1}) dx + \alpha \int_{\Omega} (v\bar{u} - \bar{v}u) dx.$$

Similarly, we have:

$$\frac{p}{(1-p)\Delta t} \int_{\Omega} v \bar{v} (v^{p-1} - \bar{v}^{p-1}) dx + \alpha \int_{\Omega} (u\bar{v} - \bar{u}v) dx = 0$$

We deduce from these equalities that $\int_{\Omega} (u\bar{v} - v\bar{u}) dx = 0$ and then $u = \bar{u}$, $v = \bar{v}$.

Theorem 3.5. *The numerical solution exists at least until the time*

$$T_1 = \min \left(\frac{m}{\alpha(1-p)} \lambda_0^{m-1}, \frac{p}{\alpha(1-p)} \lambda_0^{p-1} \right)$$

with $\lambda_0 = \max(\|u_0\|_{\infty}, \|v_0\|_{\infty})$.

Proof: We prove recurrently that the solution (u_n, v_n) satisfy the inequality: $\|u_n\|_{\infty}, \|v_n\|_{\infty} \leq \phi_n$

where ϕ_n is defined by $\phi_0 = \max \left(\frac{\alpha(1-p)}{m} \lambda_0^{m-1}, \frac{\alpha(1-p)}{p} \lambda_0^{p-1} \right)$, $\phi_n = \frac{\lambda_0}{(1 - t_n \phi_0)^{1/(1-p)}}$.

If this inequality is satisfied at the time level $t_n = n\Delta t$, if $t_{n+1} \phi_0 \leq 1$, the inequality (3.3) is verified and the solution exists at the time level t_{n+1} .

The quantity ϕ_{n+1} will be a supersolution of the system (3.1), (3.2), if we have the two inequalities:

$$-\frac{m}{(1-p)\Delta t}\phi_{n+1}^p u_n^{m-p} + \frac{m}{(1-p)\Delta t}\phi_{n+1} u_n^{m-1} \geq \alpha\phi_{n+1},$$

$$-\frac{p}{(1-p)\Delta t}\phi_{n+1}^p + \frac{p}{(1-p)\Delta t}\phi_{n+1} v_n^{p-1} \geq \alpha\phi_{n+1}.$$

This may be written:

$$\phi_{n+1}^{1-p} \geq \max \left(\frac{\|u_n\|_\infty^{1-p}}{1 - \alpha \frac{(1-p)}{m} \Delta t \|u_n\|_\infty^{1-m}}, \frac{\|v_n\|_\infty^{1-p}}{1 - \alpha \frac{(1-p)}{p} \Delta t \|v_n\|_\infty^{1-p}} \right).$$

But, from the recurrence hypothese, we get

$$\frac{\|u_n\|_\infty^{1-p}}{1 - \alpha \frac{(1-p)}{m} \Delta t \|u_n\|_\infty^{1-m}} \leq \frac{\lambda_0^{1-p}}{(1 - t_n \phi_0) \left(1 - \alpha \frac{1-p}{m} \Delta t \frac{\lambda_0^{1-m}}{(1 - t_n \phi_0)^{(1-m)/(1-p)}} \right)}$$

and it is easy to see that this quantity is bounded by ϕ_{n+1} .

In an analogous manner, we obtain that $\frac{\|v_n\|_\infty^{1-p}}{1 - \alpha \frac{1-p}{p} \Delta t \|v_n\|_\infty^{1-p}}$ is bounded by ϕ_{n+1} .

So, the solution at the time level t_{n+1} satisfies: $\|u_{n+1}\|_\infty, \|v_{n+1}\|_\infty \leq \phi_{n+1}$ and the numerical solution exists during a positive time interval.

4. PROPERTIES OF THE NUMERICAL SCHEME

In this section, we prove that if $\alpha > \lambda_1$, the numerical solution blows up in finite time. We define the functional ψ_n and F_n by:

$$\psi_n(u, v) = \left(\int_{\Omega} (m u_n^{m-p} u^{p+1} + p v^{p+1}) dx \right)^{1/(p+1)}$$

and

$$F_n(u, v) = \frac{J(u, v)}{(\psi_n(u, v))^2}.$$

Lemma 4.1. *The sequence $(F_n(u_n, v_n))_{n \geq 0}$ is nonincreasing.*

Proof: Since $u_{n+1} = \gamma \hat{u}$ and $v_{n+1} = \gamma \hat{v}$, $(\hat{u}, \hat{v}) \in K$, we get $\psi_n(u_{n+1}, v_{n+1}) = \gamma$ and $F_n(u_{n+1}, v_{n+1}) = J(\hat{u}, \hat{v})$.

Besides from (3.4), we have $J(\hat{u}, \hat{v}) = J_n(\hat{u}, \hat{v}) - \frac{1}{(1-p)\Delta t} \int_{\Omega} (m u_n^{m-1} \hat{u}^2 + p v_n^{p-1} \hat{v}^2) dx$.

