Markovian Bridges: Weak continuity and pathwise constructions

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Abstract. A Markovian bridge is a probability measure taken from a disintegration of the law of an initial part of the path of a Markov process given its terminal value. As such, Markovian bridges admit a natural parameterization in terms of the state space of the process. In the context of Feller processes with continuous transition densities, we construct by weak convergence considerations the only versions of Markovian bridges which are weakly continuous with respect to their parameter. We use this weakly continuous construction to provide an extension of the strong Markov property in which the flow of time is reversed. In the context of self-similar Feller process, the last result is shown to be useful in the construction of Markovian bridges out of the trajectories of the original process.

1. Introduction and Main Results

1.1. **Motivation.** The aim of this article is to study Markov processes on $[0, t]$, starting at x, conditioned to arrive at y at time t. Historically, the first example of such a conditional law is given by Paul Lévy's construction of the Brownian bridge: given a Brownian motion B starting at zero, let

$$b_s^{x,y,t} = x + B_s - B_t \frac{s}{t} + (y - x) \frac{s}{t}.$$

Then $b^{x,y,t}$ is a version of B started at x and conditioned on $B_t = y$ in the sense that

$$\mathbb{E}\Big(F\Big((x + B_s)_{s\in[0,t]}\Big) f(x + B_t)\Big) = \int \mathbb{E}\big(F\big(b^{x,y,t}\big)\big) f(y) \, \mathbb{P}(x + B_t \in dy) .$$

This example synthesizes the main considerations of this work: one is able to construct a specific version of the disintegration of the law of $(B_s)_{s\in[0,t]}$ given B_t which is weakly continuous, and one is able to give a pathwise construction of this conditional law out of the trajectories of B. Since there is at most one weakly continuous disintegration, it is natural to look for conditions guaranteeing its existence and to characterize it, which we do in the class of Feller processes. Kallenberg has given in [Kal81] a very general result for the existence of weakly continuous bridges of Lévy processes using the convergence criteria for processes with exchangeable increments of [Kal73]. It is a consequence of our results. The first abstract framework for the existence of particular bridge laws in the context of Markov process is [FPY93]. It is a different framework from the one adopted in this work since they rely on duality considerations while we rely on the Feller property.

P. Lévy's Gaussian construction of the Brownian bridge is of limited applicability in the context of Markov processes. However, he also gave a different pathwise construction of a Brownian bridge with a Markovian flavor: let g be the last zero of B before time 1, which is not zero since B comes back to zero at arbitrarily small times, and set

$$b_t = \frac{1}{\sqrt{g}} B_{g \cdot t}$$

for $t \leq 1$. Then b and $b^{0,0,1}$ have the same law. We will provide further examples of this type of pathwise construction, which in the case of Brownian motion is given as follows. Let $g_c = \sup \big\{t \leq 1 : B_t = c\sqrt{t}\big\}$;

Date: May 14, 2009.
2000 *Mathematics Subject Classification.* 60J25,60J65.
Key words and phrases. Markov bridges, Markov selfsimilar processes.

the positivity of g_c for any $c \in \mathbb{R}$ is an immediate consequence of the asymptotic behavior of the Brownian curve at 0. If

$$b^c_t = \frac{1}{\sqrt{g_c}} B_{g_c \cdot t}$$

for $t \leq 1$, then b^c and $b^{0,c,1}$ have the same law. To compute the law of b^c, we extend the usual strong Markov property: note that $\{g_c > t\} \in \sigma(B_u : u \geq t)$ so that g_c is a kind of backward optional time at which a verison of the strong Markov property holds, the law of $(B_{s\wedge g_c})_{t\geq 0}$ given $\sigma(B_u : u \geq g_c)$ is that of a Brownian bridge from 0 to $c\sqrt{g_c} = B_{g_c}$ of length g_c. Applying Brownian scaling gives the desired result. Looking at our results in the preliminary draft, Marc Yor noticed and pointed out to us the following short proof in the special case of Brownian motion: by time-inversion, $\tilde{B}_t = tB_{1/t}$ is a Brownian motion and

$$g_c = 1/\inf\left\{t \geq 1 : \tilde{B}_t = c\sqrt{t}\right\};$$

denote by T the stopping time appearing in the denominator. By the strong Markov property and scaling, $X_t = \tilde{B}_{T(1+t)}/\sqrt{T} - c$, $t \geq 0$ is another Brownian motion, so that

$$t\left(X_{(1-t)/t} - c\right) = B_{tg_c}/\sqrt{g_c} - tc,\ t \in [0,1]$$

has the same law as $b^{0,0,1}$ and so b^c and $b^{0,c,1}$ have the same law.

Note, however, that our methods will apply to self-similar processes which do not posses the time inversion property. In particular, we will study the case of stable Lévy processes.

1.2. **Statement of the results.** We will work on an arbitrary locally compact metric space with a countable base (or LCCB for short) denoted (S, ρ). On it we will consider the Borel σ-field denoted $\mathscr{B}_S$ and $b\mathscr{B}_S$ will stand for the set of measurable and bounded functions from S to $\mathbb{R}$. We will consider a **Markovian family** of probability measures on this space which satisfy the Feller property, by which the following is meant. Let D_∞ (D_t) stand for the Skorohod space of càdlàg functions from $[0,\infty)$ ($[0,t]$) into S and consider on it the shift operators $\theta_t : D_\infty \to D_\infty$ given by $\theta_t f : s \to f(t+s)$ (they can also be defined on $[0,t']$ if $t' > t$). Let $X = (X_s)_{s\geq 0}$ denote the canonical process, and write $\mathscr{F}$ and $(\mathscr{F}_s)_{s\geq 0}$ for the σ-field and the canonical filtration generated by X.

Definition. A **Markovian family** on (S,ρ) is a collection of probability measures $(\mathbb{P}_x)_{x\in S}$ on D_∞ indexed by the elements of S which satisfies

Starting Point Property: For all $x \in S$,

$$\mathbb{P}_x(X_0 = x) = 1.$$

Measurability Property: For all $F \in b\mathscr{F}$,

$$x \mapsto \mathbb{E}_x(F)$$

is measurable.

Markov Property: For every $F \in b\mathscr{F}_s$ and every $G \in b\mathscr{F}$,

$$\mathbb{E}_x(F \cdot G \circ \theta_s) = \mathbb{E}_x(F \cdot \mathbb{E}_{X_s}(G)).$$

A Markovian family $(\mathbb{P}_x)_{x\in S}$ is said to satisfy the Feller property (and we will therefore speak of a **Feller family**) if the operators $(P_s)_{s\geq 0}$ defined on $b\mathscr{B}_S$ by means of $P_s f(x) = \mathbb{E}_x(f(X_s))$ are an extension of a Fellerian semigroup.

Of course, Feller families are in bijection with (conservative) Feller semigroups. In this case, we even have the strong Markov property at every stopping time T: for every $F \in b\mathscr{F}_T$ and every $G \in b\mathscr{F}$,

$$\mathbb{E}_x(F \cdot G \circ \theta_T) = \mathbb{E}_x(F \cdot \mathbb{E}_{X_T}(G)).$$

We seek to build a version of the conditional law of $(X_s)_{s\leq t}$ given $X_t = y$ under $\mathbb{P}_x$, which we would call Markovian bridge from x to y of length t. One could appeal to the general theorem on existence of regular conditional distributions (see for example [Kal02, Thm. 6.3, p.107]), but that result builds the whole family of conditional laws as y varies and does not give control over individual conditional laws. Since we are working on a Polish space, we might impose further regularity conditions on conditional laws such as their weak continuity as y varies; since there is at most one weakly continuous disintegration with respect to the extremal values, this singles out specific conditional laws. This is the strategy we will follow. To that end consider a Feller family $(\mathbb{P}_x)_{x\in S}$ on (S,ρ) and its associated semigroup $P=(P_s)_{s\geq 0}$ and suppose that P_s admits a transition density $p_s(\cdot,\cdot)$ with respect to a σ-finite measure μ on (S,ρ) in the sense that

$$P_s f(x)=\int f(y)\, p_s(x,y)\ \mu(dy)\,.$$

Fix $x\in S$ and set $\mathscr{P}_t=\{y : p_t(x,y)>0\}$. Under the hypotheses

(H1): $y\mapsto p_s(x,y)$ is continuous for all $s\in(0,t]$,

(H2): The Chapman-Kolmogorov equations

$$p_t(x,y)=\int p_{t-s}(x,z)\, p_s(z,y)\ \mu(dz)$$

hold for each $y\in\mathscr{P}_t$, and for $0<s<t$, and

(H3): $s\mapsto p_s(x,y)$ is continuous for all $x,y\in S$,

which are more clearly explained in Section 3, we prove our basic existence result.

Theorem 1. *For every $y\in S$ such that $p_t(x,y)>0$, the laws $\mathbb{P}_x(\cdot\mid X_t\in B_\delta(y))$ converge weakly as $\delta\to 0$ to a law $\mathbb{P}^t_{x,y}$ such that:*

(1) $y\mapsto\mathbb{P}^t_{x,y}$ is weakly continuous, and

(2) for every $f\in b\mathscr{B}_S$ and $F\in b\mathscr{F}_t$,

$$\mathbb{E}_x(F\cdot f(X_t))=\int_{\{y:p_t(x,y)>0\}}\mathbb{E}^t_{x,y}(F)\, p_t(x,y)\ \mu(dy)\,.$$

Note. Given $x\in S$, $t>0$, $s\in(0,t)$ and y such that $p_t(x,y)>0$, the Chapman-Kolmogorov equations of **H2** hold as consquence of the continuity assumption **H1** if additionally $p_{t-s}(\cdot,y)$ is bounded.

Special cases of Theorem 1 are found in the literature: Kallenberg proves the weak continuity and the approximation for a subclass of Lévy processes in [Kal81], the special case of stable Lévy processes (when the starting and ending points are zero) is obtained by Bertoin in [Ber96, VIII.3, Proposition 11] by scaling arguments and using excursion theory by Chaumont in [Cha94, Cha97].

By the same method of proof, we can study joint weak continuity in the starting and ending point and the length. However, since bridge laws associated to different lengths are defined on different Skorohod spaces, we need to specify the interpretation of weak continuity we will use. For every $f\in D_t$, we can associate the function $f^t\in D_\infty$ given by $f^t(s)=f(s\wedge t)$. This mesurable mapping will be denoted by i_t and we will say that the sequence of measures $\mathbb{P}^{t_n}_n$ on D_{t_n} converge weakly if $\mathbb{P}^{t_n}_n\circ i_{t_n}^{-1}$ converges weakly in D_∞. To simplify notations, from this point on, we will think of bridge measures as defined on D_∞ by identifying $\mathbb{P}^t_{x,y}$ with $\mathbb{P}^t_{x,y}\circ i_t^{-1}$. Kallenberg used in [Kal81] another notion of weak continuity with respect to the temporal parameter; it differs only when one considers lengths that go to infinity.

A technical hypothesis, related to the joint weak continuity of bridge laws with respect to the ending point and the length, is the following:

(H1'): $(s,y)\mapsto p_s(x,y)$ is continuous for all $x\in S$.

Another one, related to weak continuity with respect to all variables is

(H1"): $(s,y,x)\mapsto p_s(x,y)$ is continuous.

We have:

Corollary 1. *Under* $\mathbf{H1}'$ *and* $\mathbf{H2}$*: the bridge laws $\left(\mathbb{P}^t_{x,y}\right)$ are jointly continuous in y and t. Under* $\mathbf{H1}''$ *and* $\mathbf{H2}$*, the bridge laws are weakly continuous with respect to x,y and t.*

Now we will analyze a generalization of the usual Strong Markov Property for Feller processes in which Markovian bridges play a prominent role.

Let us define, for a fixed time t, the σ-fields associated to the past before time t, $\mathscr{F}_t$, and to the future after time t, $\mathscr{F}^t=\sigma(X_s : s\geq t)$ and place ourselves under **H1-H3**; thanks to the Markov property, we obtain:

The conditional law of $X^{s,t} = \left(X_{(r+s)\wedge t}\right)_{r\geq 0}$ given X_s, X_t under $\mathbb{P}_x$ is $\mathbb{P}^{t-s}_{X_s,X_t}$.

We shall generalize the preceding conditional description to a *strong Markov property with respect to future events*. Actually the method of proof will be analogous to a known one for the Strong Markov Property: we will discretize the problem, then we shall use the local property of conditional expectation (to be stated shortly), and finally, continuity considerations will be used to transport conclusions of the discrete setup to the continuous one. The target result needs the following:

Definition. A **backward optional time** is a random variable $L : D_\infty \to [0,\infty]$ such that $\{L > t\} \in \mathscr{F}^t$ for all $t > 0$.

For a backward optional time L, the **σ-field of events occurring after** L, denoted $\mathscr{F}^L$, is defined to be $\sigma(X \circ \theta_L)$.

As a first example, let us note that if $U \subset S$ is open, then the **last visit to** U equal to zero if X is never in U and equal to

$$\sup\{s \geq 0 : X_s \in U\}$$

otherwise is a backward optional time. A second example would be the last visit to an open set (just) before a fixed time t given by

$$L^t_U = \begin{cases} 0 & \text{if } \{s < t : X_s \in U\} = \emptyset \\ \sup\{s < t : X_s \in U\} & \text{otherwise} \end{cases}.$$

The first example belongs to the following class of random times, which are all backward optional times:

Definition. A **cooptional time** is a random variable $L : D_\infty \to [0,\infty]$ such that $L \circ \theta_t = (L - t)^+$.

Cooptional times are backward optional times since, by definition they are random variables, and then

$$\{L > t\} = \left\{(L - t)^+ > 0\right\} = \theta_t^{-1}(\{L > 0\}) \in \mathscr{F}^t.$$

However, the last visit to an open set before a fixed time is an example of a backward optional time which is not cooptional.

Backward optional times are the key to opening random temporal windows in the Markov property. However, to provide a statement closer to the usual expression of the Strong Markov property, we will use the **shift and stop operators** $\sigma^s_t : D_\infty \to D_\infty$ given by

$$\sigma^s_t f(r) = \begin{cases} f(r+s) & \text{if } r+s < t \\ f(t-) & \text{if } r+s \geq t \end{cases}.$$

Since these operators were defined in terms of $f(t-)$ instead of $f(t)$, they are continuous on D_∞ (or on $D_{t'}$ if $t' > t$).

