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# PASSAGE TIME FROM FOUR TO TWO BLOCKS OF OPINIONS IN THE VOTER MODEL

IRINA KURKOVA AND KILIAN RASCHEL

ABSTRACT. We consider a voter model in which there are two candidates and initially, in the population  $\mathbf{Z}$ , four connected blocks of same opinions. We assume that a citizen changes his mind at a rate proportional to the number of its neighbors that disagree with him, and we study the passage from four to two connected blocks of same opinions. More precisely, we make explicit the generating function of the probabilities to go from four to two blocks in a given time, and we find the asymptotic of these probabilities when the time goes to infinity.

KEYWORDS. Voter model; Random walk in the quarter plane; Hitting times; Integral representations

AMS 2000 SUBJECT CLASSIFICATION: primary 82C22, 60G50; secondary 60G40, 30E20

## 1. INTRODUCTION

**Context.** Random walks with small steps in the quarter plane  $\mathbf{Z}_+^2 = \{0, 1, 2, \dots\}^2$  spatially homogeneous in the interior and on each of the two axes are now rather well studied. The analytic approach [7] elaborated by Fayolle, Iasnogorodski and Malyshev provided the generating function, say  $H(x, y)$ , of the stationary probabilities in the ergodic case, and also that of the Green functions in the transient case. Further analysis allowed to compute the asymptotic of these quantities along any path in  $\mathbf{Z}_+^2$ , see [10, 12, 15, 18, 19].

This article is an attempt to incorporate the parameter  $z$  of time into this approach, in order to derive the trivariate generating function  $H(x, y; z)$  of the probabilities  $h_{i,j,k}$  that the walk is in state  $(i, j)$  at time  $k$ . This is actually particularly important to be able to deal with this time variable  $z$ , for combinatorics (e.g., to count certain numbers of walks confined to the quarter plane, see [11]) and for probability as well (e.g., to compute the distribution of some hitting times). Our work is one of the first attempts, after [3, 8], to undertake the study of the function  $z \mapsto H(x, y; z)$  and to derive asymptotic results from it. This finds application for the voter model: indeed, this completes results of [2, 16] about the hitting time of the so-called Heaviside configuration in the voter model with initially four blocks of opinions.

**Voter model.** By the voter model we mean a continuous-time process on  $\{0, 1\}^{\mathbf{Z}}$  (here and throughout,  $\mathbf{Z} = \{\dots, -1, 0, 1, \dots\}$ ) that can be interpreted as follows: initially, there is zero or one particle at each site of  $\mathbf{Z}$ , then a particle appears (resp. disappears) at an empty (resp. occupied) site  $x$  according to an exponential law with a rate proportional to the number of nearest neighbors of  $x$  which are occupied (resp. empty). Moreover, we assume that the initial state appertains to the set of configurations having a finite number of empty (resp. occupied) sites on the left (resp. right) of the origin 0. In particular, this implies that at any time the process will belong to this set of configurations. As a consequence there is, at any time, a finite number of “01” (resp. “10”), i.e., a finite number

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I. Kurkova: Laboratoire de Probabilités et Modèles Aléatoires, Université Pierre et Marie Curie, 4 Place Jussieu, 75252 Paris Cedex 05, France. Email: [irina.kourkova@upmc.fr](mailto:irina.kourkova@upmc.fr).

K. Raschel: CNRS and Laboratoire de Mathématiques et Physique Théorique, Université François Rabelais, Parc de Grandmont, 37200 Tours, France. Email: [Kilian.Raschel@lmpt.univ-tours.fr](mailto:Kilian.Raschel@lmpt.univ-tours.fr).

of pairs of sites  $(x, x + 1)$  with zero (resp. one) particle at  $x$  and one (resp. zero) particle at  $x + 1$ .

The underlying discrete-time voter model is a Markov chain with the following dynamic: denote by  $\mathcal{C}_k$  the configuration at time  $k$  (and remember that according to the previous paragraph, there is only a finite number of “01” and “10” in  $\mathcal{C}_k$ ); next, in order to construct  $\mathcal{C}_{k+1}$ , one first chooses with a uniform distribution one of these “01” and “10” in  $\mathcal{C}_k$ , then one replaces it, with probability  $1/2$ , by “00” or “11”.

If the voter model starts from the Heaviside configuration, i.e., the configuration having only occupied (resp. empty) sites on the left (resp. right) of the origin, then at any time, the process will be a translation of it. This fact suggests to consider the following equivalence relation: two configurations are said to be equivalent if they are translations the one of the other.

From now on, we shall work on the underlying quotient space, the equivalence classes of which being identified by finite sets of positive integers  $(X_1, Y_1, \dots, X_N, Y_N)$ :

$$(1.1) \quad \dots 111 \overbrace{000000}^{X_1} \overbrace{111111}^{Y_1} \overbrace{000000}^{X_2} \overbrace{111111}^{Y_2} \dots \overbrace{000}^{X_N} \overbrace{1111}^{Y_N} 000 \dots,$$

$N$  being the number of finite blocks of zeros (or ones) and  $X_\ell$  (resp.  $Y_\ell$ ),  $\ell \in \{1, \dots, N\}$ , the size of the  $\ell$ th block of zeros (resp. ones). The number  $N$  of finite blocks of zeros is a non-increasing function of the time; furthermore,  $N = 0$  corresponds to the class of the Heaviside configuration.

We refer to [13] for additional details on the voter model and, more generally, for further information about interacting particle systems.

**Hitting time of the Heaviside configuration.** Let  $\tau$  denote the hitting time of the Heaviside configuration. It is proved in [2] that

$$(1.2) \quad \mathbf{E}[\tau^{3/2-\epsilon}] < \infty, \quad \forall \epsilon > 0,$$

$$(1.3) \quad \mathbf{E}[\tau^{3/2+\epsilon}] = \infty, \quad \forall \epsilon > 0.$$

The statement (1.2) is proved by an adequate use of some Lyapunov functions. To show (1.3), it suffices to do it only for initial states with  $N = 1$  in (1.1); that is done in [2], by applying results of [1].

