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C^* -ALGEBRAS OF PENROSE HYPERBOLIC TILINGS

HERVE OYONO-OYONO AND SAMUEL PETITE

ABSTRACT. Penrose hyperbolic tilings are tilings of the hyperbolic plane which admit, up to affine transformations a finite number of prototiles. In this paper, we give a complete description of the C^* -algebras and of the K -theory for such tilings. Since the continuous hull of these tilings have no transversally invariant measure, these C^* -algebras are traceless. Nevertheless, harmonic currents give rise to 3-cyclic cocycles and we discuss in this setting a higher-order version of the gap-labelling.

Keywords: Hyperbolic aperiodic tilings, C^* -algebras of dynamical systems, K -theory, Cyclic cohomology
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1. INTRODUCTION

The non-commutative geometry of a quasi-periodic tiling studies an appropriate C^* -algebra of a dynamical system (X, G) , for a compact metric space X , called *the Hull*, endowed with a continuous Lie group G action. This C^* -algebra is of relevance to study the space of leaves which is pathological in any topological sense. The Hull owns also a geometrical structure of *Lamination* or *foliated space*, the transverse structure being just metric [9]. The C^* -algebras and the non-commutative tools provide then topological and geometrical invariants for the tiling or the lamination. Moreover, some K -theoretical invariants of Euclidean tilings have a physical interpretation. In particular, when the tiling represents a quasi-crystal, the image of the K -theory under the canonical trace labels the gaps in the spectrum of the Schrödinger operator associated with the quasi-crystal [2].

For an Euclidean tiling, the group G is \mathbb{R}^d and \mathbb{R}^d -invariant ergodic probability measures on the Hull are in one-to-one correspondence with ergodic transversal invariant measures and also with extremal traces on the C^* -algebra [3]. These algebras are well studied and this leads, for instance, to give distinct proofs of the *gap labelling conjecture* [3, 4, 10], i.e. for minimal \mathbb{R}^d -action, the image of the K -theory under a trace is the countable subgroup of \mathbb{R} generated by the images under the corresponding transversal invariant measure of the compact-open subsets of the (Cantor) canonical transversal.

For a hyperbolic quasi-periodic tiling, the situation is quite distinct. The affine transformations act on the Hull and since this group is not unimodular, there is no transversally invariant measure [15]. This shows up a new phenomena for the C^* -algebra of the tiling: it has no trace. Nevertheless, the affine group is amenable, so the Hull admits at least one invariant probability measure. These measures are actually in one-to-one correspondence with harmonic currents [13], and they provide 3-cyclic cocycles on the smooth algebra of the tiling.

The present paper is devoted to give a complete description of the C^* -algebra and the K -theory of a specific family of hyperbolic tilings derived from the example given by Penrose in [14]. The dynamic of the Hulls under investigation, have a structure of double suspension (this make sens in term of groupoids as we shall see in section 5.1) which enables to make explicit computations. This suggests that the pairing with the 3-cyclic cocycle is closely related to the one-dimension gap-labelling for a subshift associated with the tiling. But the right setting to state an analogue of the gap-labelling seems to be Frechet algebras and a natural question is whether this bring in new computable invariants.

Background on the tiling spaces is given in the next section and we construct examples of hyperbolic quasi-periodic tilings in the third section. A description of the considered Hulls is given in Section 4. In Section 5, we recall the background on the groupoids and their C^* -algebras. Section 6 and 7 are devoted to a description of the C^* -algebras of the examples. Their K -theory is given in the section 8 then we construct 3-cyclic cocycles associated to these tilings and we discuss an odd version of the gap-labelling.

2. BACKGROUND ON TILINGS

Let \mathbb{H}_2 be the real hyperbolic 2-space, identified with the upper half complex plane: $\{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ with the metric $ds^2 = \frac{dx^2 + dy^2}{y^2}$. We denote by G the group of *affine maps* of this space: i.e. the isometries of \mathbb{H}_2 of the kind $z \mapsto az + b$ with a, b reals and $a > 0$.

