Passage time of a random walk in the quarter plane for opinions in the voter model

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PASSAGE TIME OF A RANDOM WALK IN THE QUARTER PLANE FOR OPINIONS IN THE VOTER MODEL

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Abstract. A random walk in $\mathbf{Z}_+^2$ spatially homogeneous in the interior, absorbed at the axes, starting from an arbitrary point (i_0, j_0) and with step probabilities drawn on Figure 1 is considered. The trivariate generating function of probabilities that the random walk hits a given point $(i,j) \in \mathbf{Z}_+^2$ at a given time $k \geqslant 0$ is made explicit. Probabilities of absorption at a given time k and at a given axis are found, and their precise asymptotic is derived as the time $k \to \infty$. The equivalence of two typical ways of conditioning this random walk to never reach the axes is established. The results are also applied to the analysis of the voter model with two candidates and initially, in the population $\mathbf{Z}$, four connected blocks of same opinions. Then, a citizen changes his mind at a rate proportional to the number of its neighbors that disagree with him. Namely, the passage from four to two blocks of opinions is studied.

Keywords. Voter model; Random walk in the quarter plane; Hitting times; Integral representations

AMS 2000 Subject Classification: primary 82C22, 60G50; secondary 60G40, 30E20

1. Introduction

Context. Random walks with small steps in the quarter plane $\mathbf{Z}_+^2 = \{0, 1, 2, \dots\}^2$ spatially homogeneous in the interior and on each of the two axes are now rather well studied. The analytic approach [7] elaborated by Fayolle, Iasnogorodski and Malyshev provided the generating function, say $H(x,y)$, of the stationary probabilities in the ergodic case, and also that of the Green functions in the transient case. Further analysis allowed to compute the asymptotic of these quantities along any path in $\mathbf{Z}_+^2$, see [10, 11, 14, 17, 18].

The main motivation of the present work is to develop a method for incorporating the parameter z of time into this approach, in order to derive the trivariate generating function $H(x,y;z)$ of the probabilities $h_{i,j;k}$ that the walk is in state (i,j) at time k, and to obtain asymptotic results from it. Being able to deal with this additional time variable z is actually important in combinatorics (e.g., to count certain numbers of walks confined to the quarter plane, see [16]) and in probability as well (e.g., to compute the distribution of some hitting times). This is one of the few attempts is that direction, after [3, 8]. The second motivation of this article is the application that it has for the voter model: indeed, it completes results of [2, 15] about the hitting time of the so-called Heaviside configuration in the voter model with initially four blocks of opinions.

Voter model. By the voter model we mean a continuous-time process on $\{0,1\}^{\mathbf{Z}}$ (here and throughout, $\mathbf{Z} = \{\dots, -1, 0, 1, \dots\}$) that can be interpreted as follows: initially, at each site of $\mathbf{Z}$, there is zero or one particle; then a particle appears (resp. disappears) at an empty (resp. occupied) site x according to an exponential law with a rate proportional to the number of nearest neighbors of x which are occupied (resp. empty). Moreover, we

Date: October 26, 2012.

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assume that the initial state appertains to the set of configurations having a finite number of empty (resp. occupied) sites on the left (resp. right) of the origin 0, see (1.1) for an example. In particular, this implies that at any time the process will belong to this set of configurations. As a consequence there is, at any time, a finite number of "01" (resp. "10"), i.e., a finite number of pairs of sites $(x, x+1)$ with zero (resp. one) particle at x and one (resp. zero) particle at $x+1$.

The underlying discrete-time voter model is a Markov chain with the following dynamic: denote by $\mathscr{C}_k$ the configuration at time k (and remember that according to the previous paragraph, there is only a finite number of "01" and "10" in $\mathscr{C}_k$); next, in order to construct $\mathscr{C}_{k+1}$, one first chooses with a uniform distribution one of these "01" and "10" in $\mathscr{C}_k$, then one replaces it, with probability $1/2$, by "00" or "11".

If the voter model starts from the Heaviside configuration, i.e., the configuration having only occupied (resp. empty) sites on the left (resp. right) of the origin, then at any time, the process will be a translation of it. This fact suggests to consider the following equivalence relation: two configurations are equivalent if they are translations the one of the other.

From now on, we shall work on the underlying quotient space, the equivalence classes of which being identified by finite sets of positive integers $(X_1, Y_1, \ldots, X_N, Y_N)$:

$$\ldots 111 \overbrace{000000}^{X_1} \overbrace{11111}^{Y_1} \overbrace{00000}^{X_2} \overbrace{11111}^{Y_2} \ldots\ldots \overbrace{000}^{X_N} \overbrace{1111}^{Y_N} 000 \ldots, \tag{1.1}$$

N being the number of finite blocks of zeros (or ones) and X_ℓ (resp. Y_ℓ), $\ell \in \{1, \ldots, N\}$, the size of the ℓth block of zeros (resp. ones). The number N of finite blocks of zeros is a non-increasing function of the time; furthermore, $N = 0$ corresponds to the class of the Heaviside configuration.

We refer to [12] for additional details on the voter model and, more generally, for further information about interacting particle systems.

Hitting time of the Heaviside configuration. Let τ denote the hitting time of the Heaviside configuration. It is proved in [2] that for any initial configuration,

$$\mathbf{E}[\tau^{3/2-\epsilon}] < \infty, \quad \forall \epsilon > 0, \tag{1.2}$$

$$\mathbf{E}[\tau^{3/2+\epsilon}] = \infty, \quad \forall \epsilon > 0. \tag{1.3}$$

Statement (1.2) is proved by an adequate use of Lyapunov functions. To show (1.3), it suffices to do it only for initial states with $N = 1$ in (1.1); that is done in [2], by applying results on passage time moments proved in [1].