To make sense of the following result, let us recall that we have identified bridge laws on D_t with their image on D_∞ under the embedding $i_t : (f(s))_{s\in[0,t]} \mapsto (f(s \wedge t))_{s\geq 0}$.

Theorem 2 (The Backward Strong Markov Property)**.** *Under* **H1'** *and* **H2***, $(t, x, y) \mapsto \mathbb{E}^t_{x,y}(F)$ is measurable for any measurable $F : D_\infty \to \mathbb{R}$. Let S and L be a stopping and a backward time respectively. Then for any initial distribution ν on S and any $F \in b\mathscr{F}$,*

$$\mathbb{E}_\nu\left(F \circ \sigma^S_L \,\middle|\, \mathscr{F}_S, \mathscr{F}^L, X_{L-}\right) = \mathbb{E}^{L-S}_{X_S,X_{L-}}(F)$$

almost surely on $\{S < L < \infty\}$.

The last theorem simply says that the process between a stopping time S and a backward optional time L is a Markov bridge of random length $L - S$ between its starting point X_S and its ending point X_{L-}. It also implies that σ^S_L and $\mathscr{F}_S \wedge \mathscr{F}^L$ are conditionally independent given X_S, X_{L-} and $L - S$. This result was stated by Kallenberg for Lévy processes in [Kal81] and, in a different framework, by Fitzsimmons, Pitman and Yor in [FPY93]. Our point of view is that, as for the usual Strong Markov Property for Feller processes, it is trivially true in discrete time and that to pass to continuous time, continuity considerations are useful. We can find many examples of generalizations of the strong

Markov property to random times; such generalizations consist of two parts: a statement of conditional independence of past and future with respect to the present at a given random time τ , and a description of the conditional law of the pre-τ and post-τ parts of the process given some notion of the present, which can be the σ-field generated by τ and X_τ , or only X_τ , or even more exotic ones. See for example [JP77, GS79, GS81] for examples of conditional independence (and several notions of present) and [MSW72, Jac74, Mil77] for examples where the post-τ process is also analyzed.

We now turn to a pathwise construction of bridges of self-similar Feller processes. We will focus on the state space $S = [0, \infty)$ or $\mathbb{R}$, which contains 0.

Definition. The **scaling operators** $S^\gamma_v : D_\infty \to D_\infty$ are defined by

$$S^\gamma_v f(t) = v^{1/\gamma} f(t/v) .$$

A Feller family $(\mathbb{P}_x)_{x\in S}$ is said to be **self-similar of index** γ if for every $x \in S$ and every $v > 0$, the image of $\mathbb{P}_x$ under the scaling operator S^γ_v is $\mathbb{P}_{v^{1/\gamma}x}$.

We now give a pathwise construction of bridge laws associated to a self-similar Feller family $(\mathbb{P}_x)_{x\in S}$ of index γ, the bridges going from 0 to any element of S and of any length. Suppose that $(\mathbb{P}_x)_{x\in S}$ satisfies **H1'** and **H2**; explicit examples will be given in Section 5. The hypotheses ensure the applicability of Theorems 1 and 2. Also, note that the image of $\mathbb{P}^t_{x,y}$ under the scaling operator S^γ_v is $\mathbb{P}^{tv}_{v^{1/\gamma}x, v^{1/\gamma}y}$; this can be verified by the approximation of bridge laws (Theorem 1) and the self-similarity property of $(\mathbb{P}_x)_{x\in S}$ using the continuity of the scaling operators on Skorohod space.

For $c \in S$, define the random set

$$\mathscr{Z}_c = \left\{ t \leq 1 : X_{t-} = ct^{1/\gamma} \right\}$$

as well as the random time $g_c : D_\infty \to [0, \infty)$

$$g_c = \begin{cases} 0 & \text{if } \mathscr{Z}_c = \emptyset \\ \sup \mathscr{Z}_c & \text{otherwise} \end{cases} .$$

Note that by Blumenthal's 0-1 law, the set $\{g_c > 0\}$ is trivial.

Theorem 3. *If $\mathbb{P}_0(g_c > 0) = 1$, the law of $(Y_t)_{t\in[0,1]}$ given by*

$$Y_t = \begin{cases} \frac{1}{g_c^{1/\gamma}} X_{s\cdot g_c} & \text{if } t < 1 \\ c & \text{if } t \geq 1 \end{cases}$$

under $\mathbb{P}_0$ is $\mathbb{P}^1_{0,c}$.

Note that by the scaling relationship of bridge laws, we get the following Corollary under the hypotheses of Theorem 3: let $t > 0$ and $x \in S$ be given and define $c = xt^{-1/\gamma}$, then

$$\text{the law of } (Y^x_s)_{s\in[0,t]} \text{ given by } Y^x_s = \begin{cases} \frac{t^{1/\gamma}}{g_c^\gamma} X_{s\cdot(g_c)/t} & \text{if } s < t \\ x & \text{if } s \geq t \end{cases} \text{ under } \mathbb{P}_0 \text{ is } \mathbb{P}^t_{0,x}.$$

It is therefore important to provide examples where $\mathbb{P}_0(g_c > 0) = 1$; the reader should be warned by the following one: if $(\mathbb{P}_x)_{x\in\mathbb{R}}$ is the Feller family associated to a stable Lévy process of index $\alpha \in (0, 1)$ which has jumps of both signs, then $\mathbb{P}_0(g_0 > 0) = 0$, because points are polar for them (cf. [Ber96, II.5]), while $\mathbb{P}_0(g_c > 0) = 1$ for all $c \neq 0$, as will be proved in Section 5. For symmetric stable Lévy processes of index $\alpha \in (0, 1)$, this had been proved in [Jak08, Corollary 14].

When $\mathbb{P}_x$ is the law of linear Brownian motion started at x, we have computed the moments of $1 - g_c$ in order to compare its law with the Beta type (recall that g_0 has a Beta law thanks to P. Lévy's first arcsine law, cf. Exercise III.3.20 of [RY99]). For this, we define the function

$$H_q(x) = \int_0^\infty e^{-xz - z^2/2} z^{q-1} \, dz.$$

This function can be expressed in terms of the Hermite functions of negative index, see [Leb65, Section 10.2-5], especially formula (10.5.2).

Proposition 1. *We have*

$$\mathbb{E}((1-g_c)^q) = \frac{\Gamma(2q)}{2qH_{2q}(c)\,H_{2q}(-c)} \quad \text{and} \quad \mathbb{E}((1-g_c)^q) \sim \frac{e^{-c^2/2}}{\sqrt{\pi q}}$$

as $q \to \infty$.

Note that the asymptotic behaviour of the moments of $1-g_c$ is only compatible with a Beta law whose second parameter is $1/2$. The explicit computation $H_q(0) = 2^{q/2-1}\Gamma(q/2)$ reproduces Paul Lévy's arcsine law with help of the duplication formula for the Γ function.

Our next application of Theorems 1 and 2 is related to stable subordinators and is obtained by a Doob transformation. Let $\mathbb{P}_x^\alpha$ the law of a stable subordinator of index $\alpha \in (0,1)$ starting at x. As Subsection 2.1 shows us, $(\mathbb{P}_x^\alpha)_{x\geq 0}$ is a self-similar Feller family for which hypotheses **H1'** and **H3** hold (taking μ equal to Lebesgue measure). The transition density p_t^α can be expressed in terms of the density f_t^α of X_t under $\mathbb{P}_0^\alpha$ as follows:

$$p_t^\alpha(x,y) = f_t^\alpha(y-x)\,.$$

It is possible to compute the potential density u^α given by

$$u_\alpha(a) = \int_0^\infty f_t^\alpha(a)\;dt = \frac{1}{C\Gamma(\alpha)\,a^{1-\alpha}}\mathbf{1}_{a>0}.$$

as shown in [Sat99, Example 37.19, p. 261].

For any $0<b$ we define

$$h_\alpha : [0,\infty) \to [0,\infty) = \begin{cases} u_\alpha(b-a) & \text{if } a \leq b \\ 0 & \text{otherwise} \end{cases}.$$

With it, we will consider the Doob h_α-transform of $\mathbb{P}_a^\alpha$, denoted $\mathbb{P}_a^{h_\alpha}$; it is a measure on the Skorohod space D_∞ on $[0,\infty)\cup\{\Delta\}$ (Δ is an additional isolated point called the cemetery) concentrated on trajectories with values on $[0,b)\cup\{\Delta\}$. It is the (only) probability measure such that for all $A \in \mathscr{F}_s$

$$\mathbb{P}_a^{h_\alpha}(A\cap\{s<\zeta\}) = \mathbb{E}_a^\alpha\left(\frac{h_\alpha(X_s)}{h_\alpha(x)}\mathbf{1}_A\right).$$

The family $\mathbb{P}_a^{h_\alpha}$, $a \in [0,b)\cup\{\Delta\}$ is Markovian and is associated to the Markov process termed the **stable subordinator conditioned to die at** b. The terminology is justified since Δ is absorbing and if the death-time ζ is defined as $\inf\{t : X_t = \Delta\}$, then $X_{\zeta-} = b$ $\mathbb{P}_a$-almost surely for every $a<b$ (cf. [Cha96]). Our next result is a pathwise construction of the conditioned stable subordinator in terms of the subordinator itself: let $L = \sup\{t\geq 0 : X_t < b\}$ (which is finite under $\mathbb{P}_0^\alpha$), $g = X_{L-}$ and define $Y = (Y_t)_{t\geq 0}$ as follows,

$$Y_t = \begin{cases} \frac{b}{g}X_{t(g/b)^\alpha} & \text{if } t\,(g/b)^\alpha < L \\ \Delta & \text{otherwise} \end{cases}.$$

Theorem 4. *The law of* Y *under* $\mathbb{P}_0^\alpha$ *is* $\mathbb{P}_0^{h_\alpha}$.

The paper is organized as follows: we give examples weakly continuous Markov bridges in Section 2, we then prove Theorem 1 and Corollary 1 regarding construction and weak continuity of Markovian bridges in Section 3, passing to the Backward Strong Markov Property (Theorem 2) in Section 4. The pathwise constructions of Markovian bridges for self-similar processes of Theorem 3 as well as the additional computations for the Brownian case of Proposition 1 are given in Section 5, which contains also the construction of the conditioned stable process of Theorem 4.

2. Examples of weakly continuous Markovian bridges

In this section we will meet some examples of Feller processes for which bridges can be built using Theorem 1; the emphasis is on showing how one can prove that the hypotheses enabling to use it.

We will start with a description of the probabilistic objects to consider: Brownian motion and other Lévy processes, and Bessel processes. With this, we will introduce the associated bridges. At some points, we will need facts concerning Lévy processes and Bessel processes. Although more precise information will follow, our main references will be [Ber96], [Sat99], and [RY99]. In the examples that follow, the LCCB space (S,ρ) will be either $\mathbb{R}$, $\mathbb{R}^n$ or $\mathbb{R}_+ = [0,\infty)$, endowed with the usual metrics.

2.1. **Bridges of Lévy processes.** We will now construct bridges of Lévy process and reproduce, from the point of view of our theory, the weakly continuous construction of Lévy bridges of [Kal81]; the unproved facts can be consulted in [Ber96] or [Sat99].

Consider a Lévy process ξ (that is a càdlàg process starting at zero with stationary and independent increments) and denote its law by $\mathbb{P}$. ξ is characterized by its **characteristic exponent** Ψ which satisfies

$$\mathbb{E}\left(e^{iu\xi_t}\right) = e^{-t\Psi(u)}.$$

If the trajectories of ξ are increasing, so that it is a subordinator, one can instead use its **Laplace exponent** Φ given by

$$\mathbb{E}\left(e^{-q\xi_t}\right) = e^{-t\Phi(q)}.$$

If $\mathbb{P}^\Psi_x$ denotes the law of $\xi + x$, then $\left(\mathbb{P}^\Psi_x\right)_{x\in\mathbb{R}}$ is a Feller Markov family; in the case of subordinators, $\left(\mathbb{P}^\Psi_x\right)_{x\geq 0}$ is also Feller. Suppose now that Ψ is such that $\exp(-t\Psi)$ is integrable for any $t > 0$ (this corresponds to hypothesis (C) in [Kal81]); by Fourier inversion, one can prove that the law of ξ_t is absolutely continuous and admits a jointly continuous density f^Ψ_t bounded on $[t,\infty)\times\mathbb{R}$ (the second factor is $\mathbb{R}_+$ in the subordinator case) for each $t > 0$. By independence and homogeneity of the increments, the transition density p^Ψ_t for X_t under $\mathbb{P}_x$ can be taken equal to $f^\Psi_t(\cdot - x)$, which implies the validity of hypotheses **H1'** and **H2**, where the latter holds by the bounded character of the density.

In [Sha69], it is proven that f^Ψ_t is positive on the interior of the support of the law of ξ_t, which is of the form (dt,∞) for all $t > 0$ or $(-\infty, dt)$ for all $t > 0$, where $d \in [-\infty,\infty]$; $|d| = \infty$ if the absolute value of the Lévy process is not a subordinator and it is finite otherwise.

In particular, we can apply the preceding reasoning to **stable Lévy processes of index** $\alpha \in (0,2]$ since the characteristic exponent satisfies

$$\left|e^{-t\Psi(u)}\right| = e^{-tC|u|^\alpha}$$

for some $C > 0$. Stable Lévy processes are the only Lévy processes whose Markovian family is self-similar. This includes Brownian motion, whose characteristic exponent is $\Psi(u) = u^2/2$ and the corresponding transition density is given explicitly as

$$p_t(x,y) = \frac{1}{\sqrt{2\pi t}} e^{-(y-x)^2/2t}$$

We remark that Lévy's representation gives a simpler way of deducing the existence and weak continuity of Brownian bridges both in the one-dimensional and multi-dimensional cases.

