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# A Finite Difference Element Method for thin elastic Shells

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## Abstract

We present, in this paper, a four nodes quadrangular shell element (FDEM4) based on a Finite Difference Element Method that we introduce. Its stability and robustness with respect to shear locking and membrane locking problems is discussed. Numerical tests including inhibited and non-inhibited cases of thin linear shells are presented and compared with widely used DKT and MITC4 elements.

*Key words:* Finite Element Methods, Membrane Locking, Shear Locking Elastic shells, MITC4, DKT, Finite Difference

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## 1. Introduction

The Finite Element Method based on polynomial interpolation is widely and successfully used in engineering applications. Among major problems encountered in Finite Element Methods is the numerical locking. Classical examples are incompressible continuum, shear locking in beam and plates element and shear and membrane locking in thin shells computation [3, 11, 15, 23, 27, 28]. These locking phenomenon has in common the inability of the (polynomial) shape functions to perform some constraints of the problem (such as incompressibility condition displacements or bendings in case of shell).

In the case of thin shell computations, various finite element methods claims successful (locking-free) with regard to those locking problems [2, 7, 4, 3, 12, 21], such as Partial Reduced Selective Integration method, Mixed Interpolation method, Discrete Strain Gap method and more recently Smoothed finite element method, [21]. Up to our knowledge, no mathematical proofs are available yet.

In [13], the shape functions has been identified as closely related to numerical locking for conformal finite elements. This is why, we wanted to study the possibility to implement a finite element procedure without such shape functions as an alternative.

In this paper, based on this idea, we define a *Finite Difference Element Method* concept and we develop a four nodes quadrangular Finite Difference Element procedure (FDEM4) for shell problems.

The paper is organized as follows, we first present the Finite Difference Element Method concept and we illustrate with a simple scalar problem corresponding to the static Heat or membrane problem.

The Nagdhi's model of thin elastic shells [6] is exposed in section 3. The FDEM4 shell element is based on this model, it is presented in section 4. The section 5 is devoted to numerical results. We test the robustness with respect to both shear locking and membrane locking, with regular and distorted meshes.

FDEM4's performance is compared with MITC4 and triangular DKT. Accordingly with the asymptotic analysis of thin elastic shells, [26, 10, 11], inhibited and non-inhibited cases of shells are treated .

We then conclude and discuss the perspectives of the Finite Difference Element method for thin shell computations and more.

### *Notations and conventions*

In this paper, we employ the Einstein convention of summation on repeated upper or lower indices. The Greek (resp. Latin) indices range over  $\{1, 2\}$  (resp.  $\{1, 2, 3\}$ ). The partial derivatives with respect to variables  $x_\alpha$  are denoted in lower indices preceded by a comma. The  $\nabla$  symbol represents the gradient of one function. We use overarrow to indicate space vectors. The variables  $x_1, x_2$  (resp.  $x_1, x_2, x_3$ ) live in bounded domain  $\Omega \subset \mathbb{R}^2$  (resp.  $\mathbb{R}^3$ ). The Hilbert functional space  $V$  will denote a Sobolev space and therefore injects continuously into  $H = \mathbb{L}^2(\Omega)$ . The Harpoon symbol  $\longrightarrow$  will indicate a convergence in the weak sense. The  $O$  and  $o$  symbols represent negligible quantities in the sense that

$$\lim_{h \rightarrow 0} O(h) = 0 \quad \text{and} \quad \lim_{h \rightarrow 0} \frac{o(h)}{h} = 0.$$

## 2. The Finite difference Element Method

How to avoid shape functions and in the same time apply a Galerkin method ? To this paradoxical question, one answer we found is the finite difference element method : we replace the differential operators by finite difference approximations in some Galerkin projection procedure. More precisely, we replace the differential expressions of undefined shape functions, involved in the variational form problem by some finite difference approximations at the nodes of integration schemes. Although such idea is not new, the method we propose is very similar to the finite difference energy method [3, 9, 22], we give

here a different point of view in the sense we still interpret the finite difference element method as a Galerkin method.

#### A scalar problem example

For sake of simplicity, we shall present the finite difference element method on a scalar problem (corresponding mathematically to a heat problem or a membrane) over  $\Omega \subset \mathbb{R}^2$ :

$$\begin{cases} \text{Find } u \in V \text{ such that} \\ \int_{\Omega} \nabla u \cdot \nabla u^* = \int_{\Omega} f u^* \quad \forall u^* \in V, \end{cases} \quad (1)$$

where  $V$  is the Sobolev space  $V = H_0^1(\Omega)$ . The variational (1) is the standard Poisson problem with Dirichlet boundary conditions.

#### Shape functions

Actually, we still need shape functions, at least from a theoretical point of view. The only difference with standard finite element method is that, using finite difference approximations, the explicit form or expression of shape functions need not to be explicit.

Let  $V_h$  be a subspace of  $V$ , generated by a set of shape functions  $\phi_i$  piecewise defined on  $\Omega_h$ , continuous in  $\Omega$ , where  $h$  denotes the mean size of the elements  $\Omega_i$ . We can impose some natural conditions :

$$\begin{cases} \phi_i(N_j) = \delta_{i,j}, \\ \phi_i = 0, \quad \text{on } \Omega_k \text{ if } N_i \notin \Omega_k. \end{cases} \quad (2)$$

We have then the decomposition :

$$u_h = u_i \phi_i,$$

where the coefficients  $u_i$  will represent the value of  $u$  at nodes  $N_i$ . We don't impose *a priori* that the shape functions to realize a partition of unity, but an approximate partition of unity.

Classically, as  $h \rightarrow 0$ , the subspaces  $V_h$  define a set of discrete conformal approximations of  $V$  : for any  $u \in V$ , it is easy to construct a sequence  $u_n \in V_h$  such that  $u_n$  converge almost everywhere towards  $u$  in  $\Omega$ . Let's consider now  $u_h$ , the Galerkin projection of the solution  $u$  of (1) in  $V_h$  :

$$\begin{cases} \text{Find } u_h \in V_h \text{ such that} \\ \int_{\Omega} \nabla u_h \cdot \nabla u^* = \int_{\Omega} f u^* \quad \forall u^* \in V_h \end{cases} \quad (3)$$

It classically reduces to a linear system with the 'stiffness' matrix  $K$  :

$$K \hat{u} = L,$$

where

$$\begin{aligned} \hat{u} &= [u_1, \dots, u_n]^T, \\ L &= [l_1, \dots, l_n]^T, \end{aligned}$$

and with

$$K_{ij} = \int_{\Omega} \nabla \phi_i \cdot \nabla \phi_j \quad (4)$$

$$l_j = \int_{\Omega} f \phi_j. \quad (5)$$

In classical finite element method, the computation of (4) need the explicit expression of the shape functions on the nodes of integration scheme. Here, the Finite Difference Element Method consists in simply integrate over nodes where we replace differential expressions by some consistent finite difference approximations :

#### Finite difference approximations over a 4 nodes quadrangle

If we consider a quadrangle defined by  $N_1, N_2, N_3, N_4$ , of coordinates  $(x_i, y_i)$ , and with  $N_0$  as the intersection of the two diagonals, see the Figure 1.