With the notations (1.1), consider the process starting from an initial state with $N = 1$: $(X_1, Y_1) = (X_1(k), Y_1(k))_{k \in \mathbf{Z}_+}$. We rename it here $(X, Y) = (X(k), Y(k))_{k \in \mathbf{Z}_+}$; we have:

$$\ldots 111 \overbrace{000000}^{X} \overbrace{11111}^{Y} 000 \ldots.$$

The process (X, Y) is a Markov chain on $\mathbf{Z}_+^2$ which is absorbed as it reaches the boundary, since the Heaviside configuration is an absorbing state for the voter model. Moreover, using the dynamic of the discrete-time voter model explained above, we notice that (X, Y) has homogeneous transition probabilities in the interior of $\mathbf{Z}_+^2$ equal to (with obvious notations)

$$p_{1,0} = p_{1,-1} = p_{0,-1} = p_{-1,0} = p_{-1,1} = p_{0,1} = 1/6$$

and the others to 0, see Figure 1. Further, the hitting time τ can be expressed as

$$\tau = \inf\{k \in \mathbf{Z}_+ : X(k) = 0 \text{ or } Y(k) = 0\}. \tag{1.4}$$

Define also the hitting times of the horizontal and vertical axes:

$$S = \inf\{k \in \mathbf{Z}_+ : Y(k) = 0\}, \quad T = \inf\{k \in \mathbf{Z}_+ : X(k) = 0\}, \tag{1.5}$$

so that $\tau = \inf\{S, T\}$.

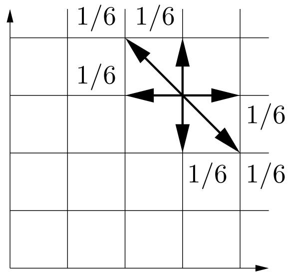

FIGURE 1. Transitions of the process (X, Y) in the interior of the quarter plane $\mathbf{Z}_+^2$; on the boundary, the process is absorbed

Main results. The present work is constituted by four main points, that we now describe.

The first one is a direct consequence of [18] and concerns explicit expressions for the probabilities that the process is absorbed at some site of the boundary in a given time, namely, $\mathbf{P}_{(i_0,j_0)}[S = k]$ and $\mathbf{P}_{(i_0,j_0)}[T = k]$, for any $k \in \mathbf{Z}_+$ and any $(i_0, j_0) \in \mathbf{Z}_+^2$. For this we shall use Proposition 5 of Section 2, taken from [18, Chapter F], which gives an integral representation of the generating functions

$$h^{i_0,j_0}(x; z) = \sum_{i\geqslant 1} \sum_{k\geqslant 0} \mathbf{P}_{(i_0,j_0)}[(X, Y) \text{ hits } (i, 0) \text{ at time } k]x^i z^k, \tag{1.6}$$

$$\widetilde{h}^{i_0,j_0}(y; z) = \sum_{j\geqslant 1} \sum_{k\geqslant 0} \mathbf{P}_{(i_0,j_0)}[(X, Y) \text{ hits } (0, j) \text{ at time } k]y^j z^k. \tag{1.7}$$

Then $\mathbf{P}_{(i_0,j_0)}[S = k]$ and $\mathbf{P}_{(i_0,j_0)}[T = k]$ can be expressed from $h^{i_0,j_0}(1; z)$ and $\widetilde{h}^{i_0,j_0}(1; z)$ via the Cauchy formulæ. Note, besides, that we also find the trivariate function

$$H^{i_0,j_0}(x, y; z) = \sum_{i,j\geqslant 1} \sum_{k\geqslant 0} \mathbf{P}_{(i_0,j_0)}[(X(k), Y(k)) = (i, j)]x^{i-1}y^{j-1}z^k \tag{1.8}$$

thanks to the functional equation (2.1). For the voter model, this means that we find explicit expressions for the probabilities that the process hits the Heaviside configuration at any fixed time, with the additional information of the size of the blocks at the time of absorption.

The second point is our main result, and is new, to the best of our knowledge. It is about the asymptotic tail distribution of the hitting time S.

Theorem 1. *As the time $k \to \infty$, we have*

$$\mathbf{P}_{(i_0,j_0)}[S = k] = \frac{9}{16}\left(\frac{3}{\pi}\right)^{1/2} \frac{i_0 j_0(i_0 + j_0)}{k^{5/2}}(1 + o(1)). \tag{1.9}$$

Theorem 1 will be a consequence of Proposition 9 and of classical singularity analysis [9, Sections 6.2–6.4], see Section 3. Theoretically, the methods developed in this paper could work for (the hitting time of the boundary of) any random walk with transitions to the eight nearest neighbors, see Remark 6.

We now introduce the third point of our work. First notice that the transition probabilities of the walk are such that $\mathbf{P}_{(i_0,j_0)}[S = k] = \mathbf{P}_{(j_0,i_0)}[T = k]$, see Figure 1, and that the quantity $i_0 j_0(i_0 + j_0)$ in (1.9) is invariant by $(i_0, j_0) \mapsto (j_0, i_0)$. Accordingly, the asymptotic of $\mathbf{P}_{(i_0,j_0)}[T = k]$ is exactly the same as that of $\mathbf{P}_{(i_0,j_0)}[S = k]$. Further,

$$\mathbf{P}_{(i_0,j_0)}[\tau = k] = \mathbf{P}_{(i_0,j_0)}[S = k] + \mathbf{P}_{(i_0,j_0)}[T = k], \tag{1.10}$$

so that Theorem 1 entails the following corollary.

Corollary 2. *The result* (1.3) *proved in* [2] *for $\epsilon > 0$ also holds for $\epsilon = 0$.*

The latter completes the results of [2, 15]. It is worth noting that this corollary can also be obtained as a consequence of results in [21, 4]. More details are provided at the end of this introduction.

Finally, the last result in our paper is the following: the precision of the asymptotic result (1.9) implies Corollary 3 below that compares two typical ways on conditioning the process (X,Y) to never reach the axes. Define $h(i_0,j_0)=i_0j_0(i_0+j_0)$ (in fact, using methods closed to [17], it could be proved that it is the unique positive harmonic function associated with the walk (X,Y) absorbed at the boundary of $\mathbf{Z}_+^2$).

Corollary 3. *The Doob h-process of (X,Y) coincides in distribution with the limit, as $k\to\infty$, of the process conditioned on $\{\tau>k\}$.*

The proof relies on the following precise asymptotic, as $k\to\infty$,

$$\mathbf{P}_{(i_0,j_0)}[\tau\geqslant k]=\frac{27}{16}\left(\frac{3}{\pi}\right)^{1/2}\frac{i_0j_0(i_0+j_0)}{k^{3/2}}(1+o(1)),$$

which is a direct consequence of Theorem 1 and (1.10). It is then carried out by a standard reasoning as in [17] or [18, Chapter F].