With the notations (1.1), consider the process starting from an initial state with  $N = 1$ :  $(X_1, Y_1) = (X_1(k), Y_1(k))_{k \geq 0}$ . We rename it here  $(X, Y) = (X(k), Y(k))_{k \geq 0}$ :

$$\dots 111 \overbrace{000000}^X \overbrace{111111}^Y 000 \dots$$

The process  $(X, Y)$  is a Markov chain on  $\mathbf{Z}_+^2$  which is absorbed as it reaches the boundary, since the Heaviside configuration is an absorbing state for the voter model. Moreover, using the dynamic of the discrete-time voter model explained above, we obtain that  $(X, Y)$  has homogeneous transition probabilities in the interior of  $\mathbf{Z}_+^2$  equal to (with obvious notations)

$$p_{1,0} = p_{1,-1} = p_{0,-1} = p_{-1,0} = p_{-1,1} = p_{0,1} = 1/6$$

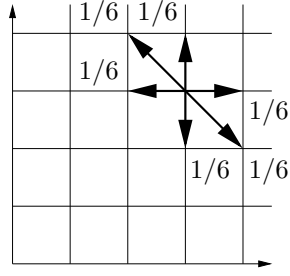
and the others to 0, see Figure 1. Further, the hitting time  $\tau$  can be expressed as

$$(1.4) \quad \tau = \inf\{k \geq 0 : X(k) = 0 \text{ or } Y(k) = 0\}.$$

Define also the hitting times of the horizontal and vertical axes:

$$(1.5) \quad S = \inf\{k \geq 0 : Y(k) = 0\}, \quad T = \inf\{k \geq 0 : X(k) = 0\},$$

so that  $\tau$ , defined in (1.4), is equal to  $\inf\{S, T\}$ .


 FIGURE 1. Transitions of the process  $(X, Y)$ 

**Main results.** In this work, we find the probabilities that the process is absorbed at some site of the boundary in a given time, namely,  $\mathbf{P}_{(i_0, j_0)}[S = k]$  and  $\mathbf{P}_{(i_0, j_0)}[T = k]$ , for any  $k \geq 0$  and any  $(i_0, j_0) \in \mathbf{Z}_+^2$ . For this we use Proposition 5 of Section 2, taken from [19, Chapter F], which even gives an integral representation of the generating functions

$$(1.6) \quad h^{i_0, j_0}(x; z) = \sum_{i \geq 1} \sum_{k \geq 0} \mathbf{P}_{(i_0, j_0)}[(X, Y) \text{ hits } (i, 0) \text{ at time } k] x^i z^k,$$

$$(1.7) \quad \tilde{h}^{i_0, j_0}(y; z) = \sum_{j \geq 1} \sum_{k \geq 0} \mathbf{P}_{(i_0, j_0)}[(X, Y) \text{ hits } (0, j) \text{ at time } k] y^j z^k.$$

Then  $\mathbf{P}_{(i_0, j_0)}[S = k]$  and  $\mathbf{P}_{(i_0, j_0)}[T = k]$  can be expressed from  $h^{i_0, j_0}(1; z)$  and  $\tilde{h}^{i_0, j_0}(1; z)$  via the Cauchy formulas. Note, besides, that we also find the trivariate function

$$(1.8) \quad H^{i_0, j_0}(x, y; z) = \sum_{i, j \geq 1} \sum_{k \geq 0} \mathbf{P}_{(i_0, j_0)}[(X(k), Y(k)) = (i, j)] x^{i-1} y^{j-1} z^k$$

thanks to the functional equation (2.1).

The main result of this paper is the following.

**Theorem 1.** *As the time  $k \rightarrow \infty$ , we have*

$$(1.9) \quad \mathbf{P}_{(i_0, j_0)}[S = k] \sim \frac{9}{16} \left( \frac{3}{\pi} \right)^{1/2} \frac{i_0 j_0 (i_0 + j_0)}{k^{5/2}}.$$

It will be a simple consequence of Proposition 8 and of classical singularity analysis [9, Sections 6.2–6.4], see Section 3.

Since the transition probabilities of the walk are such that  $\mathbf{P}_{(i_0, j_0)}[S = k] = \mathbf{P}_{(j_0, i_0)}[T = k]$ , see Figure 1, the asymptotic of  $\mathbf{P}_{(i_0, j_0)}[T = k]$  is exactly the same. Also,

$$(1.10) \quad \mathbf{P}_{(i_0, j_0)}[\tau = k] = \mathbf{P}_{(i_0, j_0)}[S = k] + \mathbf{P}_{(i_0, j_0)}[T = k],$$

so that Theorem 1 entails the following corollary that completes the results of [2, 16].

**Corollary 2.** *The result (1.3) proved in [2] for  $\epsilon > 0$  also holds for  $\epsilon = 0$ .*

Finally, the precision of the asymptotic result (1.9) implies Corollary 3 below that compares two typical ways on conditioning the process  $(X, Y)$  to never reach the axes. Define  $h(i_0, j_0) = i_0 j_0 (i_0 + j_0)$ —it is the unique positive harmonic function associated with the walk  $(X, Y)$  absorbed at the boundary of  $\mathbf{Z}_+^2$ .

**Corollary 3.** *The Doob  $h$ -process of  $(X, Y)$  coincides in distribution with the limit, as  $k \rightarrow \infty$ , of the process conditioned on  $\{\tau > k\}$ .*

The proof relies on the following precise asymptotic, as  $k \rightarrow \infty$ ,

$$\mathbf{P}_{(i_0, j_0)}[\tau \geq k] \sim \frac{27}{16} \left( \frac{3}{\pi} \right)^{1/2} \frac{i_0 j_0 (i_0 + j_0)}{k^{3/2}},$$

which is a direct consequence of Theorem 1 and (1.10). It is carried out then by a standard reasoning as in [19, Chapter F].