Hence, we get : $J(\hat{u}, \hat{v}) \leq \frac{J_n(u_n, v_n)}{\psi_n^2(u_n, v_n)} - \frac{1}{(1-p)\Delta t} \frac{\int_{\Omega} (mu_n^{m-1}u_{n+1}^2 + pv_n^{p-1}v_{n+1}^2)dx}{\psi_n^2(u_{n+1}, v_{n+1})}$.

In addition, we have the equality:

$$J_n(u_n, v_n) = J(u_n, v_n) + \frac{1}{(1-p)\Delta t} (\psi_n(u_n, v_n))^{p+1}.$$

We deduce:

$$J(\hat{u}, \hat{v}) \leq \frac{J(u_n, v_n)}{\psi_n^2(u_n, v_n)} + \frac{1}{(1-p)\Delta t} \left((\psi_n(u_n, v_n))^{p-1} - \frac{\int_{\Omega} (mu_n^{m-1}u_{n+1}^2 + pv_n^{p-1}v_{n+1}^2)dx}{\psi_n^2(u_{n+1}, v_{n+1})} \right).$$

By the Hölder inequality, we have at once:

$$\begin{aligned} \int_{\Omega} u_n^{m-p} u_{n+1}^{p+1} dx &\leq \left(\int_{\Omega} u_{n+1}^2 u_n^{m-1} dx \right)^{(p+1)/2} \left(\int_{\Omega} u_n^{m+1} dx \right)^{(1-p)/2}, \\ \int_{\Omega} v_{n+1}^{p+1} dx &\leq \left(\int_{\Omega} v_{n+1}^2 v_n^{p-1} dx \right)^{(p+1)/2} \left(\int_{\Omega} v_n^{p+1} dx \right)^{(1-p)/2}. \end{aligned}$$

Hence, we get:

$$\begin{aligned} \psi_n^{p+1}(u_{n+1}, v_{n+1}) &\leq m \left(\int_{\Omega} u_{n+1}^2 u_n^{m-1} dx \right)^{(p+1)/2} \left(\int_{\Omega} u_n^{m+1} dx \right)^{(1-p)/2} \\ &+ p \left(\int_{\Omega} v_{n+1}^2 v_n^{p-1} dx \right)^{(p+1)/2} \left(\int_{\Omega} v_n^{p+1} dx \right)^{(1-p)/2} \end{aligned}$$

and

$$(4.1) \quad \psi_n^2(u_{n+1}, v_{n+1}) \leq \left(m \int_{\Omega} u_{n+1}^2 u_n^{m-1} dx + p \int_{\Omega} v_{n+1}^2 v_n^{p-1} dx \right) \psi_n^{1-p}(u_n, v_n).$$

We deduce: $J(\hat{u}, \hat{v}) \leq \frac{J(u_n, v_n)}{\psi_n^2(u_n, v_n)}$, that is $F_n(u_{n+1}, v_{n+1}) \leq F_n(u_n, v_n)$.

Lemma 4.2. For $n \geq 0$, we have the estimate:

$$(4.2) \quad (1-p)\Delta t F_n(u_{n+1}, v_{n+1}) \leq \psi_n^{p-1}(u_{n+1}, v_{n+1}) - \psi_n^{p-1}(u_n, v_n) \leq (1-p)\Delta t F_n(u_n, v_n).$$

Proof: ι) We prove first the right inequality. We have: $\psi(u_{n+1}, v_{n+1}) = \gamma$; from (3.6), we obtain

$$\psi_n(u_{n+1}, v_{n+1}) = ((1-p)\Delta t J_n(\hat{u}, \hat{v}))^{1/(p-1)}$$

and

$$\psi_n^{p-1}(u_{n+1}, v_{n+1}) \leq (1-p)\Delta t \frac{J_n(u_n, v_n)}{\psi_n^2(u_n, v_n)},$$

that is

$$\psi_n^{p-1}(u_{n+1}, v_{n+1}) \leq (1-p)\Delta t \left(F_n(u_n, v_n) + \frac{1}{(1-p)\Delta t} \psi_n^{p-1}(u_n, v_n) \right).$$

ι) Multiplying (3.1) by u_{n+1} and (3.2) by v_{n+1} and integrating on Ω , we get:

$$\frac{m}{(1-p)\Delta t} \int_{\Omega} u_{n+1}^2 u_n^{m-1} dx - \frac{m}{(1-p)\Delta t} \int_{\Omega} u_{n+1}^{p+1} u_n^{m-p} dx + \int_{\Omega} (|\nabla u_{n+1}|^2 - \alpha u_{n+1} v_{n+1}) dx = 0,$$

$$\frac{p}{(1-p)\Delta t} \int_{\Omega} v_{n+1}^2 v_n^{p-1} dx - \frac{p}{(1-p)\Delta t} \int_{\Omega} v_{n+1}^{p+1} dx + \int_{\Omega} (|\nabla v_{n+1}|^2 - \alpha u_{n+1} v_{n+1}) dx = 0.$$

Hence , we get :

$$\frac{1}{(1-p)\Delta t} \int_{\Omega} (m u_{n+1}^2 u_n^{m-1} + p v_{n+1}^2 v_n^{p-1}) dx - \frac{1}{(1-p)\Delta t} \psi_n^{p+1}(u_{n+1}, v_{n+1}) + J(u_{n+1}, v_{n+1}) = 0.$$

By using (4.1), we deduce:

$$\psi_n^{p-1}(u_n, v_n) - \psi_n^{p-1}(u_{n+1}, v_{n+1}) + F(u_{n+1}, v_{n+1}) \leq 0.$$

This concludes the proof.