2.2. **Bridges of Bessel processes.** The next family of processes we shall consider is that of Bessel processes of dimension $\delta \in [0,\infty)$. When $\delta \in \mathbb{Z}_+$, the law of the **Bessel process** of dimension $\delta \in \mathbb{Z}_+$ starting at x, denoted $\mathbb{P}^\delta_x$, is the law of $\|\vec{x} + \vec{B}\|$ where B is a δ-dimensional Brownian motion for any vector $\vec{x}$ such that $\|\vec{x}\| = x$. In [RY99, VI.3.1, p.251] it is argued that $\left(\mathbb{P}^\delta_x\right)_{x\in[0,\infty)}$ is a Markovian family; its Feller property is immediate from that of Brownian motion. The case $\delta \notin \mathbb{Z}_+$ is handled via stochastic differential equations in [RY99, XI.1]; its law will be denoted by $\mathbb{P}^\delta_x$ and it constitutes, a Feller family on $[0,\infty)$ whose transition density with respect to Lebesgue measure, which is expressed in a simpler fashion in terms of the **index** $\nu = \delta/2 - 1$ associated to the dimension $\delta > 0$ and the **modified Bessel function of the first kind**, denoted I_ν, given by

$$I_\nu(x) = \sum_{k=0}^{\infty} \left(\frac{x}{2}\right)^{\nu+2k} \frac{1}{k!\Gamma(1+\nu+k)}. \tag{1}$$

The transition density is given by

$$p^\delta_t(x,y) = \frac{1}{t}\left(\frac{y}{x}\right)^\nu y e^{-\left(x^2+y^2\right)/2t} I_\nu\left(\frac{xy}{t}\right)$$

for $x > 0$ and $t > 0$. For $x = 0$, we have the expression

$$p_t^\delta(0, y) = \frac{y^{2\nu+1}}{2^\nu t^{\nu+1}\Gamma(\nu+1)} e^{-y^2/2t}.$$

This transition density satisfies hypotheses **H1′** and **H2**. This is because we can bound the transition density using the asymptotic equality

$$I_\nu(x) \sim \frac{1}{\sqrt{2\pi x}} e^x$$

valid as $x \to \infty$ (cf. [Leb65, 5.11.10, p. 123]), which implies

$$\sup_{x \in \mathbb{R}_+, y \leq M} p_t^\delta(x, y) < \infty$$

for any $M > 0$. We can therefore construct Bessel bridges from x to y for any $x \geq 0$ and $y > 0$. It is possible to consider $y = 0$ for a bridge law if instead of using Lebesgue measure λ, we use the σ-finite measure with density $y \mapsto y^{2\nu+1}$ with respect to Lebesgue measure, which would imply the fact that the transition density of $\left\{\mathbb{P}_x^\delta : x \in [0, \infty)\right\}$ with respect to it assigns a positive value to 0 starting from any $x \in [0, \infty)$, and satisfies hypotheses **H1'** and **H2**.

2.3. **Bridges of Bessel processes with drift.** Bessel processes are particular instances of Bessel processes in the wide sense, introduced in [Wat75], which will provide the next example of stochastic processes for which one can build bridges by weak continuity. Let $\delta > 0$, $c \geq 0$ and consider $\nu = \delta/2 - 1$ and

$$\rho_c(x) = 2^\nu \Gamma(1+\nu) \left(\sqrt{2cx}\right)^{-\nu} I_\nu\left(\sqrt{2cx}\right)$$

where I_ν is the modified Bessel function of the first kind given in (1). A **Bessel process in the wide sense** with index (δ, c) is a diffusion process on $[0, \infty)$ determined by the local generator

$$L^{\delta,c} = \frac{1}{2}\frac{\partial}{\partial x^2} + \left(\frac{\delta - 1}{2x} + \frac{\rho_c'(x)}{\rho_c(x)}\right)\frac{\partial}{\partial x};$$

the point 0 is a reflecting boundary when $0 < \alpha < 2$ and an entrance boundary for $\alpha \geq 2$. When $c = 0$, this is just an ordinary Bessel process. Their law starting at x will be denoted $\mathbb{P}_x^{\delta,c}$. Bessel processes in the wide sense can also be interpreted as Bessel processes with drift: for integer $\delta \geq 1$, $\mathbb{P}_0^{\delta,c}$ is the law of the modulus of δ-dimensional Brownian motion with a drift vector $\vec{c}$ of length c that starts at zero (cf. [PY81, Remark 5.4.iii, p.319]). The last result is actually proved through a third description of Bessel processes in the wide sense contained in [PY81, Sect. 3 & 4]: the law $\mathbb{P}_x^{\delta,c}$ is locally absolutely continuous with respect to $\mathbb{P}_x^\delta$. To describe this relationship, we introduce $\alpha = \sqrt{2c}$, the hitting-time T_y of y by the canonical process, and the functions

$$\phi_\alpha(x, y) = \mathbb{E}_x^\delta\left(e^{-\alpha T_y}\right) \quad \text{and} \quad \phi_{\alpha\uparrow}(y) = \begin{cases} \phi_\alpha(x_0, y) & y \leq x_0 \\ 1/\phi_\alpha(x_0, y) & y > x_0 \end{cases}$$

where x_0 is any element of $(0, \infty)$; the choice of x_0 affects the definition of $\phi_{\alpha\uparrow}$ by a constant factor, as can be seen from [IM74]. For any t, the restriction of $\mathbb{P}_x^{\delta,c}$ to $\mathscr{F}_t$ is absolutely continuous with respect to the restriction of $\mathbb{P}_x^\delta$ to $\mathscr{F}_t$ and the Radon-Nikodým derivative is given by

$$\frac{d\mathbb{P}_x^{\delta,c}|_{\mathscr{F}_t}}{d\mathbb{P}_x^\delta|_{\mathscr{F}_t}} = e^{-\alpha t}\frac{\phi_{\alpha\uparrow}(X_t)}{\phi_{\alpha\uparrow}(x)}.$$

From the form of the Radon-Nikodým derivative, we see that the finite-dimensional distributions of the bridges of $\mathbb{P}_x^{\delta,c}$ do not depend on c, they are just Bessel bridges. Therefore, we get not only the existence of bridge laws but also their weak continuity with respect to the parameters involved, because this is the case for $c = 0$.

To end this subsection, let us mention a pathwise construction of Bessel bridges from the trajectories of Bessel processes, contained in Theorem (5.8) of [PY81, p. 324]. It states that the law of the bridge of a Bessel process (with or without drift) from x to y of length t can be obtained as:

- the law of $\left(uX_{1/u-1/t}\right)_{u\in[0,t]}$ under $\mathbb{P}_{y/t}^{\delta,\sqrt{2x}}$ or as

- the law of $\left(\left(\frac{t-u}{t}\right) X_{tu/(t-u)}\right)_{u\in[0,t]}$ under $\mathbb{P}_x^{\delta,\sqrt{y/t}}$.

3. Construction and weak continuity of Markovian bridges

In this section we will prove Theorem 1 and Corollary 1. First we discuss the heuristic of our proof, which borrows heavily from the construction of Markovian bridge laws of [FPY93]. Recall that we are working with a Feller family $(\mathbb{P}_x)_{x\in S}$ on a LCCB space which admits a transition density $p_s(x,y)$ with respect to a σ-finite measure μ.

3.1. **Heuristics.** Let $0 < s < t$ and note that for every $F \in b\mathscr{F}_s$ and every $f \in b\mathscr{B}_S$ the Markov property and the Tonelli-Fubini theorem imply

$$\begin{aligned}\mathbb{E}_x(F\cdot f(X_t)) &= \mathbb{E}_x(F\cdot P_{t-s}f(X_s))\\ &= \int f(y)\,\mathbb{E}_x(F\cdot p_{t-s}(X_s,y))\;\mu(dy)\,.\end{aligned}$$

By restricting the last integral to

$$\mathscr{P}_t = \{y\in S : p_t(x,y) > 0\}\,,$$

we obtain our base formula

$$\mathbb{E}_x(F\cdot f(X_t)) = \int_{\mathscr{P}_t} \mathbb{E}_x\left(F\cdot\frac{p_{t-s}(X_s,y)}{p_t(x,y)}\right) f(y)\,p_t(x,y)\;\mu(dy)\,.$$

To construct a version of the conditional law of $(X_s)_{s\leq t}$ given $X_t = y$ under $\mathbb{P}_x$, one could therefore seek to build a law $\mathbb{P}^t_{x,y}$ on the Skorohod space of càdlàg trajectories of $[0,t]$ into S, denoted D_t, such that for every $s < t$, $\mathbb{P}^t_{x,y}$ is absolutely continuous with respect to $\mathbb{P}_x$ with Radon-Nikodým density $M^s_{x,y}$ given by

$$M^s_{x,y} = \frac{d\mathbb{P}^t_{x,y}|_{\mathscr{F}_s}}{d\mathbb{P}_x|_{\mathscr{F}_s}} = \frac{p_{t-s}(X_s,y)}{p_t(x,y)}, \tag{2}$$

because for such measures the equality

$$\mathbb{E}_x(F\cdot f(X_t)) = \int_{\mathscr{P}_t} \mathbb{E}^t_{x,y}(F)\,f(y)\,p_t(x,y)\;\mu(dy) \tag{3}$$

would follow for $s < t$. Equation (3) contains a disintegration of the law of $(X_r)_{r<s}$ with respect to X_t under $\mathbb{P}_x$. The laws $\mathbb{P}^t_{x,y}$ are usually called bridges since under clearly stated hypotheses, the starting point condition

$$\mathbb{P}^t_{x,y}(X_0 = x) = 1$$

as well as the **ending point condition**

$$\mathbb{P}^t_{x,y}(X_{t-} = X_t = y) = 1 \tag{4}$$

are satisfied. This explains why, even if we succeed at constructing such a law $\mathbb{P}^t_{x,y}$, the local absolute continuity relationship (2) would not hold for $s = t$, unless of course the law of X_t under $\mathbb{P}_x$ charges y and for the examples we have considered this is not the case. However, if we can build the laws $\mathbb{P}^t_{x,y}$ satisfying the local absolute continuity relationship (2) and the ending point condition (4) we can extend (3) to $s = t$ by the following argument. Let $\sigma_t : \cup_{s>t} D_s \to D_t$ be defined by

$$\sigma_t f(s) = \begin{cases} f(s) & \text{if } s < t\\ f(t-) & \text{if } s \geq t\end{cases}.$$

Then the ending point condition (4) implies that for every $F \in b\mathscr{F}_t$, $\mathbb{P}^t_{x,y}(F = F\circ\sigma_t) = 1$ and, since Feller processes do not jump at fixed times, $\mathbb{P}_x(F = F\circ\sigma_t) = 1$. The disintegration (3) can be extended to $\mathscr{F}_{t-} = \sigma(X_s : s < t)$ by a monotone class argument and if $F \in b\mathscr{F}_t$ then $F \circ \sigma_t \in b\mathscr{F}_{t-}$ so that:

$$\begin{aligned}\mathbb{E}_x(Ff(X_t)) &= \mathbb{E}_x(F \circ \sigma_t f(X_t)) \\ &= \int_{\mathscr{P}_t} \mathbb{E}^t_{x,y}(F \circ \sigma_t)\, f(y)\, p_t(x,y)\ \mu(dy) \\ &= \int_{\mathscr{P}_t} \mathbb{E}^t_{x,y}(F)\, f(y)\, p_t(x,y)\ \mu(dy)\,.\end{aligned}$$

To continue our discussion of bridges, recall that weak continuity of the bridge laws is implied by tightness and weak continuity of one-dimensional distributions. Weak continuity of one-dimensional distributions is implied by continuity in variation, which is implied by continuity of the densities by Scheffe's lemma. Hence, hypotheses **H1** and **H2** are not far fetched. Together, they imply the weak-continuity of finite dimensional distributions, at least for times $s < t$, since the first implies the almost sure convergence

$$M^s_{x,z} \to M^s_{x,y}$$

as $z \to y$ under $\mathbb{P}_x$, and the second one implies the applicability of Scheffe's lemma, since it implies that the integral of $M^s_{x,y}$ with respect to $\mathbb{P}_x$ is equal to 1. Hypothesis **H3** does not have a simple explanation but its use is very transparent in the proof of Theorem 1.

3.2. **The proof.** Under the set of hypotheses **H1-H3** we will prove the next theorem, which by the preceding discussion proves Theorem 1.

Theorem 5. *On D_t, the laws $\mathbb{P}_x(\cdot \mid X_t \in B_\delta(y))$ converge weakly as $\delta \to 0$ to a law $\mathbb{P}^t_{x,y}$ which satisfies the following three conditions*

(1) the local absolute continuity relationship (2),
(2) the ending point condition (4), *and*
(3) $y \mapsto \mathbb{P}^t_{x,y}$ is weakly continuous.

Proof of Theorem 5. From the weak convergence $\mathbb{P}_x(\cdot \mid X_t \in B_\delta(y)) \Rightarrow \mathbb{P}^t_{x,y}$, the ending point property 4 is immediately deduced. The weak convergence statement will be proved in the usual manner, by establishing tightness and the convergence of the finite-dimensional distributions, although some technical preliminaries are needed.

Let us first see that the support of μ is S: let $y \in S$ and consider $\delta > 0$. Then, there exists $t > 0$ such that

$$\mathbb{P}_y(X_t \in B_\delta(y)) > 0$$

since X_t converges in probability to y as $t \to 0$ under $\mathbb{P}_y$, because of the Feller property. Since

$$\mathbb{P}_y(X_t \in B_\delta(y)) = \int_{B_\delta(y)} p_t(y,z)\ \mu(dz)\,,$$

it follows that $\mu(B_\delta(y)) > 0$.

Now we will obtain the approximation

$$\lim_{\delta \to 0, z \to y} \frac{\mathbb{P}_x(X_s \in B_\delta(z))}{\mu(B_\delta(z))} = p_s(x,y)\,. \tag{5}$$

of the transition density p_s. Since $p_s(x,\cdot)$ is continuous at y, for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|p_s(x,y) - p_s(x,z)| < \varepsilon$ for all $z \in B_\delta(y)$. Therefore, for all $\delta' < \delta/2$ and all $z \in B_{\delta/2}(y)$:

$$\left| p_s(x,y) - \frac{1}{\mu(B_{\delta'}(z))} \int_{B_{\delta'}(z)} p_s(x,z')\ \mu(dz) \right| < \varepsilon,$$

so that (5) holds.

