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# Passage time from four to two blocks in the voter model

Kilian Raschel\*

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## Abstract

We consider a voter model in which there are two candidates and initially, in the population  $\mathbb{Z}$ , four connected blocks of same opinions. We assume that a citizen changes his mind at a rate proportional to the number of its neighbors that disagree with him, and we study the passage from four to two connected blocks of same opinions. More precisely we make explicit the generating function of the probabilities to go from four to two blocks in time  $k$  and we find the asymptotic of these probabilities when  $k$  goes to infinity.

*Keywords* : voter model, random walk, hitting times, integral representations.

*AMS 2000 Subject Classification* : primary 82C22, 60G50 ; secondary 60G40, 30E20.

## 1 Introduction

The aim of this note is to complete some results of [BFMP01] and [MMVW08] that concern the voter model, and more particularly to study some properties of the hitting time of the so-called Heaviside configuration in the voter model.

By “the voter model” we mean a continuous-time process on  $\{0, 1\}^{\mathbb{Z}}$  that can be interpreted as follows : initially there is zero or one “particle” at each site of  $\mathbb{Z}$ , then a new particle appears at an empty site  $x$  according to an exponential law with a rate proportional to the number of nearest neighbors of  $x$  occupied by a particle and a particle present at  $x$  disappears according to an exponential law with a rate proportional to the number of empty neighbors of  $x$ . Moreover, we suppose that the initial state is in  $\mathcal{D}$ , that is by definition the set of configurations with a finite number of empty sites to the left of the origin and a finite number of particles to the right of it. The fact that the initial state is in  $\mathcal{D}$  implies that at any time the process belongs to  $\mathcal{D}$ . In particular, at any time, there is a finite number of “01” (resp. “10”), i.e. a finite number of pairs of sites  $(x, x + 1)$  with 0 (resp. 1) particle at  $x$  and 1 (resp. 0) particle at  $x + 1$ .

The associated discrete-time voter model is the Markov process with state-space  $\mathcal{D}$  and the following dynamic : consider  $\mathcal{C}_n$ , the configuration at time  $n$  ; according to the previous paragraph there is in  $\mathcal{C}_n$  a finite number of “01” and “10”. In order to construct  $\mathcal{C}_{n+1}$ , one first chooses with a uniform distribution one of these “01” and “10” in  $\mathcal{C}_n$ , then one replaces it with probability 1/2 by “00” or “11”.

If the voter model starts from the configuration  $1_{\{x \leq 0\}}$ , called the Heaviside configuration, then at any time the process is a (random) translation of this configuration. This fact suggests to consider the following equivalence relation : two configurations are said to be equivalent if we can obtain both of them by a translation of the other. From now on we will work on  $\tilde{\mathcal{D}}$ ,

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the quotient space of  $\mathcal{D}$  by this equivalence relation. Clearly, any equivalence class of  $\tilde{\mathcal{D}}$  can be identified by a finite set of positive numbers  $(n_1, m_1, \dots, n_N, m_N)$  :

$$\dots \underbrace{111}_{n_1} \underbrace{0000}_{m_1} \underbrace{111}_{n_2} \underbrace{000}_{m_2} \underbrace{11111}_{n_N} \underbrace{000}_{m_N} \underbrace{1111}_{n_N} \underbrace{000}_{m_N} \dots, \quad (1)$$

$N$  being the number of finite blocks of zeros (or ones) and  $n_i$  (resp.  $m_i$ ),  $i \in \{1, \dots, N\}$ , the size of the  $i$ th block of zeros (resp. ones). The number  $N$  of finite blocks of zeros is obviously a non-increasing function of the time ; also  $N = 0$  corresponds to the class of the Heaviside configuration. We refer to [Lig85] for additional details about the voter model and more generally for further information about interacting particle systems.

Thanks to an ingenious use of Lyapunov functions, the authors of [BFMP01] prove in their article that if  $\tau$  denotes the hitting time of the Heaviside configuration, then  $\mathbb{E}[\tau^{3/2-\epsilon}] < \infty$  for any  $\epsilon > 0$  and any initial configuration of particles. Next, in order to show that  $\mathbb{E}[\tau^{3/2+\epsilon}] = \infty$  for any  $\epsilon > 0$  and any initial state, they remark that it suffices to prove it in the case of an initial state such that  $N = 1$  in (1). So, with the notations of (1), they consider the process  $(n_1(k), m_1(k))_{k \in \mathbb{N}}$ , that we rename here  $(X(k), Y(k))_{k \in \mathbb{N}}$  – by  $\mathbb{N}$  we mean  $\{0, 1, 2, \dots\}$ , the set of the non-negative integers.

$(X, Y) = (X(k), Y(k))_{k \in \mathbb{N}}$  is a Markov chain on  $\mathbb{N}^2$  that is absorbed as it reaches the boundary  $\{(0, 0)\} \cup \{(i, 0) : i \geq 1\} \cup \{(0, j) : j \geq 1\}$ , since the Heaviside configuration is an absorbing state for the voter model. Moreover, using the dynamic of the discrete-time voter model previously explained, we obtain that  $(X, Y)$  has homogeneous transition probabilities in the interior of  $\mathbb{N}^2$  ; we call these probabilities  $p_{ij} = \mathbb{P}[(X(k+1), Y(k+1)) = (X(k), Y(k)) + (i, j) | (X(k), Y(k))]$ , they verify  $p_{10} = p_{1-1} = p_{0-1} = p_{-10} = p_{-11} = p_{01} = 1/6$  and all the others are equal to zero.

Throughout the note, for the conditional probability  $\mathbb{P}[\cdot | (X(0), Y(0)) = (x_0, y_0)]$  we will write  $\mathbb{P}_{(x_0, y_0)}[\cdot]$ . Likewise, we will write  $\mathbb{E}_{(x_0, y_0)}[\cdot]$  for the associated conditional expectation.

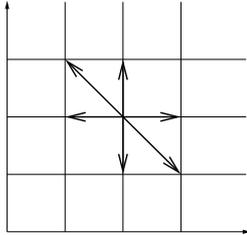


Figure 1: Each arrow above stands for a transition probability equal to  $1/6$

Then, using results of [AIM96] that concern the passage-time moments for reflected random walks in a quadrant, the authors of [BFMP01] prove that  $\mathbb{E}_{(x_0, y_0)}[\tau^{3/2+\epsilon}] = \infty$  for any  $\epsilon > 0$  and any initial state  $(x_0, y_0)$  in the interior of the quadrant,  $\tau$  being the hitting time of the boundary of  $\mathbb{N}^2$  for the process  $(X, Y)$ , namely

$$\tau = \inf \{k \in \mathbb{N} : X(k) Y(k) = 0\}, \quad (2)$$

or equivalently, in the voter model, the hitting time of the Heaviside configuration starting from an initial configuration such that  $N = 1$  in (1) such that  $n_1(0) = x_0$ ,  $m_1(0) = y_0$ .

Now we would like to emphasize some other approaches and methods that lead to precise results concerning the hitting time  $\tau$ .

Firstly, in [Var99], N. Varopoulos studies the hitting times of cones for centered and  $d$ -dimensional random walks having non-correlated components : he obtains, for large times  $k$ , rather precise lower and upper bounds for the probability that the process doesn't have left the cone at time  $k$ . By linear transforms these results can be extended to the case of centered random walks with correlated coordinates. Applying these quite general findings to our situation, we obtain the existence of two positive functions  $C_1$  and  $C_2$  such that for  $k$  large enough,  $C_1(x_0, y_0)/k^{3/2} \leq \mathbb{P}_{(x_0, y_0)}[\tau \geq k] \leq C_2(x_0, y_0)/k^{3/2}$  ; in particular this easily leads to  $\mathbb{E}_{(x_0, y_0)}[\tau^{3/2}] = \infty$ .

