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LOW FREQUENCY RESOLVENT ESTIMATES FOR LONG RANGE PERTURBATIONS OF THE EUCLIDEAN LAPLACIAN

JEAN-FRANÇOIS BONY AND DIETRICH HÄFNER

ABSTRACT. Let P be a long range metric perturbation of the Euclidean Laplacian on \mathbb{R}^d , $d \geq 3$. We prove that the following resolvent estimate holds:

$$\|\langle x \rangle^{-\alpha} (P - z)^{-1} \langle x \rangle^{-\beta}\| \lesssim 1 \quad \forall z \in \mathbb{C} \setminus \mathbb{R}, |z| < 1,$$

if $\alpha, \beta > 1/2$ and $\alpha + \beta > 2$. The above estimate is false for the Euclidean Laplacian in dimension 3 if $\alpha \leq 1/2$ or $\beta \leq 1/2$ or $\alpha + \beta < 2$.

1. INTRODUCTION

There are now many results dealing with the low frequency behavior of the resolvent of Schrödinger type operators. The methods used to obtain these results are various: one can apply the Fredholm theory to study perturbations by a potential (see *e.g.* [6]) or a short range metric (see *e.g.* [9]). The resonance theory is also useful to treat compactly supported perturbations of the flat case (see *e.g.* [3]). Using the general Mourre theory, one can obtain limiting absorption principles at the thresholds (see *e.g.* [5] or [8]). The pseudo-differential calculus of Melrose allows to describe the kernel of the resolvent at low energies for compactifiable manifolds (see *e.g.* [7]). Concerning the long range case, Bouclet [1] has obtained a uniform control of the resolvent for perturbations in divergence form. We refer to his article and to [4] for a quite exhaustive list of previous results for perturbations of the Euclidean Laplacian.

On \mathbb{R}^d with $d \geq 3$, we consider the following operator

$$(1) \quad P = -b \operatorname{div}(G \nabla b) = - \sum_{i,j=1}^d b(x) \frac{\partial}{\partial x_i} G_{i,j}(x) \frac{\partial}{\partial x_j} b(x),$$

where $b(x) \in C^\infty(\mathbb{R}^d)$ and $G(x) \in C^\infty(\mathbb{R}^d; \mathbb{R}^{d \times d})$ is a real symmetric $d \times d$ matrix. The C^∞ hypothesis is made mostly for convenience, much weaker regularity could actually be considered. We make an ellipticity assumption:

$$(H1) \quad \exists C > 0, \forall x \in \mathbb{R}^d \quad G(x) \geq CI_d \quad \text{and} \quad b(x) \geq C,$$

I_d being the identity matrix. We also assume that P is a long range perturbation of the Euclidean Laplacian:

$$(H2) \quad \exists \rho > 0, \forall \alpha \in \mathbb{N}^d \quad |\partial_x^\alpha (G(x) - I_d)| + |\partial_x^\alpha (b(x) - 1)| \lesssim \langle x \rangle^{-\rho - |\alpha|}.$$

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In particular, if $b = 1$, we are concerned with an elliptic operator in divergence form $P = -\operatorname{div}(G\nabla)$. On the other hand, if $G = (g^2 g^{i,j}(x))_{i,j}$, $b = (\det g^{i,j})^{1/4}$, $g = \frac{1}{b}$, then the above operator is unitarily equivalent to the Laplace–Beltrami $-\Delta_{\mathfrak{g}}$ on $(\mathbb{R}^d, \mathfrak{g})$ with metric

$$\mathfrak{g} = \sum_{i,j=1}^d g_{i,j}(x) dx^i dx^j,$$

where $(g_{i,j})_{i,j}$ is inverse to $(g^{i,j})_{i,j}$ and the unitary transform is just multiplication by g .

Theorem 1. *Let P be of the form (1) in \mathbb{R}^d with $d \geq 3$. Assume (H1) and (H2).*

i) For all $\varepsilon > 0$, we have

$$(2) \quad \|\langle x \rangle^{-1/2-\varepsilon} (\sqrt{P} - z)^{-1} \langle x \rangle^{-1/2-\varepsilon}\| \lesssim 1,$$

uniformly in $z \in \mathbb{C} \setminus \mathbb{R}$, $|z| < 1$.

ii) For all $\varepsilon > 0$, we have

$$(3) \quad \|\langle x \rangle^{-1/2-\varepsilon} (P - z)^{-1} \langle x \rangle^{-1/2-\varepsilon}\| \lesssim |z|^{-1/2},$$

uniformly in $z \in \mathbb{C} \setminus \mathbb{R}$, $|z| < 1$.

iii) For all $\alpha, \beta > 1/2$ with $\alpha + \beta > 2$, we have

$$(4) \quad \|\langle x \rangle^{-\alpha} (P - z)^{-1} \langle x \rangle^{-\beta}\| \lesssim 1,$$

uniformly in $z \in \mathbb{C} \setminus \mathbb{R}$, $|z| < 1$.