A tiling $T = \{t_1, \dots, t_n, \dots\}$ of \mathbb{H}_2 , is a collection of convex compact polygons t_i with geodesic borders, called *tiles*, such that their union is the whole space \mathbb{H}_2 , their interiors are pairwise disjoint and they meet full edge to full edge. A tiling is said of *G-finite type* or *finite affine type*, if there exists a finite number of polygons $\{p_1, \dots, p_n\}$ called *prototiles* such that each t_i is the image of one of these polygons by an element of G .

For instance, when F is a fundamental domain of a co-compact lattice Γ of isometries of \mathbb{H}_2 , then $\{\gamma(F), \gamma \in \Gamma\}$ is a tiling of \mathbb{H}_2 . However the set of tilings is much richer than the one given by lattices. Besides its famous Euclidean tiling, Penrose in [14] constructs a finite affine type tiling made with a single prototile which is not stable for any Fuchsian group. The construction goes as follows.

2.1. Hyperbolic Penrose's tiling. Let P be the convex polygon with vertices A_p with affix $(p-1)/2 + i$ for $1 \leq p \leq 3$ and $A_4 : 2i + 1$ and $A_5 : 2i$ P is a polygon with 5 geodesic edges. Consider the two maps:

$$R : z \mapsto 2z \text{ and } S : z \mapsto z + 1.$$

The hyperbolic Penrose's tiling is defined by $\mathcal{P} = \{R^k \circ S^n P \mid n, k \in \mathbb{Z}\}$ (see figure 1). This is an example of finite affine type tiling of \mathbb{H}_2 .

This tiling is stable under no co-compact group of hyperbolic isometries. The proof is homological: we associate with the edge A_4A_5 a positive charge and two negative charges with edges A_1A_2 , A_2A_3 . If \mathcal{P} was stable for a Fuchsian group, then P would tile a compact surface. Since the edge A_4A_5 can meet only the edges A_1A_2 or A_2A_3 , the surface has a neutral charge. This is in contradiction with the fact P is negatively charged.

G. Margulis and S. Mozes [12] have generalized this construction to build a family of prototiles which cannot be used to tile a compact surface. Notice the group of

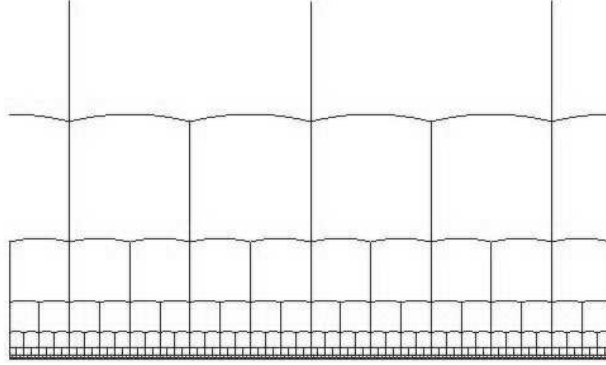


FIGURE 1. The hyperbolic Penrose's tiling

isometries which preserves \mathcal{P} is not trivial and is generated by the transformation R . In order to break this symmetry, it is possible to decorate prototiles to get a new finite type tiling which is not stable for any non trivial isometry (we say in this case that the tiling is *aperiodic*).

3. BACKGROUND ON TILING SPACES

In this section, we recall some basic definitions and properties on dynamical systems associated with tilings. We refer to [5], [11] and [20] for the proofs. We give then a description of the dynamical system associated to the hyperbolic Penrose's tiling.