Other approaches. We close this introduction by mentioning other possible approaches for analyzing asymptotic tail distribution of hitting times for random walks in cones of $\mathbf{Z}^d$. First, as already quoted, methods using Lyapunov functions in [1, 2, 15] show the finiteness or infiniteness of hitting times' moments. A series of tail distribution estimates for hitting times is presented in [21], by using potential theory. They result in upper and lower bounds for $\mathbf{P}_{(i_0,j_0)}[\tau\geqslant k]$, which are enough to obtain Corollary 2 in this paper. In a recent work [4], the tail asymptotic of the hitting time up to a multiplicative factor is obtained by comparison with Brownian motion. This is another way to deduce Corollary 2. All these methods are powerful for rather general random walks in conic domains of $\mathbf{Z}^d$, but do not give as much accurate results as Theorem 1 in this paper. Finally, let us mention the paper [5], where an approach based on an extension of the Karlin-McGregor formula is applied to the family of the so-called non-colliding random walks. The latter leads to precise results, but it exploits a particular independence property of this family, and therefore seems to be restricted to this class of models.

2. Exact distribution of the hitting times of both axes

This section contains preliminary material, which is needed for Section 3, where we prove our main results.

A functional equation and the kernel of the walk. With the notations (1.6), (1.7) and (1.8) of Section 1, we can state on $\{(x,y,z)\in\mathbf{C}^3:|x|,|y|,|z|\leqslant 1\}$ (here and throughout, $\mathbf{C}$ denotes the complex plane) the following crucial functional equation:

$$K(x,y;z)H^{i_0,j_0}(x,y;z)=h^{i_0,j_0}(x;z)+\widetilde{h}^{i_0,j_0}(y;z)-x^{i_0}y^{j_0},\tag{2.1}$$

where $K(x,y;z)$ is the following polynomial—called the kernel of the walk—, depending only on the walk's transition probabilities:

$$K(x,y;z)=xyz[\textstyle\sum_{-1\leqslant i,j\leqslant 1}p_{i,j}x^iy^j-1/z].\tag{2.2}$$

For $z=0$, Equation (2.1) simply becomes $\mathbf{P}_{(i_0,j_0)}[(X(0),Y(0))=(i_0,j_0)]=1$. For $z=1$, it becomes a functional equation between the Green functions generating function and the absorption probabilities generating functions; the latter is studied in [11, 17, 18]. For the proof of (2.1), we exactly use the same arguments as in [18, Chapter F].

We now study the set of the zeros of the kernel $K(x,y;z)$ defined in (2.2). For this we start by remarking that it can be written alternatively

$$K(x,y;z)=a(x;z)y^2+b(x;z)y+c(x;z)=\widetilde{a}(y;z)x^2+\widetilde{b}(y;z)x+\widetilde{c}(y;z),\tag{2.3}$$

where

$$a(x;z)=z(x+1)/6,\quad b(x;z)=zx^2/6-x+z/6,\quad c(x;z)=zx(x+1)/6,$$

and

$$\widetilde{a}(y;z)=z(y+1)/6,\quad \widetilde{b}(y;z)=zy^2/6-y+z/6,\quad \widetilde{c}(y;z)=zy(y+1)/6.$$

Next, we introduce the algebraic function $Y(x;z)$ defined by $K(x,Y(x;z);z)=0$. Note that $K(x,y;z)=0$ is equivalent to $[b(x;z)+2a(x;z)y]^2=d(x;z)$, where

$$d(x;z)=b(x;z)^2-4a(x;z)c(x;z),$$

so that the construction of the function $Y(x;z)$ is equivalent to that of the square roots of the polynomial $d(x;z)$, namely, $\pm d(x;z)^{1/2}$. For this we need the following:

Lemma 4. *Let $z\in]0,1[$. The four roots of $x\mapsto d(x;z)$ are positive and mutually distinct. We call them $x_1(z)<x_2(z)<x_3(z)<x_4(z)$. They satisfy $x_1(z)x_4(z)=x_2(z)x_3(z)=1$. In particular, $x_1(z),x_2(z)\in]0,1[$ and $x_3(z),x_4(z)\in]1,\infty[$. Further, $x_1(0)=x_2(0)=0$, $x_3(0)=x_4(0)=\infty$, $x_2(1)=x_3(1)=1$ and $x_1(1)=7-4\sqrt{3}$, $x_4(1)=7+4\sqrt{3}$.*

Proof. As we can easily verify, the polynomial $d(x;z)$ is reciprocal, in other words it satisfies $x^4d(1/x;z)=d(x;z)$. This property allows us to write it as a second degree polynomial in the variable $x+1/x$. Following this way we obtain the explicit expression of its roots: if $s_1(z)=3/z+1$ and $s_2(z)=(6/z+3)^{1/2}$, then $x_1(z)=s_1(z)+s_2(z)+[(s_1(z)+s_2(z))^2-1]^{1/2}$ and $x_2(z)=s_1(z)-s_2(z)+[(s_1(z)-s_2(z))^2-1]^{1/2}$, $x_3(z)=1/x_2(z)$ and $x_4(z)=1/x_1(z)$. All the properties of Lemma 4 immediately follow from these explicit expressions. □

There are two branches of the square root of $d(x;z)$. Each determination leads to a single-valued and meromorphic function on the complex plane $\mathbf{C}$ appropriately cut, that is, in our case, on $\mathbf{C}\setminus([x_1(z),x_2(z)]\cup[x_3(z),x_4(z)])$. We have

$$Y(x;z)=\frac{-b(x;z)\pm d(x;z)^{1/2}}{2a(x;z)},$$

and we fix the notations of the branches $Y_0(x;z)$ and $Y_1(x;z)$ by (arbitrarily) choosing that $|Y_0(x;z)|<|Y_1(x;z)|$ on the whole of $\mathbf{C}\setminus([x_1(z),x_2(z)]\cup[x_3(z),x_4(z)])$. For more details about the construction of algebraic functions, see, e.g., [19].

In a similar way, the functional equation (2.1) defines an algebraic function $X(y;z)$. But it turns out that $K(x,y;z)=K(y,x;z)$, see (2.2), so that $X(y;z)=Y(y;z)$; in particular, all properties proved for $Y(x;z)$ immediately result in similar ones for $X(y;z)$.