The next step is to note that if $y \in \mathscr{P}_t$ then for all $\delta > 0$,

$$\mathbb{P}_x(X_t \in B_\delta(y)) > 0.$$

This is because, by hypothesis **H1**, there exists δ_0 such that $p_t(x,z) > 0$ for all $z \in B_{\delta_0}(y)$. Therefore, for all $\delta \leq \delta_0$,

$$\mathbb{P}_x(X_t \in B_\delta(y)) = \int_{B_\delta(y)} p_t(x,z)\ \mu(dz) > 0$$

since otherwise, $\mu(B_\delta(y)) = 0$.

We will now take care of Property 1. For any $F \in b\mathscr{F}_s$ where $s < t$, the Markov property implies the equality

$$\mathbb{E}_x(\, F \mid X_t \in B_\delta(y)) = \mathbb{E}_y\left(F \cdot \frac{\mathbb{P}_{X_s}(X_{t-s} \in B_\delta(y))}{\mathbb{P}_x(X_t \in B_\delta(y))} \right),$$

the right-hand side of which converges to

$$\mathbb{E}_x\left(F \cdot \frac{p_{t-s}(X_s, y)}{p_t(x,y)} \right)$$

because of (5), and Scheffe's lemma. The latter is applicable because of the Chapman-Kolmogorov equations. From this, we conclude something quite a bit stronger than the convergence of finite-dimensional distributions: for any $s < t$, the law of $(X_r)_{r \leq s}$ converges in variation (hence weakly) to a law $\mathbb{P}^{t,s}_{x,y}$ on D_s such that

$$\mathbb{P}^{t,s}_{x,y}(A) = \mathbb{E}_x\left(\mathbf{1}_A \cdot \frac{p_{t-s}(X_s, y)}{p_t(x,y)} \right).$$

In particular, if $\tilde{\omega}(f,t,h)$ denotes the so-called modified modulus of continuity on D_t given by

$$\tilde{\omega}(f,t,h) = \inf_{\{t_i\}} \max_i \max_{s,s' \in [t_{i-1}, t_i)} \rho(f(s)\,, f(s'))$$

where the infimum extends over all partitions

$$0 = t_0 < t_1 < \cdots < t_n = t$$

such that $t_i - t_{i-1} > h$, then the above functional weak convergence implies the following condition: for all $\varepsilon > 0$ and $s < t$

$$\lim_{h \to 0} \limsup_{\delta \to 0} \mathbb{P}_x(\tilde{\omega}(X,s,h) > \varepsilon \mid X_t \in B_\delta(y)) = 0. \tag{6}$$

We will use (6) to study the tightness of our approximations

$$\mathbb{P}_x(\cdot \mid X_t \in B_\delta(y))$$

as $\delta \to 0$. Let $Z_h = \sup_{s,s' \in [0,h]} \rho(X_s, X_{s'})$. It suffices, in view of the convergence of finite-dimensional distributions on $[0,s)$ and the fact that the law of X_t under the approximating law converges weakly to unit mass at y so that all finite-dimensional distributions converge, to verify the following for all $\varepsilon > 0$:

$$\lim_{h \to 0} \lim_{\delta \to 0} \mathbb{P}_x(\, Z_h \circ \theta_{t-h} > \varepsilon \mid X_t \in B_\delta(y)) = 0.$$

To that end, we will now prove a technical result displayed in (7). By the Feller property, for any compact set $K \subset S$, the laws $(\mathbb{P}_z)_{z \in K}$ are weakly continuous on D_h with respect to z. Since for each individual law

$$\lim_{h \to 0} \mathbb{P}_z(Z_h > \varepsilon) \to 0$$

and $z \mapsto \mathbb{P}_z(Z_h > \varepsilon)$ is continuous (because Feller processes do not jump at fixed times and Z_h seen as a functional on D_∞ is continuous at f if f is continuous at h) and increasing in h, then

$$\lim_{h \to 0} \sup_{z \in K} \mathbb{P}_z(Z_h > \varepsilon) \to 0. \tag{7}$$

Otherwise, there would be two sequences, (z_n) in K and (h_n) decreasing to zero, such that

$$\liminf_{n \to 0} \mathbb{P}_{z_n}(Z_{h_n} > \varepsilon) > 0.$$

However, since K is compact, there exists a subsequence (z_{n_k}) converging to $z \in K$ and because Feller processes do not admit fixed-time discontinuities and have càdlàg paths:

$$\begin{aligned} 0 < \liminf_{k\to\infty} \mathbb{P}_{z_{n_k}}\left(Z_{h_{n_k}} > \varepsilon\right) &\leq \liminf_{m\to\infty} \lim_{k\to\infty} \mathbb{P}_{z_{n_k}}(Z_{h_m} > \varepsilon) \\ &= \lim_{m\to\infty} \mathbb{P}_z(Z_{h_m} > \varepsilon) = 0, \end{aligned}$$

which is a contradiction.

To continue our main line of argument, note that by local compactness, there exists a $\delta > 0$ such that $B_\delta(y)$ has compact closure. We will write

$$\begin{aligned} &\mathbb{P}_x(Z_h \circ \theta_{t-h} > \varepsilon \mid X_t \in B_\delta(y)) \\ &= \mathbb{P}_x(Z_h \circ \theta_{t-h} > \varepsilon, X_{t-h} \in B_\delta(y) \mid X_t \in B_\delta(y)) \\ &+ \mathbb{P}_x(Z_h \circ \theta_{t-h} > \varepsilon, X_{t-h} \notin B_\delta(y) \mid X_t \in B_\delta(y)) \end{aligned}$$

and bound each one of the summands of the right-hand side. For the first one, use Bayes rule

$$\begin{aligned} &\mathbb{P}_x(Z_h \circ \theta_{t-h} > \varepsilon, X_{t-h} \in B_\delta(y) \mid X_t \in B_\delta(y)) \\ &= \mathbb{P}_x(Z_h \circ \theta_{t-h} > \varepsilon, X_t \in B_\delta(y) \mid X_{t-h} \in B_\delta(y)) \frac{\mathbb{P}_x(X_{t-h} \in B_\delta(y))}{\mathbb{P}_x(X_t \in B_\delta(y))}. \end{aligned}$$

However, in view of the Markov property, the technical result of the last paragraph, hypothesis **H3** and the transition density approximation (5):

$$\begin{aligned} &\mathbb{P}_x(Z_h \circ \theta_{t-h} > \varepsilon, X_t \in B_\delta(y) \mid X_{t-h} \in B_\delta(y)) \\ &\leq \frac{\mathbb{P}_x(Z_h \circ \theta_{t-h} > \varepsilon, X_{t-h} \in B_\delta(y))}{\mathbb{P}_x(X_{t-h} \in B_\delta(y))} \\ &\leq \sup_{z\in B_\delta(y)} \mathbb{P}_z(Z_h > \varepsilon) \end{aligned}$$

and

$$\lim_{h\to 0} \lim_{\delta\to 0} \frac{\mathbb{P}_x(X_{t-h} \in B_\delta(y))}{\mathbb{P}_x(X_t \in B_\delta(y))} = \lim_{h\to 0} \frac{p_{t-h}(x,y)}{p_t(x,y)} = 1,$$

so that

$$\lim_{h\to 0} \limsup_{\delta\to 0} \mathbb{P}_x(Z_h \circ \theta_{t-h} > \varepsilon, X_{t-h} \in B_\delta(y) \mid X_t \in B_\delta(y)) = 0.$$

We will now obtain a second bound by means of

$$\begin{aligned} &\mathbb{P}_x(Z_h \circ \theta_{t-h} > \varepsilon, X_{t-h} \notin B_\delta(y) \mid X_t \in B_\delta(y)) \\ &\leq \mathbb{P}_x(X_{t-h} \notin B_\delta(y) \mid X_t \in B_\delta(y)) \\ &= 1 - \frac{\mathbb{P}_x(X_{t-h} \in B_\delta(y)\,, X_t \in B_\delta(y))}{\mathbb{P}_x(X_{t-h} \in B_\delta(y))} \frac{\mathbb{P}_x(X_{t-h} \in B_\delta(y))}{\mathbb{P}_x(X_t \in B_\delta(y))}; \end{aligned}$$

we have already seen that if $\delta \to 0$ and we then let $h \to 0$, the second factor in the right-hand side of the last equality converges to 1. To study the first factor, write it as

$$1 - \frac{\mathbb{P}_x(X_{t-h} \in B_\delta(y)\,, X_t \notin B_\delta(y))}{\mathbb{P}_x(X_{t-h} \in B_\delta(y))}$$

and use the Feller property in the following manner: for δ small enough (so that $B_\delta(y)$ has compact closure) and $\delta' \in (0, \delta)$, let $\phi : S \to [0,1]$ be a continuous function which is equal to 1 on $B_{\delta'}(y)$ and vanishes outside $B_\delta(y)$, since ϕ is continuous and vanishes at infinity, the Feller property implies that for all $z \in B_{\delta'}(y)$

$$\mathbb{P}_z(X_h \notin B_\delta(y)) \leq \mathbb{E}_z(1 - \phi(X_h)) = |\mathbb{E}_z(\phi(X_h)) - \phi(z)| \leq \|P_h - \mathrm{Id}\,\|.$$

Since the previous estimation does not depend on $\delta' < \delta$, our conclusion is that it holds for all $z \in B_\delta(y)$ and so, by the Markov property:

$$\frac{\mathbb{P}_x(X_{t-h} \in B_\delta(y)\,, X_t \notin B_\delta(y))}{\mathbb{P}_x(X_{t-h} \in B_\delta(y))} \le \|P_h - \mathrm{Id}\,\|.$$

We finally obtain

$$\lim_{h\to 0}\lim_{\delta\to 0}\mathbb{P}_x(\,Z_h \circ \theta_{t-h} > \varepsilon, X_{t-h} \notin B_\delta(y) \mid X_t \in B_\delta(y)) = 0,$$

which implies the existence of a law $\mathbb{P}^t_{x,y}$ on D_t to which $\mathbb{P}_x(\,\cdot \mid X_t \in B_\delta(y))$ converges weakly as $\delta \to 0$. As we have already remarked, $\mathbb{P}^t_{x,y}$ satisfies the local absolute continuity relationship (2). It also satisfies the ending point condition since the law of X_t under $\mathbb{P}_x$ conditionally on $\{X_t \in B_\delta(y)\}$ is concentrated on $B_\delta(y)$.

To conclude the proof of the theorem, we must examine the weak continuity of $\mathbb{P}^t_{x,y}$ as y varies. To do it, we will prove that if $K \subset S$ is compact in $\mathscr{P}_t$ then $(\mathbb{P}_x(\,\cdot \mid X_t \in B_\delta(z)))_{z\in K,\delta>0}$ is tight in D_t. If this is true then $\left(\mathbb{P}^t_{x,z}\right)_{z\in K}$ will be tight and because as $z \to y \in \mathscr{P}_t$, $\mathbb{P}^t_{x,z}$ converges in variation to $\mathbb{P}^t_{x,y}$ on D_s and the ending point condition is satisfied, then the finite-dimensional distributions of $\mathbb{P}^t_{x,z}$ converge to those of $\mathbb{P}^t_{x,y}$ and therefore, there is also weak convergence. To analyze the tightness of $(\mathbb{P}_x(\,\cdot \mid X_t \in B_\delta(z)))_{z\in K,\delta>0}$, we note that tightness holds on D_s for each $s < t$, so that it suffices to prove, for all $\varepsilon > 0$:

$$\lim_{h\to 0}\lim_{\delta\to 0, z\to y}\mathbb{P}_x(\,Z_h \circ \theta_{t-h} > \varepsilon \mid X_t \in B_\delta(z)) = 0.$$

Our previous arguments can be extended to this case, since by the density approximation (5):

$$\lim_{h\to 0}\lim_{\delta\to 0, z\to y}\frac{\mathbb{P}_x(X_{t-h} \in B_\delta(z))}{\mathbb{P}_x(X_t \in B_\delta(z))} = 1$$

and for sufficiently small δ (so that $B_{2\delta}(y)$ has compact closure) and $z \in B_\delta(y)$, we have that

$$\lim_{h\to 0}\sup_{z'\in B_\delta(z)}\mathbb{P}_{z'}(X_h \notin B_\delta(z)) \le \lim_{h\to 0}\sup_{z'\in B_{2\delta}(y)}\mathbb{P}_{z'}(X_h \notin B_{2\delta}(y)) = 0$$

by (7) and

$$\mathbb{P}_x(\,X_t \notin B_\delta(z) \mid X_{t-h} \in B_\delta(z)) \le \|P_h - \mathrm{Id}\,\|.$$

□

Proof of Corollary 1. Let us prove that as $t' \to t$ and $z \to y$ (in $\mathscr{P}_t$), $\mathbb{P}^{t'}_{x,z}$ converges in law to $\mathbb{P}^t_{x,y}$. As in the proof of Theorem 5, under **H1'** we have convergence in variation of $\mathbb{P}^{t'}_{x,z}|_{\mathscr{F}_s}$ to $\mathbb{P}^t_{x,y}|_{\mathscr{F}_s}$ if $s < t$ and, because of the ending point condition, this implies not only the convergence of the finite-dimensional distributions but also a tightness criterion on the compact intervals of $[0,\infty)\setminus\{t\}$. Hence, we must only prove the following for all $\varepsilon > 0$:

$$\lim_{h\to 0}\lim_{\delta\to 0, z\to y, t'\to t}\mathbb{P}_x(\,Z_h \circ \theta_{t'-h} > \varepsilon \mid X_{t'} \in B_\delta(z)) = 0.$$

Again, we can use the same arguments as in the proof of Theorem 5 since under **H1′**, (5) can be generalized to:

$$\lim_{\delta\to 0, z\to y, s'\to s}\frac{\mathbb{P}_x(X_{s'} \in B_\delta(z))}{\mu(B_\delta(z))} = p_s(x,y)\,.$$

The other bounds needed did not depend on the length parameter t'.

We can extend the preceding reasoning by imposing the joint continuity of the density in all variables to obtain the joint weak continuity of bridge laws $\mathbb{P}^t_{x,y}$ in all variables. □

4. The backward strong Markov property

In this section we will prove Theorem 2. We begin with a basic summary of the properties of conditional independence which we will need. We use the notation $\mathscr{G}_1 \perp_{\mathscr{H}} \mathscr{G}_2$ to mean that $\mathscr{G}_1$ and $\mathscr{G}_2$ are conditionally independent given a σ-field $\mathscr{H}$.