Secondly, in [EK08], motivated by constructing some processes conditioned on staying in strict order at all times as  $h$ -Doob transformed processes, the authors consider "ordered random walks" and study the time  $\tilde{\tau}$  when for the first time the processes become not ordered. In this way and among other things, they consider  $W_1, \dots, W_d$ ,  $d$  independent copies of the simple random walk on  $\mathbb{Z}$  such that  $0 < W_1(0) < \dots < W_d(0)$ , and they show that if  $\tilde{\tau} = \inf\{k \in \mathbb{N} : W_i(k) = W_{i+1}(k) \text{ for some } i\}$ , then  $\mathbb{P}_{(W_1(0), \dots, W_d(0))}[\tilde{\tau} \geq k] \sim_{k \rightarrow \infty} \tilde{K}_d \prod_{i < j} (W_j(0) - W_i(0))/k^{d(d-1)/4}$ , where  $\tilde{K}_d$  is some – made explicit – constant. In the case  $d = 3$ , if we set  $U = W_2 - W_1$  and  $V = W_3 - W_2$ , we obtain that  $(U, V)$  is a random walk on  $\mathbb{N}^2$ , having, in the interior of  $\mathbb{N}^2$ , homogeneous transition probabilities such that  $p_{20} = p_{2-2} = p_{0-2} = p_{-20} = p_{-22} = p_{02} = p_{00}/2 = 1/8$ . After some manipulations, we can apply these results of [EK08] to our situation and we obtain  $\mathbb{P}_{(x_0, y_0)}[\tau \geq k] \sim_{k \rightarrow \infty} K x_0 y_0 (x_0 + y_0)/k^{3/2}$ , where  $K$  is some constant independent of  $x_0, y_0$  and  $\tau$  the hitting time defined in (2).

In this note we will carry out our analysis by following an analytic approach initiated in [FIM99]. This book studies random walks in the quarter plane  $\mathbb{N}^2$  with a negative drift and non-zero jump probabilities from the boundary to the interior of  $\mathbb{N}^2$ . Moreover, the jumps from the boundary are supposed such that the Markov chain is ergodic. G. Fayolle, R. Iasnogorodski and V. Malyshev elaborate in this book a quite interesting and profound analytic approach in order to compute the generating functions of stationary probabilities of these random walks.

This approach has already been adapted to the study of some random walks in the quarter plane absorbed at the boundary. In this way we have, in [KR09], obtained the generating function of the absorption probabilities in space for all the walks having a positive drift and non-zero transition probabilities at distance at most one. Also, in [Ras09], we have got and made use of the generating function of the absorption probabilities, in time and in space, for the walks having a drift positive or zero and transition probabilities such that  $p_{10} + p_{0-1} + p_{-10} + p_{01} = 1$  – in fact, more generally, for all the walks having a "group of random walk" of order four, the "group of random walk" being a group of automorphisms, of order even and larger than four, associated very naturally with the random walk, see Subsection 2.4 of [FIM99] for a precise definition.

In this note, this analytic approach is resumed in preparatory Subsections 2.1 and 2.2 ; the facts directly inspired from [FIM99] are gathered in Subsection 2.2.

The aim of this note is to illustrate in which extent the analytic approach of [FIM99] leads to precise results concerning the random walk of  $\mathbb{N}^2$  absorbed at the boundary and having in the interior of  $\mathbb{N}^2$  the homogeneous transition probabilities  $p_{10} = p_{1-1} = p_{0-1} = p_{-10} = p_{-11} = p_{01} = 1/6$ , see Figure 1 ; what in particular gives for the first time an example of the use of the analytic methods of [FIM99] in the case a random walk having a drift equal to zero, i.e.  $\sum_{i,j} i p_{ij} = 0$  and  $\sum_{i,j} j p_{ij} = 0$ , and a associated "group of the random walk" of order six, see Remark 5 of Section 2.

In this work, we will start, in Section 2, by finding the explicit expression of the probabilities

that the process is absorbed at some site and in some time. Indeed, in Theorem 6 of Section 2, we will find an integral representation of the generating functions

$$h(x, z) = \sum_{i \geq 1, k \geq 0} \mathbb{P}_{(x_0, y_0)} [(X, Y) \text{ hits } (i, 0) \text{ at time } k] x^i z^k, \quad (3)$$

$$\tilde{h}(y, z) = \sum_{j \geq 1, k \geq 0} \mathbb{P}_{(x_0, y_0)} [(X, Y) \text{ hits } (0, j) \text{ at time } k] y^j z^k. \quad (4)$$

In particular, if we define the hitting times of the  $x$ -axis and of the  $y$ -axis :

$$S = \inf \{k \in \mathbb{N} : Y(k) = 0\}, \quad T = \inf \{k \in \mathbb{N} : X(k) = 0\}, \quad (5)$$

– in such a way that  $\tau$ , the hitting time defined in (2), is equal to  $\inf\{S, T\}$  – then having the explicit formulation of the generating functions (3) and (4) also allows us to obtain the explicit expression of  $\mathbb{P}_{(x_0, y_0)}[S = k]$  and  $\mathbb{P}_{(x_0, y_0)}[T = k]$  for any  $k \in \mathbb{N}$  and any  $x_0, y_0 \in \mathbb{N}$ . In other words, we will obtain, for any time  $k$ , an explicit formulation for the probability to hit at time  $k$  the Heaviside configuration in the voter model starting from an initial state such that  $N = 1$  in (1) ; we will also obtain the additional information of the size of the blocks at the time of the absorption.

Then, in Section 3, by studying closely the functions  $h(1, z)$  and  $\tilde{h}(1, z)$  – notably their singularities – we will obtain the asymptotic as  $k$  goes to infinity of  $\mathbb{P}_{(x_0, y_0)}[S = k]$  and  $\mathbb{P}_{(x_0, y_0)}[T = k]$ , hence also of  $\mathbb{P}_{(x_0, y_0)}[\tau = k]$ , since  $\mathbb{P}_{(x_0, y_0)}[\tau = k] = \mathbb{P}_{(x_0, y_0)}[S = k] + \mathbb{P}_{(x_0, y_0)}[T = k]$ . More precisely we will show in Proposition 8 that  $\mathbb{P}_{(x_0, y_0)}[S = k] \sim_{k \rightarrow \infty} (9/16)(3/\pi)^{1/2} x_0 y_0 (x_0 + y_0) / k^{5/2}$  ; in particular this yields  $\mathbb{P}_{(x_0, y_0)}[\tau \geq k] \sim_{k \rightarrow \infty} (27/16)(3/\pi)^{1/2} x_0 y_0 (x_0 + y_0) / k^{3/2}$ , since the transition probabilities of the walk (see Figure 1 above) are such that  $\mathbb{P}_{(x_0, y_0)}[S = k] = \mathbb{P}_{(y_0, x_0)}[T = k]$ .

We first note that the knowledge of the asymptotic of  $\mathbb{P}_{(x_0, y_0)}[\tau \geq k]$  allows us, as in [EK08] or in [Ras09], to construct the process conditioned never to reach the boundary as a  $h$ -Doob transformed. We also remark that these results are more precise than the ones we can obtain starting from [EK08] : indeed, the asymptotic of  $\mathbb{P}_{(x_0, y_0)}[S = k]$  and  $\mathbb{P}_{(x_0, y_0)}[T = k]$  allow to know that of  $\mathbb{P}_{(x_0, y_0)}[\tau = k]$ , but the reverse is not true.