Explicit expression of distributions. They are obtained in the result that follows.

Proposition 5. *The function $h^{i_0,j_0}(x;z)$ is equal to*

$$\begin{aligned} h^{i_0,j_0}(x;z) &= x^{i_0}Y_0(x;z)^{j_0}\\ &+\int_{x_1(z)}^{x_2(z)} t^{i_0}\mu_{j_0}(t;z)\left[\frac{\partial_t w(t;z)}{w(t;z)-w(x;z)}-\frac{\partial_t w(t;z)}{w(t;z)-w(0;z)}\right][-d(t;z)]^{1/2}\mathrm{d}t, \end{aligned}\tag{2.4}$$

where

$$\mu_{j_0}(t;z)=\frac{1}{[2a(t;z)]^{j_0}}\sum_{k=0}^{(j_0-1)/2}\binom{2k+1}{j_0}d(t;z)^k[-b(t;z)]^{j_0-(2k+1)},\tag{2.5}$$

and

$$w(t;z)=\frac{t(1+t)}{(t-x_2(z))(t-x_3(z)^{1/2})^2}.\tag{2.6}$$

Remark 6. Equations (2.4) and (2.5) are obtained in [18, Chapter F], while (2.6) is found in [16]. It is worth noting that Equations (2.4) and (2.5) are valid not only for the random walk under consideration in this paper, but for all random walks with jumps to the eight nearest neighbors, see [7] and [18, Chapter F].

On the other hand, finding an expression for the function $w(t;z)$ happens to be quite complex in general, and dependent on the particular model. Further, in general, there is no reason for this function to be rational (in t) as in (2.6), or even algebraic. To be complete, we note that for our model, $w(t;z)$ is rational because a certain group of automorphisms is finite. We refer to [6, 7, 11, 16, 17, 18] for any details on this group, and more generally on how finding expressions as in Proposition 5.

The algebraicity of $w(t;z)$ happens to be crucial for the proof of Theorem 1 (see Section 3), and this is why this article focuses on one particular model. It remains an open problem to determine, for all random walks with jumps to the eight nearest neighbors, the asymptotic tail distribution of the hitting time of the boundary, by using analytic methods as in this paper.

Now, using the partial fraction expansion (direct consequence of (2.6))

$$\frac{\partial_t w(t;z)}{w(t;z)-w(x;z)}-\frac{\partial_t w(t;z)}{w(t;z)-w(0;z)}= \\ \frac{x}{t(t-x)}+\frac{1}{t-X_1(Y_0(x;z);z)}+\frac{1}{t-X_1(Y_1(x;z);z)}-\frac{1}{t+1},$$

we immediately obtain the following.

Corollary 7. *The function* $h^{i_0,j_0}(x;z)$ *can be split as* $h^{i_0,j_0}(x;z) = h_1^{i_0,j_0}(x;z)+h_2^{i_0,j_0}(x;z)+h_3^{i_0,j_0}(x;z)$, *where*

$$h_1(x;z) = x^{i_0}Y_0(x;z)^{j_0}, \tag{2.7}$$

$$h_2(x;z) = \frac{x}{\pi}\int_{x_1(z)}^{x_2(z)}\frac{t^{n_0-1}}{t-x}\mu_{j_0}(t;z)[-d(t;z)]^{1/2}\mathrm{d}t, \tag{2.8}$$

$$h_3(x;z) = \frac{1}{\pi}\int_{x_1(z)}^{x_2(z)} t^{i_0}\left[\frac{1}{t-X_1(Y_0(x;z);z)}+ \frac{1}{t-X_1(Y_1(x;z);z)}-\frac{1}{t+1}\right]\mu_{j_0}(t;z)[-d(t;z)]^{1/2}\mathrm{d}t. \tag{2.9}$$

The end of Section 2 aims at obtaining an expression of $h^{i_0,j_0}(1;z)$ which is efficient—in the sense of computing the asymptotic of its coefficients (that we shall do in Section 3). In order to achieve this, we shall make the change of variable $\widehat{b}(t;z)=b(t;z)/[4a(t;z)c(t;z)]^{1/2}$ in the integrals (2.8) and (2.9) of Corollary 7. The main reason of this is that using (2.5) yields

$$\mu_{j_0}(t;z)[-d(t;z)]^{1/2}=\left(\frac{c(t;z)}{a(t;z)}\right)^{j_0/2}U_{j_0-1}(-\widehat{b}(t;z))[1-\widehat{b}(t;z)^2]^{1/2}, \tag{2.10}$$

where the $(U_n)_{n\in\mathbf{Z}_+}$ are the Chebyshev polynomials of the second kind. We refer to [20] for a complete exposition on these polynomials. We recall that they are the orthogonal polynomials associated with the weight $t\mapsto[1-t^2]^{1/2}\mathbf{1}_{]-1,1[}(t)$, and that their explicit expression is

$$U_n(u)=\frac{(u+[u^2-1]^{1/2})^{n+1}-(u-[u^2-1]^{1/2})^{n+1}}{2[u^2-1]^{1/2}},\quad \forall u\in\mathbf{C},\quad \forall n\in\mathbf{Z}_+.$$

We also recall two properties of the Chebyshev polynomials of the second kind that we will especially use here (see [20] for their proof):

- They have the parity of their order, in other words, for all $u\in\mathbf{C}$ and all $n\in\mathbf{Z}_+$, $U_n(-u)=(-1)^nU_n(u)$;
- Their expansion in the neighborhood of 1 is $U_n(u)=(n+1)[1+n(n+2)(u-1)/3+O(u-1)^2]$.