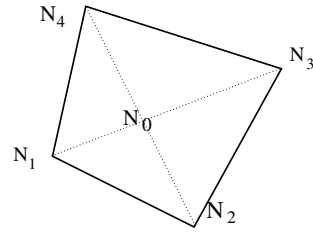


Figure 1: A four nodes quadrangle

By definition, we have, for  $j = 1, 2, 3, 4$  :

$$u(N_j) = u(N_0) + (\nabla u(N_0), \overrightarrow{N_0 N_j}) + O(\overrightarrow{N_0 N_j}).$$

This gives :

$$\begin{aligned} u(N_3) - u(N_1) &= (\nabla u(N_0), \overrightarrow{N_1 N_3}) + O(\overrightarrow{N_0 N_3}) - O(\overrightarrow{N_0 N_1}), \\ u(N_4) - u(N_2) &= (\nabla u(N_0), \overrightarrow{N_2 N_4}) + O(\overrightarrow{N_0 N_4}) - O(\overrightarrow{N_0 N_2}). \end{aligned} \quad (6)$$

We can then solve the linear system (6) to obtain the *consistent* finite difference central approximation :

$$\nabla u(N_0) = B \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} + \max_j (O(\overrightarrow{N_0 N_j})), \quad (7)$$

with

$$B = \frac{1}{\Delta} \begin{bmatrix} y_2 - y_4 & y_3 - y_1 & y_4 - y_2 & y_1 - y_3 \\ x_4 - x_2 & x_1 - x_3 & x_2 - x_4 & x_3 - x_1 \end{bmatrix}$$

and

$$\Delta = (x_3 - x_1)(y_4 - y_2) - (x_4 - x_2)(y_3 - y_1).$$

Similarly, it is also possible to write the finite difference on each nodes  $N_k$  instead of  $N_0$ , for instance :

$$u(N_2) = u(N_1) + (\nabla u(N_1), \overrightarrow{N_1 N_2}) + O(\overrightarrow{N_1 N_2}),$$

$$u(N_4) = u(N_1) + (\nabla u(N_1), \overrightarrow{N_1 N_4}) + O(\overrightarrow{N_1 N_4}).$$

Then

$$\nabla u(N_1) = B^* \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} + \max_{i \neq j} (O(\overrightarrow{N_i N_j})) \quad (8)$$

with

$$B^* = \frac{1}{\Delta^*} \begin{bmatrix} y_2 - y_4 & y_4 - y_1 & 0 & y_1 - y_2 \\ x_4 - x_2 & x_1 - x_4 & 0 & x_2 - x_1 \end{bmatrix}$$

and

$$\Delta^* = (x_2 - x_1)(y_4 - y_1) - (x_4 - x_1)(y_2 - y_1).$$

With those two finite difference approximation schemes, we can propose 2 different integration scheme to compute the 'stiffness' matrix  $K$ : one node integration with (7) and four nodes integration with (8). In this paper, we shall exclusively focus on the central finite difference approximation (7).

With the central finite difference approximation (7), on each quadrangular element  $E_k$ , the elementary 'stiffness' matrix writes simply :

$$K_{el} = B' * B |E_k|.$$

where  $|E_k|$  is the measure of the element.

*Galerkin or not ?*

Our interpretation of the Finite Difference Element Method, is that the rough integration scheme (only one node) is nevertheless exact for some class of shape functions. A description of such class is still under investigation, but we can reasonably conjecture such class is not void in the space of admissible solutions. From this point of view, the Finite Difference Element Method is a conformal finite element method.

*Numerical results on the scalar model problem (1)*

We apply the finite difference method to the model problem (3) with  $\Omega = [-\frac{1}{2}, \frac{1}{2}]^2$ . Due to the symmetries of the problem we compute on a quarter of the domain only, that is  $[0, \frac{1}{2}]^2$ . With a set of regular and distorted  $k \times k$  meshes where the parameter  $k$  denotes the number of subdivision of the domain, see Figure 2.

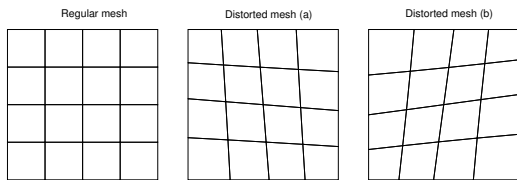


Figure 2: Regular and distorted 4x4 meshes

We consider the finite difference element procedure with one degree of freedom (DOF) per node, for which we solely use the (7) central finite difference approximation coupled with one node integration scheme; we call this element FDEM4.

In Figure 3, we display the relative error of the solution at the center of  $\Omega$ , where it takes its maximal value, compared

to the exact solution (obtained through Fourier series) and the computed solution by a standard four node quadrangle polynomial element, QUA4.

We can see that FDEM4 element seems to converge with an apparent nearly quadratic rate, even though it is clearly outperformed by QUA4. The convergence rate for FDEM4 does not show much sensitivity to mesh distortion, whereas it deteriorates for QUA4 in the case of distorted meshes (b). This is not surprising since the finite difference scheme (7) is independent of the shape of the quadrangular and the P2 polynomial interpolation of QUA4 is incomplete. Those numerical results already indicates the behavior of FDEM4 element : it will have average to poor performance on very coarse meshes, but the convergence rate should be nearly quadratic. More interesting FDEM4 should not be too sensitive on mesh distortion.

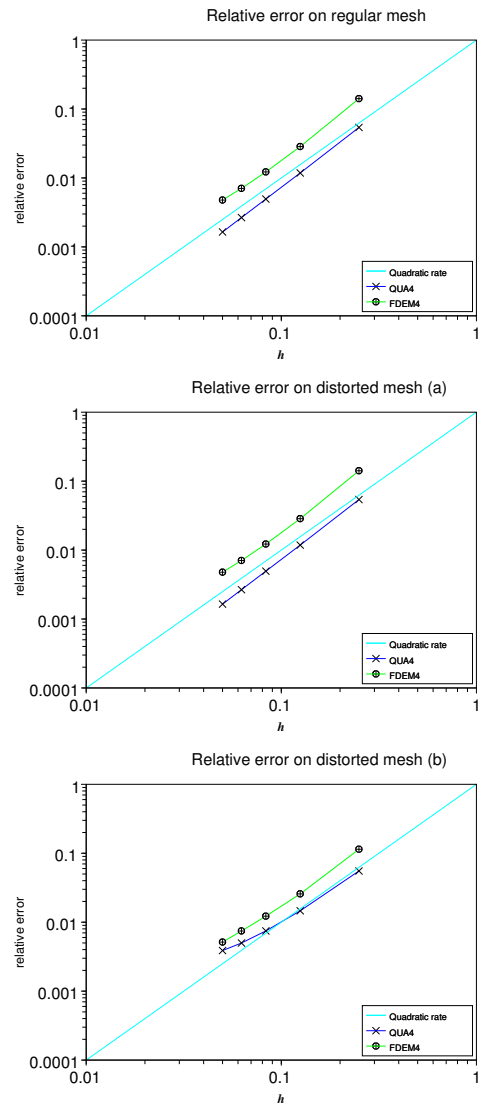


Figure 3: Relative error  $\frac{u_{exact} - u}{u_{exact}}$  at  $(0, 0)$  for regular and distorted meshes.  $h$  is the mean size of the quadrangular elements.

### 3. Thin elastic shells : the Naghdi's model

We refer to [6, 11] for a full description and presentation of the theory on thin shells in the Naghdi's model. We only recall the mechanical problem in static linear elasticity for a homogeneous and isotropic shell with Young's modulus  $E$  and coefficient of Poisson  $\nu$ .