**Other approaches.** We close this introduction by mentioning other possible approaches for analyzing asymptotic tail distribution of hitting times for random walks in cones of  $\mathbf{Z}^d$ . First, as already quoted, methods using Lyapunov functions in [1, 2, 16] show the finiteness or infiniteness of hitting times' moments. A series of tail distribution estimates for hitting times is presented in [22], by using potential theory. In a very recent work [4] the tail asymptotic of the hitting time up to a multiplicative factor is obtained by comparison with Brownian motion. All these methods are powerful for rather general random walks in conic domains of  $\mathbf{Z}^d$ , but do not give as much accurate results as the analytic approach in this paper. Finally, let us mention the paper [5], where an approach based on an extension of the Karlin-McGregor formula is applied to the family of the so-called non-colliding random walks. The latter leads to precise results, but it exploits a particular independence property of this family, and therefore seems to be restricted to this class of models.

## 2. EXACT DISTRIBUTION OF THE HITTING TIMES OF BOTH AXES

**A functional equation and the kernel of the walk.** With the notations (1.6), (1.7) and (1.8) of Section 1, we can state on  $\{(x, y, z) \in \mathbf{C}^3 : |x|, |y|, |z| \leq 1\}$  (here and throughout,  $\mathbf{C}$  denotes the complex plane) the following crucial functional equation:

$$(2.1) \quad K(x, y, z)H^{i_0, j_0}(x, y, z) = h^{i_0, j_0}(x, z) + \tilde{h}^{i_0, j_0}(y, z) - x^{i_0}y^{j_0},$$

where  $K(x, y, z)$  is the following polynomial—called the kernel of the walk—, depending only on the walk's transition probabilities:

$$(2.2) \quad K(x, y, z) = xyz[\sum_{-1 \leq i, j \leq 1} p_{i, j}x^i y^j - 1/z].$$

For  $z = 0$ , Equation (2.1) simply becomes  $\mathbf{P}_{(i_0, j_0)}[(X(0), Y(0)) = (i_0, j_0)] = 1$ . For  $z = 1$ , it becomes a functional equation between the Green functions generating function and the absorption probabilities generating functions, studied in [12, 18, 19]. For the proof of (2.1), we exactly use the same arguments as in [19, Chapter F].

We now study the set of the zeros of the kernel  $K(x, y, z)$  defined in (2.2). For this we start by remarking that it can be written alternatively

$$(2.3) \quad K(x, y, z) = a(x, z)y^2 + b(x, z)y + c(x, z) = \tilde{a}(y, z)x^2 + \tilde{b}(y, z)x + \tilde{c}(y, z),$$

where  $a(x, z) = z(x+1)/6$ ,  $b(x, z) = zx^2/6 - x + z/6$ ,  $c(x, z) = zx(x+1)/6$ ,  $\tilde{a}(y, z) = z(y+1)/6$ ,  $\tilde{b}(y, z) = zy^2/6 - y + z/6$  and  $\tilde{c}(y, z) = zy(y+1)/6$ . Next, we introduce the algebraic function  $Y(x, z)$  defined by  $K(x, Y(x, z), z) = 0$ . Note that  $K(x, y, z) = 0$  is equivalent to  $[b(x, z) + 2a(x, z)y]^2 = d(x, z)$ , where

$$d(x, z) = b(x, z)^2 - 4a(x, z)c(x, z),$$

so that the construction of the function  $Y(x, z)$  is equivalent to that of the square root of the polynomial  $d(x, z)$ . For this we need the following:

**Lemma 4.** *Let  $z \in ]0, 1[$ . The four roots of  $x \mapsto d(x, z)$  are positive and mutually distinct. We call them  $x_1(z) < x_2(z) < x_3(z) < x_4(z)$ . They satisfy  $x_1(z)x_4(z) = x_2(z)x_3(z) = 1$ . In particular,  $x_1(z), x_2(z) \in ]0, 1[$  and  $x_3(z), x_4(z) \in ]1, \infty[$ . Further,  $x_1(0) = x_2(0) = 0$ ,  $x_3(0) = x_4(0) = \infty$ ,  $x_2(1) = x_3(1) = 1$  and  $x_1(1) = 7 - 4\sqrt{3}$ ,  $x_4(1) = 7 + 4\sqrt{3}$ .*

*Proof.* As we can easily verify, the polynomial  $d(x, z)$  is reciprocal, in other words it satisfies  $x^4d(1/x, z) = d(x, z)$ . This property allows us to write it as a second degree polynomial in the variable  $x + 1/x$ . Following this way we obtain the explicit expression of its roots: if  $s_1(z) = 3/z + 1$  and  $s_2(z) = (6/z + 3)^{1/2}$ , then  $x_1(z) = s_1(z) + s_2(z) + [(s_1(z) + s_2(z))^2 - 1]^{1/2}$  and  $x_2(z) = s_1(z) - s_2(z) + [(s_1(z) - s_2(z))^2 - 1]^{1/2}$ ,  $x_3(z) = 1/x_2(z)$  and  $x_4(z) = 1/x_1(z)$ . All the properties of Lemma 4 follow immediately from these explicit expressions.  $\square$

There are two branches of the square root of  $d(x; z)$ . Each determination leads to a single valued and meromorphic function on the complex plane  $\mathbf{C}$  appropriately cut, that is, in our case, on  $\mathbf{C} \setminus ([x_1(z), x_2(z)] \cup [x_3(z), x_4(z)])$ . We have

$$Y(x; z) = \frac{-b(x; z) \pm d(x; z)^{1/2}}{2a(x; z)},$$

and we fix the notations of the branches  $Y_0(x; z)$  and  $Y_1(x; z)$  by (arbitrarily) choosing that  $|Y_0(x; z)| < |Y_1(x; z)|$  on the whole of  $\mathbf{C} \setminus ([x_1(z), x_2(z)] \cup [x_3(z), x_4(z)])$ . For more details about the construction of algebraic functions, see, e.g., [20].