Further, function $t\mapsto\widehat{b}(t;z)$ is clearly a diffeomorphism between $]x_1(z),x_2(z)[$ and $]-1,1[$; in addition, $\widehat{b}(t;z)=u$ implies $b(t;z)^2-4u^2a(t;z)c(t;z)=0$, which, as a polynomial in the variable t, is reciprocal, so that we can quite easily obtain and write the explicit expression of its roots, called the $t_\ell(u;z)$, $\ell\in\{1,\ldots,4\}$. Defining $T(u;z)=3/z+u^2-u[2+u^2+6/z]^{1/2}$, then $t_2(u;z)=T(u;z)-[T(u;z)^2-1]^{1/2}$, $t_3(u;z)=T(u;z)+[T(u;z)^2-1]^{1/2}$, $t_1(u;z)=t_2(-u;z)$ and $t_4(u;z)=t_3(-u;z)$. Notice that we have enumerated the $t_\ell(u;z)$ in such a way that $t_\ell(1;z)=x_\ell(z)$ for any $\ell\in\{1,\ldots,4\}$. Moreover, it turns out that for $u\in]-1,1[$, $\widehat{b}(t_2(u;z);z)=-u$, so that the following result is an immediate consequence of the change of variable $t=t_2(u;z)$ in Corollary 7 as well as of the identity (2.10).

Corollary 8. *We have* $h^{i_0,j_0}(1;z)=h_1^{i_0,j_0}(1;z)+h_2^{i_0,j_0}(1;z)+h_3^{i_0,j_0}(1;z)$, *where*

$$\begin{aligned}
h_1^{i_0,j_0}(1;z) &= \left(\frac{(3-z-3[(1-z)(1+z/3)]^{1/2}}{(2z)}\right)^{j_0},\\
h_2^{i_0,j_0}(1;z) &= \frac{1}{\pi}\int_{-1}^{1}\frac{U_{j_0-1}(u)t_2(u;z)^{i_0+j_0/2-1}}{t_2(u;z)-1}\partial_u t_2(u;z)[1-u^2]^{1/2}\mathrm{d}u,\\
h_3^{i_0,j_0}(1;z) &= \frac{1}{\pi}\int_{-1}^{1}U_{j_0-1}(u)t_2(u;z)^{i_0+j_0/2}\left[\frac{1}{t_2(u;z)-X_1(Y_0(1;z);z)}+\right.\\
&\qquad\left.+\frac{1}{t_2(u;z)-X_1(Y_1(1;z);z)}-\frac{1}{t_2(u;z)+1}\right]\partial_u t_2(u;z)[1-u^2]^{1/2}\mathrm{d}u.
\end{aligned}$$

3. Asymptotic tail distribution of the hitting times

In this section we prove Theorem 1, which is the main result in this paper. Let $\mathscr{D}$ denote the open unit disc: $\mathscr{D}=\{z\in\mathbf{C}:|z|<1\}$. In order to prove Theorem 1, we are going to prove that $h^{i_0,j_0}(1;z)$ is holomorphic in $(1+\epsilon)\mathscr{D}\setminus[1,1+\epsilon[$ and that in the neighborhood of 1,

$$h^{i_0,j_0}(1;z)=(3/4)3^{1/2}i_0j_0(i_0+j_0)[1-z]^{3/2}[1+o(1)]+h_0^{i_0,j_0}(z), \tag{3.1}$$

where $h_0^{i_0,j_0}$ is holomorphic at 1; it will then be enough to use classical singularity analysis (see [9, Sections 6.2–6.4]).

For this, according to Corollary 8, we shall consider successively $h_1^{i_0,j_0}(1;z)$, $h_2^{i_0,j_0}(1;z)$ and $h_3^{i_0,j_0}(1;z)$ in Proposition 9. Equation (3.1) and Theorem 1 will then be a direct consequence of these three results.

Proposition 9. *The functions* $h_1^{i_0,j_0}(1;z)$, $h_2^{i_0,j_0}(1;z)$ *and* $h_3^{i_0,j_0}(1;z)$ *are holomorphic in* $(1+\epsilon)\mathscr{D}\setminus[1,1+\epsilon[$. *Moreover, in the neighborhood of* 1, *we have*

$$\begin{aligned}
h_1^{i_0,j_0}(1;z) &= -j_0 3^{1/2}[1-z]^{1/2}[1+(3+4j_0{}^2)(1-z)/8+f_{1,1}^{i_0,j_0}(z)(z-1)^2]+f_{1,2}^{i_0,j_0}(z),\\
h_2^{i_0,j_0}(1;z) &= \frac{3^{1/2}j_0}{2}[1-z]^{1/2}[1+(1/2)(3/4+j_0{}^2)(1-z)+f_{2,1}^{i_0,j_0}(z)(1-z)^2]\\
&\quad+\frac{3^{1/2}j_0}{2\pi}(i_0+j_0/2-1/2)\ln(1-z)[1+(1-z)f_{2,2}^{i_0,j_0}(z)]+f_{2,3}^{i_0,j_0}(z),\\
h_3^{i_0,j_0}(1;z) &= \frac{3^{1/2}j_0}{16}(1-z)^{1/2}[8+(3+4j_0{}^2+12i_0(i_0+j_0))(1-z)+f_{3,1}^{i_0,j_0}(z)(1-z^2)]\\
&\quad-\frac{3^{1/2}j_0}{4\pi}(2i_0+j_0-1)\ln(1-z)[1+(1-z)f_{3,2}^{i_0,j_0}(z)]+f_{3,3}^{i_0,j_0}(z),
\end{aligned}$$

where the $f_{k,\ell}^{i_0,j_0}$ are holomorphic at 1*.*

Proof. The proof of the facts dealing with $h_1^{i_0,j_0}(1;z)$ directly follows from the expression of this function written in Corollary 8.

Let us now focus on $h_2^{i_0,j_0}(1;z)$. We recall from Corollary 8 that

$$h_2^{i_0,j_0}(1;z)=\frac{1}{\pi}\int_{-1}^{1}\frac{U_{j_0-1}(u)t_2(u;z)^{i_0+j_0/2-1}}{t_2(u;z)-1}\partial_u t_2(u;z)[1-u^2]^{1/2}\mathrm{d}u, \tag{3.2}$$

where $t_2(u;z)=T(u;z)-[T(u;z)^2-1]^{1/2}$ and $T(u;z)=3/z+u^2-u[2+u^2+6/z]^{1/2}$. In particular, the fact that $h_2^{i_0,j_0}(1;z)$ is holomorphic in $(1+\epsilon)\mathscr{D}\setminus[1,1+\epsilon[$ is clear, since making the change of variable $u\mapsto -u$ in (3.2) allows us to write it as the integral on $[0,1]$ of some function of (u,z) holomorphic in $\mathscr{D}\times((1+\epsilon)\mathscr{D}\setminus[1,1+\epsilon[)$—note that although $T(u;z)$ has algebraic singularities, any function symmetrical in $(T(u;z),T(-u;z))$ is meromorphic w.r.t. the variable z.