Let's define first a shell with its geometry, see Figure 4. Let  $S$  be the middle surface defined with the map  $(\Omega, \vec{r}) : \Omega \in \mathbb{R}^2$  and

$$\vec{r} = x(x_1, x_2)\vec{e}_1 + y(x_1, x_2)\vec{e}_2 + z(x_1, x_2)\vec{e}_3. \quad (9)$$

The surface map defines the tangent vectors

$$\vec{d}_\alpha = \vec{r}_{,\alpha},$$

and the unit normal vector

$$\vec{d}_3 = \frac{\vec{d}_1 \wedge \vec{d}_2}{\|\vec{d}_1 \wedge \vec{d}_2\|}.$$

the three vectors  $\vec{d}_i$  constitute the covariant basis of  $S$ . We keep in mind that the covariant basis is not a priori orthonormal nor orthogonal, we thus define the dual basis  $\vec{d}^i$  in order to make scalar products.

Let  $\varepsilon$  be the constant thickness the shell, then any of its point can be given by the position vector,  $\forall (x_1, x_2) \in \Omega, x_3 \in [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]$ ,

$$\vec{p}(x_1, x_2, x_3) = \vec{r}(x_1, x_2) + x_3\vec{d}_3.$$

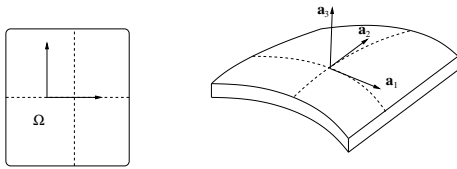


Figure 4: A Shell, the covariant basis of its midsurface

The Naghdi's model involves the displacement  $\vec{u}$  and the rotation  $\vec{\theta}$  to define the deformation of the shell under external loadings. Although the theory involves covariant components, we shall represent the displacement with its Cartesian component :

$$\vec{u} = u_i \vec{e}_i.$$

The rotation is tangent to  $S$  and thus can conveniently be represented by its covariant components:

$$\vec{\theta} = \theta_\alpha \vec{d}^\alpha.$$

Under external loadings, represented by a linear form  $L^\varepsilon$ , we define the mechanical problem of thin elastic shells in its variational form as follow, where  $V$  denotes a Sobolev space of kinematically admissible displacements and rotations :

$$\left\{ \begin{array}{l} \text{Find } U = (\vec{u}, \vec{\theta}) \in V \text{ such that} \\ a_\varepsilon(U, U^*) = L^\varepsilon(U^*) \quad \forall U^* \in V. \end{array} \right. \quad (10)$$

The bilinear form of deformation energy,  $a_\varepsilon$ , writes for any  $U = (\vec{u}, \vec{\theta})$  and  $U^* = (\vec{u}^*, \vec{\theta}^*)$  :

$$a_\varepsilon(U, U^*) = \varepsilon^3 a^b(U, U^*) + \varepsilon a^s(U, U^*) + \varepsilon a^m(U, U^*) \quad (11)$$

where we decomposed the energy form respectively in bending, shear and membrane parts :

$$a^b(U, U^*) = \int_\Omega \frac{1}{12} A^{\alpha\beta\lambda\mu} \rho_{\alpha\beta} \rho_{\lambda\mu} \quad (12)$$

$$a^s(U, U^*) = \int_\Omega C^{\alpha\beta} \Lambda_\alpha \Lambda_\beta \quad (13)$$

$$a^m(U, U^*) = \int_\Omega A^{\alpha\beta\lambda\mu} \gamma_{\alpha\beta} \gamma_{\lambda\mu} \quad (14)$$

where  $\gamma_{\alpha\beta}$  and  $\rho_{\alpha\beta}$  represent respectively the variation tensors of the first and second fundamental form of the surface  $S$ , measuring the variation of length and variation of curvature, the  $\Lambda_\alpha$  measure the shear :

$$\gamma_{\alpha\beta} = \frac{1}{2} (\vec{u}_{,\alpha} \cdot \vec{d}_\beta + \vec{u}_{,\beta} \cdot \vec{d}_\alpha) \quad (15)$$

$$\rho_{\alpha\beta} = \frac{1}{2} (\vec{\theta}_{,\alpha} \cdot \vec{d}_\beta + \vec{\theta}_{,\beta} \cdot \vec{d}_\alpha - \vec{d}_{3,\alpha} \cdot \vec{u}_{,\beta} - \vec{d}_{3,\beta} \cdot \vec{u}_{,\alpha}) \quad (16)$$

$$\Lambda_\alpha = \theta_\alpha + \vec{u}_{,\alpha} \cdot \vec{d}_3. \quad (17)$$

Since the shell is homogeneous and isotropic, the constitutive law is given by :

$$A^{\alpha\beta\lambda\mu} = \frac{E}{2(1+\nu)} \left[ a^{\alpha\lambda} a^{\beta\mu} + a^{\alpha\mu} a^{\beta\lambda} + \frac{2\nu}{1-\nu} a^{\alpha\beta} a^{\lambda\mu} \right] \quad (18)$$

$$C^{\alpha\beta} = \frac{E}{4(1+\nu)} a^{\alpha\beta}. \quad (19)$$

with the contravariant components of the first fundamental form

$$a^{\alpha\beta} = \vec{d}^\alpha \cdot \vec{d}^\beta.$$

#### Asymptotic behavior and Numerical difficulties

The natural trend for a very thin elastic shell is to perform bendings as we can see from (10) that shear and membrane deformation energies are penalized then the thickness  $\varepsilon$  is very small, see also [26, 11, 25]. But, unlike the bendings in beams problems, bending deformation on a surface is not always admissible: it depends on the boundary conditions and the geometry [17]. Thin shells are thus classified as [with bendings] *inhibited* or *not-inhibited*, following the terminology introduced by Sanchez-Palencia[26].

This classification is also referred as *membrane dominated* and *bending dominated* [23, 16]. These two very distinct asymptotic behaviors lead to strong difficulties in the numerical approximation of very thin elastic shells by finite element procedures : boundary layers, propagation and reflexion of singularities, sensitivity within the inhibited case and numerical (membrane) locking in the non-inhibited or bending dominated case, see [23, 24, 15, 11, 19, 25].

Classically, we have the following asymptotic behavior valid for thin shells linear models such as Koiter's or Naghdi's [25, 16, 11]:

**Theorem 3.1.** *If the (scaled) load  $L^\varepsilon(\vec{v})$  depends on  $\varepsilon$  as  $L^\varepsilon(\vec{v}) = \varepsilon^3 L(\vec{v})$ , then the solutions  $\vec{u}^\varepsilon$  of (10) converge (weakly in  $\vec{V}$ ) towards the solution  $\vec{u}^0$  of the asymptotic (limit) bending problem*

$$\left\{ \begin{array}{l} \text{find } \vec{u}^0 \in \vec{G} \text{ such that} \\ \int_S \frac{A^{\alpha\beta\lambda\mu}}{12} \rho_{\alpha\beta}(\vec{u}^0) \rho_{\lambda\mu}(\vec{v}), = L(\vec{v}^*), \quad \forall \vec{v}^* \in \vec{G}, \end{array} \right. \quad (20)$$

where  $\vec{G}$  is the set of kinematically admissible infinitesimal bendings (also called inextensional displacement) :

$$\vec{G} = \left\{ \vec{u} \in V \text{ such that } \gamma_{\alpha\beta}(\vec{u}) = 0 \right\}.$$

and where we replaced  $\theta_\alpha$  in  $\rho_{\alpha\beta}$  with  $-\vec{u}_{,\alpha} \cdot \vec{a}_3$ .