## 2 Analytic approach

### 2.1 A functional equation and the algebraic curve $Q(x, y, z) = 0$

Define the function  $G(x, y, z) = \sum_{i, j \geq 1, k \geq 0} \mathbb{P}_{(x_0, y_0)} [(X(k), Y(k)) = (i, j)] x^{i-1} y^{j-1} z^k$ . Then, with the notations (3) and (4) of Section 1, we can state, on  $\{(x, y, z) \in \mathbb{C}^3 : |x| < 1, |y| < 1, |z| \leq 1\}$ , the following functional equation :

$$Q(x, y, z) G(x, y, z) = h(x, z) + \tilde{h}(y, z) - x^{x_0} y^{y_0}, \quad (6)$$

where  $Q$  is the following polynomial, depending only on the walk's transition probabilities :

$$Q(x, y, z) = xy \left( \sum_{i, j} z p_{ij} x^i y^j - 1 \right). \quad (7)$$

For  $z = 0$ , Equation (6) simply becomes  $\mathbb{P}_{(x_0, y_0)} [(X(0), Y(0)) = (x_0, y_0)] = 1$ . For  $z = 1$ , it becomes a functional equation between the Green functions generating function and the

absorption probabilities generating functions, studied in [KR09] in the case of random walks with a positive drift. For the proof of (6), we exactly use the same arguments as in Subsection 2.1 of [Ras09].

We will now study the algebraic curve  $Q(x, y, z) = 0$ ,  $Q$  being defined in (7). For this we start by remarking that  $Q$  can be written alternatively

$$Q(x, y, z) = a(x, z)y^2 + b(x, z)y + c(x, z) = \tilde{a}(y, z)x^2 + \tilde{b}(y, z)x + \tilde{c}(y, z), \quad (8)$$

where  $a(x, z) = z(x+1)/6$ ,  $b(x, z) = zx^2/6 - x + z/6$ ,  $c(x, z) = zx(x+1)/6$ ,  $\tilde{a}(y, z) = z(y+1)/6$ ,  $\tilde{b}(y, z) = zy^2/6 - y + z/6$  and  $\tilde{c}(y, z) = zy(y+1)/6$ .

We will now build the algebraic function  $Y(x, z)$  defined by  $Q(x, y, z) = 0$ . Note first that  $Q(x, y, z) = 0$  is equivalent to  $(b(x, z) + 2a(x, z)y)^2 = b(x, z)^2 - 4a(x, z)c(x, z)$ , so that the construction of the function  $Y$  is equivalent to that of the square root of the polynomial  $d = b^2 - 4ac$ . For this we need the following :

**Lemma 1.** *For  $z \in ]0, 1[$ , the four roots of  $x \mapsto d(x, z)$  are real, positive and mutually distinct. We call them in such a way that for any  $z \in ]0, 1[$ ,  $x_1(z) < x_2(z) < x_3(z) < x_4(z)$ . They verify  $x_1(z)x_4(z) = x_2(z)x_3(z) = 1$ . In particular,  $x_1(z), x_2(z) \in ]0, 1[$  and  $x_3(z), x_4(z) \in ]1, \infty[$  for all  $z \in ]0, 1[$ . Also,  $x_1(0) = x_2(0) = 0$ ,  $x_3(0) = x_4(0) = \infty$ ,  $x_2(1) = x_3(1) = 1$  and  $x_1(1) = 7 - 4\sqrt{3}$ ,  $x_4(1) = 7 + 4\sqrt{3}$ . At last, the  $x_i$ ,  $i \in \{1, \dots, 4\}$ , vary continuously and monotonously with respect to  $z \in [0, 1]$ .*

*Proof.* As we can easily verify, the polynomial  $d$  is reciprocal, in other words it satisfies  $x^4d(1/x, z) = d(x, z)$ . This property allows us to write  $d$  as a second degree polynomial in the variable  $x + 1/x$ . Following this way we get the explicit expression of the roots of  $d$  : if  $s_1(z) = 3/z + 1$  and  $s_2(z) = (6/z + 3)^{1/2}$ , then  $x_1(z) = s_1(z) + s_2(z) + ((s_1(z) + s_2(z))^2 - 1)^{1/2}$  and  $x_2(z) = s_1(z) - s_2(z) + ((s_1(z) - s_2(z))^2 - 1)^{1/2}$ ,  $x_3(z) = 1/x_2(z)$  and  $x_4(z) = 1/x_1(z)$ . All the properties written in Lemma 1 follow immediately from these explicit expressions.  $\square$

There are two branches of the square root of  $d$ . Each determination leads to a well defined (i.e. single valued) and meromorphic function on the complex plane  $\mathbb{C}$  appropriately cut, that is, in our case, on  $\mathbb{C} \setminus [x_1(z), x_2(z)] \cup [x_3(z), x_4(z)]$ . We can write the analytic expression of these two branches  $Y_0$  and  $Y_1$  of  $Y$  :  $Y_0(x, z) = Y_-(x, z)$  and  $Y_1(x, z) = Y_+(x, z)$  where :

$$Y_{\pm}(x, z) = \frac{-b(x, z) \pm d(x, z)^{1/2}}{2a(x, z)}. \quad (9)$$

Just above, and in fact throughout the whole paper, we chose the principal determination of the logarithm as soon as we use the complex logarithm ; in this case to define the square root.

For more details about the construction of algebraic functions, see e.g. Book [SG69].

In a similar way, the functional equation (6) defines also an algebraic function  $X(y, z)$ . But it turns out that  $Q(x, y, z) = Q(y, x, z)$ , see (7), so that  $X(y, z) = Y(y, z)$  ; in particular all the properties proved for  $Y$  result immediately in similar properties for  $X$ .

## 2.2 Riemann boundary value problem with shift

Using the notations of Subsection 2.1, we define, for any  $z \in [0, 1]$ , the two following curves :

$$\mathcal{L}_z = Y_0 \left( \left[ \overrightarrow{x_1(z), x_2(z)} \right], z \right), \quad \mathcal{M}_z = X_0 \left( \left[ \overrightarrow{y_1(z), y_2(z)} \right], z \right). \quad (10)$$

Just above, we use the notation  $\overleftarrow{[u, v]}$  for the contour  $[u, v]$  traversed from  $u$  to  $v$  along the upper edge of the slit  $[u, v]$  and then back to  $u$  along the lower edge of the slit.

We make now some remarks about  $\mathcal{L}_z$  and  $\mathcal{M}_z$ . For any  $z \in [0, 1]$ , (i)  $\mathcal{L}_z = \mathcal{M}_z$ , (ii)  $\mathcal{L}_z$  is a simple closed curve, symmetrical about the real axis, (iii)  $\mathcal{L}_z$  is in fact a quartic, whose analytic expression we can easily find, (iv) for all  $z \in [0, 1]$ ,  $\mathcal{L}_z \subset \{s \in \mathbb{C} : |s| \leq 1\}$ .

Just a word about these remarks : (i) is straightforward since  $X(y, z) = Y(y, z)$  ; for (ii) and (iii), we can easily adapt some arguments of Subsection 5.3 of [FIM99] ; as for (iv), note that if  $u \in [x_1(z), x_2(z)]$ , then  $|Y(u, z)|^2 = c(u, z)/a(u, z) = u$ , moreover with Lemma 1  $x_1(z), x_2(z) \in [0, 1]$ , so that we have  $|Y(u, z)| \leq 1$ .