Proposition 2. *The σ-fields $\mathscr{G}_1$ and $\mathscr{G}_2$ are conditionally independent given $\mathscr{H}$ if and only if for all $G \in b\mathscr{G}_1$:*

$$\mathbb{E}(G|\mathscr{G}_2, \mathscr{H}) = \mathbb{E}(G|\mathscr{H}) .$$

Furthermore for any σ-fields $\mathscr{H}, \mathscr{G}, \mathscr{G}_1, \mathscr{G}_2, \ldots$, the following conditions are equivalent:

(i) $\mathscr{G} \underset{\mathscr{H}}{\perp} \mathscr{G}_1, \mathscr{G}_2, \ldots$.

(ii) For any $n \geq 1$, $\mathscr{G} \underset{\mathscr{H}, \mathscr{G}_1, \ldots, \mathscr{G}_n}{\perp} \mathscr{G}_{n+1}$.

Finally, if $\mathscr{G}_1 \perp_{\mathscr{H}} \mathscr{G}_2$ and $\mathscr{G}_1' \subset \mathscr{G}_1$ then

$$\mathscr{G}_1' \underset{\mathscr{H}}{\perp} \mathscr{G}_2 \text{ and } \mathscr{G}_1 \underset{\mathscr{H}, \mathscr{G}_1'}{\perp} \mathscr{G}_2$$

The first property is the **asymmetric expression of conditional independence** and is the link between conditional independence and the Markov property, as has been expanded upon. The second of the above properties will be refered to as the **chain rule for conditional independence**. Proofs of them are found in [Kal02]. The third property consists of the **downwards monotone character of conditional independence** in the non-conditioning σ-fields and a **partial upwards monotone character in the conditioning σ-field**. It is a trivial application of the chain rule since under the conditions stated, $\sigma(\mathscr{G}_1, \mathscr{G}_1') = \sigma(\mathscr{G}_1)$. We cannot expect a general upwards monotone character to hold: for example, if X and Y are two independent random variables on $\{-1, 1\}$ which take the two values with equal probability, and $Z = XY$, then X and Y are independent but they are not conditionally independent given Z, since the conditional law of Y given X, Z is concentrated at XZ and the conditional law of Y given Z is the same as that of Y since Y and Z are independent. The following formulation of the preceding example might be more impressive. Let $\mathscr{H}_1 \subset \mathscr{H}_2 \subset \mathscr{H}_3$ then

$$\mathscr{G}_1 \underset{\mathscr{H}_1}{\perp} \mathscr{G}_2 \text{ and } \mathscr{G}_1 \underset{\mathscr{H}_3}{\perp} \mathscr{G}_2 \text{ do not imply } \mathscr{G}_1 \underset{\mathscr{H}_2}{\perp} \mathscr{G}_2;$$

just take $\mathscr{H}_1 = \{\Omega, \emptyset\}$, $\mathscr{H}_2 = \sigma(Z)$, $\mathscr{H}_3 = \sigma(X, Y)$, $\mathscr{G}_1 = \sigma(X)$ and $\mathscr{G}_2 = \sigma(Y)$. During the course of the proof of the backward strong Markov property, we will use:

Definition. Given **two σ-fields** $\mathscr{G}$ and $\mathscr{G}'$, we say that they **agree on a set** A, written $\mathscr{G} = \mathscr{G}'$ on A, if $A \in \mathscr{G} \cap \mathscr{G}'$ and $A \cap \mathscr{G} = A \cap \mathscr{G}'$.

Proposition 3 (Local property of conditional expectation)**.** *On a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, let $\mathscr{G}$ and $\mathscr{G}'$ be sub-σ-fields of $\mathscr{F}$ and consider two integrable random variables ξ, ξ'. Suppose that $\mathscr{G} = \mathscr{G}'$ on A and that $\xi = \xi'$ almost surely on A. Then*

$$\mathbb{E}(\xi|\mathscr{G}) = \mathbb{E}(\xi'|\mathscr{G}') \text{ almost surely on } A.$$

The preceding proposition is proved in [Kal02].

Proof of Theorem 2. We begin by discussing the measurability of $(t, x, y) \mapsto \mathbb{P}^t_{x,y}(F)$ for any measurable $F : D_\infty \to \mathbb{R}$. First, let us note that the set $\{(t, x, y) : p_t(x, y) > 0\}$ is measurable because $(t, x, y) \mapsto p_t(x, y)$ is measurable since it is jointly continuous in (t, y) for fixed x and measurable in x for fixed (t, y). The latter is true since for all $\delta > 0$,

$$x \mapsto \frac{\mathbb{P}_x(X_t \in B_\delta(y))}{\mu(B_\delta(y))}$$

is measurable by the measurability property of Markovian famiies and its limit as $\delta \to 0$ is $p_t(x, y)$ by the density approximation (5) implied by hypothesis **H1'**.

For the rest of the argument, we will work on the set $\{(t, x, y) : p_t(x, y) > 0\}$. Let us note that if $F \in b\mathscr{F}_s$ and $s < t$, then the local absolute continuity relationship (2) implies that $x \mapsto \mathbb{P}^t_{x,y}(F)$ is measurable and by the monotone class theorem, we see that the measurability extends first to any

$F \in b\mathscr{F}_t$ and then to any measurable F. Since by Corollary 1, $(t, y) \mapsto \mathbb{P}^t_{x,y}(F)$ is continuous if F is, we see that $(t, x, y) \mapsto \mathbb{P}^t_{x,y}(F)$ is measurable whenever F is continuous. By a monotone class argument, the preceding measurability extends to measurable F.

We now turn to the computation of the conditional expectation of Theorem 2. Because of the strong Markov property, it suffices to prove the theorem when $S = 0$; we will simplify the notation for σ^0_L to σ_L.

Let

$$L^n = \sum_{k=0}^{\infty} \frac{k}{2^n} \mathbf{1}_{(\frac{k}{2^n}, \frac{k+1}{2^n}]}(L) .$$

Then L^n is a random time strictly smaller than L which increases with n towards L. Since L is a backward optional time:

$$\left\{ L^n = \frac{k}{2^n} \right\} = \left\{ \frac{k}{2^n} < L \leq \frac{k+1}{2^n} \right\} \in \mathscr{F}^{k/2^n}.$$

Furthermore, the σ-fields $\mathscr{F}^{k/2^n}$ and $\mathscr{F}^{L^n}$ agree on the set $\{L^n = k/2^n\}$ since θ_{L^n} coincides with $\theta_{k/2^n}$ on that set. For every bounded and measurable $H : D_\infty \to \mathbb{R}$

$$\mathbb{E}_\nu\Big(H \circ \sigma_{k/2^n} \mathbf{1}_{L^n = k/2^n} \,\Big|\, \mathscr{F}^{k/2^n} \Big) = \mathbb{P}^{k/2^n}_{X_0, X_{k/2^n}}(H)\, \mathbf{1}_{L^n = k/2^n},$$

so that by the local property of conditional expectation:

$$\mathbb{E}_\nu\Big(H \circ \sigma_{L^n} \,\Big|\, \mathscr{F}^{L^n} \Big) = \mathbb{P}^{L^n}_{X_0, X_{L^n}}(H) \text{ a.s. on } \{L^n > 0\}. \tag{8}$$

If H is actually continuous and bounded then $H \circ \sigma_{L^n} \to H \circ \sigma_L$. If $A \in \mathscr{F}^L$ and $B \in \mathscr{B}_S$ then $A \cap \{X_{L-} \in B\} \cap \{L_n > 0\} \in \mathscr{F}^{L^n}$, and so (8) implies

$$\begin{aligned} &\mathbb{E}_\nu\Big(H \circ \sigma^{L^n} \mathbf{1}_A \mathbf{1}_{X_{L-} \in B} \mathbf{1}_{L^n > 0} \Big) \\ &= \mathbb{E}_\nu\Big(\mathbf{1}_A \mathbf{1}_{X_{L-} \in B} \mathbb{P}^{L^n}_{X_0, X_{L^n-}}(H)\, \mathbf{1}_{L^n > 0} \Big) . \end{aligned}$$

The left-hand side of the preceding expression converges to

$$\mathbb{E}_\nu\big(H \circ \sigma^L \mathbf{1}_A \mathbf{1}_{X_{L-} \in B} \mathbf{1}_{L > 0} \big)$$

as $n \to \infty$, while the right-hand side converges to

$$\mathbb{E}_\nu\Big(\mathbf{1}_A \mathbf{1}_{X_{L-} \in B} \mathbb{P}^L_{X_0, X_{L-}}(H)\, \mathbf{1}_{L > 0} \Big)$$

by Corollary 1, so that

$$\mathbb{E}_\nu\big(H \circ \sigma_L \,\big|\, \mathscr{F}^L, X_{L-} \big) = \mathbb{P}^L_{X_0, X_{L-}}(H) \text{ a.s. on } \{L > 0\}.$$

□

5. Self-similarity and pathwise construction of Markovian bridge laws

In this section we will discuss examples for which the pathwise construction of bridges of self-similar Feller processes of Theorem 3 works and we will verify the pathwise construction of the stable subordinator conditioned to die at a given level of Theorem 4. The latter is found in Subsection 5.1 while the former is included in Subsection 5.2.

5.1. **Pathwise construction of bridges of self-similar Markov processes.** Note that Theorem 3 is trivial from Theorem 2; however, the real problem lies in identifying processes for which the hypothesis holds. In this section we give several (general) examples and a word of caution against the impression that the hypothesis should hold trivially because of self-similarity.

Example 1. Consider first a self-similar Feller family $(\mathbb{P}_x)_{x\in S}$ of index γ, and suppose that under each $\mathbb{P}_x$, the jumps of X have the same sign. As we now see, the set

$$\mathscr{Z}_c = \left\{ t \in (0,1] : X_{t-} = ct^{1/\gamma} \right\}$$

is almost surely not empty under $\mathbb{P}_0$ if $\mathbb{P}_0(X_1 > c)$ and $\mathbb{P}_0(X_1 < c)$ are both positive.

To see this, consider a sequence (t_n) decreasing to zero and define the set

$$A = \limsup_{n\to\infty} \left\{ X_{t_n} \text{ or } X_{t_n-} > ct_n^{1/\gamma} \right\}.$$

By Blumenthal's 0-1 law, A is $\mathbb{P}_x$ trivial for every x. However, under $\mathbb{P}_0$ we can apply scaling to give:

$$\mathbb{P}_0(A) \geq \limsup_{n\to\infty} \mathbb{P}_0\Big(X_{t_n} \text{ or } X_{t_n-} > ct_n^{1/\gamma} \Big) = \mathbb{P}_0(X_1 \text{ or } X_{1-} > c).$$

We see that $\mathbb{P}_0(A) = 1$ when $\mathbb{P}_0(X_1 > c)$ is positive. By the same argument, if

$$B = \limsup_{n\to\infty} \left\{ X_{t_n} \text{ or } X_{t_n-} < ct_n^{1/\gamma} \right\}$$

then $\mathbb{P}_0(B) = 1$ when $\mathbb{P}_0(X_1 < c) > 0$. If $\mathbb{P}_0(X_1 > c)$ and $\mathbb{P}_0(X_1 < c)$ are both positive then X will cross the curve $t \mapsto ct^{1/\gamma}$ an infinite number of times near zero, in the sence that for every $t \in (0,1)$ there will exist $s \in (0,t)$ such that $\operatorname{sgn}\big(X_s - ct^{1/\gamma}\big) \neq \operatorname{sgn}\big(X_s - cs^{1/\gamma}\big)$. However either the downcrossings or the upcrossings will touch the curve, since X either decreases or increases continuously, which implies the existence of $t \in (0,1)$ such that $X_t = ct^{1/\gamma} = X_{t-}$.

This reasoning implies that Thereom 3 holds for Brownian motion and Bessel processes. It also holds for spectrally asymmetric stable Lévy processes: these are stable Lévy processes whose jumps have almost surely the same sign.

Example 2. We continue with the special case of stable Lévy processes which are not spectrally asymmetric with following result:

Theorem 6. *Let $\mathbb{P}$ be the law of a stable Lévy process of index $\alpha \in (0,2]$ started at 0; then $\mathbb{P}(g_c > 0) = 1$if and only if either $\alpha > 1$ or $\alpha < 1$ and $c \neq 0$.*

When the process is spectrally asymmetric we use Example 1. When the process has jumps of both signs, our proof of Theorem 6 passes through associated Ornstein-Uhlenbeck process, this is the process Y defined by $Y_t = e^{-t/\alpha} X_{e^t}$ for $t \in \mathbb{R}$ under $\mathbb{P}_0$. Then Y is a stationary (time-homogeneous) Markov process whose semigroup is described as follows (cf. [Bre68] or [BW96]): let f_t be the density of X_t under $\mathbb{P}_0$ with respect to Lebesgue measure (as in Subsection 2.1) and set

$$p_t(x,y) = f_t(y-x).$$

Then the semigroup of Y admits transition densities $q_t, t \geq 0$ given by

$$q_t(x,y) = p_{e^t-1}\Big(x, e^{t/\alpha} y\Big)\, e^{t/\alpha}. \tag{9}$$

Note the equality

$$\left\{ t > 0 : X_{t-} = ct^{1/\alpha} \right\} = \exp(\{ t \in \mathbb{R} : Y_{t-} = c\}).$$

The positivity of g_c under $\mathbb{P}_0$ would follow if the set $\{t \in \mathbb{R} : Y_{t-} = c\}$ had no lower bound almost surely; since Y is Feller, it is sufficient to prove this for $A = \{t \in \mathbb{R} : Y_{t-} \text{ or } Y_t = c\}$, by using their quasi-continuity as in [Ber96, Cor. 8, p. 22]. This would in turn be obtained if the set $A \cap (0,\infty)$ were non-empty with positive probability, since the stationary character of Y under $\mathbb{P}_0$ implies that

$$\mathbb{P}(A \text{ has no lower bound}) = \lim_{M\to-\infty} \mathbb{P}(A \cap (-\infty, M) \neq \emptyset) = \mathbb{P}(A \cap (-\infty, M) \neq \emptyset),$$

where the last equality is a consequence of the translation invariance of A under $\mathbb{P}_0$. This same translation invariance also shows us that

$$\mathbb{P}(A \cap (-\infty, M) \neq \emptyset) = \lim_{M\to\infty} \mathbb{P}(A \cap (-\infty, M) \neq \emptyset) = \mathbb{P}(A \cap (-\infty, \infty) \neq \emptyset)$$

and the same reasoning gives us

$$\mathbb{P}(A \cap (-\infty, \infty) \neq \emptyset) = \mathbb{P}(A \cap (0, \infty) \neq \emptyset).$$