In a similar way, the functional equation (2.1) defines also an algebraic function  $X(y; z)$ . But it turns out that  $K(x, y; z) = K(y, x; z)$ , see (2.2), so that  $X(y; z) = Y(y; z)$ ; in particular, all properties proved for  $Y(x; z)$  immediately result in similar ones for  $X(y; z)$ .

**Explicit expression of distributions.** They are obtained in the result that follows.

**Proposition 5.** *The function  $h^{i_0, j_0}(x; z)$  is equal to*

$$(2.4) \quad h^{i_0, j_0}(x; z) = x^{i_0} Y_0(x; z)^{j_0} + \int_{x_1(z)}^{x_2(z)} t^{i_0} \mu_{j_0}(t; z) \left[ \frac{\partial_t w(t; z)}{w(t; z) - w(x; z)} - \frac{\partial_t w(t; z)}{w(t; z) - w(0; z)} \right] [-d(t; z)]^{1/2} dt,$$

where

$$(2.5) \quad \mu_{j_0}(t; z) = \frac{1}{[2a(t; z)]^{j_0}} \sum_{k=0}^{(j_0-1)/2} \binom{2k+1}{j_0} d(t; z)^k [-b(t; z)]^{j_0-(2k+1)},$$

and

$$(2.6) \quad w(t; z) = \frac{t(1+t)}{(t-x_2(z))(t-x_3(z)^{1/2})^2}.$$

Equations (2.4) and (2.5) are obtained in [19, Chapter F], while (2.6) is found in [17]. It is worth noting that Equations (2.4) and (2.5) are valid not only for our random walk under consideration, but in fact for all random walks with jumps to the eight nearest neighbors, see [7] and [19, Chapter F]. On the other hand, finding an expression for the function  $w(t; z)$  happens to be quite complex in general, and dependent on the particular model. Further, in general, there is no reason for this function to be rational (in  $t$ ) as in (2.6), or even algebraic. To be complete, we note that for our model,  $w(t; z)$  is rational because a certain group of automorphisms is finite. We refer to [6, 7, 11, 12, 17, 18, 19] for any details on this group, and more generally on how finding such expressions.

Now, using the partial fraction expansion (direct consequence of (2.6))

$$\begin{aligned} \frac{\partial_t w(t; z)}{w(t; z) - w(x; z)} - \frac{\partial_t w(t; z)}{w(t; z) - w(0; z)} = \\ \frac{x}{t(t-x)} + \frac{1}{t - X_1(Y_0(x; z); z)} + \frac{1}{t - X_1(Y_1(x; z); z)} - \frac{1}{t+1}, \end{aligned}$$

we immediately obtain the following.

**Corollary 6.** *Function  $h^{i_0, j_0}(x; z)$  can be split as  $h^{i_0, j_0}(x; z) = h_1^{i_0, j_0}(x; z) + h_2^{i_0, j_0}(x; z) + h_3^{i_0, j_0}(x; z)$ , where*

$$(2.7) \quad h_1(x; z) = x^{i_0} Y_0(x; z)^{j_0},$$

$$(2.8) \quad h_2(x; z) = \frac{x}{\pi} \int_{x_1(z)}^{x_2(z)} \frac{t^{n_0-1}}{t-x} \mu_{j_0}(t; z) [-d(t; z)]^{1/2} dt,$$

$$(2.9) \quad h_3(x; z) = \frac{1}{\pi} \int_{x_1(z)}^{x_2(z)} t^{i_0} \left[ \frac{1}{t - X_1(Y_0(x; z); z)} + \frac{1}{t - X_1(Y_1(x; z); z)} - \frac{1}{t+1} \right] \mu_{j_0}(t; z) [-d(t; z)]^{1/2} dt.$$

The end of Section 2 aims at obtaining an expression of  $h^{i_0, j_0}(1; z)$  which is efficient—in the sense of computing the asymptotic of its coefficients. In order to achieve this, we are going to make the change of variable  $\widehat{b}(t; z) = b(t; z)/[4a(t; z)c(t; z)]^{1/2}$  in the integrals (2.8) and (2.9) of Corollary 6. The main reason of this is that using (2.5) yields

$$(2.10) \quad \mu_{j_0}(t; z) [-d(t; z)]^{1/2} = \left( \frac{c(t; z)}{a(t; z)} \right)^{j_0/2} U_{j_0-1}(-\widehat{b}(t; z)) [1 - \widehat{b}(t; z)^2]^{1/2},$$

where the  $(U_n)_{n \in \mathbf{Z}_+}$  are the Chebyshev polynomials of the second kind, see [21]. We recall that they are the orthogonal polynomials associated with the weight  $t \mapsto [1-t^2]^{1/2} \mathbf{1}_{]-1,1[}(t)$  and that their explicit expression is

$$U_n(u) = \frac{(u + [u^2 - 1]^{1/2})^{n+1} - (u - [u^2 - 1]^{1/2})^{n+1}}{2[u^2 - 1]^{1/2}}, \quad \forall u \in \mathbf{C}, \quad \forall n \in \mathbf{Z}_+.$$

We also recall two properties of the Chebyshev polynomials of the second kind that we will especially use here (see [21] for their proof):

- They have the parity of their order, in other words, for all  $u \in \mathbf{C}$  and all  $n \in \mathbf{Z}_+$ ,  $U_n(-u) = (-1)^n U_n(u)$ ;
- Their expansion in the neighborhood of 1 is  $U_n(u) = (n+1)[1 + n(n+2)(u-1)/3 + O(u-1)^2]$ .