The reason why we have introduced these curves appears now : the functions  $h$  and  $\tilde{h}$ , defined in (3) and (4), verify the following boundary value conditions on  $\mathcal{M}_z$  and  $\mathcal{L}_z$  :

$$\begin{aligned} \forall t \in \mathcal{M}_z : \quad h(t, z) - h(\bar{t}, z) &= t^{x_0} Y_0(t, z)^{y_0} - \bar{t}^{x_0} Y_0(\bar{t}, z)^{y_0}, \\ \forall t \in \mathcal{L}_z : \quad \tilde{h}(t, z) - \tilde{h}(\bar{t}, z) &= X_0(t, z)^{x_0} t^{y_0} - X_0(\bar{t}, z)^{x_0} \bar{t}^{y_0}. \end{aligned} \quad (11)$$

The way to obtain these boundary conditions is described in Subsection 5.1 of [FIM99], so we refer to this book for the details.

The function  $h$  of the variable  $x$ , as a generating function, is well defined on the closed unit disc, domain that contains  $\mathcal{M}_z$  – thanks to the property (iv) of the curve  $\mathcal{M}_z$  stated above. Now we have the problem *to find  $h$  holomorphic inside  $\mathcal{M}_z$ , continuous up to the boundary  $\mathcal{M}_z$  and verifying the boundary condition (11) ; in addition  $h(0, z) = 0$ .*

Problems with boundary conditions like (11) are called Riemann boundary value problems with shift. The classical way to study this kind of problems consists in reducing them into Riemann-Hilbert problems, for which there exists a suitable and complete theory. The conversion between Riemann problems with shift and Riemann-Hilbert problems is done thanks to the use of conformal gluing functions, notion defined just below. For details about boundary value problems, we refer to [Lu93].

**Definition 2.** *Let  $\mathcal{C}$  be a simple closed curve, symmetrical about the real axis. Denote by  $\mathcal{G}_{\mathcal{C}}$  the interior of the bounded domain delimited by  $\mathcal{C}$ .  $w$  is called a conformal gluing function (CGF) for  $\mathcal{C}$  if (i)  $w$  is meromorphic in  $\mathcal{G}_{\mathcal{C}}$ , continuous in  $\mathcal{G}_{\mathcal{C}} \cup \mathcal{C}$ , (ii)  $w$  establishes a conformal mapping of  $\mathcal{G}_{\mathcal{C}}$  onto the complex plane cut along an arc, (iii) for all  $t \in \mathcal{C}$ ,  $w(t) = w(\bar{t})$ .*

Suppose temporarily that there exists a CGF  $w_z$  for the curve  $\mathcal{M}_z$  – we will make explicit a such function in Proposition 4. Then, starting from (11) and according to Subsection 5.4 of [FIM99], we obtain the following integral representation of the function  $h$ , defined in (3).

**Proposition 3.** *Let  $z$  be in  $]0, 1[$ . Consider  $\mathcal{M}_z$  the curve defined in (10),  $\mathcal{M}_z^+ = \mathcal{M}_z \cap \{t \in \mathbb{C} : \text{Im}(t) \geq 0\}$  and  $w_z$  a CGF associated to the curve  $\mathcal{M}_z$ . Then the function  $h$  admits for  $x \in \mathcal{G}_{\mathcal{M}_z}$  the following integral representation :*

$$h(x, z) = \frac{1}{2\pi i} \int_{\mathcal{M}_z^+} \left( t^{x_0} Y_0(t, z)^{y_0} - \bar{t}^{x_0} Y_0(\bar{t}, z)^{y_0} \right) \left[ \frac{w'_z(t)}{w_z(t) - w_z(x)} - \frac{w'_z(t)}{w_z(t) - w_z(0)} \right] dt.$$

### 2.3 Integral representation of the generating function

The purpose of Subsection 2.3 is twofold. We will first, in Proposition 4, make explicit a suitable CGF for the curve  $\mathcal{M}_z$  ; in particular this will complete Proposition 3 and will allow us to know totally the function  $h$ . Then, in Theorem 6, we will transform the explicit expression of  $h$  obtained in Proposition 3 into a new integral representation, that will be both clearer and

more suitable, notably for the proof of results of Section 3 concerning the asymptotic of the hitting probabilities.

**Proposition 4.** *The following function is a CGF (in the sense of Definition 2) for the curve  $\mathcal{M}_z$ , defined in (10) :*

$$w_z(t) = \frac{t(1+t)}{(t-x_2(z))(t-x_3(z)^{1/2})^2}.$$

*Proof.* The proof of Proposition 4 simply consists in verifying the different item of Definition 2. Firstly,  $w_z$  is clearly meromorphic on  $\mathbb{C}$ . Secondly, we remark – by a direct calculation – that for all  $y \in \mathbb{C}$ ,  $w_z(X_0(y, z)) = w_z(X_1(y, z))$ . In particular, for all  $t \in \mathcal{M}_z$ ,  $w_z(t) = w_z(\bar{t})$ . Thirdly, solving, for  $t \in \mathcal{G}_{\mathcal{M}_z}$ ,  $w_z(u) = w_z(t)$  leads to find the roots of a third degree polynomial ; after simplifications we obtain  $u = t$ ,  $u = X_1(Y_0(t, z), z)$  or  $u = X_1(Y_1(t, z), z)$ . Moreover, adapting an argument of Subsection 5.3 of [FIM99], we immediately obtain that  $X_1(\mathbb{C}, z) \cap \mathcal{G}_{\mathcal{M}_z} = \emptyset$ . In particular  $X_1(Y_i(\mathcal{G}_{\mathcal{M}_z}, z), z) \cap \mathcal{G}_{\mathcal{M}_z} = \emptyset$ ,  $i = 0, 1$ , and  $w_z$  is therefore one to one in  $\mathcal{G}_{\mathcal{M}_z}$ . Finally,  $w_z$  is a conformal mapping of  $\mathcal{G}_{\mathcal{M}_z}$  onto  $\mathbb{C} \setminus w_z(\mathcal{M}_z) = \mathbb{C} \setminus [-x_1(z)^{1/2}, x_2(z)^{1/2}]$ .  $\square$

We have therefore verified that the function suggested in Proposition 4 is actually a CGF for the curve  $\mathcal{M}_z$ . On the other hand, we don't have explained how we have found this CGF, what is in fact the real difficulty. This is the purpose of the next remark ; note that the fact of not reading it doesn't hinder the global understanding of the text.