We emphasize on the fact that infinitesimal bendings are solution of a partial differential equation, the nature of which depends on the nature of the surface, in particular if a surface is defined by a position vector

$$r(x_1, x_2) = x_1 \vec{e}_1 + x_2 \vec{e}_2 + \varphi(x_1, x_2) \vec{e}_3,$$

then the vertical component  $u_3$  of any infinitesimal bendings on the surface satisfies the second order PDE :

$$\varphi_{,22} u_{3,11} + 2\varphi_{,12} u_{3,12} + \varphi_{,11} u_{3,22} = 0, \quad (21)$$

the characteristics of which corresponds to the *asymptotic lines* of the surface [17]. Namely, the nature of the equation (21) is respectively hyperbolic, parabolic, elliptic when the surface is hyperbolic, developpable, elliptic.

This essential property underlines the fundamental importance of the asymptotic lines and therefore the topology of a meshing when dealing with the membrane locking, see [1, 26, 11].

In the inhibited shell cases, that is, when the space of infinitesimal bendings  $\vec{G}$  reduces to  $\{0\}$  or rigid displacements, which is the most common situation in practice, the theorem 3.1 simply indicates that the solutions  $\vec{u}^\varepsilon$  converge to 0, as  $\varepsilon \searrow 0$ .

To fully analyze the *membrane dominated* asymptotic behavior, we need to change the scale of the loading from  $\varepsilon^3 L$  to  $\varepsilon^\alpha L$  with  $1 < \alpha < 3$ , see [11, 19] depending on the case. We shall not recall here all the subtleties of membrane dominated case, we simply recall that singularities and boundary layers appear, which make accurate computations difficult :

**Theorem 3.2.** *With the loading scaled as  $L^\varepsilon = \varepsilon L$ , if the shell is inhibited, then as  $\varepsilon$  decreases towards 0, the solution  $\vec{u}^\varepsilon$  of problem (1) converge weakly to  $\vec{u}$  the solution of the membrane limit problem :*

$$\left\{ \begin{array}{l} \text{find } \vec{u} \in \vec{V}^m \text{ such that} \\ \int_S A^{\alpha\beta\lambda\mu} \gamma_{\alpha\beta}(\vec{u}) \gamma_{\lambda\mu}(\vec{v}^*), = L(\vec{v}^*), \quad \forall \vec{v}^* \in \vec{V}^m, \end{array} \right. \quad (22)$$

for any  $L \in \vec{V}^{m'}$ ,

where  $\vec{V}^m$  is the completion of  $V$  for the norm associated with  $a^m$ . In some cases  $\vec{V}^m$  may be so big that it is not contained in the space of distributions. The shell is then said *ill-inhibited*, see [25]. The corollary is that the dual space  $\vec{V}^{m'}$  can be very small : a uniform loading may even not be admissible in most cases with a free edges. This explains the singularities and boundary layer appearing in the inhibited or membrane dominated shells or even the instabilities such as the *chancelling* shells [20]. The asymptotic behavior of inhibited (membrane dominated) shells thus may vary depending on the loading [25, 11].

#### 4. The FDEM4 shell element

The FDEM4 shell element is based on the Naghdi's Shell Model presented in section 3. The element is simply defined as replacing all first order differentials involved in the variational problem of Naghdi's shell (10) by the central finite difference scheme (7). The integration is then obtained through a single node integration at the intersection of the diagonals of each quadrangular element. It is indeed a very simple element, very cheap from a computational point of view, since the elementary stiffness matrix is only evaluated once per element. We believe no shell element can be more simple.

Each node have 5 degrees of freedom (DOF) : the 3 Cartesian components of the displacement and 2 covariant components of the rotation. We remark from (16), it is not necessary to compute the derivatives of the covariant component of the rotation, we simply need to compute the derivatives of the rotation by their expressions in the Cartesian basis.

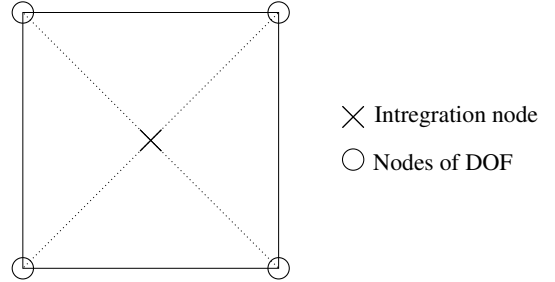


Figure 5: The FDEM4 shell element

The use of such 'reduced' integration scheme, as in the scalar example of section 2, implies we don't expect high performance of the FDEM4 element with coarse meshes but the FDEM4 should be robust with respect to distorted meshes since the finite difference scheme since it is not dependent on any fixed direction. It is remarkable that we make the computation directly on each element without dealing with a reference element.

In an algebraic way, such 'reduced' integration scheme reduces the number of constraints coming from shear and bending condition, that is, two conditions from the shear constraints and another three from the bending condition when in bending dominated situations. Thus a simple count of the number of

available DOF, allow us to hope the FDEM4 to be locking free, or at least be less sensitive to the locking effect.

To this (necessary) algebraic count, we prefer another interpretation : we believe there is a class of shape functions such that the finite difference element schemes coincide exactly with the integral involved in (3). We can hope then this class of functions is large enough to 'contain' infinitesimal bendings. Of course, such a reduced constraints could lead to spurious modes in membrane dominated cases, but our idea is that since the chosen finite difference scheme is consistent, the stability *should* necessarily follow.

The convergence of the method is not yet mathematically established and is still under investigation. We can just trivially show that stability and consistency are necessary and sufficient condition for convergence.

Reciprocally, we think that a non-conformal finite element method using reduced integration scheme should be equivalent in some sense to a finite difference element procedure when it is convergent and stable.

In this paper, we chose to use exact geometry for the definition of the surface, that is, we use the exact expressions of the covariant basis and theirs derivatives. The development of FDEM shell procedure using only the coordinates of the nodes of the surface is still in progress, we may simply follow the path designed for MITC shell elements.

## 5. Numerical results

In this section, we follow [10, 23, 13, 14] to test the performance of the quadrangular finite difference element, FDEM4, on several cases of inhibited and non-inhibited shells and compare with widely used shell elements, namely the 3-nodes triangular plane facet DKT shell element within the CasT3M software and the 4-nodes quadrangular MITC4 element within the software Scilab and the package OpenFEM, [5, 3].

We used the same meshes for FDEM4 and MITC4, but for the triangular DKT element, we divided each quadrangle in the meshes in 4 triangular parts, thus our computations with DKT involve significantly more nodes and DOF than their quadrangular counterparts.

### 5.1. A plate as a plane shell

We start the numerical tests with the case of a plate, for which the Nagdhi's model reduces to Reissner-Mindlin plate model. The test is interesting from the numerical locking point of view because it first evaluate the robustness with respect to shear locking.

We consider the classical example of a circular plate clamped all along its boundary. It is submitted to a uniform vertical loading. To test the shear locking we have taken a very small thickness,  $\varepsilon = 10^{-4}$ . Taking advantage of the axial symmetries of the problem, the computational domain is reduced to a quarter of the domain, see figure 6.

In Figure 7, we plot the relative error on the central deflection, compared to exact analytical solution of the problem within Kirchhoff-Love plate model. We display the errors for

various regular  $k \times k$  mesh,  $k$  being the number of element subdivision of one side of the quarter of the domain. The size of the mesh being  $h = \frac{1}{k}$ .

The FDEM4 plate element seems convergent with quadratic rate but is outperformed by both DKT and MITC elements. This is similar to the scalar model case. (1). Nevertheless, the shear locking is not present in FDEM4 for this test.