In conclusion, $\{A$ has no lower bound$\}$ and $\{A \cap (0, \infty) \neq \emptyset\}$ have the same probability; however, because of Blumenthal's 0-1 law, the former event is trivial, and so its probability is 1 once it is positive. Note also that

$$\{A \cap (0, \infty) \neq \emptyset\} = \{\exists t > 0, Y_{t-} \text{ or } Y_t = c\}$$

and that if we let $T_c = \inf\{t \geq 0 : Y_{t-} \text{ or } Y_t = c\}$ then $\mathbb{P}(T_c < \infty) > 0$ implies $\mathbb{P}(g_c > 0) = 1$. Hence, the question of knowing whether g_c is positive or not has been recast as a question concerning the finitude of the hitting time of $\{c\}$ for the associated Ornstein-Uhlenbeck process Y. The latter problem can be solved explicitly by use of polarity criteria using the resolvent density v_λ of Y given by

$$v_\lambda(x, y) = \int_0^\infty e^{-\lambda t} q_t(x, y)\, dt$$

for $\lambda > 0$. Note that v_λ can take the value ∞; since q_t is continuous, v_λ is lower semicontinuous and so it is continuous at (x, y) (as a function with values on $[0, \infty]$) if $v_\lambda(x, y) = \infty$. We will see that:

Proposition 4. *When the stable process has jumps of both signs, v_λ is bicontinuous and*

$$v_\lambda(x, y) < \infty \Leftrightarrow \begin{cases} \alpha \in (1, 2), \\ \alpha = 1 \text{ and } x \neq y \text{ or} \\ \alpha \in (0, 1) \text{ and } x \text{ or } y \text{ are not zero.} \end{cases}$$

Theorem 6 follows from Proposition 4 by the following well known method: let $(\mathbb{Q}_x)_{x\in\mathbb{R}}$ be the Markovian family associated to the semigroup of Y, introduce the resolvent operator defined by

$$V_\lambda(x, A) = \int_0^\infty e^{-\lambda t} \mathbb{Q}_x(X_t \in A)\, dt = \int_A v_\lambda(x, z)\, dz,$$

as well as the stopping time

$$H_\varepsilon = \inf\{t \geq 0 : X_t \in B_\varepsilon(c)\},$$

so that by the strong Markov property,

$$V_\lambda(x, B_\varepsilon(c)) = \mathbb{E}_{\mathbb{Q}_x}\left(e^{-\lambda H_\varepsilon} V_\lambda(X_{H_\varepsilon}, B_\varepsilon(c))\right).$$

Note that as $\varepsilon \to 0$, H_ε converges to T_c.

If $x \neq c$, then in any case $v_\lambda(x, c) < \infty$ and

$$v_\lambda(x, c) = \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} V_\lambda(x, B_\varepsilon(c)).$$

If $v_\lambda(c, c) < \infty$, then

$$v_\lambda(c, c) = \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon} V_\lambda(X_{H_\varepsilon}, B_\varepsilon(c))$$

and the bounded convergence theorem tells us that

$$v_\lambda(x, c) = \mathbb{E}_{\mathbb{Q}_x}\left(e^{-\lambda T_c}\right) v_\lambda(c, c),$$

so that

$$\mathbb{E}_{\mathbb{Q}_x}\left(e^{-\lambda T_c}\right) > 0$$

implying the almost sure finitude of T_c under $\mathbb{Q}_x$ for any $x \neq c$ so that

$$\mathbb{P}(\exists t > 0 : Y_{t-} \text{ or } Y_t = c) = \int \mathbb{Q}_x(T_c < \infty)\, f_1(x)\, dx = 1.$$

If, on the other hand, $v_\lambda(c, c) = \infty$, then Fatou's lemma tells us that

$$\mathbb{E}_{\mathbb{Q}_x}\left(\infty \cdot e^{-\lambda T_c}\right) = \mathbb{E}_{\mathbb{Q}_x}\left(\liminf_{\varepsilon\to 0} e^{-\lambda H_\varepsilon} \frac{1}{2\varepsilon} V_\lambda(X_{H_\varepsilon}, B_\varepsilon(c))\right) \leq v_\lambda(x, c) < \infty,$$

so that $\mathbb{Q}_x(T_c = \infty) = 1$ for all $x \neq c$ and so

$$\mathbb{P}(\exists t > 0 : Y_{t-} \text{ or } Y_t = c) = \int \mathbb{Q}_x(T_c < \infty)\, f_1(x)\ dx = 0.$$

Even though the proof of Proposition (4) requires only elementary analysis, it is long and technical; it is therefore presented in the Appendix.

Note. It is also a consequence of Proposition 4 that for the Ornstein-Uhlenbeck process driven by an α-stable Lévy process which has jumps of both signs, x is polar if and only if $\alpha = 1$ or $\alpha \in (0,1)$ and $x = 0$. Also, the resolvent density v_λ has been explicitly computed by Patie in [Pat07] in the spectrally asymmetric case of index $\alpha \in (1,2)$. It is expressed in terms of Novikov's generalization of Hermite's function introduced in [Nov81].

Suppose now that $\mathbb{P}$ is the law of Brownian motion. We now study the law of the random variable g_c by proving Proposition 1.

Proof of Proposition 1. Again we pass through the stationary Ornstein-Uhlenbeck process $Y_t = e^{-t/2} X_{e^t}$. Then

$$g_c = 1 - \log(\inf \{r \geq 0 : Y_{-r} = c\})\,.$$

By time inversion, we see that $(Y_{-t})_{t\in\mathbb{R}}$ has the same law as $(Y_t)_{t\in\mathbb{R}}$, so that

$$1 - g_c \text{ has the same law as } \log(\inf \{t \geq 0 : Y_t = c\})\,.$$

Let $\mathbb{Q}_x, x \in \mathbb{R}$ stand for the Feller family of Y. The (extended) generator A of Y is given by

$$Af(x) = \left.\frac{d}{dt}\right|_{t=0} \mathbb{E}_{\mathbb{Q}_x}(f(X_t)) = \frac{f''(x)}{2} - \frac{1}{2} x f'(x)$$

if $f : \mathbb{R} \to \mathbb{R}$ has two bounded continuous derivatives. The Laplace transform of the hitting times T_c of c under $\mathbb{Q}_x$ can be expressed in terms of monotone eigenfunctions of the preceding generator, which are in turn, expressible in terms of parabolic cylinder functions; integrating with respect to the law of Y_0 will then give us an expression of the Mellin transform of $1 - g_c$. We now provide a streamlined exposition of this suited to our needs.

Consider the nonnegative function on $\mathbb{R}$ given by

$$H_q(x) = \int_0^\infty e^{-xz - z^2/2} z^{q-1}; \tag{10}$$

integrating by parts in the preceding expression leads to

$$H_{q+2}(x) = qH_q(x) - xH_{q+1}(x)$$

while differentiating under the integral in (10) gives

$$H_q'(x) = -H_{q+1}(x)\,,$$

so that H_q is decreasing and

$$H_{2q}''(x) - xH_{2q}'(x) = 2qH_{2q}(x)\,.$$

The same equation is satisfied by $x \mapsto H_{2q}(-x)$. Itô's formula then tells us that $e^{-qt}H_{2q}(X_t)$ and $e^{-qt}H_{2q}(-X_t)$ (for $t \geq 0$) are local martingales under $\mathbb{P}_x$ for any x, the first one of which is bounded up to time T_c if $x \geq c$ while the second one is bounded up to T_c if $x \leq c$. By optional stopping, we see that

$$\mathbb{E}_{\mathbb{Q}_x}\left(e^{-qT_c}\right) = \frac{H_{2q}(x \operatorname{sgn}(x-c))}{H_{2q}(c \operatorname{sgn}(x-c))}.$$

Hence

$$\mathbb{E}((1-g_c)^q) = \int_{-\infty}^\infty \frac{e^{-x^2/2}}{\sqrt{2\pi}} \frac{H_{2q}(x \operatorname{sgn}(x-c))}{H_{2q}(c \operatorname{sgn}(x-c))}\, dx. \tag{11}$$

To evaluate the last integral, use (10) to obtain

$$\int_c^\infty \frac{e^{-x^2/2}}{\sqrt{2\pi}} \frac{H_{2q}(x)}{H_{2q}(c)}\, dx = \frac{e^{-c^2/2}}{\sqrt{2\pi}2q} \frac{H_{2q+1}(c)}{H_{2q}(c)}$$

so that

$$\mathbb{E}((1-g_c)^q) = \frac{e^{-c^2/2}}{\sqrt{2\pi}2q}\left(\frac{H_{2q+1}(c)}{H_{2q}(c)} + \frac{H_{2q+1}(-c)}{H_{2q}(-c)}\right). \tag{12}$$

To obtain the required result, recall that the Wronskian W of the two solutions $c \mapsto H_q(c)$ and $c \mapsto H_q(-c)$ of the differential equation $f''(c) - cf'(c) - qf(c) = 0$ is given by Abel's identity

$$W(c) = W(0)\, e^{-c^2/2}.$$

However, since $H_q'(c) = -H_{q+1}(c)$, W can also be expressed as

$$W(c) = H_q(c)\, H_{q+1}(-c) + H_q(-c)\, H_{q+1}(c)\,.$$

Substituting in (12), we get the following expression expression for the q-th moment of $(1-g_c)$:

$$\frac{1}{2q\sqrt{2\pi}} W_{2q}(0)\, \frac{1}{H_{2q}(c)\, H_{2q}(-c)}.$$

Since

$$H_q(0) = 2^{q/2-1}\Gamma\left(\frac{q}{2}\right),$$

the quantity $W(0)$ is explicitly evaluated as follows:

$$W(0) = 2H_{2q}(0)\, H_{2q+1}(0) = 2^{2q-1/2}\Gamma\left(\frac{q}{2}\right)\Gamma\left(\frac{q}{2}+\frac{1}{2}\right) = \sqrt{2\pi}\Gamma(2q)$$

where we have used the duplication formula for the Γ function in the last equality. This gives

$$\mathbb{E}((1-g_c)^q) = \frac{\Gamma(2q)}{2qH_{2q}(c)\, H_{2q}(-c)}.$$

For the asymptotic behaviour of the moments of $1-g_c$, it suffices to apply Laplace's method (as found in [dB81] or [Olv97]) to obtain, as $q \to \infty$:

$$H_q(x) \sim \sqrt{\pi} q^{q/2} e^{-x\sqrt{q}-q/2}.$$

□

5.2. **Pathwise construction of the stable subordinator conditioned to die at a given level.** The aim of this subsection is to prove Theorem 4.

First of all, note that the effect of the scaling operator S_v^α to the stable subordinator started at zero and conditioned to die at $b > 0$ gives the stable subordinator started at zero and conditioned to die at $v^{1/\alpha}b$; this is proved by the scaling properties of $\mathbb{P}_0^\alpha$ and its relationship to $\mathbb{P}_0^{h_\alpha}$.

We need four additional elements to verify the desired pathwise construction:

(1) The law of ζ under $\mathbb{P}_a^{h_\alpha}$. This is obtained from the fact that

$$\mathbb{P}_a^{h_\alpha}(\zeta > t) = \frac{1}{h_\alpha(a)}\mathbb{E}_a^\alpha(h_\alpha(X_t)) = \frac{1}{h_\alpha(a)}\int f_t^\alpha(x)\, h_\alpha(x)\, dx = \frac{1}{h_\alpha(a)}\int_t^\infty f_s^\alpha(b-a)\, ds$$

where the last equality follows by the definition of h_α and the Chapman-Kolmogorov equations. The density of ζ under $\mathbb{P}_a^{h_\alpha}$ is then equal to

$$t \mapsto \frac{f_t(b-a)}{h_\alpha(a)}.$$

(2) The computation of the law of the conditioned stable subordinator given its death time ζ when it starts at zero. Using the preceding expression of the density of ζ and writing down the finite-dimensional distributions, we see that given $\zeta = t$ the stable subordinator started at 0 and conditioned to die at b has law $\mathbb{P}_{0,b}^{\alpha,t}$.

(3) The computation of the law of Y given its death time, equal to $L\,(b/g)^{\alpha}$. This is accomplished by use of the backward strong Markov property: the law of $X_{[0,L)}$ given $g = X_{L-}$ and $\mathscr{F}^L$ under $\mathbb{P}_0^\alpha$ is $\mathbb{P}_{0,g}^{\alpha,L}$. Note that Y is obtained from X (on $[0, L-)$) by applying the scaling operator $S^\alpha_{(b/g)^\alpha}$. By self-similarity, the law of $Y_{[0,L(b/g)^\alpha)}$ given g and $\mathscr{F}^L$ is $\mathbb{P}_{0,b}^{\alpha,L(b/g)^\alpha}$, which only depends on $L\,(b/g)^{\alpha}$. It follows that the law of $Y_{[0,L(g/b)^\alpha)}$ given that its death time is t is $\mathbb{P}_{0,b}^{0,t}$.

(4) The density of (L, g). This will be performed using the Poisson process description of the stable subordinator and will be postponed. We prove that the law of L given $g = x$ (where $x < b$) is the law of the death time of the stable subordinator started at zero and conditioned to die at x (which does not depend on b); then $L\,(b/g)^{\alpha}$ has the law of the death time of the stable subordinator started at zero and conditioned to die at b.

Summarizing, the law of the absorption time of Y is equal to the law of the absorption time of the stable subordinator conditioned to die at b started at zero, and, conditionally on the absorption times, Y and the conditioned subordinator are bridges of the stable subordinator which start at 0, end at b, and whose length is the corresponding absorption time. We conclude that Y is a stable subordinator, of index α, conditioned to die at b and started at zero.