**Remark 5.** *Set  $S_z = \{(x, y) \in \mathbb{C}^2 : Q(x, y, z) = 0\}$ . With (8), we have that if  $(x, y) \in S_z$  then  $(x, x/y) \in S_z$  and  $(y/x, y) \in S_z$ . For this reason,  $\xi$  and  $\eta$ , defined by  $\xi(x, y) = (x, x/y)$  and  $\eta(x, y) = (y/x, y)$ , map  $S_z$  on  $S_z$  and are in fact automorphisms of order two of  $S_z$ . Let  $\mathcal{H}_z = \langle \xi, \eta \rangle$  be the subgroup of the automorphisms of  $S_z$  generated by  $\xi$  and  $\eta$ .*

*More generally, for any homogeneous random walk such that  $p_{11} + p_{10} + p_{1-1} + p_{0-1} + p_{-1-1} + p_{-10} + p_{-11} + p_{01} = 1$ , we can define the associated group of the random walk. It turns out that this group – notably its order, eventually infinite – is quite important in the study of the corresponding random walk, see, in this perspective, Section 3 of [FIM99] and Paper [KR09].*

*In our case, for all  $z \in ]0, 1[$ ,  $\mathcal{H}_z$  is of order six – for instance this can be deduced from the fact that  $\langle \xi, \eta \rangle$  is of order six as a subgroup of the automorphisms of  $\mathbb{C}^2$ . As it can be deduced from Section 4 of [KR09], the fact that this group is finite entails that there exists a rational CGF for the curve  $\mathcal{M}_z$ , and the fact that this group is of order six yields that this rational function is of order three. Then, to get the explicit expression of the CGF  $w_z$  written in Proposition 4, we adapt Subsection 5.5 of [FIM99] – that concerns  $z = 1$  in the case of a positive drift – ; we obtain the expression of  $w_z$  in terms of Weierstrass functions. Using the theory of transformation of elliptic functions, we can simplify the expression of  $w_z$  and we finally obtain Proposition 4.*

We will need the following function in the statement of the forthcoming Theorem 6 :

$$\mu_{y_0}(t, z) = \frac{1}{(2a(t, z))^{y_0}} \sum_{k=0}^{\lfloor (y_0-1)/2 \rfloor} \binom{y_0}{2k+1} d(t, z)^k (-b(t, z))^{y_0-(2k+1)}. \quad (12)$$

The function  $\mu_{y_0}$  is such that for all  $t \downarrow [x_1(z), x_2(z)]$  (resp.  $t \uparrow [x_1(z), x_2(z)]$ ),  $Y_0(t, z)^{y_0} - \overline{Y_0(t, z)^{y_0}}$  is equal to  $-2\iota(-d(t, z))^{1/2}\mu_{y_0}(t, z)$  (resp.  $2\iota(-d(t, z))^{1/2}\mu_{y_0}(t, z)$ ).

**Theorem 6.** *The function  $h$  can be split as  $h(x, z) = h_1(x, z) + h_2(x, z) + h_3(x, z)$ , where :*

$$h_1(x, z) = x^{x_0} Y_0(x, z)^{y_0}, \quad h_2(x, z) = \frac{x}{\pi} \int_{x_1(z)}^{x_2(z)} \frac{t^{n_0-1}}{t-x} \mu_{y_0}(t, z) (-d(t, z))^{1/2} dt, \quad (13)$$

$$h_3(x, z) = \frac{1}{\pi} \int_{x_1(z)}^{x_2(z)} t^{x_0} \left[ \frac{1}{t - X_1(Y_0(x, z), z)} + \frac{1}{t - X_1(Y_1(x, z), z)} - \frac{1}{t+1} \right] \mu_{y_0}(t, z) (-d(t, z))^{1/2} dt. \quad (14)$$

If  $z$  is fixed in  $]0, 1[$ ,  $x \mapsto h_1(x, z)$  is holomorphic on  $\mathbb{C} \setminus [x_1(z), x_2(z)] \cup [x_3(z), x_4(z)]$ ,  $x \mapsto h_2(x, z)$  on  $\mathbb{C} \setminus [x_1(z), x_2(z)]$  and  $x \mapsto h_3(x, z)$  on  $\mathbb{C}$ .

Moreover, we could prove – but it will not be useful here – that  $x \mapsto h_1(x, z) + h_2(x, z)$  is in fact holomorphic in the neighborhood of  $[x_1(z), x_2(z)]$ , following Subsection 2.4 of [Ras09].

*Proof.* We start by expressing the integral obtained in Proposition 3 as an integral on a closed contour, namely  $\mathcal{M}_z$ . To do this, we make the change of variable  $t \mapsto \bar{t}$  and we use that on  $\mathcal{M}_z$ ,  $w_z(t) = w_z(\bar{t})$ . Also, with the partial fraction expansion of  $w'_z(t)/(w_z(t) - w_z(x)) - w'_z(t)/(w_z(t) - w_z(0)) = x/(t(t-x)) + 1/(t - X_1(Y_0(x, z), z)) + 1/(t - X_1(Y_1(x, z), z)) - 1/(t+1)$ , we obtain that  $h(x, z)$  is equal to

$$\frac{1}{2\pi i} \int_{\mathcal{M}_z} t^{x_0} Y_0(t, z)^{y_0} \left[ \frac{x}{t(t-x)} + \frac{1}{t - X_1(Y_0(x, z), z)} + \frac{1}{t - X_1(Y_1(x, z), z)} - \frac{1}{t+1} \right] dt.$$

Finally, using that  $X_1(Y_i(\mathcal{G}_{\mathcal{M}_z}, z), z) \cap \mathcal{G}_{\mathcal{M}_z} = \emptyset$ ,  $i = 0, 1$  – see the proof of Proposition 4 – and following Subsection 3.2 of [KR09], we get Theorem 6.  $\square$

We close the study of the integral representations of  $h$  by making a quite natural change of variable in the integrals (13) and (14). In this perspective, define  $\hat{b}(t, z) = b(t, z)/(4a(t, z)c(t, z))^{1/2}$ . Remark that using (12) we get :

$$\mu_{y_0}(t, z) (-d(t, z))^{1/2} = \left( \frac{c(t, z)}{a(t, z)} \right)^{y_0/2} U_{y_0-1}(-\hat{b}(t, z)) \left( 1 - \hat{b}(t, z)^2 \right)^{1/2}, \quad (15)$$

where the  $U_n$ ,  $n \in \mathbb{N}$ , are the Chebyshev polynomials of the second kind. We recall that they are the orthogonal polynomials associated to the weight  $t \mapsto (1-t^2)^{1/2} 1_{]-1, 1[}(t)$  and that their explicit expression is  $U_n(u) = [(u + (u^2 - 1)^{1/2})^{n+1} - (u - (u^2 - 1)^{1/2})^{n+1}] / [2(u^2 - 1)^{1/2}]$ ,  $u \in \mathbb{C}$ ,  $n \in \mathbb{N}$ . We also recall two properties of the Chebyshev polynomials of the second kind that we will especially use : firstly that they have the parity of their order, in other words, for all  $n \in \mathbb{N}$  and all  $u \in \mathbb{C}$ ,  $U_n(-u) = (-1)^n U_n(u)$  ; secondly their expansion in the neighborhood of 1 :  $U_n(u) = (n+1)(1+n(n+2)(u-1)/3) + \mathcal{O}((u-1)^2)$ . For all the facts concerning the Chebyshev polynomials used here, we refer to [Sze75].

Moreover,  $t \mapsto \hat{b}(t, z)$  is clearly a diffeomorphism between  $]x_1(z), x_2(z)[$  and  $] - 1, 1[$ , besides also between  $]x_3(z), x_4(z)[$  and  $] - 1, 1[$ . In addition,  $\hat{b}(t, z) = u$  implies  $b(t, z)^2 - 4u^2 a(t, z) c(t, z) = 0$ , which, as a polynomial in the variable  $t$ , is reciprocal, so that we can quite easily obtain and write the explicit expression of its roots, called the  $t_i(u, z)$ ,  $i \in \{1, \dots, 4\}$  : if we define  $T(u, z) = 3/z + u^2 - u(2 + u^2 + 6/z)^{1/2}$ , then  $t_2(u, z) = T(u, z) - (T(u, z)^2 - 1)^{1/2}$ ,  $t_1(u, z) = t_2(-u, z)$ ,  $t_3(u, z) = T(u, z) + (T(u, z)^2 - 1)^{1/2}$  and  $t_4(u, z) = t_3(-u, z)$ . Note that we have enumerated the  $t_i(u, z)$  in such a way that  $t_i(1, z) = x_i(z)$ , the  $x_i(z)$  being the branch points defined in Lemma 1.