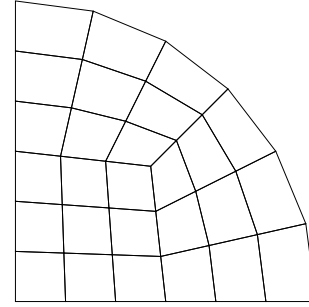


Figure 6: A quadrangular mesh of the quarter of the circular plate with refinement parameter  $k = 6$ .

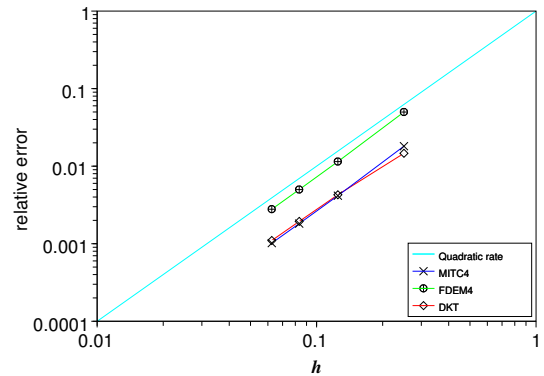


Figure 7: The relative error of central displacement  $\frac{u_{exact} - u}{u_{exact}}$  compared to the exact solution

### 5.2. An inhibited cylindrical shell, the Scoderlis-Lo roof

We start the numerical test on shells with a classic: the Scoderlis-Lo roof. It is a cylindrical shell supported by rigid diaphragms at his end edges, the two other boundaries along the generatrices are free. The shell is submitted to a uniform vertical loading. Due to the plane symmetries of the problem, the computed domain is reduced to one fourth of the cylinder, see Figure 8. The original benchmark proposed takes a fixed relative thickness  $\varepsilon = \frac{3}{300} = 10^{-2}$ , the Young's modulus  $E = 3.10^6$ , the Poisson coefficient  $\nu = 0$  and the vertical load  $L = 0.625$ , the coherent units being inches and pounds.

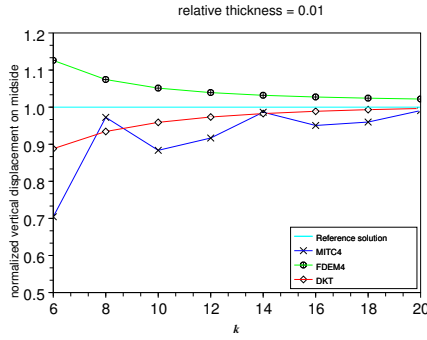


Figure 9: Vertical deflection on midside of Scordelis-Lo roof for regular  $k \times k$  meshes, normalized by a reference solution.

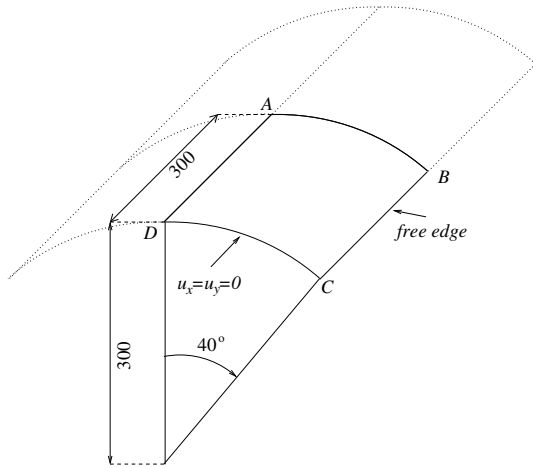


Figure 8: Scordelis-Lo roof

We plot the value of the vertical component at midside  $B$  normalized by a reference value and for  $k \times k$  regular meshes, see Figure 9. The reference values are the exact analytical solution based on Flugge's shell theory, see [8] and are given in Table 1. We note that the FDEM4 seems to converge but is overestimating the deflection, unlike to DKT and MITC4. We shall see from the results with adapted mesh, that this depends on the mesh. We also remark the slight irregularities of MITC4 convergence in this test.

We push forward this example by decreasing the relative thicknesses from  $\varepsilon = 10^{-2}$  to  $10^{-5}$ . We refer to [6, 11] for a complete asymptotic analysis of the test, but we emphasize on the fact it is a case of inhibited shell, more precisely *ill-inhibited* shell. The vertical loading does not belong to the space  $\bar{V}^{m'}$  and thus is not admissible : boundary layers appear along the free edges if the thickness is chosen very small. The width of this boundary layer was shown, in [18], to be proportional to  $\varepsilon^{\frac{1}{4}}$ . We then take account of this information, choosing  $\frac{3}{2}\varepsilon^{\frac{1}{4}}$  as the width of the boundary layer to define adapted meshes equally refined inside and outside the boundary layer, see Figure 10. This can be seen as a test on whether the FDEM4 is suitable for adapted meshes with stretched elements. We scaled the loading

$\varepsilon$	$10^{-2}$	$10^{-3}$	$10^{-4}$	$10^{-5}$
$\tilde{u}_3$	3.6285	3.9528	4.0882	4.0805

Table 1: Reference value of vertical deflection on midside of Scordelis-Lo roof for relative thickness from  $10^{-2}$  to  $10^{-5}$ , based on Flugge's shell theory.

by a  $\frac{\varepsilon^2}{9}$  factor. In Figure 10 we plot the deflection on midside of the free edge normalized by a reference value, see Table 1. The FDEM4 performs here very well with a satisfactory convergence rate, and clearly better than both DKT and MITC4 when the relative thickness  $\varepsilon = 10^{-4}$  or  $\varepsilon = 10^{-5}$ .

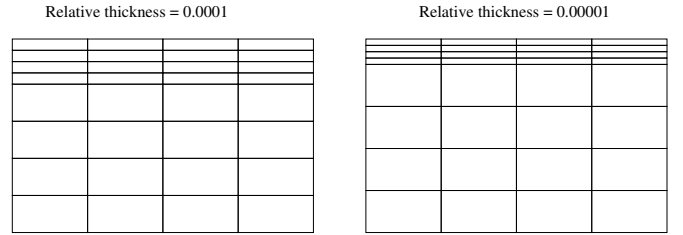


Figure 10: Adapted  $(k+k) \times k$  meshes with  $k = 4$ .

### 5.3. A partially clamped non inhibited cylindrical shell

We continue with a cylindrical shell, but we choose the boundary conditions in order the infinitesimal bendings to be admissible. We roughly take a simplified Scordelis-Lo cylinder partially clamped along a generatrix (straight edge) while the other edges are free. We impose a localized loading on a corner, see Figure 12. This test is not classical, we preferred it over the full cylinder with free ends and submitted to periodic pressure, see for instance [10]. The main reason is the more complex shape of the solution in our test; it makes the test tougher than the full cylinder one. There is no analytic solution to this problem but it is possible, in this case to solve the associated asymptotic bending problem (20), see [14]. The asymptotic solution can then be used as a reference solution as we decrease the thickness.

Let us consider the geometry :

$$x = x_1, \quad y = x_2, \quad z = \varphi(x), \quad \forall (x_1, x_2) \in \Omega, \quad (23)$$

with  $\varphi(x_1) = \sqrt{1 - x_1^2}$  and  $\Omega = [0, \frac{\sqrt{2}}{2}] \times [-0.5, 0.5]$ . The shell is submitted to a vertical force  $\vec{F}^\varepsilon = \varepsilon^3 \vec{e}_3$  localized on the point  $P = (\frac{\sqrt{2}}{2}, 0.5, \frac{\sqrt{2}}{2})$ ; note the  $\varepsilon^3$  scaling corresponding to a non-inhibited shell.