It remains to prove that if $L = \sup\{s \geq 0 : X_s < b\}$ and $g = X_{L-}$, then under the law of stable subordinator of index α started at zero, $\mathbb{P}_0^\alpha$, the conditional law of L given g has density $s \mapsto f_s^\alpha(x)/u_\alpha(x)$, where f_s^α is the density of X_s under $\mathbb{P}_0^\alpha$ and u_α is the potential density associated to $\mathbb{P}_x^\alpha, x \geq 0$. Thanks to the Lévy-Itô decomposition of Lévy processes (see [Ber96, I.1,Thm. 1]), a stable subordinator increases only by jumps, so that under $\mathbb{P}_0^\alpha$, $X_t = \sum_{s\leq t} \Delta X_s$. (In the preceding sum, there is at most a countable quantity of non-zero terms.) Note that if $f : \mathbb{R}_+ \times [0, 1] \to \mathbb{R}_+$ is measurable, then

$$f(L, g) = \sum_s f(s, X_{s-})\,\mathbf{1}_{X_{s-}<b<X_{s-}+\Delta X_s}$$

since only one term is positive. Since under $\mathbb{P}_0^\alpha$ the jump process of X, given by $(\Delta X_t)_{t\geq 0}$ is a Poisson point process whose characteristic measure π^α is absolutely continuous with respect to Lebesgue measure with a density given by

$$x \mapsto \frac{\alpha C}{\Gamma(1-\alpha)\,x^{1+\alpha}}\mathbf{1}_{x>0},$$

we can use the additive formula (cf. [RY99, XII.1.10, p.475]) to compute

$$\begin{aligned}
\mathbb{E}_0^\alpha(f(L, g)) &= \mathbb{E}_0^\alpha\left(\sum_s f(s, X_{s-})\,\mathbf{1}_{X_{s-}<b<X_{s-}+\Delta X_s}\right)\\
&= \int_0^\infty \mathbb{E}_0^\alpha\big(f(s, X_{s-})\,\mathbf{1}_{X_{s-}<b}\pi^\alpha([b - X_{s-}, \infty))\big)\,ds\\
&= \int_0^\infty \mathbb{E}_0^\alpha\left(f(s, X_{s-})\,\mathbf{1}_{X_{s-}<b}\frac{C}{\Gamma(1-\alpha)\,(b - X_{s-})^\alpha}\right)\,ds.
\end{aligned}$$

We can substitute X_{s-} with X_s in the preceding computation, since $\mathbb{P}_0^\alpha(X_{s-} = X_s) = 1$, to obtain

$$\mathbb{E}_0^\alpha(f(L,g)) = \int_0^\infty \int_0^1 f(s, x)\,f_s^\alpha(x)\,\frac{C}{\Gamma(1-\alpha)\,(1-x)^\alpha}\,dx\,ds.$$

We therefore see that the joint law of (L, g) under $\mathbb{P}_0^\alpha$ is absolutely continuous with respect to Lebesgue measure, with a version of the density given by

$$(s, x) \mapsto f_s^\alpha(x)\,\frac{C}{\Gamma(1-\alpha)\,(1-x)^\alpha}\mathbf{1}_{0<x<1}.$$

Using the explicit value of the potential density u_α, we see that the law of g under $\mathbb{P}_0^\alpha$ has the density

$$\frac{1}{\Gamma(1-\alpha)\,\Gamma(\alpha)\,x^{1-\alpha}\,(1-x)^\alpha},$$

so that g has the generalized arc-sine law with parameter α. We see then that the conditional density of L given $g = x$ can be taken equal to

$$s \mapsto f_s^{\alpha}(x)\, C\Gamma(1-\alpha)\, x^{1-\alpha} = \frac{f_s^{\alpha}(x)}{u_{\alpha}(x)}$$

as announced.

Appendix A. Proof of Proposition 4

Recall that f_t denotes the (bounded and positive) density of the Lévy process at time t; since the process has jumps of both signs, it is known that $f_1(x) \,/\, |x|^{\,1+\alpha}$ converges as $x \to \pm\infty$ to positive constants $c_{\pm}$. We also have the scaling identity

$$f_t(x) = f_1\left(xt^{-1/\alpha}\right) t^{-1/\alpha},$$

from which we can deduce the asymptotic behaviour of $f_t(x)$ as $x \to \infty$ and $t \to 0$:

$$\text{if } t^{-1/\alpha}x \to \pm\infty \text{ then } f_t(x) \sim \frac{c_{\pm} t}{x^{1+\alpha}}. \tag{13}$$

We begin by analyzing the finitude of the resolvent density

$$v_{\lambda}(x,y) = \int_0^{\infty} q_t(x,y)\, e^{-\lambda t}\, dt.$$

From equations (9) and (13)

$$\lim_{t\to\infty} q_t(x,y) = f_1(y)\,,$$

as expected since Y has a stationary distribution with density f_1, and so

$$\int_a^{\infty} q_t(x,y)\, e^{-\lambda t}\, dt < \infty$$

for all $a > 0$. On the other hand, if $x, y = 0$ then

$$q_t(0,0) = p_{e^t-1}(0,0) = f_{e^t-1}(0) = f_1(0)\, \frac{1}{(e^t-1)^{1/\alpha}} \sim f_1(0)\, \frac{1}{t^{1/\alpha}}$$

so that

$$\int_0^{a} q_t(0,0)\, e^{-\lambda t}\, dt < \infty \text{ if and only if } \alpha \in (1,2).$$

If $x \neq y$, then

$$\left|\left(ye^{t/\alpha} - x\right)\left(e^t - 1\right)^{-1/\alpha}\right| \to \infty \text{ as } t \to 0+,$$

so that

$$q_t(x,y) = f_1\Big(\left(ye^{t/\alpha} - x\right)\left(e^t-1\right)^{-1/\alpha}\Big)\, e^{t/\alpha}\left(e^t-1\right)^{-1/\alpha} \sim \frac{c_{\pm}}{|y-x|^{\,1+\alpha}}\left(e^t-1\right) \to 0 \text{ as } t \to 0+\,.$$

Hence

$$\int_0^{a} q_t(x,y)\, e^{-\lambda t}\, dt < \infty \text{ if } x \neq y.$$

The only remaining case is $x = y \neq 0$, and we have

$$q_t(x,x) = f_1\Big(x\left(e^{t/\alpha}-1\right)\left(e^t-1\right)^{-1/\alpha}\Big)\, e^{t/\alpha}\left(e^t-1\right)^{-1/\alpha}.$$

Since

$$\lim_{t\to 0+}\left(e^{t/\alpha}-1\right)\left(e^t-1\right)^{-1/\alpha} = \begin{cases} 0 & \text{if } \alpha > 1 \\ 1 & \text{if } \alpha = 1\,, \\ \infty & \text{if } \alpha < 1 \end{cases}$$

then

$$q_t(x,x) \sim \begin{cases} f_1(0)\left(e^t-1\right)^{-1/\alpha} & \text{if } \alpha\in(1,2) \\ f_1(x)\left(e^t-1\right)^{-1} & \text{if } \alpha=1 \\ \frac{c_\pm}{|x|^{\alpha+1}} t^{-\alpha}\alpha^{1+\alpha} & \text{if } \alpha\in(0,1) \end{cases} \quad \text{as } t\to 0+\,.$$

We conclude that if $x\neq 0$:

$$v_\lambda(x,x)<\infty \Leftrightarrow \alpha\neq 1.$$

We proceed by analyzing the continuity of v_λ. We argued, using the lower semicontinuity of v_λ, that it is continuous when it is infinite. It therefore remains to see if it is continuous where it is finite. For $\alpha\in(1,2)$, we should show that v_λ is continuous everywhere, an assertion which is easily handled: since f_1 is bounded, say by M, then

$$q_t(x,y)\leq M e^{t/\alpha}\left(e^t-1\right)^{-1/\alpha}; \tag{14}$$

the right-hand side multiplied by $e^{-\lambda t}$ is integrable on $[0,\infty)$ for every $\lambda>0$ and so, by dominated convergence, v_λ is continuous and bounded when $\alpha\in(1,2)$. Actually, for any $\alpha\in(0,2)$, $\lambda>0$ and $a>0$,

$$(x,y)\mapsto\int_a^\infty q_t(x,y)\,e^{-\lambda t}\,dt$$

is continuous and bounded. This happens since $\sup_{t\geq\varepsilon,\ x,y\in\mathbb{R}} q_t(x,y)<\infty$ by (14). It is therefore sufficient to study the behaviour of

$$(x,y)\mapsto\int_0^a q_t(x,y)\,e^{-\lambda t}\,dt$$

for some (judiciously chosen) $a>0$. To do this, note that since f_1 is bounded and because of its asymptotic behaviour recalled in (13), there exist constants $D,M>0$ such that for any $b>0$:

$$f_1(x)\leq\begin{cases} M & |x|\leq b \\ \frac{D}{|x|^{1+\alpha}} & |x|>b \end{cases}.$$

By scaling, it follows that

$$f_t(x)\leq\begin{cases} Mt^{-1/\alpha} & |x|\leq bt^{1/\alpha} \\ \frac{Dt}{|x|^{1+\alpha}} & |x|>bt^{1/\alpha} \end{cases}.$$

Hence:

$$q_t(x,y)\leq\begin{cases} M\left(e^t-1\right)^{-1/\alpha} & \left|ye^{t/\alpha}-x\right|/\left(e^t-1\right)^{1/\alpha}\leq b \\ D\frac{e^t-1}{\left|ye^{t/\alpha}-x\right|^{1+\alpha}} & \left|ye^{t/\alpha}-x\right|/\left(e^t-1\right)^{1/\alpha}>b \end{cases}. \tag{15}$$

To study the continuity of v_λ, we will make a careful analysis implementing (15). Let us first show that v_λ is bicontinuous at (x,y) if $x\neq y$. First, consider $\varepsilon,a>0$ such that

$$\inf_{\substack{|x'-x|,|y'-y|\leq\varepsilon \\ t\leq a}}\left|y'e^{t/\alpha}-x'\right|=\rho>0, \tag{16}$$

and then use the second bound of (15), with b small enough, together with (16) to obtain

$$\sup_{\substack{|x'-x|,|y'-y|\leq\varepsilon \\ t\leq a}} q_t(x',y')\leq D\frac{e^a-1}{\rho^{1+\alpha}}.$$

From the above, we conclude the continuity of $(x',y')\mapsto\int_0^a q_t(x',y')\,e^{-\lambda t}\,dt$ at (x,y).

It remains to verify the continuity of v_λ at (y,y) if $\alpha\in(0,1)$ and $y\neq 0$; for concreteness we will assume that $y>0$. We have to argue separately that

$$\lim_{\substack{x,z\to y \\ x\leq z}} v_\lambda(x,z)=v_\lambda(y,y) \quad\text{and}\quad \lim_{\substack{x,z\to y \\ x>z}} v_\lambda(x,z)=v_\lambda(y,y)\,.$$

$x \leq z$: Choose $\varepsilon > 0$ such that $y - \varepsilon > 0$; if $x \leq z$ and $z \geq y - \varepsilon$ then

$$\left| z e^{t/\alpha} - x \right| \geq z \left(e^{t/\alpha} - 1 \right) \geq (y - \varepsilon) \left(e^{t/\alpha} - 1 \right).$$

Since $\alpha \in (0,1)$, then

$$\lim_{t \to 0} \frac{e^{t/\alpha} - 1}{\left(e^t - 1\right)^{1/\alpha}} \to 0$$

and so there exists $a > 0$ such that

$$\inf_{\substack{y-\varepsilon \leq z, x \leq z \\ 0 \leq t \leq a}} \frac{\left| z e^{t/\alpha} - x \right|}{\left(e^t - 1\right)^{1/\alpha}} \geq D.$$

We can then continue from (16).

$x > z$: Here is where we have to be most careful since

$$\inf_{\substack{x,z \in B_\varepsilon(y) \\ t \leq a}} \frac{\left| z e^{t/\alpha} - x \right|}{\left(e^t - 1\right)^{1/\alpha}} = 0.$$

for all $\varepsilon, a > 0$ and so the bounds used previously no longer work.

For $x > z$, let us introduce the function

$$t \mapsto \frac{z e^{t/\alpha} - x}{\left(e^t - 1\right)^{1/\alpha}};$$

which tends to $-\infty$ as t decreases to zero, tends to z when t goes to infinity, touches 0 at $\alpha \log(x/z)$, and since its derivative is given by

$$\frac{x e^t - z e^{t/\alpha}}{\alpha \left(e^t - 1\right)^{1+1/\alpha}},$$

it is increasing on $(0, \alpha \log(x/z) \,/\, (1-\alpha)]$ and decreasing on $[\alpha \log(x/z) \,/\, (1-\alpha), \infty)$. The function

$$\phi : t \mapsto \frac{\left| z e^{t/\alpha} - x \right|}{\left(e^t - 1\right)^{1/\alpha}}$$

will be important in what follows because it governs, by means of (15), the choice of the bound on $q_t(x,z)$.

If $b \leq z$ then ϕ equals b at two points, say t_1 and t_2, delimiting the three regions where on which we can bound q_t:

$$q_t(x,z)\, e^{-\lambda t} \leq \begin{cases} \frac{D\left(e^t - 1\right)}{\left(x - z e^{t/\alpha}\right)^{1+\alpha}} & \text{if } t \leq t_1 \\ M \left(e^t - 1\right)^{-1/\alpha} & \text{if } t \in [t_1, t_2] \\ \frac{D\left(e^t - 1\right)}{\left(z e^{t/\alpha} - x\right)^{1+\alpha}} & \text{if } t \geq t_2. \end{cases} \tag{17}$$

There is an obvious problem with the second region since there the upper bound is asymptotic to $M t^{-1/\alpha}$ which is not integrable on $(0, \varepsilon)$ for any positive ε since $\alpha \in (0,1)$.