Moreover, it turns out that for all  $u \in [-1, 1]$ ,  $\hat{b}(t_2(u, z), z) = -u$ , so that the following result is an immediate consequence of (9), Theorem 6, (15) and of the change of variable  $t = t_2(u, z)$ .

**Corollary 7.** *For all  $z$  in the unit disc,  $h(1, z)$  is the sum of the three following quantities :*

$$\begin{aligned} h_1(1, z) &= \left( (3 - z - 3((1 - z)(1 + z/3))^{1/2}) / (2z) \right)^{y_0}, \\ h_2(1, z) &= \frac{1}{\pi} \int_{-1}^1 \frac{U_{y_0-1}(u) t_2(u, z)^{x_0+y_0/2-1}}{t_2(u, z) - 1} \partial_u t_2(u, z) (1 - u^2)^{1/2} du, \\ h_3(1, z) &= \frac{1}{\pi} \int_{-1}^1 U_{y_0-1}(u) t_2(u, z)^{x_0+y_0/2} \left[ \frac{1}{t_2(u, z) - U_0(1, z)} + \right. \\ &\quad \left. + \frac{1}{t_2(u, z) - U_1(1, z)} - \frac{1}{t_2(u, z) + 1} \right] \partial_u t_2(u, z) (1 - u^2)^{1/2} du. \end{aligned}$$

### 3 Asymptotic of the hitting probabilities

#### 3.1 Statement of the result

**Proposition 8.** *Let  $S$  be the hitting time of the  $x$ -axis, defined in (5). The probability that the process is absorbed exactly at time  $k$  on the  $x$ -axis starting from  $(x_0, y_0)$ ,  $\mathbb{P}_{(x_0, y_0)}[S = k]$ , admits the following asymptotic :*

$$\mathbb{P}_{(x_0, y_0)}[S = k] \sim_{k \rightarrow \infty} \frac{9}{16} \sqrt{\frac{3}{\pi}} x_0 y_0 (x_0 + y_0) \frac{1}{k^{5/2}}. \quad (16)$$

Furthermore, the probabilities of transition of the random walk are such that  $\mathbb{P}_{(x_0, y_0)}[T = k] = \mathbb{P}_{(y_0, x_0)}[S = k]$ ,  $T$  being the hitting time of the  $y$ -axis, defined in (5). Moreover, since  $\tau < \infty$  almost surely,  $\mathbb{P}_{(x_0, y_0)}[\tau \geq k] = \mathbb{P}_{(x_0, y_0)}[S \geq k] + \mathbb{P}_{(x_0, y_0)}[T \geq k] \sim_{k \rightarrow \infty} (27/16)(3/\pi)^{1/2} x_0 y_0 (x_0 + y_0) / k^{3/2}$ .

#### 3.2 Sketch of the proof

In order to prove Proposition 8, we will study the function  $h(1, z)$ , which is equal to  $\sum_{k=1}^{\infty} \mathbb{P}_{(x_0, y_0)}[S = k] z^k$ , see (3). More precisely we will show that  $h(1, z)$  is holomorphic in the unit disc, continuable holomorphically through every point of the unit circle except 1 and we will prove that in the neighborhood of 1,

$$h(1, z) = (3/4) \sqrt{3} x_0 y_0 (x_0 + y_0) (1 - z)^{3/2} (1 + \mathcal{O}(1 - z)^{1/4}) + h_0(z), \quad (17)$$

where  $h_0$  is holomorphic at 1. Then, using (17) and Pringsheim's result, see Lemma 9 below, we immediately obtain Proposition 8.

**Lemma 9** (Pringsheim's Theorem). *Let  $f(z) = \sum_{k=0}^{\infty} f_k z^k$  be a function holomorphic in the unit disc. Furthermore, suppose that  $f$  is continuable holomorphically through every point of the unit disc except 1, in the neighborhood of which it can be written as  $f(z) = (1 - z)^\theta (c + \mathcal{O}(1 - z)^\epsilon) + f_0(z)$ , where  $f_0$  is holomorphic at 1,  $\theta$  is not an integer,  $c \neq 0$  and  $\epsilon > 0$ . Then  $f_k \sim_{k \rightarrow \infty} c / (\Gamma(-\theta) k^{\theta+1})$ ,  $\Gamma$  being the classical Gamma function.*

#### 3.3 Proof

For the reasons explained in Subsection 3.2, it suffices, in order to prove Proposition 8, firstly to show that  $h(1, z)$  is holomorphic in the unit disc, continuable holomorphically through

every point of the unit circle except 1, secondly to prove (17). For this we will, according to Corollary 7, consider successively  $h_1(1, z)$ ,  $h_2(1, z)$  and  $h_3(1, z)$  in respectively Propositions 10, 11 and 12. Proposition 8 will be then an immediate consequence of these three results.

**Proposition 10.**  $z \mapsto h_1(1, z)$  is holomorphic in the unit disc, continuable holomorphically through every point of the unit circle except 1 and is in the neighborhood of 1 equal to :

$$h_1(1, z) = -y_0\sqrt{3}(1-z)^{1/2} \left[ 1 + (3+4y_0^2)(1-z)/8 + (z-1)^2 f_{1,1}(z) \right] + f_{1,2}(z),$$

where  $f_{1,1}$  and  $f_{1,2}$  are holomorphic at 1.

**Proposition 11.**  $z \mapsto h_2(1, z)$  is holomorphic in the unit disc, continuable holomorphically through every point of the unit circle except 1 and is in the neighborhood of 1 equal to :

$$\begin{aligned} h_2(1, z) = & \frac{\sqrt{3}y_0}{2} (1-z)^{1/2} \left[ 1 + (1/2)(3/4+y_0^2)(1-z) + (1-z)^2 f_{2,1}(z) \right] \\ & + \frac{\sqrt{3}y_0}{2\pi} (x_0 + y_0/2 - 1/2) \ln(1-z) \left[ 1 + (1-z) f_{2,2}(z) \right] + f_{2,3}(z), \end{aligned}$$

where  $f_{2,1}$ ,  $f_{2,2}$  and  $f_{2,3}$  are holomorphic at 1.

**Proposition 12.**  $z \mapsto h_3(1, z)$  is holomorphic in the unit disc, continuable holomorphically through every point of the unit circle except 1 and is in the neighborhood of 1 equal to :

$$\begin{aligned} h_3(1, z) = & \frac{\sqrt{3}y_0}{16} (1-z)^{1/2} \left[ 8 + (3+4y_0^2+12x_0(x_0+y_0))(1-z) + (1-z)^2 f_{3,1}(z) \right] \\ & - \frac{\sqrt{3}y_0}{4\pi} (2x_0 + y_0 - 1) \ln(1-z) \left[ 1 + (1-z) f_{3,2}(z) \right] + f_{3,3}(z), \end{aligned}$$

where  $f_{3,1}$ ,  $f_{3,2}$  and  $f_{3,3}$  are holomorphic at 1.

*Proof of Proposition 10.* It is clear from the expression of  $h_1(1, z)$  written in Corollary 7.  $\square$

To prove Propositions 11 and 12 we will need the two following results that concern the singularities of some integrals with parameters.