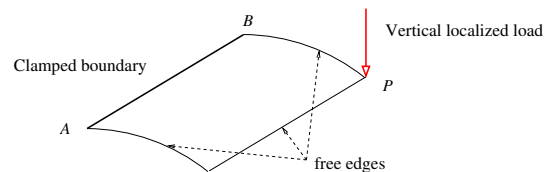


Figure 12: A Cylinder clamped along  $AB$  and submitted to a vertical load localized on  $P$ .



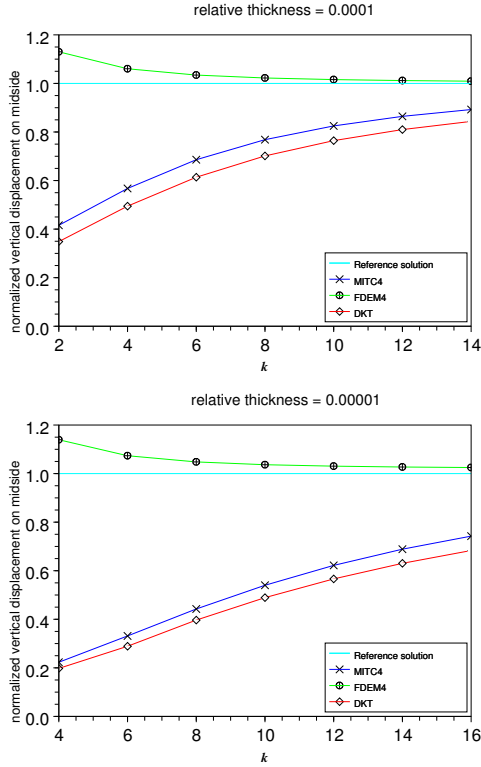


Figure 11: Normalized deflection on midside on Scordelis-Lo roof with adapted  $(k+k) \times k$  meshes.

The equation (21) characterizing the vertical component of an infinitesimal bending reduces to

$$\varphi_{22}u_{3,11} = 0.$$

The vertical component of an infinitesimal bending on the cylinder are then the polynomials in  $x_2$  with arbitrary functions  $g$  and  $h$  of the variable  $x_1$  :

$$u_3 = g(x_1) + x_2h(x_1).$$

The shell being clamped along the straight line  $x_1 = 0.$ , all vertical displacements  $u_3 = g(x_1) + x_2h(x_1)$  satisfying the boundary conditions  $g(0) = g'(0) = h(0) = h'(0) = 0$  define admissible infinitesimal bendings. The shell is thus non-inhibited.

The bending limit problem (20) reduces to a one dimensional differential problem :

**Proposition 5.1.** *The vertical component  $u_3^0$  of the asymptotic bending problem (20) can be computed by solving a variational problem of two functions of one variable :*

$$\left\{ \begin{array}{l} \text{Find } u_3^0 = g(x_1) + x_2h(x_1) \in V \\ \int_S \frac{1}{12} [A^{1111}\rho_{11}\rho_{11}^* + 4A^{1212}\rho_{12}\rho_{12}^*] = L(u_3^*), \quad (24) \\ \forall u_3^* = g^*(x_1) + x_2h^*(x_1) \in V. \end{array} \right.$$

We don't have the exact analytical solution of the reduced asymptotic bending problem (24), but it can be solved using

standard one dimensional finite element procedure. To fix the idea, the value of the vertical component on  $P$  for  $E = 1$ ,  $\nu = 1/3$  and unitary vertical load is  $u_3^0 = 2.1168107$ .

We note that in this case, the vertical displacement on  $P$  of the limit bending solution equals to the limit flexion energy or the asymptotic energy :

$$u_3^0(P) = a^b(\vec{u}^0, \vec{u}^0).$$

It is also remarkable that from minimizing energy results, if  $u^\varepsilon$  is the solution of (10) with the scaled loading  $L^\varepsilon$  then, we have necessarily :

$$u_3^0(P) \leq u_3^\varepsilon(P) \quad \forall \varepsilon > 0.$$

The value of the vertical component at  $P$  is then a good criteria to evaluate a finite element computation in this case. We study the convergence for regular and distorted meshes, see Figure 13, where the distortion is defined as a geometric sequence of common ratio 1.1. We must keep in mind that with such distorted meshes, the asymptotic lines are no more represented by some sides of the elements. To test the locking we ran computations for thicknesses from  $10^{-2}$  to  $10^{-4}$ .



Figure 13: Regular and distorted  $k \times k$  meshes, with  $k = 4$ .

In the Figure 14, we plot the vertical displacement normalized by the solution of the asymptotic (bending) limit problem (24) using regular meshes for various values of mesh refinement parameter  $k$ , the number of subdivision of  $[0, \sqrt{2}/2]$ . The mean size of an element of the mesh is then  $h = \frac{\sqrt{2}}{2k}$ . For the thickness  $\varepsilon = 10^{-2}$  we see that DKT, MITC4 and FDEM4 are coherent and converge to the same solution. As the thickness decreases, we clearly see that DKT underestimates the real solution whereas the MITC4 and FDEM4 are both coherent with the asymptotic analysis and seem to converge. In this case, the membrane locking is absent for both MITC4 and FDEM4 whereas present with DKT.

In the Figure 15, we plot the vertical displacement normalized by a reference solution using quasi uniform distorted meshes, see Figure 13, for various values of mesh refinement parameter  $k$ , the number of subdivision of  $[0, \sqrt{2}/2]$ . As a reference solution, we used the value obtained with MITC4 element with a  $28 \times 28$  regular mesh. We note in this case that the reference solution is not very different from the asymptotic solution.

The locking is still present for DKT, while not in a different manner. On the contrary MITC4 shows signs of significant deterioration of the approximation for very small thicknesses ( $\varepsilon = 5 \cdot 10^{-4}$ ), while FDEM4 seems completely unaffected.