We now study the behavior of $h_2^{i_0,j_0}(1;z)$ in the neighborhood of 1. For this, we first transform (3.2), until obtaining an expression that makes clearly appear the singularities of $h_2^{i_0,j_0}(1;z)$.

An easy calculation entails that $\partial_u t_2=\partial_u T/(1-t_3^2)$. Moreover, by definition of the t_ℓ (see Section 2), $(z^2/36)\prod_{\ell=1}^4(t-t_\ell(u;z))$ is equal to $b(t;z)^2-4u^2a(t;z)c(t;z)$. In particular, $\prod_{\ell=1}^4(1-t_\ell(u;z))=(36/z^2)(1-z(1+2u)/3)(1-z(1-2u)/3)$. So we have

$$\frac{\partial_u t_2(u;z)}{t_2(u;z)-1}=\frac{z^2\partial_u T(u;z)(1-t_1(u;z))(1-t_4(u;z))(1-t_2(u;z))}{18(1-z(1-2u)/3)(t_2(u;z)-t_3(u;z))(1-z(1+2u)/3)}. \tag{3.3}$$

We now expand the quantity $(1-t_2(u;z))t_2(u;z)^{i_0+j_0/2-1}$ according to the powers of $[T(u;z)^2-1]^{1/2}$, say $(1-t_2(u;z))t_2(u;z)^{i_0+j_0/2-1}=\sum_{k\geqslant 0}F_k^{i_0,j_0}(u;z)[T(u;z)^2-1]^{k/2}$. With these notations, (3.2) and (3.3), we obtain

$$\begin{aligned}h_2^{i_0,j_0}(1;z)=\sum_{k\geqslant 0}\int_{-1}^{1}&\frac{z^2\partial_u T(u;z)(1-t_1(u;z))(1-t_4(u;z))}{18(1-z(1-2u)/3)}F_k^{i_0,j_0}(u;z)\times\\&\times\frac{[T(u;z)^2-1]^{k/2}}{(t_2(u;z)-t_3(u;z))(1-z(1+2u)/3)}U_{j_0-1}(u)[1-u^2]^{1/2}\mathrm{d}u.\end{aligned} \tag{3.4}$$

In what follows, we analyze the behavior at 1 of each integral in the sum (3.4), first for $k\in\{0,1,2\}$, then for $k\geqslant 3$.

Integrals corresponding to $k\in\{0,1,2\}$ in the sum (3.4). First, note that

$$\begin{aligned}F_0^{i_0,j_0}&=T^{i_0+j_0/2-1}(1-T),\\F_1^{i_0,j_0}&=T^{i_0+j_0/2-2}[T-(i_0+j_0/2-1)(1-T)],\\F_2^{i_0,j_0}&=T^{i_0+j_0/2-3}(i_0+j_0/2-1)[(1-T)(i_0+j_0/2-2)/2-T].\end{aligned}$$

Now we set $F^{j_0}(u;z)=-z^2\partial_u T(u;z)(1-t_1(u;z))(1-t_4(u;z))U_{j_0-1}(u)/(36(1-z(1-2u)/3))$, as well as,

$$\begin{aligned}G_0^{i_0,j_0}(u;z)&=[F^{j_0}(u;z)F_0^{i_0,j_0}(u;z)z^2[T(-u;z)^2-1]^{1/2}]/[3(z+3)(1-z(1-2u)/3)^{1/2}],\\G_1^{i_0,j_0}(u;z)&=F^{j_0}(u;z)F_1^{i_0,j_0}(u;z),\\G_2^{i_0,j_0}(u;z)&=[F^{j_0}(u;z)F_2^{i_0,j_0}(u;z)3(z+3)(1-z(1-2u)/3)^{1/2}]/[z^2[T_1(-u;z)^2-1]^{1/2}].\end{aligned}$$

Since $t_2(u;z)-t_3(u;z)=-2[T(u;z)^2-1]^{1/2}$ and since $(t_2(u;z)-t_3(u;z))(t_1(u;z)-t_4(u;z))=12(z+3)^2[(1-z(1+2u)/3)(1-z(1-2u)/3)]^{1/2}/z^2$, the sum of the three terms for $k \in \{0,1,2\}$ in (3.4) is equal to

$$\sum_{k=0}^{2} \int_{-1}^{1} \frac{G_k^{i_0,j_0}(u;z)[1-u^2]^{1/2}}{[1-z(1+2u)/3]^{(3-k)/2}} \mathrm{d}u. \tag{3.5}$$

Using now the expansion of the Chebyshev polynomials at 1 (see [20]), we obtain the expansion $G_0^{i_0,j_0}(u;z) = -2j_0(u-1)/9 - j_0(z-1)/3 + \sum_{k+\ell\geqslant 2} G_{0,k,\ell}^{i_0,j_0}(u-1)^k(z-1)^\ell$. Then, with a repeated use of (A.7) of Lemma 11, we get

$$\int_{-1}^{1} \frac{G_0^{i_0,j_0}(u;z)[1-u^2]^{1/2}}{[1-z(1+2u)/3]^{3/2}} \mathrm{d}u = j_0 3^{1/2} \ln(1-z)[(1-z)/4 + (1-z)^2 g_0^{i_0,j_0}(z)] + f_0^{i_0,j_0}(z),$$

$f_0^{i_0,j_0}$ and $g_0^{i_0,j_0}$ being holomorphic at 1.

In the same way, $G_1^{i_0,j_0}(u;z) = \sum_{k+\ell\geqslant 2} G_{1,k,\ell}^{i_0,j_0}(u-1)^k(z-1)^\ell - j_0/3 - j_0(6{j_0}^2+35-48i_0-24j_0)(u-1)/54 + j_0(-53+48i_0+24j_0)(z-1)/36$. A repeated application of Lemma 10 then gives that

$$\begin{aligned}\int_{-1}^{1} \frac{G_1^{i_0,j_0}(u;z)[1-u^2]^{1/2}}{1-z(1+2u)/3} \mathrm{d}u &= f_1^{i_0,j_0}(z) + j_0 3^{1/2}[1-z]^{1/2} \times \\ &\times [1/2 + (3/4+{j_0}^2)(1-z)/4 + (1-z)^2 g_1(z)],\end{aligned}$$

$f_1^{i_0,j_0}$ and $g_1^{i_0,j_0}$ being holomorphic at 1.