Remark 2. *i) The estimate (4) is not far from optimal. Indeed, this estimate is false for the Euclidean Laplacian $-\Delta$ in dimension 3 if $\alpha \leq 1/2$ or $\beta \leq 1/2$ or $\alpha + \beta < 2$.*

ii) One can interpret (4) in the following way: one needs a $\langle x \rangle^{-1/2}$ on the left and on the right to assure that the resolvent is continuous on $L^2(\mathbb{R}^d)$ and one needs an additional $\langle x \rangle^{-1}$ (distributed, as we want, among the left and the right) to guarantee that its norm is uniform with respect to z .

iii) By interpolation of (3) and (4), for $\alpha, \beta > 1/2$ with $\alpha + \beta \leq 2$, one obtains estimates like (4) with $|z|^{-1+\frac{\alpha+\beta}{2}-\varepsilon}$ on the right hand side.

iv) In dimension 1, the kernel of $(-\Delta - z)^{-1}$ is given by $\frac{ie^{i\sqrt{z}|x-y|}}{2\sqrt{z}}$. In particular, this operator satisfies (3) but not (4) (for any α, β). Therefore it seems that (3) is more general than (4). It could perhaps be possible to prove (3) in lower dimensions (at least, in dimension 2 and when P is of divergence form $P = -\operatorname{div}(G\nabla)$).

v) For large z , the estimate (3) coincides with the high energy estimate in the non-trapping case. In particular, if we suppose in addition a non trapping condition for P , then (2) and (3) hold uniformly in $z \in \mathbb{C} \setminus \mathbb{R}$.

The proof of the above theorem is based on the low frequency estimates of [2]. Concerning the square root of P , they are used to treat the wave equation. Note that in [2] they are formulated for the Laplace–Beltrami operator $-\Delta_{\mathfrak{g}}$, but they obviously hold for the operators studied in the present paper. Essentially, we will show that (2) \Rightarrow (3) \Rightarrow (4).

2. PROOF OF THE RESULTS

We begin by recalling some results of [2]. For $\lambda \geq 1$, we set

$$\mathcal{A}_\lambda = \varphi(\lambda P)A_0\varphi(\lambda P),$$

where

$$A_0 = \frac{1}{2}(xD + Dx), \quad D(A_0) = \{u \in L^2(\mathbb{R}^d); A_0u \in L^2(\mathbb{R}^d)\},$$

is the generator of dilations and $\varphi \in C_0^\infty(\]0, +\infty[; [0, +\infty[)$ satisfies $\varphi(x) > 1$ on some open bounded interval $I = [1 - \tilde{\varepsilon}, 1 + \tilde{\varepsilon}]$, $0 < \tilde{\varepsilon} < 1$ sufficiently small. As usual, we define the multi-commutators $\text{ad}_A^j B$ inductively by $\text{ad}_A^0 B = B$ and $\text{ad}_A^{j+1} B = [A, \text{ad}_A^j B]$. We recall [2, Proposition 3.1]:

Proposition 3. *i) We have $(\lambda P)^{1/2} \in C^2(\mathcal{A}_\lambda)$. The commutators $\text{ad}_{\mathcal{A}_\lambda}^j (\lambda P)^{1/2}$, $j = 1, 2$, can be extended to bounded operators and we have, uniformly in $\lambda \geq 1$,*

$$\begin{aligned} \|\llbracket \mathcal{A}_\lambda, (\lambda P)^{1/2} \rrbracket\| &\lesssim 1, \\ \|\text{ad}_{\mathcal{A}_\lambda}^2 (\lambda P)^{1/2}\| &\lesssim \begin{cases} 1 & \rho > 1, \\ \lambda^\delta & \rho \leq 1, \end{cases} \end{aligned}$$

where $\delta > 0$ can be chosen arbitrary small.