Lemma 4.3. *The sequence $(J(u_n, v_n))_{n \geq 0}$ is nonincreasing*

Proof: In [3] , we have proved the inequality:

$$\forall a, b \in \mathbb{R}^+, a^{p-1}(b-a)^2 \leq a^{p+1} - b^{p+1} - \frac{1+p}{1-p} b^2 (b^{p-1} - a^{p-1}).$$

We deduce from this inequality:

$$\int_{\Omega} (u_{n+1} - u_n)^2 u_n^{m-1} dx \leq \int_{\Omega} u_n^{m+1} dx - \int_{\Omega} u_{n+1}^{p+1} u_n^{m-p} dx - \frac{1+p}{1-p} \int_{\Omega} u_{n+1}^2 (u_{n+1}^{p-1} - u_n^{p-1}) u_n^{m-p} dx$$

and

$$\int_{\Omega} (v_{n+1} - v_n)^2 v_n^{p-1} dx \leq \int_{\Omega} v_n^{p+1} dx - \int_{\Omega} v_{n+1}^{p+1} dx - \frac{1+p}{1-p} \int_{\Omega} v_{n+1}^2 (v_{n+1}^{p-1} - v_n^{p-1}) dx.$$

Since u_{n+1} and v_{n+1} are solutions of (3.1), (3.2), we get:

$$\int_{\Omega} u_{n+1}^2 (u_{n+1}^{p-1} - u_n^{p-1}) u_n^{m-p} dx = \frac{(1-p)\Delta t}{m} \int_{\Omega} (|\nabla u_{n+1}|^2 - \alpha u_{n+1} v_{n+1}) dx$$

and

$$\int_{\Omega} v_{n+1}^2 (v_{n+1}^{p-1} - v_n^{p-1}) dx = \frac{(1-p)\Delta t}{p} \int_{\Omega} (|\nabla v_{n+1}|^2 - \alpha u_{n+1} v_{n+1}) dx.$$

So, we obtain:

$$\begin{aligned} & m \int_{\Omega} (u_{n+1} - u_n)^2 u_n^{m-1} dx + p \int_{\Omega} (v_{n+1} - v_n)^2 v_n^{p-1} dx \\ & \leq \psi_n^{p+1}(u_n, v_n) - \psi_n^{p+1}(u_{n+1}, v_{n+1}) - (1+p)\Delta t J(u_{n+1}, v_{n+1}). \end{aligned}$$

From the inequality [3] : $\forall a, b \in \mathbb{R}^+$, $a^{p+1} - b^{p+1} \leq \frac{1+p}{1-p} a^2(b^{p-1} - a^{p-1})$, we deduce:

$$\psi_n^{p+1}(u_n, v_n) - \psi_n^{p+1}(u_{n+1}, v_{n+1}) \leq \frac{1+p}{1-p} \psi_n^2(u_n, v_n) (\psi_n^{p-1}(u_{n+1}, v_{n+1}) - \psi_n^{p-1}(u_n, v_n))$$

and by using (4.2), we get:

$$(4.3) \quad \psi_n^{p+1}(u_n, v_n) - \psi_n^{p+1}(u_{n+1}, v_{n+1}) \leq (1+p)\Delta t J(u_n, v_n).$$

So, we get

$$m \int_{\Omega} (u_{n+1} - u_n)^2 u_n^{m-1} dx + p \int_{\Omega} (v_{n+1} - v_n)^2 v_n^{p-1} dx \leq (1+p)\Delta t (J(u_n, v_n) - J(u_{n+1}, v_{n+1})).$$

We deduce: $J(u_n, v_n) \geq J(u_{n+1}, v_{n+1})$.

We note

$$\Phi_n = \int_{\Omega} \left(\frac{m}{m+1} u_n^{m+1} + \frac{p}{p+1} v_n^{p+1} \right) dx$$

Lemma 4.4. *If $J(u_0, v_0) < 0$, the sequence $(\Phi_n)_{n \geq 0}$ is increasing.*

Proof: We have the equality :

$$(4.4) \quad \psi_n^{p+1}(u_n, v_n) = \int_{\Omega} (m u_n^{m+1} + p v_n^{p+1}) dx = (p+1)\Phi_n + \frac{m(m-p)}{m+1} \int_{\Omega} u_n^{m+1} dx.$$

Besides, we get:

$$(4.5) \quad \psi_n^{p+1}(u_{n+1}, v_{n+1}) = \int_{\Omega} (m u_n^{m-p} u_{n+1}^{p+1} + p v_{n+1}^{p+1}) dx \leq (p+1)\Phi_{n+1} + \frac{m(m-p)}{m+1} \int_{\Omega} u_n^{m+1} dx.$$

and we deduce:

$$\Phi_n - \Phi_{n+1} \leq \frac{1}{p+1} (\psi_n^{p+1}(u_n, v_n) - \psi_n^{p+1}(u_{n+1}, v_{n+1})).$$

By using (4.3) , we obtain $\Phi_n - \Phi_{n+1} \leq \Delta t J(u_n, v_n)$.