Finally, we have $G_2^{i_0,j_0}(u;z) = 2j_0(i_0+j_0/2-1)/3 + \sum_{k+\ell\geqslant 1} G_{2,k,\ell}^{i_0,j_0}(u-1)^k(z-1)^\ell$. So with a repeated use of (A.6) of Lemma 11, we reach the conclusion that

$$\begin{aligned}\int_{-1}^{1} \frac{G_2^{i_0,j_0}(u;z)[1-u^2]^{1/2}}{[1-z(1+2u)/3]^{1/2}} \mathrm{d}u &= f_2^{i_0,j_0}(z) + j_0(i_0+j_0/2-1)3^{1/2} \times \\ &\times \ln(1-z)[(1-z)/2 + (1-z)^2 g_2^{i_0,j_0}(z)],\end{aligned}$$

$f_2^{i_0,j_0}$ and $g_2^{i_0,j_0}$ being holomorphic at 1.

Integrals corresponding to $k \geqslant 3$ in the sum (3.4). Note first that if k is odd and larger than 3, the associated function in (3.4) is in fact holomorphic in the neighborhood of 1: indeed, for this it is enough to notice that $t_2(u;z) - t_3(u;z) = -2[T(u;z)^2-1]^{1/2}$. For this reason, all the terms in (3.4) associated with values of k which are odd and larger than 3 do not have any singularity at 1.

On the other hand, if k is even and larger than 3, then the underlying term in the sum (3.4) can be written as

$$\int_{-1}^{1} [1-z(1+2u)/3]^{(k-3)/2} H_k^{i_0,j_0}(u;z)[1-u^2]^{1/2} \mathrm{d}u,$$

where the function $H_k^{i_0,j_0}(u;z)$ is holomorphic in the neighborhood of $(1,1)$. The last integral is obviously equal to

$$\int_{-1}^{1} [1-z(1+2u)/3]^{(k-2)/2} H_k^{i_0,j_0}(u;z)[1-u^2]^{1/2}[1-z(1+2u)/3]^{-1/2} \mathrm{d}u.$$

Then, expanding $[1-z(1+2u)/3]^{(k-2)/2} H_k^{i_0,j_0}(u;z)$ w.r.t. the powers of $(u-1)^k(z-1)^\ell$ and using (A.6) of Lemma 11, we obtain that the integral above equals $\ln(1-z)(z-1)^{k-2} g_k^{i_0,j_0}(z) + f_k^{i_0,j_0}(z)$, $f_k^{i_0,j_0}$ and $g_k^{i_0,j_0}$ being holomorphic at 1.

Finally, the sum of all the terms corresponding in (3.4) to values of k which are even and larger than 3 can be written, in the neighborhood of 1, as $\ln(1-z)(1-z)^2 g^{i_0,j_0}(z) + f^{i_0,j_0}(z)$, where f^{i_0,j_0} and g^{i_0,j_0} are holomorphic at 1.

End of the proof of Proposition 9. Putting the latter fact together with (3.5) concludes the proof of the expansion for $h_2^{i_0,j_0}(1;z)$ stated in Proposition 9. Finally, the proof of the facts regarding $h_3^{i_0,j_0}(1;z)$, via a repeated use of Lemmas 10 and 11, is totally similar to that for $h_2^{i_0,j_0}(1;z)$, so we omit it. $\square$

References

[1] Aspandiiarov, S., Iasnogorodski, R., and Menshikov, M.: Passage-time moments for nonnegative stochastic processes and an application to reflected random walks in a quadrant. *Ann. Probab.* **24** 932–960 (1996)

[2] Belitsky, V., Ferrari, P., Menshikov, M., and Popov, S.: A mixture of the exclusion process and the voter model. *Bernoulli* **7** 119–144 (2001)

[3] Blanc, J.P.C.: The relaxation time of two queueing systems in series. *Comm. Statist. Stochastic Models* **1** 1–16 (1985)

[4] Denisov, D., and Wachtel, V.: Random walks in cones. *Preprint* (2011)

[5] Eichelsbacher, P., and König, W.: Ordered random walks. *Electron. J. Probab.* **13** 1307–1336 (2008)

[6] Fayolle, G., and Iasnogorodski, R.: Two coupled processors: the reduction to a Riemann-Hilbert problem. *Z. Wahrsch. Verw. Gebiete* **47** 325–351 (1979)

[7] Fayolle, G., Iasnogorodski, R., and Malyshev, V.: *Random walks in the quarter-plane.* Springer-Verlag, Berlin (1999)

[8] Fayolle, G., Iasnogorodski, R., and Mitrani, I.: The distribution of sojourn times in a queueing network with overtaking: Reduction to a boundary problem. *Performance '83: Proceedings of the 9th International Symposium on Computer Performance Modelling, Measurement and Evaluation.* North-Holland Publishing Co., Amsterdam 477–486 (1983)

[9] Flajolet, P., and Sedgewick, R.: *Analytic combinatorics.* Cambridge University Press, Cambridge (2009)

[10] Kurkova, I., and Malyshev, V.: Martin boundary and elliptic curves. *Markov Process. Related Fields* **4** 203–272 (1998)

[11] Kurkova, I., and Raschel, K.: Random walks in $\mathbb{Z}_+^2$ with non-zero drift absorbed at the axes. *Bull. Soc. Math. France* **139** 341–387 (2011)

[12] Liggett, T.: *Interacting Particle Systems.* Springer-Verlag, New York (1985)

[13] Lu, J.: *Boundary value problems for analytic functions.* World Scientific Publishing, River Edge (1993)

[14] Malyshev, V.: An analytical method in the theory of two-dimensional positive random walks. *Siberian Math. J.* **13** 1314–1329 (1972)

[15] MacPhee, I., Menshikov, M., Volkov, S., and Wade, A.: Passage-time moments and hybrid zones for the exclusion-voter model. *Bernoulli* **16** 1312–1342 (2010)

[16] Raschel, K.: Counting walks in a quadrant: a unified approach via boundary value problems. *J. Eur. Math. Soc.* **14** 749–777 (2012)

[17] Raschel, K.: Green functions for killed random walks in the Weyl chamber of Sp(4). *Ann. Inst. H. Poincaré Probab. Statist.* **47** 1001–1019 (2011)

[18] Raschel, K.: *Paths confined to a quadrant.* PhD Thesis of Pierre et Marie Curie University (2010)

[19] Sansone, G., and Gerretsen, J.: *Lectures on the theory of functions of a complex variable II: Geometric theory.* Wolters-Noordhoff Publishing, Groningen (1969)

[20] Szegő, G.: *Orthogonal polynomials.* American Mathematical Society, Providence (1975)

[21] Varopoulos, N.: Potential theory in conical domains. *Math. Proc. Cambridge Philos. Soc.* **125** 335–384 (1999)

Appendix A.