ii) For λ large enough, we have the following Mourre estimate:

$$\mathbb{1}_I(\lambda P)[i(\lambda P)^{1/2}, \mathcal{A}_\lambda]\mathbb{1}_I(\lambda P) \geq \frac{\sqrt{\inf I}}{2}\mathbb{1}_I(\lambda P).$$

iii) For $0 \leq \mu \leq 1$ and $\psi \in C_0^\infty(\]0, +\infty[)$, we have

$$\|\langle \mathcal{A}_\lambda \rangle^\mu \psi(\lambda P) \langle x \rangle^{-\mu}\| \lesssim \lambda^{-\mu/2+\delta},$$

for all $\delta > 0$.

We will also need [2, Lemma B.12]:

Lemma 4. *Let $\chi \in C_0^\infty(\mathbb{R})$ and $\beta, \gamma \geq 0$ with $\gamma + \beta/2 \leq d/4$. Then, for all $\delta > 0$, we have*

$$\|\langle x \rangle^\beta \chi(\lambda P)u\| \lesssim \lambda^{-\gamma+\delta} \|\langle x \rangle^{\beta+2\gamma}u\|$$

uniformly in $\lambda \geq 1$.

By Mourre theory (see Theorem 2.2 and Remark 2.3 of [2] for example) and Proposition 3, we obtain the following limiting absorption principle:

$$(5) \quad \sup_{\text{Re } z \in I, \text{Im } z \neq 0} \|\langle \mathcal{A}_\lambda \rangle^{-1/2-\varepsilon} ((\lambda P)^{1/2} - z)^{-1} \langle \mathcal{A}_\lambda \rangle^{-1/2-\varepsilon}\| \lesssim \lambda^\delta,$$

for all $\varepsilon, \delta > 0$. This entails the following

Lemma 5. *For $\Psi \in C_0^\infty(\]0, +\infty[)$ and $\varepsilon > 0$, we have*

$$(6) \quad \|\langle x \rangle^{-1/2-\varepsilon} \Psi(\lambda P) (\sqrt{P} - \lambda^{-1/2}z)^{-1} \langle x \rangle^{-1/2-\varepsilon}\| \lesssim 1,$$

$$(7) \quad \|\langle x \rangle^{-1/2-\varepsilon} \Psi(\lambda P) (P - \lambda^{-1}z^2)^{-1} \langle x \rangle^{-1/2-\varepsilon}\| \lesssim \frac{\sqrt{\lambda}}{|z|},$$

uniformly in $\lambda \geq 1$ and $z \in \mathbb{C} \setminus \mathbb{R}$ with $\text{Re } z \in I$.

Proof. Let $\tilde{\Psi} \in C_0^\infty(]0, +\infty[)$ be such that $\Psi\tilde{\Psi} = \Psi$.

To prove the first identity, we write

$$\begin{aligned} & \left\| \langle x \rangle^{-1/2-\varepsilon} \Psi(\lambda P) (P^{1/2} - \lambda^{-1/2} z)^{-1} \langle x \rangle^{-1/2-\varepsilon} \right\| \\ & \lesssim \left\| \langle x \rangle^{-1/2-\varepsilon} \Psi(\lambda P) \langle \mathcal{A}_\lambda \rangle^{1/2+\varepsilon} \right\| \left\| \langle \mathcal{A}_\lambda \rangle^{-1/2-\varepsilon} (P^{1/2} - \lambda^{-1/2} z)^{-1} \langle \mathcal{A}_\lambda \rangle^{-1/2-\varepsilon} \right\| \\ & \quad \times \left\| \langle \mathcal{A}_\lambda \rangle^{1/2+\varepsilon} \tilde{\Psi}(\lambda P) \langle x \rangle^{-1/2-\varepsilon} \right\| \\ & \lesssim \lambda^{-\frac{1}{4}-\frac{\varepsilon}{2}+\delta} \lambda^{\frac{1}{2}+\delta} \lambda^{-\frac{1}{4}-\frac{\varepsilon}{2}+\delta} \lesssim 1. \end{aligned}$$

Here we have used Proposition 3 *iii*), Lemma 4 as well as the fact that δ can be chosen arbitrary small.