**Lemma 13.** For any non-negative integer  $k$  let  $P_k$  be the principal part at infinity of  $(Z^2 - 1)^{1/2}(1 - Z)^k$ , i.e. the only polynomial such that  $(Z^2 - 1)^{1/2}(1 - Z)^k - P_k(Z)$  goes to zero as  $|Z|$  goes to infinity. Then

$$\int_{-1}^1 \frac{(1-u)^k (1-u^2)^{1/2}}{1-z(1+2u)/3} du = \frac{3\pi}{2z} \left[ (1+z/3)^{1/2} \left( \frac{-3}{2z} \right)^{k+1} (1-z)^{k+1/2} + P_k \left( \frac{3}{2z} - \frac{1}{2} \right) \right].$$

*Proof.* Consider, for  $\epsilon > 0$ , the closed contour  $C_\epsilon^+ \cup C_\epsilon^- \cup D_\epsilon^+ \cup D_\epsilon^-$ , where  $C_\epsilon^\pm = \{\pm 1 \mp i\epsilon \exp(it), t \in [0, \pi]\}$  and  $D_\epsilon^\pm = \{\pm i\epsilon \mp t, t \in [-1, 1]\}$ . Then apply the residue theorem at infinity to the function  $(1-u)^k(1-u^2)^{1/2}/(1-z(1+2u)/3)$  and let  $\epsilon$  going to zero.  $\square$

**Lemma 14.** For any non-negative integer  $k$ , the functions written in the left side of the equalities (18) and (19) are holomorphic in the unit disc, continuable holomorphically through every point of the unit circle except 1, in the neighborhood of which they have a logarithmic

singularity as follows :

$$\int_{-1}^1 \frac{(1-u)^k (1-u^2)^{1/2}}{(1-z(1+2u)/3)^{1/2}} du = \ln(1-z)(1-z)^{k+1} \alpha_k(z) + \beta_k(z), \quad (18)$$

$$\int_{-1}^1 \frac{(1-u)^k (1-u^2)^{1/2}}{(1-z(1+2u)/3)^{3/2}} du = \ln(1-z)(1-z)^k \gamma_k(z) + \delta_k(z), \quad (19)$$

where  $\alpha_k, \beta_k, \gamma_k, \delta_k$  are functions holomorphic at 1 and  $\alpha_k(1) \neq 0, \gamma_k(1) \neq 0$ . Moreover,  $\alpha_0(1) = 3\sqrt{3}/4, \gamma_0(1) = -3\sqrt{3}/2, \gamma'_0(1) = -99\sqrt{3}/32$  and  $\gamma_1(1) = 27\sqrt{3}/8$ .

*Proof.* The fact that the functions (18) and (19) are holomorphic in the unit disc, continuable holomorphically through every point of the unit circle except 1 is clear from their expression. Let us now study their behavior in the neighborhood of 1 and start by considering (18).

Replacing the lower bound  $-1$  by  $-1/2$  in the integral (18) is equivalent to add a function holomorphic in some disc of center 0 and radius greater than 1 ; that will change  $\beta_k$  but not  $\alpha_k$  in the right member of (18). Then, the change of variable  $v^2 = (1+2u)/3$  gives :

$$\int_{-1/2}^1 \frac{(1-u)^k (1-u^2)^{1/2}}{(1-z(1+2u)/3)^{1/2}} du = \sqrt{3} \left(\frac{3}{2}\right)^{k+1} \int_0^1 \frac{(1-v^2)^{k+1/2}}{(1-zv^2)^{1/2}} (1+3v^2)^{1/2} v dv. \quad (20)$$

The function  $v \mapsto (1+3v^2)^{1/2}v$  admits a holomorphic expansion according to the powers of  $v^2 - 1$  (using for this the expansion of  $v \mapsto v^{1/2}$  in the neighborhood of 1) :  $(1+3v^2)^{1/2}v = 2 + (7/4)(v^2 - 1) + \dots$ . But in Subsection 3.2 of [Ras09] we have proved, using elliptic integrals theory, that for any non-negative integer  $k$  there exist two functions,  $\phi_k$  and  $\psi_k$ , holomorphic in the neighborhood of 1,  $\phi_k(1) \neq 0$ , such that :

$$\int_0^1 \frac{(1-v^2)^{k+1/2}}{(1-zv^2)^{1/2}} dv = \ln(1-z)(1-z)^{k+1} \phi_k(z) + \psi_k(z), \quad (21)$$

we have there also proved that  $\phi_0(1) = 1/4$ . The equality (18) is then an immediate consequence of (20), of the expansion of  $(1+3v^2)^{1/2}v$  according to the powers of  $v^2 - 1$  and of the repeated use of (21). The fact that  $r_0(1) = 3\sqrt{3}/4$  comes from the equality  $\phi_0(1) = 1/4$ .

Likewise, we prove the equality (19) and we obtain the values of  $\gamma_0(1), \gamma'_0(1)$  and  $\gamma_1(1)$ .  $\square$

*Proof of Proposition 11.* We recall from Corollary 7 that

$$h_2(1, z) = \frac{1}{\pi} \int_{-1}^1 \frac{\partial_u t_2(u, z)}{t_2(u, z) - 1} t_2(u, z)^{x_0+y_0/2-1} U_{y_0-1}(u) (1-u^2)^{1/2} du, \quad (22)$$

where  $t_2(u, z) = T(u, z) - (T(u, z)^2 - 1)^{1/2}$ ,  $T(u, z) = 3/z + u^2 - u(2+u^2+6/z)^{1/2}$ . In particular, the fact that  $z \mapsto h_2(1, z)$  is holomorphic in the unit disc, continuable holomorphically through every point of the unit circle except 1 is clear, since making the change of variable  $u \mapsto -u$  in (22) allows us to write  $h_2(1, z)$  as the integral on  $[0, 1]$  of a function verifying these properties for all  $u \in [-1, 1]$  – indeed, any symmetrical function of  $(T(u, z), T(-u, z))$  satisfies these properties for all  $u \in [-1, 1]$ . Let us now study the behavior of  $h_2(1, z)$  in the neighborhood of 1. For this we will transform (22), until obtaining an expression that makes clearly appear the singularities of  $h_2(1, z)$ .

An easy calculation shows that  $\partial_u t_2 = \partial_u T / (1 - t_3^2)$ . Moreover, by definition of the  $t_i$ , see Subsection 2.3,  $(z^2/36) \prod_{i=1}^4 (t - t_i(u, z))$  is equal to  $b(t, z)^2 - 4u^2 a(t, z) c(t, z)$ . In particular,

$\prod_{i=1}^4(1 - t_i(u, z)) = (36/z^2)(1 - z(1 + 2u)/3)(1 - z(1 - 2u)/3)$ . So we have :

$$\frac{\partial_u t_2(u, z)}{t_2(u, z) - 1} = \frac{z^2 \partial_u T_1(u, z) (1 - t_1(u, z)) (1 - t_4(u, z)) (1 - t_2(u, z))}{18 (1 - z(1 - 2u)/3) (t_2(u, z) - t_3(u, z)) (1 - z(1 + 2u)/3)}. \quad (23)$$

We will now expand  $(1 - t_2(u, z))t_2(u, z)^{x_0+y_0/2-1}$  according to the powers of  $(T(u, z)^2 - 1)^{1/2}$  : we set  $(1 - t_2(u, z))t_2(u, z)^{x_0+y_0/2-1} = \sum_{k \geq 0} F_k(u, z)(T(u, z)^2 - 1)^{k/2}$ . With (22) and (23),  $h_2(1, z)$  is therefore simply equal to

$$h_2(1, z) = \sum_{k \geq 0} \int_{-1}^1 \frac{z^2 \partial_u T(u, z) (1 - t_1(u, z)) (1 - t_4(u, z))}{18 (1 - z(1 - 2u)/3)} F_k(u, z) \times \quad (24)$$

$$\times \frac{(T(u, z)^2 - 1)^{k/2}}{(t_2(u, z) - t_3(u, z)) (1 - z(1 + 2u)/3)} U_{y_0-1}(u) (1 - u^2)^{1/2} du.$$

Let us now study the behavior in the neighborhood of 1 of each integral in the sum (24).