If $J(u_0, v_0) < 0$, since the sequence $(J(u_n, v_n))_{n \geq 0}$ is nonincreasing, we deduce that the sequence $(\Phi_n)_{n \geq 0}$ is increasing.

Lemma 4.5. *If $J(u_0, v_0) < 0$, for $n \geq 0$, we have the inequality:*

$$(4.6) \quad \Phi_n^{2/(p+1)} \left(\Phi_{n+1}^{(p-1)/(p+1)} - \Phi_n^{(p-1)/(p+1)} \right) \leq \frac{1}{m+1} \psi_n^2(u_n, v_n) \left(\psi_n^{p-1}(u_{n+1}, v_{n+1}) - \psi_n^{p-1}(u_n, v_n) \right).$$

Proof: This inequality may be written:

$$\psi_n^{p+1}(u_n, v_n) + (m+1)\Phi_n^{2/(p+1)}\Phi_{n+1}^{(p-1)/(p+1)} \leq \psi_n^2(u_n, v_n)\psi_n^{p-1}(u_{n+1}, v_{n+1}) + (m+1)\Phi_n.$$

By using (4.4) and (4.5), we obtain that a sufficient condition to satisfy this inequality is:

$$\begin{aligned} & (p+1)\Phi_n + (m-p)\mu_n + (m+1)\Phi_n^{2/(p+1)}\Phi_{n+1}^{(p-1)/(p+1)} \\ & \leq ((p+1)\Phi_n + (m-p)\mu_n)^{2/(p+1)} ((p+1)\Phi_{n+1} + (m-p)\mu_n)^{(p-1)/(p+1)} + (m+1)\Phi_n \end{aligned}$$

$$\text{with } \mu_n = \frac{m}{m+1} \int_{\Omega} u_n^{m+1} dx.$$

If $J(u_0, v_0) < 0$, the sequence $(\Phi_n)_{n \geq 0}$ is increasing, so we get: $\frac{(p+1)\Phi_n + (m-p)\mu_n}{(p+1)\Phi_{n+1} + (m-p)\mu_n} \geq \frac{\Phi_n}{\Phi_{n+1}}$.

Hence, it is sufficient to prove:

$$\begin{aligned} & (p+1)\Phi_n + (m-p)\mu_n + (m+1)\Phi_n^{2/(p+1)}\Phi_{n+1}^{(p-1)/(p+1)} \\ & \leq ((p+1)\Phi_n + (m-p)\mu_n) \Phi_{n+1}^{(p-1)/(p+1)} \Phi_n^{(1-p)/(1+p)} + (m+1)\Phi_n \end{aligned}$$

$$\text{that is } \mu_n (\Phi_{n+1}^{(1-p)/(1+p)} - \Phi_n^{(1-p)/(1+p)}) \leq \Phi_n (\Phi_{n+1}^{(1-p)/(1+p)} - \Phi_n^{(1-p)/(1+p)})$$

and this inequality is satisfied since $\mu_n \leq \Phi_n$.

Lemma 4.6. *If $J(u_0, v_0) < 0$, we have the estimate:*

$$(4.7) \quad \Phi_n^{2/(p+1)} \left(\Phi_{n+1}^{(p-1)/(p+1)} - \Phi_n^{(p-1)/(p+1)} \right) \leq \frac{1-p}{1+m} \Delta t J(u_n, v_n)$$

Proof: We deduce the estimate immediately from (4.2) and (4.6)

Theorem 4.7. *If $J(u_0, v_0) < 0$, the numerical solution blows up in a finite time T_* such that*

$$T_* < \frac{1+m}{1-p} \frac{\int_{\Omega} \left(\frac{m}{m+1} u_0^{m+1} + \frac{p}{p+1} v_0^{p+1} \right) dx}{-J(u_0, v_0)}.$$

Proof: From (4.7), we get $\Phi_{n+1}^{(p-1)/(p+1)} \leq \Phi_n^{(p-1)/(p+1)} + \frac{1-p}{1+m} \Delta t \Phi_n^{-2/(p+1)} J(u_n, v_n)$

and since the sequence $(J(u_n, v_n))_{n \geq 0}$ is decreasing and the sequence (Φ_n) increasing, we get :

$$\Phi_{n+1}^{(p-1)/(p+1)} \leq \Phi_0^{(p-1)/(p+1)} \left(1 + \frac{1-p}{1+m} t_n \Phi_0^{-1} J(u_0, v_0) \right)$$

and we deduce the estimate.