Further, function  $t \mapsto \widehat{b}(t; z)$  is clearly a diffeomorphism between  $]x_1(z), x_2(z)[$  and  $] -1, 1[$ ; in addition,  $\widehat{b}(t; z) = u$  implies  $b(t; z)^2 - 4u^2 a(t; z) c(t; z) = 0$ , which, as a polynomial in the variable  $t$ , is reciprocal, so that we can quite easily obtain and write the explicit expression of its roots, called the  $t_\ell(u; z)$ ,  $\ell \in \{1, \dots, 4\}$ . Defining  $T(u; z) = 3/z + u^2 - u[2 + u^2 + 6/z]^{1/2}$ , then  $t_2(u; z) = T(u; z) - [T(u; z)^2 - 1]^{1/2}$ ,  $t_3(u; z) = T(u; z) + [T(u; z)^2 - 1]^{1/2}$ ,  $t_1(u; z) = t_2(-u; z)$  and  $t_4(u; z) = t_3(-u; z)$ . Notice that we have enumerated the  $t_\ell(u; z)$  in such a way that  $t_\ell(1; z) = x_\ell(z)$  for any  $\ell \in \{1, \dots, 4\}$ . Moreover, it turns out that for  $u \in [-1, 1]$ ,  $\widehat{b}(t_2(u; z); z) = -u$ , so that the following result is an immediate consequence of the change of variable  $t = t_2(u; z)$  in Corollary 6 as well as of the identity (2.10).

**Corollary 7.** *We have  $h^{i_0, j_0}(1; z) = h_1^{i_0, j_0}(1; z) + h_2^{i_0, j_0}(1; z) + h_3^{i_0, j_0}(1; z)$ , where*

$$\begin{aligned} h_1^{i_0, j_0}(1; z) &= [(3 - z - 3[(1 - z)(1 + z/3)]^{1/2}) / (2z)]^{j_0}, \\ h_2^{i_0, j_0}(1; z) &= \frac{1}{\pi} \int_{-1}^1 \frac{U_{j_0-1}(u) t_2(u; z)^{i_0 + j_0/2 - 1}}{t_2(u; z) - 1} \partial_u t_2(u; z) [1 - u^2]^{1/2} du, \\ h_3^{i_0, j_0}(1; z) &= \frac{1}{\pi} \int_{-1}^1 U_{j_0-1}(u) t_2(u; z)^{i_0 + j_0/2} \left[ \frac{1}{t_2(u; z) - X_1(Y_0(1; z); z)} + \frac{1}{t_2(u; z) - X_1(Y_1(1; z); z)} - \frac{1}{t_2(u; z) + 1} \right] \partial_u t_2(u; z) [1 - u^2]^{1/2} du. \end{aligned}$$

## 3. ASYMPTOTIC TAIL DISTRIBUTION OF THE HITTING TIMES

Let  $\mathcal{D}$  denote the open unit disc:  $\mathcal{D} = \{z \in \mathbf{C} : |z| < 1\}$ . In order to prove Theorem 1, we are going to prove that  $h^{i_0, j_0}(1; z)$  is holomorphic in  $(1 + \epsilon)\mathcal{D} \setminus [1, 1 + \epsilon[$  and that in the neighborhood of 1,

$$(3.1) \quad h^{i_0, j_0}(1; z) = (3/4)3^{1/2}i_0j_0(i_0 + j_0)[1 - z]^{3/2}[1 + o(1)] + h_0^{i_0, j_0}(z),$$

where  $h_0^{i_0, j_0}$  is holomorphic at 1; it will then be enough to use the results of [9, Sections 6.2–6.4].

For this, according to Corollary 7, we shall consider successively  $h_1^{i_0, j_0}(1; z)$ ,  $h_2^{i_0, j_0}(1; z)$  and  $h_3^{i_0, j_0}(1; z)$  in Proposition 8. Theorem 1 will then be a direct consequence of these three results.

**Proposition 8.** *The functions  $h_1^{i_0, j_0}(1; z)$ ,  $h_2^{i_0, j_0}(1; z)$  and  $h_3^{i_0, j_0}(1; z)$  are holomorphic in  $(1 + \epsilon)\mathcal{D} \setminus [1, 1 + \epsilon[$ . Moreover, in the neighborhood of 1, we have*

$$\begin{aligned} h_1^{i_0, j_0}(1; z) &= -j_0 3^{1/2} [1 - z]^{1/2} [1 + (3 + 4j_0^2)(1 - z)/8 + f_{1,1}^{i_0, j_0}(z)(z - 1)^2] + f_{1,2}^{i_0, j_0}(z), \\ h_2^{i_0, j_0}(1; z) &= \frac{3^{1/2} j_0}{2} [1 - z]^{1/2} [1 + (1/2)(3/4 + j_0^2)(1 - z) + f_{2,1}^{i_0, j_0}(z)(1 - z)^2] \\ &\quad + \frac{3^{1/2} j_0}{2\pi} (i_0 + j_0/2 - 1/2) \ln(1 - z) [1 + (1 - z)f_{2,2}^{i_0, j_0}(z)] + f_{2,3}^{i_0, j_0}(z), \\ h_3^{i_0, j_0}(1; z) &= \frac{3^{1/2} j_0}{16} (1 - z)^{1/2} [8 + (3 + 4j_0^2 + 12i_0(i_0 + j_0))(1 - z) + f_{3,1}^{i_0, j_0}(z)(1 - z^2)] \\ &\quad - \frac{3^{1/2} j_0}{4\pi} (2i_0 + j_0 - 1) \ln(1 - z) [1 + (1 - z)f_{3,2}^{i_0, j_0}(z)] + f_{3,3}^{i_0, j_0}(z), \end{aligned}$$

where  $f_{1,1}^{i_0, j_0}$ ,  $f_{1,2}^{i_0, j_0}$ ,  $f_{2,1}^{i_0, j_0}$ ,  $f_{2,2}^{i_0, j_0}$ ,  $f_{2,3}^{i_0, j_0}$ ,  $f_{3,1}^{i_0, j_0}$ ,  $f_{3,2}^{i_0, j_0}$  and  $f_{3,3}^{i_0, j_0}$  are holomorphic at 1.