We start by studying the terms in (24) associated to  $k = 0, 1, 2$ . Since  $t_2(u, z) - t_3(u, z) = -2(T(u, z)^2 - 1)^{1/2}$  and since  $(t_2(u, z) - t_3(u, z))(t_1(u, z) - t_4(u, z)) = 12(z + 3)^2((1 - z(1 + 2u)/3)(1 - z(1 - 2u)/3))^{1/2}/z^2$ , we have

$$F_0 = T^{x_0+y_0/2-1} (1 - T), \quad F_1 = T^{x_0+y_0/2-2} (T - (x_0 + y_0/2 - 1) (1 - T)),$$

$$F_2 = T^{x_0+y_0/2-3} (x_0 + y_0/2 - 1) ((1 - T) (x_0 + y_0/2 - 2) / 2 - T).$$

Now we set  $F(u, z) = -z^2 \partial_u T(u, z)(1 - t_1(u, z))(1 - t_4(u, z))U_{y_0-1}(u)/(36(1 - z(1 - 2u)/3))$  and

$$G_0(u, z) = [F(u, z) z^2 F_0(u, z) (T(-u, z)^2 - 1)^{1/2}] / [3(z + 3) (1 - z(1 - 2u)/3)^{1/2}],$$

$$G_1(u, z) = F(u, z) F_1(u, z),$$

$$G_2(u, z) = [F(u, z) F_2(u, z) 3(z + 3) (1 - z(1 - 2u)/3)^{1/2}] / [z^2 (T_1(-u, z)^2 - 1)^{1/2}],$$

in such a way that the sum of the three terms for  $k = 0, 1, 2$  in (24) is equal to  $\sum_{i=0}^2 \int_{-1}^1 G_i(u, z) (1 - z(1 + 2u)/3)^{(-3+i)/2} (1 - u^2)^{1/2} du$ .

Using the expansion of the Chebyshev polynomials in the neighborhood of 1 recalled in Subsection 2.3, we obtain the expansion of  $G_0$  in the neighborhood of  $(u, z) = (1, 1)$  :  $G_0(u, z) = -2y_0(u - 1)/9 - y_0(z - 1)/3 + \sum_{k+l \geq 2} G_{0,k,l}(u - 1)^k (z - 1)^l$ . Then, with a repeated use of (19) of Lemma 14, we get  $\int_{-1}^1 G_0(u, z) (1 - z(1 + 2u)/3)^{-3/2} (1 - u^2)^{1/2} du = y_0 \sqrt{3} \ln(1 - z) ((1 - z)/4 + (1 - z)^2 g_0(z)) + f_0(z)$ ,  $f_0$  and  $g_0$  being holomorphic at 1.

Similarly we have  $G_2(u, z) = 2y_0(x_0 + y_0/2 - 1)/3 + \sum_{k+l \geq 1} G_{2,k,l}(u - 1)^k (z - 1)^l$ . So with a repeated use of (18) of Lemma 14, we get  $\int_{-1}^1 G_2(u, z) (1 - z(1 + 2u)/3)^{-1/2} (1 - u^2)^{1/2} du = y_0(x_0 + y_0/2 - 1) \sqrt{3} \ln(1 - z) ((1 - z)/2 + (1 - z)^2 g_2(z)) + f_2(z)$ ,  $f_2$  and  $g_2$  being holomorphic in the neighborhood of 1.

As for  $G_1$ , it admits the following expansion :  $G_1(u, z) = -y_0/3 - y_0(6y_0^2 + 35 - 48x_0 - 24y_0)(u - 1)/54 + y_0(-53 + 48x_0 + 24y_0)(z - 1)/36 + \sum_{k+l \geq 2} G_{1,k,l}(u - 1)^k (z - 1)^l$ . A repeated application of Lemma 13 gives that  $\int_{-1}^1 G_1(u, z) (1 - z(1 + 2u)/3)^{-1} (1 - u^2)^{1/2} du$  has an algebraic singularity at 1 and is equal in a neighborhood of 1 to  $y_0 \sqrt{3} (1 - z)^{1/2} (1/2 + (3/4 + y_0^2)(1 - z)/4 + (1 - z)^2 g_1(z)) + f_1(z)$ ,  $f_1$  and  $g_1$  being holomorphic at 1.

Let us now consider the terms in the sum (24) for  $k \geq 3$ . Note first that if  $k$  is odd and larger than 3, then the corresponding function in (24) is in fact holomorphic in the neighborhood of 1 ; indeed, we recall that  $t_2(u, z) - t_3(u, z) = -2(T(u, z)^2 - 1)^{1/2}$ . For this reason, the terms associated to odd values of  $k$  in (24) don't have any singularity at 1.

On the other hand, if  $k \geq 3$  is even, then the associated term in the sum (24) can be written as  $\int_{-1}^1 (1 - z(1 + 2u)/3)^{(k-3)/2} H_k(u, z)(1 - u^2)^{1/2} du$ , where  $H_k(u, z)$  is some function holomorphic in the neighborhood of  $(1, 1)$ . This last integral is obviously equal to  $\int_{-1}^1 (1 - z(1 + 2u)/3)^{(k-2)/2} H_k(u, z)(1 - u^2)^{1/2} / (1 - z(1 + 2u)/3)^{1/2} du$ . Then, by expanding the function  $(1 - z(1 + 2u)/3)^{(k-2)/2} H_k(u, z)$  – holomorphic in the neighborhood of  $(1, 1)$  – according to the powers of  $(u - 1)^k(z - 1)^l$  and using (18) of Lemma 14, we obtain that  $\int_{-1}^1 (1 - z(1 + 2u)/3)^{(k-3)/2} (1 - u^2)^{1/2} H_k(u, z) du = \ln(1 - z)(z - 1)^{k-2} g_k(z) + f_k(z)$ ,  $f_k$  and  $g_k$  being holomorphic at 1. So that the sum of all the terms corresponding to even  $k$  in (24) can finally be written as  $\ln(1 - z)(1 - z)^2 g(z) + f(z)$  in the neighborhood of 1, where  $f$  and  $g$  are holomorphic at 1.  $\square$

In order to prove Proposition 12, we proceed as in the proof of Proposition 11. Namely we start by simplifying the integral representation of  $h_3(1, z)$  found in Corollary 7, using for this the root-coefficient relationships. In this way, the quantity  $1 - z(4u^2 - 1)/3$  appears in the denominator of the integrand of  $h_3(1, z)$ , instead of  $1 - z(1 + 2u)/3$  in (23). Then, as in (24), we expand the integrand of  $h_3(1, z)$  according to the powers of  $(T(u, z)^2 - 1)^{1/2}$ . Lastly we could state analogous results as Lemma 13 and 14 and finally we obtain the expansion of  $h_3(1, z)$  in the neighborhood of 1 and thus Proposition 12, we omit the details.

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