To obtain (7), it is sufficient to write

$$\begin{aligned} & \left\| \langle x \rangle^{-1/2-\varepsilon} \Psi(\lambda P) (P - \lambda^{-1} z^2)^{-1} \langle x \rangle^{-1/2-\varepsilon} \right\| \\ & \lesssim \lambda^{1/2} \left\| \langle x \rangle^{-1/2-\varepsilon} \Psi(\lambda P) ((\lambda P)^{1/2} + z)^{-1} \langle x \rangle^{1/2+\varepsilon/2} \right\| \\ & \quad \times \left\| \langle x \rangle^{-1/2-\varepsilon/2} \tilde{\Psi}(\lambda P) (P^{1/2} - \lambda^{-1/2} z)^{-1} \langle x \rangle^{-1/2-\varepsilon} \right\| \\ & \lesssim \frac{\lambda^{1/2}}{|z|}. \end{aligned}$$

Here we have used (6) and Lemma 4. It is clear from the proof of Lemma 4 in [2] that we can apply it to $\Psi(\lambda P) ((\lambda P)^{1/2} + z)^{-1}$ and that we gain $\frac{1}{|z|}$. Indeed, as an almost analytic extension, we can just take the almost analytic extension of Ψ multiplied by the analytic function $\frac{1}{\sqrt{x+z}}$. \square

Proof of Theorem 1. We only show the third part of the theorem, the proof of the other parts is analogous. Also it is clearly sufficient to replace z by $\lambda^{-1}\tilde{z}^2$ with $\operatorname{Re} \tilde{z} = 1 \in I$ and $\lambda \geq 1$ (for instance, $\lambda = (\operatorname{Re} \sqrt{z})^{-2}$ and $\tilde{z} = \sqrt{z}/(\operatorname{Re} \sqrt{z})$). Let $\varphi, \tilde{\varphi} \in C_0^\infty([\frac{1}{3}, 3])$ and $f \in C^\infty(\mathbb{R})$ be such that $\tilde{\varphi} = 1$ on the support of φ , $f(x) = 0$ for $x < 2$ and

$$f(x) + \sum_{\mu=2^n, n \geq 0} \varphi(\mu x) = 1,$$

for all $x > 0$. Since 0 is not an eigenvalue of P , we can write

$$\langle x \rangle^{-\alpha} (P - z)^{-1} \langle x \rangle^{-\beta} = \langle x \rangle^{-\alpha} f(P) (P - z)^{-1} \langle x \rangle^{-\beta} + \sum_{\mu=2^n, n \geq 0} \langle x \rangle^{-\alpha} \varphi(\mu P) (P - \lambda^{-1}\tilde{z}^2)^{-1} \langle x \rangle^{-\beta}.$$

Of course, since $|z| < 1$, the functional calculus gives

$$\left\| \langle x \rangle^{-\alpha} f(P) (P - z)^{-1} \langle x \rangle^{-\beta} \right\| \lesssim 1.$$

Let $\tilde{\alpha} = \min(\alpha, \frac{d}{2})$ and $\tilde{\beta} = \min(\beta, \frac{d}{2})$. Note that $\tilde{\alpha} + \tilde{\beta} > 2$ since $d \geq 3$. Let $\Psi \in C_0^\infty(]0, +\infty[)$ be such that $\Psi = 1$ near $[\frac{1}{12}, 12]$. Then, for $\frac{\mu}{4} \leq \lambda \leq 4\mu$, we have

$$\begin{aligned} & \left\| \langle x \rangle^{-\alpha} \varphi(\mu P) (P - \lambda^{-1}\tilde{z}^2)^{-1} \langle x \rangle^{-\beta} \right\| \\ & \lesssim \left\| \langle x \rangle^{-\alpha} \varphi(\mu P) \langle x \rangle^{1/2+\varepsilon} \right\| \left\| \langle x \rangle^{-1/2-\varepsilon} \Psi(\lambda P) (P - \lambda^{-1}\tilde{z}^2)^{-1} \langle x \rangle^{-1/2-\varepsilon} \right\| \left\| \langle x \rangle^{1/2+\varepsilon} \tilde{\varphi}(\mu P) \langle x \rangle^{-\beta} \right\| \\ & \lesssim \lambda^{\frac{1}{4}+\frac{\varepsilon}{2}-\frac{\tilde{\alpha}}{2}+\delta} \lambda^{\frac{1}{2}} |\tilde{z}|^{-1} \lambda^{\frac{1}{4}+\frac{\varepsilon}{2}-\frac{\tilde{\beta}}{2}+\delta} \lesssim \lambda^{1+\varepsilon+2\delta-\frac{\tilde{\alpha}+\tilde{\beta}}{2}} \lesssim 1, \end{aligned}$$