*Proof of Proposition 8.* The proof of the facts dealing with  $h_1^{i_0, j_0}(1; z)$  follows directly from the expression of this function written in Corollary 7.

Let us now focus on  $h_2^{i_0, j_0}(1; z)$ . We recall from Corollary 7 that

$$(3.2) \quad h_2^{i_0, j_0}(1; z) = \frac{1}{\pi} \int_{-1}^1 \frac{U_{j_0-1}(u)t_2(u; z)^{i_0+j_0/2-1}}{t_2(u; z) - 1} \partial_u t_2(u; z) [1 - u^2]^{1/2} du,$$

where  $t_2(u; z) = T(u; z) - [T(u; z)^2 - 1]^{1/2}$  and  $T(u; z) = 3/z + u^2 - u[2 + u^2 + 6/z]^{1/2}$ . In particular, the fact that  $h_2^{i_0, j_0}(1; z)$  is holomorphic in  $(1 + \epsilon)\mathcal{D} \setminus [1, 1 + \epsilon[$  is clear, since making the change of variable  $u \mapsto -u$  in (3.2) allows us to write it as the integral on  $[0, 1]$  of some function holomorphic in  $\mathcal{D} \times ((1 + \epsilon)\mathcal{D} \setminus [1, 1 + \epsilon[)$ —note that any function symmetrical in  $(T(u; z), T(-u; z))$  is holomorphic w.r.t.  $z$ .

Let us now study the behavior of  $h_2^{i_0, j_0}(1; z)$  in the neighborhood of 1. For this, we are first going to transform (3.2), until obtaining an expression that makes clearly appear the singularities of  $h_2^{i_0, j_0}(1; z)$ .

An easy calculation entails that  $\partial_u t_2 = \partial_u T / (1 - t_3^2)$ . Moreover, by definition of the  $t_\ell$  (see Section 2),  $(z^2/36) \prod_{\ell=1}^4 (t - t_\ell(u; z))$  is equal to  $b(t; z)^2 - 4u^2 a(t; z) c(t; z)$ . In particular,  $\prod_{\ell=1}^4 (1 - t_\ell(u; z)) = (36/z^2)(1 - z(1 + 2u)/3)(1 - z(1 - 2u)/3)$ . So we have

$$(3.3) \quad \frac{\partial_u t_2(u; z)}{t_2(u; z) - 1} = \frac{z^2 \partial_u T(u; z) (1 - t_1(u; z)) (1 - t_4(u; z)) (1 - t_2(u; z))}{18(1 - z(1 - 2u)/3)(t_2(u; z) - t_3(u; z))(1 - z(1 + 2u)/3)}.$$

Let us now expand the quantity  $(1 - t_2(u; z))t_2(u; z)^{i_0+j_0/2-1}$  according to the powers of  $[T(u; z)^2 - 1]^{1/2}$ , say  $(1 - t_2(u; z))t_2(u; z)^{i_0+j_0/2-1} = \sum_{k \geq 0} F_k^{i_0, j_0}(u; z) [T(u; z)^2 - 1]^{k/2}$ .



With these notations, (3.2) and (3.3), we get

$$(3.4) \quad h_2^{i_0, j_0}(1; z) = \sum_{k \geq 0} \int_{-1}^1 \frac{z^2 \partial_u T(u; z) (1 - t_1(u; z)) (1 - t_4(u; z))}{18(1 - z(1 - 2u)/3)} F_k^{i_0, j_0}(u; z) \times \\ \times \frac{[T(u; z)^2 - 1]^{k/2}}{(t_2(u; z) - t_3(u; z))(1 - z(1 + 2u)/3)} U_{j_0-1}(u) [1 - u^2]^{1/2} du.$$

Below, we are going to study the behavior near 1 of each integral in the sum (3.4).

**Integrals corresponding to  $k \in \{0, 1, 2\}$  in the sum (3.4).** First, note that

$$\begin{aligned} F_0^{i_0, j_0} &= T^{i_0 + j_0/2 - 1}(1 - T), \\ F_1^{i_0, j_0} &= T^{i_0 + j_0/2 - 2} [T - (i_0 + j_0/2 - 1)(1 - T)], \\ F_2^{i_0, j_0} &= T^{i_0 + j_0/2 - 3} (i_0 + j_0/2 - 1) [(1 - T)(i_0 + j_0/2 - 2)/2 - T]. \end{aligned}$$

Now we set  $F^{j_0}(u; z) = -z^2 \partial_u T(u; z) (1 - t_1(u; z)) (1 - t_4(u; z)) U_{j_0-1}(u) / (36(1 - z(1 - 2u)/3))$  and

$$\begin{aligned} G_0^{i_0, j_0}(u; z) &= [F^{j_0}(u; z) F_0^{i_0, j_0}(u; z) z^2 [T(-u; z)^2 - 1]^{1/2}] / [3(z + 3)(1 - z(1 - 2u)/3)^{1/2}], \\ G_1^{i_0, j_0}(u; z) &= F^{j_0}(u; z) F_1^{i_0, j_0}(u; z), \\ G_2^{i_0, j_0}(u; z) &= [F^{j_0}(u; z) F_2^{i_0, j_0}(u; z) 3(z + 3)(1 - z(1 - 2u)/3)^{1/2}] / [z^2 [T_1(-u; z)^2 - 1]^{1/2}]. \end{aligned}$$