Remark 4.8. *In the case $p = m$, we obtain the same bound for the numerical blow-up time and for the blow-up time of the exact solution. In the case $p < m$, the bound obtained for the numerical blow-up time is inferior to the estimate obtained for the blow-up time of the exact solution*

5. THE CASE $p = m$

In the case $p = m$, the functionals ψ_n and F_n are independent of n . We shall note them respectively ψ and F :

$$\psi(u, v) = \left(\int_{\Omega} m(u^{m+1} + v^{m+1}) dx \right)^{\frac{1}{m+1}} \text{ and } F(u, v) = \frac{J(u, v)}{\psi^2(u, v)}.$$

Besides, we get: $\psi^{m+1}(u_n, v_n) = (m+1)\Phi_n$, $n \geq 0$.

The estimate of the numerical blow-up is in that case the same as the estimate of the exact blow-up.

5.1. Properties of the scheme.

Proposition 5.1. *If $\alpha > \lambda_1$ and T^* is the numerical blow-up time, we get the estimate:*

$$\Phi_n^{\frac{1}{m+1}} \leq \left(\frac{T_*}{T_* - t} \right)^{\frac{1}{1-m}} \Phi_0^{\frac{1}{m+1}}$$

Proof: The estimate (4.2) may be written in this case:

$$(1-m)\Delta t F(u_{n+1}, v_{n+1}) \leq \psi^{m-1}(u_{n+1}, v_{n+1}) - \psi^{m-1}(u_n, v_n) \leq (1-m)\Delta t F(u_n, v_n).$$

We deduce, since $(F(u_n, v_n))_{n \geq 0}$ is a nonincreasing sequence:

$$\psi_n^{m-1}(u_{n+j}, v_{n+j}) - \psi_n^{m-1}(u_n, v_n) \leq (1-m)j\Delta t F(u_n, v_n), \quad \forall j \geq 0,$$

$$\psi_n^{m-1}(u_{n-i}, v_{n-i}) - \psi_n^{m-1}(u_n, v_n) \leq -(1-m)i\Delta t F(u_n, v_n), \quad \forall i \geq 0.$$

So, we obtain:

$$i\psi_n^{m-1}(u_{n+j}, v_{n+j}) + j\psi_n^{m-1}(u_{n-i}, v_{n-i}) \leq (i+j)\psi_n^{m-1}(u_n, v_n).$$

If $T_* = N\Delta t$ is the numerical blow-up time, by choosing $i = n$, $j = N - n$, we get :

$$(N-n)\psi^{m-1}(u_0, v_0) \leq N\psi^{1-m}(u_n, v_n) \text{ or, } \psi^{1-m}(u_n, v_n) \leq \frac{T_*}{T_* - t} \psi^{1-m}(u_n, v_n).$$

This is the same estimate as for the exact solution.

Proposition 5.2. *If $\alpha < \lambda_1$, the numerical solution tends to 0 when $t \rightarrow +\infty$.*

Proof: In that case, we get: $J(u, v) \geq (\lambda_1 - \alpha) \int_{\Omega} (u^2 + v^2) dx \geq 0$.

Besides, since $m < 1$, for any $\phi \in L^2(\Omega)$, we have: $\int_{\Omega} \phi^2 dx \geq C(\Omega) \left(\int_{\Omega} \phi^{m+1} dx \right)^{\frac{2}{m+1}}$ and by using Young inequality, we get

$$\begin{aligned} 2^{-\frac{1-m}{m+1}} \left(\int_{\Omega} (u^{m+1} + v^{m+1}) dx \right)^{\frac{2}{m+1}} &\leq \left(\int_{\Omega} u^{m+1} dx \right)^{\frac{2}{m+1}} + \left(\int_{\Omega} v^{m+1} dx \right)^{\frac{2}{m+1}} \\ &\leq \frac{1}{C(\Omega)} \left(\int_{\Omega} (u^2 + v^2) dx \right). \end{aligned}$$

So, we get : $F(u, v) \geq \frac{\lambda_1 - \alpha}{C(\Omega)}$.

Besides, from (4.2), we obtain $\psi^{m-1}(u_{n+1}, v_{n+1}) \geq \psi^{m-1}(u_n, v_n) \geq \frac{\lambda_1 - \alpha}{C(\Omega)} \Delta t$ and

$$\psi^{m-1}(u_n, v_n) \geq \psi^{m-1}(u_0, v_0) + \frac{\lambda_1 - \alpha}{C(\Omega)} t_n.$$

We deduce: $\lim_{n \rightarrow +\infty} \int_{\Omega} (u^{m+1} + v^{m+1}) dx = 0$.