In this appendix, we state and prove two technical lemmas, which concern the behavior of some integrals with parameters near their singularities. These lemmas are crucial for the proof of our main results, but are independent of the rest of the paper.

Lemma 10. *For any $k \in \mathbf{Z}_+$, let P_k be the principal part at infinity of $[Z^2-1]^{1/2}(1-Z)^k$, i.e., the unique polynomial such that $[Z^2-1]^{1/2}(1-Z)^k - P_k(Z)$ goes to zero as $|Z|$ goes to infinity. Then*

$$\int_{-1}^{1} \frac{(1-u)^k[1-u^2]^{1/2}}{1-z(1+2u)/3}\mathrm{d}u = \frac{3\pi}{2z}\left[(1+z/3)^{1/2}\left(\frac{-3}{2z}\right)^{k+1}(1-z)^{k+1/2} + P_k\left(\frac{3}{2z}-\frac{1}{2}\right)\right].$$

Proof. For $\epsilon > 0$, we consider the closed contour $\mathscr{A}_\epsilon^+ \cup \mathscr{A}_\epsilon^- \cup \mathscr{B}_\epsilon^+ \cup \mathscr{B}_\epsilon^-$, where $\mathscr{A}_\epsilon^\pm = \{\pm 1 \mp i\epsilon \exp(it), t \in [0,\pi]\}$ and $\mathscr{B}_\epsilon^\pm = \{\pm i\epsilon \mp t, t \in [-1,1]\}$. Then we apply on it the residue theorem at infinity to the function $(1-u)^k[1-u^2]^{1/2}/[1-z(1+2u)/3]$ and we let ϵ going to zero. $\square$

Lemma 11. *For any $k \in \mathbf{Z}_+$, the integrals written in the left hand side of (A.6) and (A.7) below are holomorphic in $(1+\epsilon)\mathscr{D} \setminus [1, 1+\epsilon[$ for $\epsilon > 0$ small enough. In the neighborhood of 1, they are equal to*

$$\int_{-1}^{1} \frac{(1-u)^k[1-u^2]^{1/2}}{[1-z(1+2u)/3]^{1/2}} \mathrm{d}u = \ln(1-z)(1-z)^{k+1}\alpha_k(z) + \beta_k(z), \tag{A.6}$$

$$\int_{-1}^{1} \frac{(1-u)^k[1-u^2]^{1/2}}{[1-z(1+2u)/3]^{3/2}} \mathrm{d}u = \ln(1-z)(1-z)^{k}\gamma_k(z) + \delta_k(z), \tag{A.7}$$

where α_k, β_k, γ_k and δ_k are holomorphic at 1, $\alpha_k(1) \neq 0$ and $\gamma_k(1) \neq 0$. Furthermore, $\alpha_0(1) = 3^{3/2}/4$, $\gamma_0(1) = -3^{3/2}/2$, $\gamma_0'(1) = -3^{1/2}99/32$ and $\gamma_1(1) = 3^{1/2}27/8$.

Proof. The fact that the integrals written in the left hand side of (A.6) and (A.7) are, for $\epsilon > 0$ small enough, holomorphic in $(1+\epsilon)\mathscr{D} \setminus [1, 1+\epsilon[$ is clear from their expression.

Let us now study their behavior near 1 and start by considering (A.6). Replace first the lower bound -1 by $-1/2$ in the integral (A.6). This is equivalent to add a function holomorphic in some $(1+\epsilon)\mathscr{D}$ and this will eventually change β_k but not α_k in the right hand side member of (A.6). Then, the change of variable $v^2 = (1+2u)/3$ gives

$$\int_{-1/2}^{1} \frac{(1-u)^k[1-u^2]^{1/2}}{[1-z(1+2u)/3]^{1/2}} \mathrm{d}u = 3^{1/2}\left(\frac{3}{2}\right)^{k+1} \int_0^1 \frac{[1-v^2]^{k+1/2}}{[1-zv^2]^{1/2}}[1+3v^2]^{1/2} v \mathrm{d}v. \tag{A.8}$$

By using the expansion of $v^{1/2}$ in the neighborhood of 1, we can develop the function $[1+3v^2]^{1/2}v$ according to the powers of v^2-1: $[1+3v^2]^{1/2}v = 2 + (7/4)[v^2-1] + \cdots$. Further, in [18, Chapter F], we have proved, using the elliptic integrals theory, that for any $k \in \mathbf{Z}_+$ there exist two functions ϕ_k and ψ_k, holomorphic in the neighborhood of 1 and satisfying $\phi_k(1) \neq 0$, such that

$$\int_0^1 \frac{[1-v^2]^{1/2+k}}{[1-zv^2]^{1/2}} \mathrm{d}v = \ln(1-z)(z-1)^{k+1}\phi_k(z) + \psi_k(z), \tag{A.9}$$

we have there also proved that $\phi_0(1) = 1/4$. The equality (A.6) is then an obvious outcome of (A.8), of the expansion of $[1+3v^2]^{1/2}v$ according to the powers of v^2-1, and of the repeated use of (A.9). The fact that $\alpha_0(1) = 3^{3/2}/4$ comes from the equality $\phi_0(1) = 1/4$.

Likewise, we prove the equality (A.7) and we obtain the announced values of $\gamma_0(1)$, $\gamma_0'(1)$ and $\gamma_1(1)$. $\square$