Since  $t_2(u; z) - t_3(u; z) = -2[T(u; z)^2 - 1]^{1/2}$  and since  $(t_2(u; z) - t_3(u; z))(t_1(u; z) - t_4(u; z)) = 12(z + 3)^2 [(1 - z(1 + 2u)/3)(1 - z(1 - 2u)/3)]^{1/2} / z^2$ , the sum of the three terms for  $k \in \{0, 1, 2\}$  in (3.4) is equal to

$$(3.5) \quad \sum_{k=0}^2 \int_{-1}^1 \frac{G_k^{i_0, j_0}(u; z) [1 - u^2]^{1/2}}{[1 - z(1 + 2u)/3]^{(3-k)/2}} du.$$

Using now the expansion of the Chebyshev polynomials at 1 [21], we obtain the expansion  $G_0^{i_0, j_0}(u; z) = -2j_0(u - 1)/9 - j_0(z - 1)/3 + \sum_{k+l \geq 2} G_{0, k, l}^{i_0, j_0}(u - 1)^k (z - 1)^l$ . Then, with a repeated use of (A.7) of Lemma 10, we get

$$\int_{-1}^1 \frac{G_0^{i_0, j_0}(u; z) [1 - u^2]^{1/2}}{[1 - z(1 + 2u)/3]^{3/2}} du = j_0 3^{1/2} \ln(1 - z) [(1 - z)/4 + (1 - z)^2 g_0^{i_0, j_0}(z)] + f_0^{i_0, j_0}(z),$$

$f_0^{i_0, j_0}$  and  $g_0^{i_0, j_0}$  being holomorphic at 1.

In the same way,  $G_1^{i_0, j_0}(u; z) = \sum_{k+l \geq 2} G_{1, k, l}^{i_0, j_0}(u - 1)^k (z - 1)^l - j_0/3 - j_0(6j_0^2 + 35 - 48i_0 - 24j_0)(u - 1)/54 + j_0(-53 + 48i_0 + 24j_0)(z - 1)/36$ . A repeated application of Lemma 9 gives then that

$$\begin{aligned} \int_{-1}^1 \frac{G_1^{i_0, j_0}(u; z) [1 - u^2]^{1/2}}{1 - z(1 + 2u)/3} du &= f_1^{i_0, j_0}(z) + j_0 3^{1/2} [1 - z]^{1/2} \times \\ &\times [1/2 + (3/4 + j_0^2)(1 - z)/4 + (1 - z)^2 g_1(z)], \end{aligned}$$

$f_1^{i_0, j_0}$  and  $g_1^{i_0, j_0}$  being holomorphic at 1.

At last, we have  $G_2^{i_0, j_0}(u; z) = 2j_0(i_0 + j_0/2 - 1)/3 + \sum_{k+l \geq 1} G_{2, k, l}^{i_0, j_0}(u - 1)^k (z - 1)^l$ . So with a repeated use of (A.6) of Lemma 10, we get

$$\begin{aligned} \int_{-1}^1 \frac{G_2^{i_0, j_0}(u; z) [1 - u^2]^{1/2}}{[1 - z(1 + 2u)/3]^{1/2}} du &= f_2^{i_0, j_0}(z) + j_0(i_0 + j_0/2 - 1) 3^{1/2} \times \\ &\times \ln(1 - z) [(1 - z)/2 + (1 - z)^2 g_2^{i_0, j_0}(z)], \end{aligned}$$

$f_2^{i_0, j_0}$  and  $g_2^{i_0, j_0}$  being holomorphic at 1.

**Integrals corresponding to  $k \geq 3$  in the sum (3.4).** Note first that if  $k$  is odd and larger than 3, then the associated function in (3.4) is in fact holomorphic in the neighborhood of 1: indeed, for this it is enough to notice that  $t_2(u; z) - t_3(u; z) = -2[T(u; z)^2 - 1]^{1/2}$ . For this reason, all the terms associated in (3.4) with values of  $k$  odd and larger than 3 don't have any singularity at 1.

On the other hand, if  $k$  is even and larger than 3, then the underlying term in the sum (3.4) can be written as

$$\int_{-1}^1 [1 - z(1 + 2u)/3]^{(k-3)/2} H_k^{i_0, j_0}(u; z) [1 - u^2]^{1/2} du,$$

where  $H_k^{i_0, j_0}(u; z)$  is some function holomorphic in the neighborhood of  $(1, 1)$ . The last integral is obviously equal to

$$\int_{-1}^1 [1 - z(1 + 2u)/3]^{(k-2)/2} H_k^{i_0, j_0}(u; z) [1 - u^2]^{1/2} [1 - z(1 + 2u)/3]^{-1/2} du.$$

Then, expanding  $[1 - z(1 + 2u)/3]^{(k-2)/2} H_k^{i_0, j_0}(u; z)$  w.r.t. the powers of  $(u - 1)^k (z - 1)^l$  and using (A.6) of Lemma 10, we obtain that the integral above equals  $\ln(1 - z)(z - 1)^{k-2} g_k^{i_0, j_0}(z) + f_k^{i_0, j_0}(z)$ ,  $f_k^{i_0, j_0}$  and  $g_k^{i_0, j_0}$  being holomorphic at 1.

Finally, the sum of all the terms corresponding in (3.4) to even  $k$  larger than 3 can be written, in the neighborhood of 1, as  $\ln(1 - z)(1 - z)^2 g^{i_0, j_0}(z) + f^{i_0, j_0}(z)$ , where  $f^{i_0, j_0}$  and  $g^{i_0, j_0}$  are holomorphic at 1.

**End of the proof of Proposition 8.** Putting the latter fact together with (3.5) concludes the proof of the expansion for  $h_2^{i_0, j_0}(1; z)$  stated in Proposition 8.