Proposition 5.3. *If $\alpha = \lambda_1$, then $\lim_{n \rightarrow +\infty} u_n = \lim_{n \rightarrow +\infty} v_n = \theta \rho_1$ where θ is depending on the initial conditions.*

Proof: We have the equality: $\Phi_{n+1} - \Phi_n = \frac{1}{m+1} (\psi^{m+1}(u_{n+1}, v_{n+1}) - \psi^{m+1}(u_n, v_n))$

and from the inequality $\forall a, b \in \mathbb{R}^+$, $a^{m+1} - b^{m+1} \leq \frac{1+m}{1-m} a^2 (b^{m-1} - a^{m-1})$,

we obtain:

$$\Phi_{n+1} - \Phi_n \leq \frac{1}{1-m} \psi^2(u_{n+1}, v_{n+1}) (\psi^{m-1}(u_n, v_n) - \psi^{m-1}(u_{n+1}, v_{n+1})) \leq -\Delta t J(u_{n+1}, v_{n+1}).$$

In the same manner as in proposition (2.5), we deduce that $\lim_{n \rightarrow +\infty} u_n = \lim_{n \rightarrow +\infty} v_n = \theta \rho_1$.

We now obtain an estimate for θ :

By multiplying the equations (3.1), (3.2) by ρ_1 and integrating on Ω , we get:

$$\begin{aligned} \frac{m}{1-m} \int_{\Omega} u_{n+1} u_n^{m-1} \rho_1 dx - \frac{m}{1-m} \int_{\Omega} u_{n+1}^m \rho_1 dx + \lambda_1 \Delta t \int_{\Omega} (u_{n+1} - v_{n+1}) \rho_1 dx &= 0, \\ \frac{m}{1-m} \int_{\Omega} v_{n+1} v_n^{m-1} \rho_1 dx - \frac{m}{1-m} \int_{\Omega} v_{n+1}^m \rho_1 dx + \lambda_1 \Delta t \int_{\Omega} (v_{n+1} - u_{n+1}) dx &= 0. \end{aligned}$$

By using the inequality: $u_{n+1}^m \leq (1-m)u_n^m + m u_{n+1} u_n^{m-1}$ and the same inequality applied to the function v , we get

$$\begin{aligned} \int_{\Omega} u_{n+1}^m \rho_1 dx &\leq \int_{\Omega} u_n^m \rho_1 dx + \Delta t \frac{\lambda_1}{m} \int_{\Omega} (v_{n+1} - u_{n+1}) \rho_1 dx, \\ \int_{\Omega} v_{n+1}^m \rho_1 dx &\leq \int_{\Omega} v_n^m \rho_1 dx + \Delta t \frac{\lambda_1}{m} \int_{\Omega} (u_{n+1} - v_{n+1}) \rho_1 dx, \end{aligned}$$

and then: $\int_{\Omega} (u_{n+1}^m + v_{n+1}^m) \rho_1 dx \leq \int_{\Omega} (u_0^m + v_0^m) \rho_1 dx$, $n \geq 0$.

We deduce: $\theta^m \leq \frac{\int_{\Omega} (u_0^m + v_0^m) \rho_1 dx}{2 \int_{\Omega} \rho_1^{m+1} dx}$.

5.2. Convergence of the scheme.

In this section, we obtain estimates on the numerical solution so we can extract by compactness a convergent subsequence. In order to prove that the limit is solution of the system (1.1), we need an hypothesis on the initial condition, (this is due to the fact that we have a negative power in the scheme and we may not use a Hölder inequality). If the initial condition does not satisfy the hypothesis, we observe numerically that this hypothesis is satisfied after a few times steps and the scheme again converges.

Let us denote by $T_1^* = \inf_{0 < \Delta t < \Delta t_0} T_*(\Delta t)$ if T_* is the existence time of the numerical solution. It follows from theorem 3.5 that $T_1^* \geq T_1 > 0$. Let $T \in [0, T_1^*]$, ($T = N\Delta t$). We denote $u_{\Delta t}$ and $v_{\Delta t}$ the approximation of u and v defined by:

$$u_{\Delta t}(t) = \left(u_n^m + \frac{t - t_n}{\Delta t} (u_{n+1}^m - u_n^m) \right)^{1/m},$$

$$v_{\Delta t}(t) = \left(v_n^m + \frac{t - t_n}{\Delta t} (v_{n+1}^m - v_n^m) \right)^{1/m}.$$

Theorem 5.4. *The sequences $(u_{\Delta t})$ and $(v_{\Delta t})$ are uniformly bounded in $C(0, T; H_0^1(\Omega))$ and $H^1(0, T; L^2(\Omega))$.*

Proof: Since $T < T_1^*$, the functions $(u_n)_{n \geq 0}$ and $(v_n)_{n \geq 0}$ are uniformly bounded in $C(0, T; \Omega)$ and since $J(u_n, v_n)$ is nonincreasing, we get: $J(u_n, v_n) \leq J(u_0, v_0)$. We deduce that the sequences $(\nabla u_n)_{n \geq 0}$ and $(\nabla v_n)_{n \geq 0}$ are uniformly bounded in $L^2(\Omega)$.