3.1. Action on tilings space. First, note that the group G acts transitively, freely (without a fixed point) and preserving the orientation of the surface \mathbb{H}_2 , thus G is a Lie group homeomorphic to \mathbb{H}_2 . The metric on \mathbb{H}_2 gives a left multiplicative invariant metric on G . We fix the point O in \mathbb{H}_2 with affix i that we call *origin*. For a tiling T of G finite type and an isometry p in G , the image of T by p is again a tiling of \mathbb{H}_2 of G finite type. We denote by $G.T$ the set of tilings which are image of T by isometries in G . The group G acts on this set by the left action:

$$\begin{aligned} G \times G.T &\longrightarrow G.T \\ (p, T') &\longmapsto p.T' = p(T'). \end{aligned}$$

We equip $G.T$ with a metrizable topology, finer as one induced by the metric on \mathbb{H}_2 . A base of neighborhoods is defined as follows: two tilings are close one of the other if they agree, on a big ball of \mathbb{H}_2 centered at the origin, up to an isometry in G close to the identity. This topology can be generated by the metric δ on $G.T$ defined by (see [5]):

For T and T' be two tilings of $G.T$, let

$$A = \left\{ \epsilon \in \left(0, \frac{1}{\sqrt{2}}\right] \mid \exists g \in B_\epsilon(Id) \subset G \text{ s.t. } g.T \cap B_{1/\epsilon} = T' \cap B_{1/\epsilon} \right\}$$

where $B_{1/\epsilon}$ is the set of points $x \in \mathbb{H}_2$ such that $d(x, O) < 1/\epsilon$.

We define:

$$\delta(T, T') = \inf A \text{ if } A \neq \emptyset$$

$$\delta(T, T') = \frac{1}{\sqrt{2}} \text{ else.}$$

The *continuous hull* of the tiling T , is the metric completion of $G.T$ for the metric δ . We denote it by X_T^G . Actually this space is a set of tilings of \mathbb{H}_2 of G -finite type. A *patch* of a tiling T is a finite set of tiles of T . It is straightforward to show that patches of tilings in X_T^G are copies of patches of T . The set X_T^G is then a compact metric set and the action of G can be extended to a continuous left action on this space. The dynamical system (X_T^G, G) has a dense orbit: the orbit of T . Some combinatorial properties can be interpreted in a dynamical way like, for instance, the following.

Definition 3.1. *A tiling T satisfies the repetitivity condition if for each patch P , there exists a real $R(P)$ such that every ball of \mathbb{H}_2 with radius $R(P)$ intersected with the tiling T contains a translated by an element G of the patch P .*

This definition can be interpreted from a dynamical point of view (for a proof see for instance [8]).

Proposition 3.2 (Gottschalk). *The dynamical system (X_T^G, G) is minimal (any orbit is dense) if and only if the tiling T satisfies the repetitivity condition.*

We call a tiling *aperiodic* if the action of G on X_T^G is free: for all $p \neq Id$ of G and all tilings T' of X_T^G we have $p.T' \neq T'$.

As we have seen in the former section the hyperbolic Penrose's tiling is not aperiodic, however, using this example, we shall construct in Section 4 uncountably many examples of repetitive and aperiodic affine finite type tilings.

When the tiling T is aperiodic and repetitive, the hull X_T^G has also a geometric structure of a specific lamination called a *G -solenoid* (see [5]). Locally at any point x , there exists a *vertical germ* which is a Cantor set included in X_T^G , transverse to the local G -action and which is defined independently of the neighborhood of the point x . This implies that X_T^G is locally homeomorphic to the Cartesian product of a Cantor set with an open subset (called a *slice*) of the Lie group G . The connected component of the slices that intersect is called a *leaf* and has a manifold structure. Globally, X_T^G is a disjoint union of uncountably many leaves, and it turns out that each leaf is a G -orbit. Since the action is free, each leaf is homeomorphic to \mathbb{H}_2 .

In the aperiodic case, the G -action is *expansive*: There exists a positive real ϵ such that for every points T_1 and T_2 in the same vertical in X_T^G , if $\delta(T_1.g, T_2.g) < \epsilon$ for every $g \in G$, then $T_1 = T_2$.

Furthermore this action has locally constant return times: if an orbit (or a leaf) intersects two verticals V and V' at points v and $v.g$ where $g \in G$, then for any point w of V close enough to v , $w.g$ belongs to V' .