Let us start with the first region: we had assumed that $y > 0$ and so $z > 0$ if it is close enough to y. Let $d > 0$ and set $r = \alpha \log\left(x/z \left(1 - d\,(x-z)^{1/\alpha} \right) \right)$. Note that

$$\lim_{\substack{x,z \to y \\ x > z}} \phi(r) = \frac{d y^{1+1/\alpha}}{\alpha^{1/\alpha}}$$

so that $t_1 \leq r$ when d is small enough. We would like to see that

$$\limsup_{\substack{x,z \to y \\ x > z}} \int_0^r \frac{e^t - 1}{\left(x - z e^{t/\alpha}\right)^{1+\alpha}}\, dt = 0$$

or equivalently

$$\limsup_{\substack{x,z\to y\\ x>z}} \int_0^r \frac{t}{\left(x-ze^{t/\alpha}\right)^{1+\alpha}}\,dt=0. \tag{18}$$

Since

$$\frac{d}{dt}\frac{t}{\left(x-ze^{t/\alpha}\right)^{1+\alpha}}=\frac{1}{\alpha\left(x-ze^{t/\alpha}\right)^{2+\alpha}}\left(\alpha x-\alpha ze^{t/\alpha}+tze^{t/\alpha}\left(1+\alpha\right)\right),$$

we see that the integrand in (18), denoted ψ, is increasing on $[0,r]$, going from 0 to

$$\frac{r}{x^{1+\alpha}d^{1+\alpha}\left(x-z\right)^{1+1/\alpha}}\leq\frac{\alpha}{zx^{1+\alpha}d\left(x-z\right)^{1/\alpha}},$$

where the upper bound follows from $\log(1+t)\leq t$. Note that if $r_0=0$ and

$$r_n=\alpha\log\Big(x/z(1-d_n\left(x-z\right)^{1/(1+\alpha)})\Big),$$

where (d_n) decreases to zero then

$$\psi(r_n)\leq\frac{\alpha}{zx^{1+\alpha}d_n^{1+\alpha}},\quad \lim_{\substack{x,z\to y\\ x>z}}\psi(r_n)=\frac{\alpha}{y^{2+\alpha}d_n^{1+\alpha}},$$

and

$$r_n-r_{n-1}\leq\alpha\left(x-z\right)^{1/(1+\alpha)}\left(d_{n-1}-d_n\right)\frac{1}{\left(1-d_{n-1}\left(x-z\right)^{1/(1+\alpha)}\right)}$$

so that

$$\int_{r_{n-1}}^{r_n}\frac{t}{\left(x-ze^{t/\alpha}\right)^{1+\alpha}}\,dt\leq\frac{\alpha^2}{zx^{1+\alpha}}\left(x-z\right)^{1/(1+\alpha)}\frac{d_{n-1}-d_n}{d_n^{1+\alpha}}\frac{1}{\left(1-d_{n-1}\left(x-z\right)^{1/(1+\alpha)}\right)}.$$

If N is such that $r\leq r_N$ then

$$\limsup_{\substack{x,z\to y\\ x>z}}\int_0^r\psi(t)\,dt\leq\frac{\alpha^2}{y^{2+\alpha}}\limsup_{\substack{x,z\to y\\ x>z}}\left(x-z\right)^{1/(1+\alpha)}\sum_{1\leq n\leq N-1}\frac{d_n-d_{n+1}}{d_{n+1}^{1+\alpha}}.$$

Let $d_n=\delta/\sqrt{n}$, which is so chosen so that

$$\psi(r_n)\leq\frac{\alpha}{z}\left(\frac{\sqrt{n}}{x\delta}\right)^{1+\alpha};$$

it also implies that

$$\frac{d_n-d_{n+1}}{d_n^{1+\alpha}}\sim\frac{\delta^{-\alpha}}{2n^{1-\alpha/2}}\text{ as }n\to\infty.$$

Finally, note that if N is bigger than $\delta/d\left(x-z\right)^{-1/\alpha(1+\alpha)}$ but taken asymptotic to it as $x,z\to y$, then $r\leq r_N$ and

$$\limsup_{\substack{x,z\to y\\ x>z}}\int_0^r\psi(t)\,dt\leq\frac{\alpha^2\delta^{-\alpha}}{y^{2+\alpha}\alpha}\limsup_{\substack{x,z\to y\\ x>z}}\left(x-z\right)^{1/(1+\alpha)}N^{\alpha/2}/2.$$

Since $N^\alpha\sim\left(\delta/d\right)^{\alpha/2}\left(x-z\right)^{1/(1+\alpha)}$ then

$$\limsup_{\substack{x,z\to y\\ x>z}}\int_0^r\psi(t)\,dt\leq\frac{\alpha^2\delta^{-\alpha/2}}{2y^{2+\alpha}d^{\alpha/2}\alpha},$$

which can be made as small as we want by taking δ big enough.

We now consider the second region showing that for any $y>0$

$$\limsup_{b\to0}\limsup_{\substack{x,z\to y\\ x>z}}\int_{t_1}^{t_2}\left(e^t-1\right)^{-1/\alpha}\,dt=0\text{ or equivalently }\lim_{\substack{x,z\to y\\ x>z}}\int_{t_1}^{t_2}t^{-1/\alpha}\,dt\leq\frac{2\alpha b}{y}.$$

This will accomplished by means of a lower bound on t_1 and an upper bound for t_2, valid as $x, z \downarrow y$; the bounds on t_1 and t_2 are obtained as in the analysis of the first region: recall that if

$$r_\pm = \alpha \log\Big(\frac{x}{z}\left(1 \pm d\,(x-z)^{1/\alpha}\right)\Big) \text{ then } \lim_{\substack{x,z\to y\\ x>z}} \phi(r_\pm) = \frac{dy^{1/\alpha+1}}{\alpha^{1/\alpha}}.$$

Hence, for arbitrary $d > b\alpha^{1/\alpha}/y^{1/\alpha+1}$

$$r_- \le t_1 \le t_2 \le r_+$$

for x, z close enough to y. We then obtain

$$\lim_{\substack{x,z\to y\\ x>z}} \int_{t_1}^{t_2} t^{-1/\alpha}\,dt \le \lim_{\substack{x,z\to y\\ x>z}} \frac{r_+ - r_-}{r_-^{1/\alpha}} = \frac{2dy^{1/\alpha}}{\alpha^{1/\alpha-1}},$$

so that

$$\limsup_{x\to y+} \int_{t_1}^{t_2} t^{-1/\alpha}\,dt \le \frac{2\alpha b}{y}.$$

On the third region,

$$\lim_{\substack{x,z\to y\\ x>z}} \int_{t_2}^{\infty} q_t(x,z)\,e^{-\lambda t}\,dt = v_\lambda(y,y)\,,$$

as we now see. This implies the bicontinuity of v_λ at (y, y). It suffices to prove that for any $a > 0$,

$$\lim_{\substack{x,z\to y\\ x>z}} \int_{t_2}^{a} q_t(x,z)\,e^{-\lambda t}\,dt = \int_0^a q_t(y,y)\,e^{-\lambda t}\,dt,$$

which we achieve by arguing as for the first region. First, note that on $[t_2, a]$, for small enough a, we have

$$q_t(x,z) \le \frac{D\,(e^t-1)}{\left(ze^{t/\alpha}-x\right)^{1+\alpha}} \le \frac{2Dt}{\left(ze^{t/\alpha}-x\right)^{1+\alpha}}.$$

The rightmost bound, denoted ψ, is decreasing on $[t_2, a]$ and by setting

$$r_1 = \alpha \log\Big(x/z\left(1 - d_1\,(x-z)^{1/(1+\alpha)}\right)\Big)\,,$$

we see that $\psi(t)$ is uniformly bounded on $[r_1, a]$ as $x, z \to y$, so that by dominated convergence:

$$\lim_{\substack{x,z\to y\\ x>z}} \int_{r_1}^{a} q_t(x,z)\,e^{-\lambda t}\,dt = \int_0^a q_t(y,y)\,e^{-\lambda t}\,dt.$$

It remains to see that

$$\lim_{\substack{x,z\to y\\ x>z}} \int_{t_2}^{r_1} \frac{t}{\left(ze^{t/\alpha}-x\right)^{1+\alpha}}\,dt = 0;$$

for this we adapt the analysis of the first region from Equation (18), using

$$r_n = \alpha \log\Big(x/z\left(1 + d_n\,(x-z)^{1/(1+\alpha)}\right)\Big)\,,$$

where d_n decreases to zero.

Acknowledgements

The authors would like to thank Marc Yor for his comments regarding a preliminary version of this work and for his "Gaussian" proof of Theorem 3 in the Brownian case. GUB would like to thank his PhD supervisors, J. Bertoin and M.E. Caballero, for their help, support, and encouragement during the development of this work. GUB's research was supported by CoNaCyT grant No. 174498 and a postdoctoral fellowship from UNAM, and partially conducted at UNAM's *Instituto de Matemáticas* and Paris VI's *Laboratoire des Probabilités et Modèles Aléatoires*. Collaboration was possible through proyect ECOS grant No. M01 M07 with the *Université d'Angers* as a host.

References

[Ber96] Jean Bertoin, *Lévy processes*, Cambridge Tracts in Mathematics, vol. 121, Cambridge University Press, Cambridge, 1996. MR MR1406564 (98e:60117)

[Bre68] Leo Breiman, *A delicate law of the iterated logarithm for non-decreasing stable processes*, Ann. Math. Statist. **39** (1968), 1818–1824. MR MR0233420 (38 #1742)

[BW96] Jean Bertoin and Wendelin Werner, *Stable windings*, Ann. Probab. **24** (1996), no. 3, 1269–1279. MR MR1411494 (98a:60105)

[Cha94] Loïc Chaumont, *Processus de Lévy et conditionnement*, Ph.D. thesis, Université Paris VI, 1994.

[Cha96] ———, *Conditionings and path decompositions for Lévy processes*, Stochastic Process. Appl. **64** (1996), no. 1, 39–54. MR MR1419491 (98b:60131)

[Cha97] ———, *Excursion normalisée, méandre et pont pour les processus de Lévy stables*, Bull. Sci. Math. **121** (1997), no. 5, 377–403. MR MR1465814 (99a:60077)

[dB81] N. G. de Bruijn, *Asymptotic methods in analysis*, third ed., Dover Publications Inc., New York, 1981. MR MR671583 (83m:41028)

[FPY93] Pat Fitzsimmons, Jim Pitman, and Marc Yor, *Markovian bridges: construction, Palm interpretation, and splicing*, Seminar on Stochastic Processes, 1992 (Seattle, WA, 1992), Progr. Probab., vol. 33, Birkhäuser Boston, Boston, MA, 1993, pp. 101–134. MR MR1278079 (95i:60070)

[GS79] R. K. Getoor and M. J. Sharpe, *The Markov property at co-optional times*, Z. Wahrsch. Verw. Gebiete **48** (1979), no. 2, 201–211. MR MR534845 (80m:60083)

[GS81] ———, *Markov properties of a Markov process*, Z. Wahrsch. Verw. Gebiete **55** (1981), no. 3, 313–330. MR MR608025 (82j:60130)

[IM74] Kiyosi Itô and Henry P. McKean, Jr., *Diffusion processes and their sample paths*, Springer-Verlag, Berlin, 1974, Second printing, corrected, Die Grundlehren der mathematischen Wissenschaften, Band 125. MR MR0345224 (49 #9963)

[Jac74] Martin Jacobsen, *Splitting times for Markov processes and a generalised Markov property for diffusions*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **30** (1974), 27–43. MR MR0375477 (51 #11670)

[Jak08] Tomasz Jakubowski, *On Harnack inequality for α-stable Ornstein-Uhlenbeck processes*, Math. Z. **258** (2008), no. 3, 609–628. MR MR2369047

[JP77] Martin Jacobsen and Jim Pitman, *Birth, death and conditioning of Markov chains*, Ann. Probability **5** (1977), no. 3, 430–450. MR MR0445613 (56 #3949)

[Kal73] Olav Kallenberg, *Canonical representations and convergence criteria for processes with interchangeable increments*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **27** (1973), 23–36. MR MR0394842 (52 #15641)

[Kal81] ———, *Splitting at backward times in regenerative sets*, Ann. Probab. **9** (1981), no. 5, 781–799. MR MR628873 (84h:60103)

[Kal02] ———, *Foundations of modern probability*, second ed., Probability and its Applications (New York), Springer-Verlag, New York, 2002.

[Leb65] N. N. Lebedev, *Special functions and their applications*, Revised English edition. Translated and edited by Richard A. Silverman, Prentice-Hall Inc., Englewood Cliffs, N.J., 1965. MR MR0174795 (30 #4988)

[Mil77] P. W. Millar, *Zero-one laws and the minimum of a Markov process*, Trans. Amer. Math. Soc. **226** (1977), 365–391. MR MR0433606 (55 #6579)

[MSW72] P. A. Meyer, R. T. Smythe, and J. B. Walsh, *Birth and death of Markov processes*, Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability (Univ. California, Berkeley, Calif., 1970/1971), Vol. III: Probability theory (Berkeley, Calif.), Univ. California Press, 1972, pp. 295–305. MR MR0405600 (53 #9392)

[Nov81] A. A. Novikov, *The martingale approach in problems on the time of the first crossing of nonlinear boundaries*, Trudy Mat. Inst. Steklov. **158** (1981), 130–152, 230, Analytic number theory, mathematical analysis and their applications. MR MR662841 (84j:60082)

[Olv97] Frank W. J. Olver, *Asymptotics and special functions*, AKP Classics, A K Peters Ltd., Wellesley, MA, 1997, Reprint of the 1974 original [Academic Press, New York; MR0435697 (55 #8655)]. MR MR1429619 (97i:41001)

[Pat07] Pierre Patie, *Two-sided exit problem for a spectrally negative α-stable Ornstein-Uhlenbeck process and the Wright's generalized hypergeometric functions*, Electron. Comm. Probab. **12** (2007), 146–160 (electronic). MR MR2318162

[PY81] Jim Pitman and Marc Yor, *Bessel processes and infinitely divisible laws*, Stochastic integrals (Proc. Sympos., Univ. Durham, Durham, 1980), Lecture Notes in Math., vol. 851, Springer, Berlin, 1981, pp. 285–370. MR MR620995 (82j:60149)

[RY99] Daniel Revuz and Marc Yor, *Continuous martingales and Brownian motion*, third ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 293, Springer-Verlag, Berlin, 1999. MR 2000h:60050

[Sat99] Ken-iti Sato, *Lévy processes and infinitely divisible distributions*, Cambridge Studies in Advanced Mathematics, vol. 68, Cambridge University Press, Cambridge, 1999, Translated from the 1990 Japanese original, Revised by the author. MR MR1739520 (2003b:60064)

[Sha69] Michael Sharpe, *Zeroes of infinitely divisible densities*, Ann. Math. Statist. **40** (1969), 1503–1505. MR MR0240850 (39 #2195)

[Wat75] Shinzo Watanabe, *On time inversion of one-dimensional diffusion processes*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **31** (1974/75), 115–124. MR MR0365731 (51 #1983)

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