Finally, the proof of the facts regarding  $h_3^{i_0, j_0}(1; z)$ , via the repeated use of Lemmas 9 and 10, is totally similar to that for  $h_2^{i_0, j_0}(1; z)$ , so we omit it.  $\square$

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## APPENDIX A.

In order to prove Proposition 8, we need the two following results, that deal with the behavior of some integrals with parameters near their singularities.

**Lemma 9.** *For any  $k \in \mathbf{Z}_+$ , let  $P_k$  be the principal part at infinity of  $[Z^2 - 1]^{1/2}(1 - Z)^k$ , i.e., the only polynomial such that  $[Z^2 - 1]^{1/2}(1 - Z)^k - P_k(Z)$  goes to zero as  $|Z|$  goes to infinity. Then*

$$\int_{-1}^1 \frac{(1-u)^k [1-u^2]^{1/2}}{1-z(1+2u)/3} du = \frac{3\pi}{2z} \left[ (1+z/3)^{1/2} \left( \frac{-3}{2z} \right)^{k+1} (1-z)^{k+1/2} + P_k \left( \frac{3}{2z} - \frac{1}{2} \right) \right].$$

*Proof.* For  $\epsilon > 0$ , we consider the closed contour  $\mathcal{A}_\epsilon^+ \cup \mathcal{A}_\epsilon^- \cup \mathcal{B}_\epsilon^+ \cup \mathcal{B}_\epsilon^-$ , where  $\mathcal{A}_\epsilon^\pm = \{\pm 1 \mp i\epsilon \exp(it), t \in [0, \pi]\}$  and  $\mathcal{B}_\epsilon^\pm = \{\pm i\epsilon \mp t, t \in [-1, 1]\}$ . Then we apply on it the residue theorem at infinity to the function  $(1-u)^k [1-u^2]^{1/2} / [1-z(1+2u)/3]$  and we let  $\epsilon$  going to zero.  $\square$

**Lemma 10.** *For any  $k \in \mathbf{Z}_+$ , the integrals written in the left hand side of (A.6) and (A.7) below are holomorphic in  $(1+\epsilon)\mathcal{D} \setminus [1, 1+\epsilon[$  for  $\epsilon > 0$  small enough. In the neighborhood of 1, they are equal to*

$$(A.6) \quad \int_{-1}^1 \frac{(1-u)^k [1-u^2]^{1/2}}{[1-z(1+2u)/3]^{1/2}} du = \ln(1-z)(1-z)^{k+1} \alpha_k(z) + \beta_k(z),$$

$$(A.7) \quad \int_{-1}^1 \frac{(1-u)^k [1-u^2]^{1/2}}{[1-z(1+2u)/3]^{3/2}} du = \ln(1-z)(1-z)^k \gamma_k(z) + \delta_k(z),$$

where  $\alpha_k, \beta_k, \gamma_k$  and  $\delta_k$  are holomorphic at 1,  $\alpha_k(1) \neq 0$  and  $\gamma_k(1) \neq 0$ . Furthermore,  $\alpha_0(1) = 3^{3/2}/4$ ,  $\gamma_0(1) = -3^{3/2}/2$ ,  $\gamma'_0(1) = -3^{1/2}99/32$  and  $\gamma_1(1) = 3^{1/2}27/8$ .

*Proof.* The fact that the integrals written in the left hand side of (A.6) and (A.7) are, for  $\epsilon > 0$  small enough, holomorphic in  $(1+\epsilon)\mathcal{D} \setminus [1, 1+\epsilon[$  is clear from their expression.

Let us now study their behavior near 1 and start by considering (A.6). Replace first the lower bound  $-1$  by  $-1/2$  in the integral (A.6). This is equivalent to add a function holomorphic in some  $(1+\epsilon)\mathcal{D}$  and this will eventually change  $\beta_k$  but not  $\alpha_k$  in the right hand side member of (A.6). Then, the change of variable  $v^2 = (1+2u)/3$  gives

$$(A.8) \quad \int_{-1/2}^1 \frac{(1-u)^k [1-u^2]^{1/2}}{[1-z(1+2u)/3]^{1/2}} du = 3^{1/2} \left( \frac{3}{2} \right)^{k+1} \int_0^1 \frac{[1-v^2]^{k+1/2}}{[1-zv^2]^{1/2}} [1+3v^2]^{1/2} v dv.$$

By using the expansion of  $v^{1/2}$  in the neighborhood of 1, we can develop the function  $[1+3v^2]^{1/2}v$  according to the powers of  $v^2 - 1$ :  $[1+3v^2]^{1/2}v = 2 + (7/4)[v^2 - 1] + \dots$ . But in [19, Chapter F], we have proved, using the elliptic integrals theory, that for any  $k \in \mathbf{Z}_+$

there exist two functions  $\phi_k$  and  $\psi_k$ , holomorphic in the neighborhood of 1 and satisfying  $\phi_k(1) \neq 0$ , such that

$$(A.9) \quad \int_0^1 \frac{[1-v^2]^{1/2+k}}{[1-zv^2]^{1/2}} dv = \ln(1-z)(z-1)^{k+1} \phi_k(z) + \psi_k(z),$$

we have there also proved that  $\phi_0(1) = 1/4$ . The equality (A.6) is then an obvious outcome of (A.8), of the expansion of  $[1+3v^2]^{1/2}v$  according to the powers of  $v^2-1$ , and of the repeated use of (A.9). The fact that  $\alpha_0(1) = 3^{3/2}/4$  comes from the equality  $\phi_0(1) = 1/4$ .

Likewise, we could prove the equality (A.7) and we could obtain the announced values of  $\gamma_0(1)$ ,  $\gamma'_0(1)$  and  $\gamma_1(1)$ .  $\square$