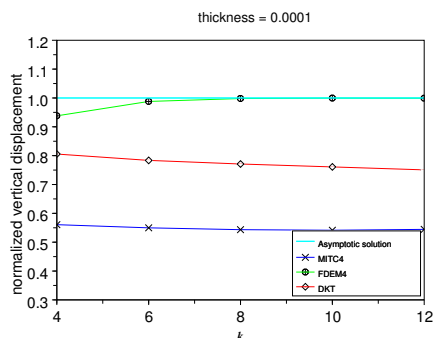
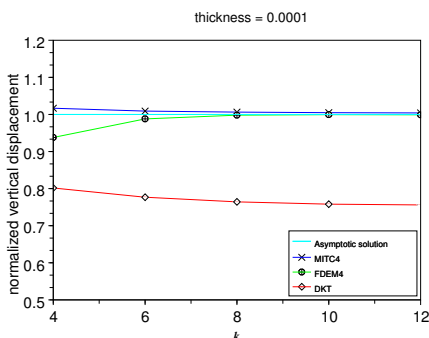
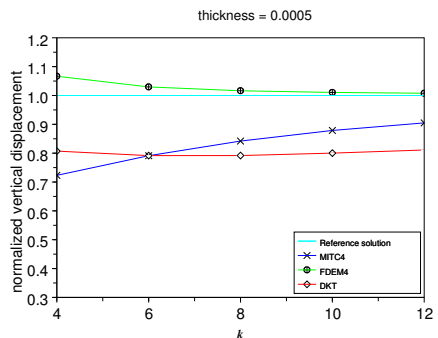
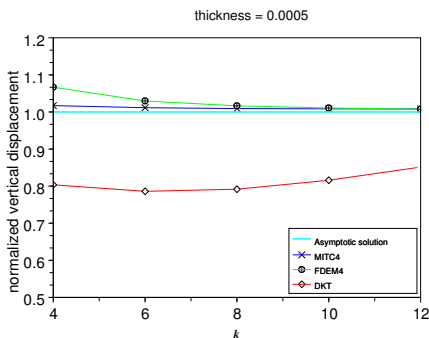
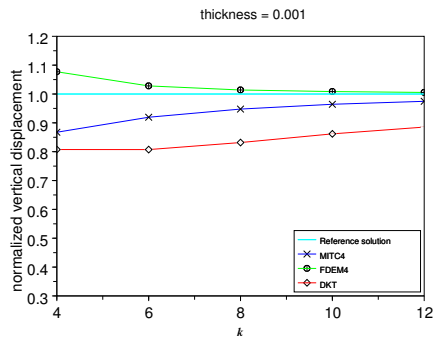
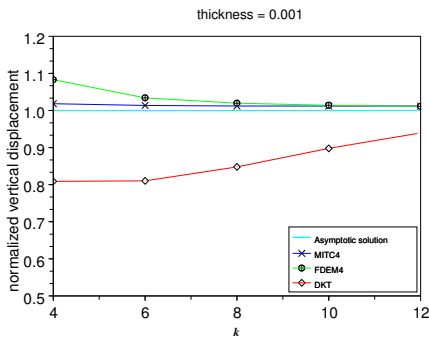
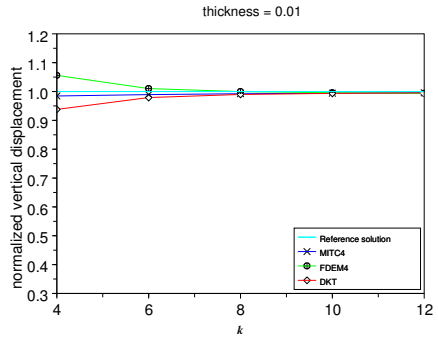
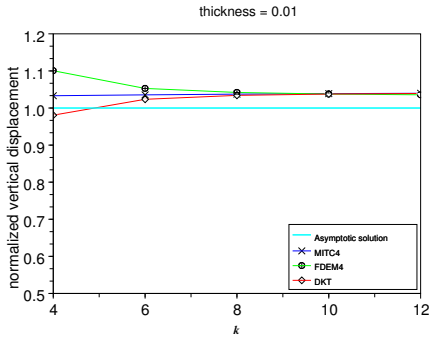


Figure 14: Non-inhibited partially clamped cylinder submitted to vertical localized load, with regular meshes,  $\frac{u_3^e(P)}{u_3^0(P)}u$  for thicknesses from  $10^{-2}$  to  $10^{-4}$ .

Figure 15: Non-inhibited partially clamped cylinder submitted to vertical localized load, with distorted meshes,  $\frac{u_3^e(P)}{u_3^0(P)}u$  for thicknesses from  $10^{-2}$  to  $10^{-4}$ .

#### 5.4. A partially clamped non-inhibited hyperbolic paraboloid

We study here the case of a partially clamped hyperbolic paraboloid defined for any  $(x_1, x_2) \in \Omega = \left[-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right]^2$

$$x = x_1, \quad y = x_2, \quad z = \frac{1}{2}(x_1^2 - x_2^2). \quad (25)$$

Let us remark that the last surface is equivalently defined by

$$x = x_1, \quad y = x_2, \quad z = x_1 x_2, \quad \forall (x_1, x_2) \in \Omega_l \quad (26)$$

where  $\Omega_l$  is the quadrangle defined by the four points  $A : (-1, 0)$ ,  $B : (0, -1)$ ,  $C : (1, 0)$ ,  $D : (0, 1)$ , see figure 16

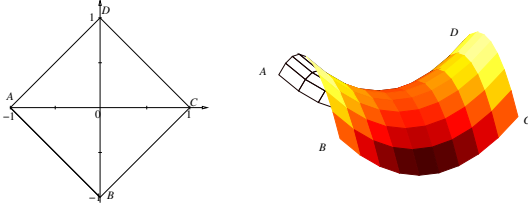


Figure 16: A hyperbolic paraboloid, partially clamped, along  $AB$ , and submitted to uniform vertical load.

Here again we compare FDEM4 with MITC4 and DKT with respect to the solution of the bending limit problem/ In the case of hyperbolic paraboloid defined by the map (26), the equation (21) characterizing the vertical component of an infinitesimal bending reduces to

$$\varphi_{,12} u_{3,12} = 0.$$

Then any arbitrary functions  $f$  and  $g$  of one variable define the vertical components of any inextensional displacements (or infinitesimal bendings) on the paraboloid :

$$u_3(x_1, x_2) = g(x_1) + h(x_2) \quad \forall (x_1, x_2) \in \Omega_l$$

We recall that when the shell is clamped along the  $AB$  boundary, it is not inhibited although bendings necessarily vanishes in the subdomain  $\{(x_1, x_2) \in \Omega_l / x_1 \leq 0 \text{ and } x_2 \leq 0\}$ . It suffices then that  $g$  and  $h$  and their derivatives vanishes on  $[-1, 0]$  and be arbitrary on  $]0, 1]$ .

On such shell problem, the limit bending limit problem can be reduced to a more simple formulation, see [13] :

$$\left\{ \begin{array}{l} \text{Find } u_3 = g(x_1) + h(x_2)/g, h \in V^2 \\ \frac{1}{12} \int_{\Omega_l} \begin{bmatrix} g'' \\ h'' \end{bmatrix}^\top \begin{bmatrix} A^{1111} & A^{1122} \\ A^{2211} & A^{1222} \end{bmatrix} \begin{bmatrix} g'' \\ h'' \end{bmatrix} a^{\frac{3}{2}} \\ = \int_{\Omega_l} (g^*(x_1) + h^*(x_2)) a^{\frac{1}{2}}, \\ \forall g^*, h^* \in V^2 \end{array} \right. \quad (27)$$

where

$$V = \{f \in \mathbb{H}^2([0, 1]) / f(0) = f'(0) = 0\}.$$

In Figure 18 and Figure 19 we normalized the vertical displacement on point  $D$  with respect to the solution of the bending

limit problem. We used regular and distorted meshes, see Figure 17, for various values of mesh refinement parameter  $k$ , the size of the mesh being  $h = \frac{\sqrt{2}}{k}$ .

As we can see no membrane locking arises on regular meshes and the results are consistent with the asymptotic behavior, we observe that the computed solutions converge to the asymptotic bending limit solution as the thickness decreases towards zero. The convergence rate seems good but FDEM4 is underperforming compared to both DTK and MITC4.



Figure 17: regular and distorted  $k \times k$  meshes, with  $k = 4$ .

The approximations turn bad for very small thickness with distorted meshes, see Figure 19. We plot the computed vertical component on node  $D$  normalized with respect to a reference solution obtained with the MITC4 solution and a very thin regular mesh (28x28), see table 2, except for the smallest thickness  $\varepsilon = 10^{-5}$  for which we normalized by the asymptotic solution.

It appears that the computations from both DKT and MITC4 presents severe signs of deterioration for thicknesses below  $10^{-3}$ . FDEM4 also presents slight signs of computational deterioration but in a very different magnitude and overall correct except for the smallest thickness for which the error in vertical deflection is nearly 20%. This is a tough test.