We prove now that the sequences $(\frac{du_{\Delta t}}{dt})$ and $(\frac{dv_{\Delta t}}{dt})$ are uniformly bounded in $L^2(0, T; L^2(\Omega))$.

We have:

$$\begin{aligned} \left\| \frac{du_{\Delta t}}{dt} \right\|_{L^2(0, T; L^2(\Omega))}^2 &= \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \int_{\Omega} \frac{1}{m^2} u_{\Delta t}^{2(1-m)} \left(\frac{u_{n+1}^m - u_n^m}{\Delta t} \right)^2 dx dt \\ &\leq \frac{1}{m^2 \Delta t} \sum_{n=0}^{N-1} \int_{\Omega} \left(u_{n+1}^{2(1-m)} + u_n^{2(1-m)} \right) (u_{n+1}^m - u_n^m)^2 dx. \end{aligned}$$

From the following inequalities, $u_{n+1}^{1-m} |u_{n+1}^m - u_n^m| \leq |u_{n+1} - u_n|$ and $u_n^{1-m} |u_{n+1}^m - u_n^m| \leq |u_{n+1} - u_n|$, we deduce:

$$\begin{aligned} \left\| \frac{du_{\Delta t}}{dt} \right\|_{L^2(0, T; L^2(\Omega))}^2 &\leq \frac{1}{m^2 \Delta t} \sum_{n=0}^{N-1} \int_{\Omega} (u_{n+1} - u_n)^2 dx \\ &\leq \frac{1}{m^2 \Delta t} \sum_{n=0}^{N-1} \left(\int_{\Omega} u_n^{m-1} (u_{n+1} - u_n)^2 dx \right) \|u_n\|_{\infty}^{1-m} \end{aligned}$$

and we obtain analogous inequality for $v_{\Delta t}$.

Hence, we get:

$$\left\| \frac{du_{\Delta t}}{dt} \right\|_{L^2(0, T; L^2(\Omega))}^2 + \left\| \frac{dv_{\Delta t}}{dt} \right\|_{L^2(0, T; L^2(\Omega))}^2 \leq C (J(u_0, v_0) - J(u_N, v_N))$$

since $(\|u_n\|_{\infty})_{0 \leq n \leq N}$ and $(\|v_n\|_{\infty})_{0 \leq n \leq N}$ are uniformly bounded.

If $\alpha \leq \lambda_1$, we have: $J(u_N, v_N) \geq 0$,

if $\alpha > \lambda_1$, we get $J(u_N, v_N) \geq (\lambda_1 - \alpha) \int_{\Omega} (u_N^2 + v_N^2) dx$ and this quantity is bounded from below; we deduce that the sequences are uniformly bounded in $L^2(0, T; L^2(\Omega))$.

Since the sequences $(u_{\Delta t})$ and $(v_{\Delta t})$ are uniformly bounded in $C(0, T; H_0^1(\Omega)) \cap H^1(0, T; L^2(\Omega))$, we can extract subsequences which converge to functions u and v in $C(0, T; \Omega)$ if $d = 1$ and in $C(0, T; L^r(\Omega))$, ($r < 2d/(d-2)$) if $d > 2$, $r = \infty$ if $d = 2$, (Simon [6]).

In order to prove that the limits u , v are solutions of the system (1.1), we use the same proof as in [3]; so we need to estimate the quantities $|(u_{n+1}^{1-m} - u_n^{1-m})u_n^{m-1}|$ and $|(v_{n+1}^{1-m} - v_n^{1-m})v_n^{m-1}|$. This is the object of the two next lemmas.

Lemma 5.5. *For $n \geq 0$, we have the inequalities:*

$$(5.1) \quad t_{n+1} u_{n+1}^{1-m} \geq t_n u_n^{1-m}, \quad t_{n+1} v_{n+1}^{1-m} \geq t_n v_n^{1-m}.$$

Proof: These inequalities are proved recurrently; it is true for $n = 0$. Assume it is true at the order $n - 1$, that is :

$$t_n u_n^{1-m} \geq t_{n-1} u_{n-1}^{1-m}, \quad t_n v_n^{1-m} \geq t_{n-1} v_{n-1}^{1-m}.$$

The functions $\left(\frac{t_n}{t_{n+1}}\right)^{1/(1-m)} u_n$, $\left(\frac{t_n}{t_{n+1}}\right)^{1/(1-m)} v_n$ will be subsolutions of (1.1) if :

$$-\frac{m}{1-m} \frac{\Delta t}{t_n} u_n^m + \Delta t A u_n - \alpha \Delta t v_n \leq 0,$$

$$-\frac{m}{1-m} \frac{\Delta t}{t_n} v_n^m + \Delta t A v_n - \alpha \Delta t u_n \leq 0.$$

But u_n, v_n are the solutions at the time level t_n , so we get:

$$-\frac{m}{1-m} \frac{\Delta t}{t_n} u_n^m + \Delta t (Au_n - \alpha v_n) = -\frac{m}{1-m} \left(\frac{\Delta t}{t_n} u_n^m + u_n u_{n-1}^{m-1} - u_n^m \right) = -\frac{m}{1-m} u_n^m \left(u_n^{1-m} u_{n-1}^{m-1} - \frac{t_{n-1}}{t_n} \right)$$

$$\text{and } -\frac{m}{1-m} \frac{\Delta t}{t_n} v_n^m + \Delta t (Av_n - \alpha u_n) = -\frac{m}{1-m} v_n^m \left(v_n^{1-m} v_{n-1}^{m-1} - \frac{t_{n-1}}{t_n} \right).$$

From the recurrence hypotheses, the second members of these two inequalities are non positive and the inequalities (5.1) are satisfied at the order n .