for all $\varepsilon, \delta > 0$ small enough. Here we have used (7) and two times Lemma 4. On the other hand, for $\lambda \notin [\frac{\mu}{4}, 4\mu]$, the functional calculus and Lemma 4 yield

$$\begin{aligned} \|\langle x \rangle^{-\alpha} \varphi(\mu P)(P - \lambda^{-1} \tilde{z}^2)^{-1} \langle x \rangle^{-\beta}\| &\lesssim |\mu^{-1} - \lambda^{-1}|^{-1} \|\langle x \rangle^{-\alpha} \varphi(\mu P)\| \|\tilde{\varphi}(\mu P) \langle x \rangle^{-\beta}\| \\ &\lesssim |\mu^{-1} - \lambda^{-1}|^{-1} \mu^{-\frac{\tilde{\alpha} + \tilde{\beta}}{2} + \varepsilon}, \end{aligned}$$

for all $\varepsilon > 0$. Splitting the sum into two, we get

$$\begin{aligned} \sum_{4\mu < \lambda} |\mu^{-1} - \lambda^{-1}|^{-1} \mu^{-\frac{\tilde{\alpha} + \tilde{\beta}}{2} + \varepsilon} &\lesssim \sum_{4\mu < \lambda} \mu \mu^{-\frac{\tilde{\alpha} + \tilde{\beta}}{2} + \varepsilon} \lesssim 1, \\ \sum_{\mu > 4\lambda} |\mu^{-1} - \lambda^{-1}|^{-1} \mu^{-\frac{\tilde{\alpha} + \tilde{\beta}}{2} + \varepsilon} &\lesssim \sum_{\mu > 4\lambda} \lambda \mu^{-\frac{\tilde{\alpha} + \tilde{\beta}}{2} + \varepsilon} \lesssim 1. \end{aligned}$$

This finishes the proof of the theorem. \square

Proof of Remark 2 i). Let us recall that the kernel of the resolvent of the flat Laplacian in \mathbb{R}^3 at $z = 0$ is given by

$$K(x, y, 0) = \frac{1}{4\pi|x - y|}.$$

Assume that $\langle x \rangle^{-\alpha} (-\Delta)^{-1} \langle x \rangle^{-\beta}$ is bounded on $L^2(\mathbb{R}^3)$. Applying to $\chi \in C_0^\infty(\mathbb{R}^3) \subset L^2(\mathbb{R}^3)$, we find

$$(\langle x \rangle^{-\alpha} (-\Delta)^{-1} \langle x \rangle^{-\beta} \chi)(x) = \int \frac{1}{4\pi|x - y|} \langle x \rangle^{-\alpha} \langle y \rangle^{-\beta} \chi(y) dy \gtrsim \langle x \rangle^{-\alpha-1},$$

for $|x| \gg 1$. But $\langle x \rangle^{-1-\alpha} \in L^2(\mathbb{R}^3)$ if and only if $\alpha > 1/2$. The condition $\beta > 1/2$ is checked in the same way. We now apply the resolvent to $f(x) = \langle x \rangle^{-3/2-\varepsilon} \in L^2(\mathbb{R}^3)$ and find

$$\begin{aligned} (\langle x \rangle^{-\alpha} (-\Delta)^{-1} \langle x \rangle^{-\beta} f)(x) &= \int \frac{1}{4\pi|x - y|} \langle x \rangle^{-\alpha} \langle y \rangle^{-\beta} \langle y \rangle^{-3/2-\varepsilon} dy \\ &\geq \int_{|y| \leq \frac{|x|}{2}} \frac{1}{4\pi|x - y|} \langle x \rangle^{-\alpha} \langle y \rangle^{-\beta} \langle y \rangle^{-3/2-\varepsilon} dy \\ &\gtrsim \langle x \rangle^{-\alpha-1} \int_{|y| \leq \frac{|x|}{2}} \langle y \rangle^{-3/2-\varepsilon-\beta} dy \gtrsim \langle x \rangle^{3/2-\alpha-\beta-1-\varepsilon}. \end{aligned}$$

This leads to the condition $2(3/2 - \alpha - \beta - 1) \leq -3$ which implies $\alpha + \beta \geq 2$. \square

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