$\varepsilon$	$10^{-2}$	$10^{-3}$	$10^{-4}$	Asymp. solution
$\tilde{u}_3$	0.86	0.60580	0.52349	0.5006

Table 2: Reference value of vertical deflection at  $D$  compared to the asymptotic solution.

## 6. Conclusion and perspective

We have proposed an original shell finite element, FDEM4, based on finite difference approximation rather than polynomial interpolation. The FDEM4 shell element derives directly from the Naghdi's shell model without any corrections. The element is simple and very cost effective due one node integration scheme in the construction of the stiffness matrix.

The numerical tests on plates, inhibited and non-inhibited shells, are very encouraging : the FDEM4 has constantly showed convergence behavior in every cases we have tested, even though it is underperforming in some cases compared to others FEM procedures such as DKT or MITC4 when using coarse meshes.

From these tests, the FDEM4 seems almost free of shear or membrane locking. Only some serious deteriorations occur in the non-inhibited partially clamped paraboloid case when using distorted meshes which is not uncommon for 'locking free

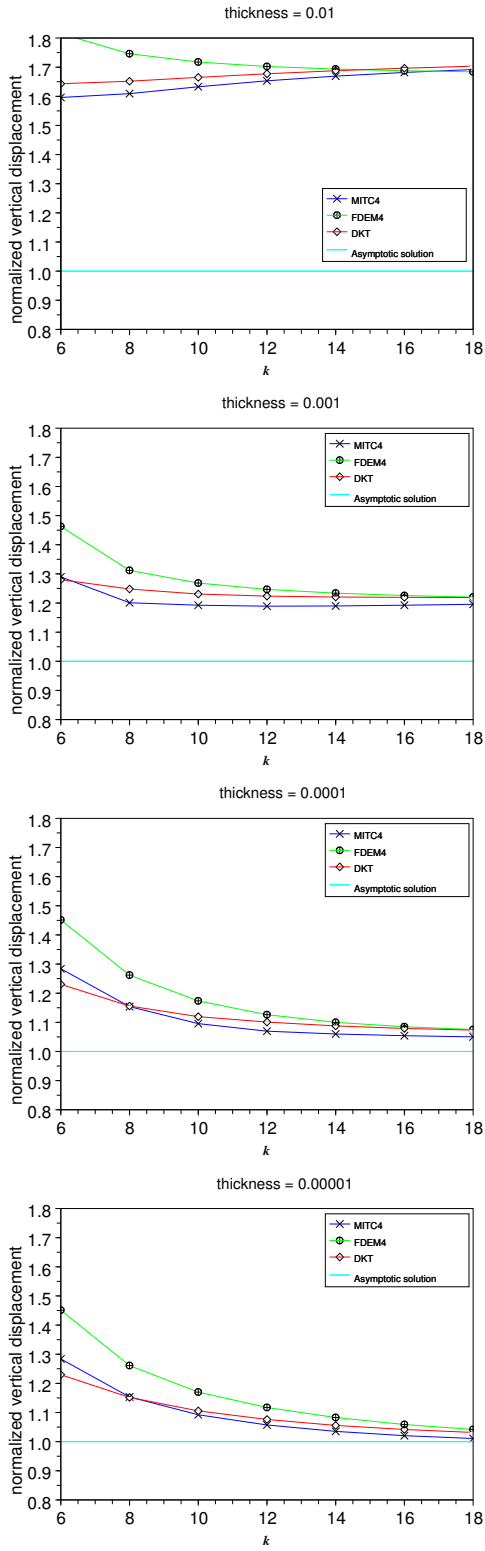


Figure 18: Non-inhibited partially clamped hyperbolic paraboloid. Normalized vertical displacement on  $k \times k$  regular meshes

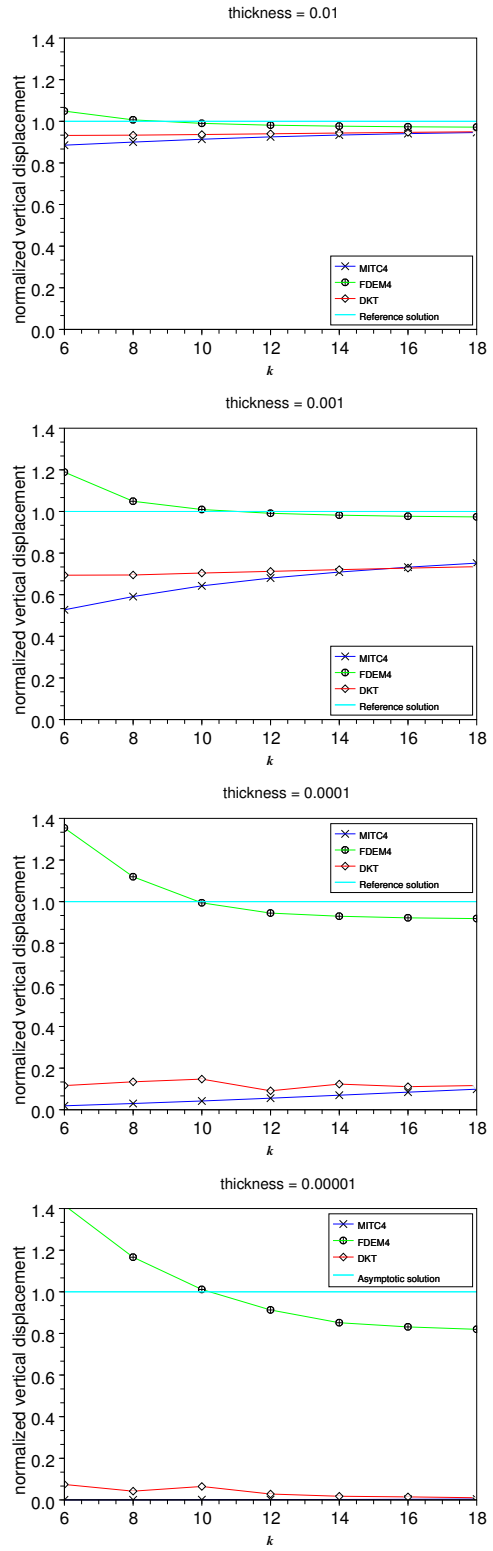


Figure 19: Non-inhibited partially clamped hyperbolic paraboloid. Normalized vertical displacement on  $k \times k$  distorted meshes

but only for very small thickness ( $\varepsilon = 10^{-5}$ ). The overall behavior of FDEM4 on distorted meshes is not perfect, yet it is very satisfactory, especially compared to MITC4 or DKT shell elements on the tests we conducted.

The FDEM4 is far from being the perfect element, we detect instabilities in the computations of some inhibited shell tests (clamped hyperbolic paraboloid and clamped spherical cap with uniform loading) which is not uncommon for 'locking free' shell elements or low order shell elements. In Case of FDEM4 though, the instabilities appear but for very small relative thicknesses ( $< 10^{-4}$ ). The good robustness to distortion and membrane locking makes the FDEM4 an interesting and mostly reliable element. It marks the path for more performant element for shells based on this concept.

Much work still need to be done on such element, in particular a theoretical study of convergence and on the interpretation of finite difference element method as part of the Galerkin method's family. We need also to develop facet based FDEM element instead of an exact definition of the geometry.

Of course, one of the most important, of utmost engineering interest, is the development of triangular shell elements as quadrangular discretisation cannot always be easily available.

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