Lemma 5.6. *Assume that the initial conditions u_0, v_0 satisfy:
There exists a constant C_0 such that*

$$(5.2) \quad Au_0 - \alpha v_0 + C_0 u_0^m \geq 0, \quad Av_0 - \alpha u_0 + C_0 v_0^m \geq 0$$

Then, we have the estimates:

$$(5.3) \quad u_{n+1} \leq \left(\frac{T_2 - t_n}{T_2 - t_{n+1}} \right)^{1/(1-m)} u_n, \quad v_{n+1} \leq \left(\frac{T_2 - t_n}{T_2 - t_{n+1}} \right)^{1/(1-m)} v_n$$

with $T_2 = \frac{m}{(1-m)C_0}$

Proof: This lemma is proved recursively. First, we prove the inequalities for $n = 0$.

$\left(\frac{T_2}{T_2 - \Delta t} \right)^{1/(1-m)} u_0, \left(\frac{T_2}{T_2 - \Delta t} \right)^{1/(1-m)}$ will be supersolutions of (3.1), (3.2) if

$$-\frac{m}{1-m} \left(\frac{T_2}{T_2 - \Delta t} \right)^{m/(1-m)} u_0^m + \frac{m}{1-m} \left(\frac{T_2}{T_2 - \Delta t} \right)^{1/(1-m)} u_0^m + \Delta t \left(\frac{T_2}{T_2 - \Delta t} \right)^{1/(1-m)} (Au_0 - \alpha v_0) \geq 0,$$

$$-\frac{m}{1-m} \left(\frac{T_2}{T_2 - \Delta t} \right)^{m/(1-m)} v_0^m + \frac{m}{1-m} \left(\frac{T_2}{T_2 - \Delta t} \right)^{1/(1-m)} v_0^m + \Delta t \left(\frac{T_2}{T_2 - \Delta t} \right)^{1/(1-m)} (Av_0 - \alpha u_0) \geq 0,$$

that is, $\frac{m}{(1-m)T_2} u_0^m + Au_0 - \alpha v_0 \geq 0$ and $\frac{m}{(1-m)T_2} v_0^m + Av_0 - \alpha u_0 \geq 0$.

From (5.2), these inequalities are satisfied.

Assume now that the estimates are satisfied at the order n . We prove it is satisfied at the order $n + 1$.

The functions $\left(\frac{T_2 - t_n}{T_2 - t_{n+1}} \right)^{1/(1-m)} u_n, \left(\frac{T_2 - t_n}{T_2 - t_{n+1}} \right)^{1/(1-m)} v_n$ will be supersolutions of (3.1), (3.2) if

$$\frac{m}{1-m} \frac{1}{T_2 - t_n} u_n^m + Au_n - \alpha v_n \geq 0, \quad \frac{m}{1-m} \frac{1}{T_2 - t_n} v_n^m + Av_n - \alpha u_n \geq 0.$$

Since u_n, v_n are the solutions at the time level t_n , we get :

$$\frac{m}{1-m} \frac{u_n^m}{T_2 - t_n} + Au_n - \alpha v_n = \frac{m}{(1-m)\Delta t} \left(u_n^m \frac{T_2 - t_{n-1}}{T_2 - t_n} - u_n u_{n-1}^{m-1} \right),$$

$$\frac{m}{1-m} \frac{v_n^m}{T_2 - t_n} + Av_n - \alpha u_n = \frac{m}{(1-m)\Delta t} \left(v_n^m \frac{T_2 - t_{n-1}}{T_2 - t_n} - v_n v_{n-1}^{m-1} \right).$$

By using the recurrence hypothesis, we obtain the estimates (5.3).

Remark 5.7. *If $Au_0 - \alpha v_0 \geq 0$ and $Av_0 - \alpha u_0 \geq 0$, then $T_2 = +\infty$ and we obtain: $u_{n+1} \leq u_n$, $v_{n+1} \leq v_n$.*

From these two lemmas, we obtain the inequalities

$$-\frac{\Delta t}{t_{n+1}} \leq (u_{n+1}^{1-m} - u_n^{1-m})u_n^{m-1} \leq \frac{\Delta t}{T_2 - t_{n+1}}$$

and analogous inequalities for v_n . Then, we obtain the convergence of the scheme as in [3].

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