3.2. Structure of the hull of the Penrose Hyperbolic tilings. First recall the notion of suspension action for X a compact metric space and $f : X \rightarrow X$ an homeomorphism. The group \mathbb{Z} acts diagonally on the product space $X \times \mathbb{R}$ by the following homeomorphism denoted \mathcal{A}_f

$$\begin{aligned} \mathcal{A}_f : X \times \mathbb{R} &\rightarrow X \times \mathbb{R} \\ (x, t) &\mapsto (f(x), t - 1) \end{aligned}$$

The quotient space of $(X \times \mathbb{R})/\mathcal{A}_f$, where two points are identified if they belong to the same orbit, is a compact set for the product topology and is called *the*

suspension of (X, f) . The group \mathbb{R} acts also diagonally on $X \times \mathbb{R}$: trivially on X and by translation on \mathbb{R} . Since this action commutes with \mathcal{A}_f , this induces a continuous \mathbb{R} action on the suspension space $(X \times \mathbb{R})/\mathcal{A}_f$ that we call: the *suspension action* of the system (X, f) and we denote it by $((X \times \mathbb{R})/\mathcal{A}_f, \mathbb{R})$.

We recall here, the construction of the dyadic completion of the integers. On the set of integers \mathbb{Z} , we consider the dyadic norm defined by

$$|n|_2 = 2^{-\sup\{p \in \mathbb{N}, 2^p \text{ divides } |n|\}} \quad n \in \mathbb{Z}.$$

Let Ω be the completion of the set \mathbb{Z} for the metric given by $|\cdot|_2$. The set Ω has a commutative group structure where \mathbb{Z} is a dense subgroup, and Ω is a Cantor set. The continuous action given by the map $o : x \mapsto x + 1$ on Ω is called *adding-machine* or *odometer* and is known to be minimal and equicontinuous. We denote by $((\Omega \times \mathbb{R})/\mathcal{A}_o, \mathbb{R})$ the suspension action of this homeomorphism.

Recall that a conjugacy map between two dynamical systems is a homeomorphism which commutes with the actions. Let \mathcal{N} be the group of maps $\{z \mapsto z + t, t \in \mathbb{R}\}$ isomorphic to \mathbb{R} .

Proposition 3.3. *Let $X_{\mathcal{P}}^{\mathcal{N}}$ be the completion (for the tiling topology) of the orbit $\mathcal{N} \cdot \mathcal{P}$.*

Then the dynamical system $(X_{\mathcal{P}}^{\mathcal{N}}, \mathcal{N})$ is conjugate to the suspension action of the odometer $((\Omega \times \mathbb{R})/\mathcal{A}_o, \mathbb{R})$.

Proof. Let $\phi : \mathcal{N} \cdot \mathcal{P} \rightarrow (\Omega \times \mathbb{R})/\mathcal{A}_o$ be the map defined by $\phi(\mathcal{P} + t) = [0, t]$ where $[0, t]$ is the \mathcal{A}_o -class in of $(0, t) \in \Omega \times \mathbb{R}$. Since the tiling is invariant under no translations, the application ϕ is well defined. It is straightforward to check that ϕ is continuous for the tiling topology and for the dyadic topology on Ω . So the map ϕ extends by continuity to $X_{\mathcal{P}}^{\mathcal{N}}$. Let us check that ϕ is a homeomorphism by constructing its inverse. Let $\psi : \mathbb{Z} \times \mathbb{R} \rightarrow X_{\mathcal{P}}^{\mathcal{N}}$ defined by $\psi(n, t) = \mathcal{P} + n + t$. This application is continuous for the dyadic topology, so it extends by continuity to $\Omega \times \mathbb{R}$. Notice $\psi(n, t)$ is constant along the orbits of the \mathcal{A}_o action on $\mathbb{Z} \times \mathbb{R}$ which is dense in $\Omega \times \mathbb{R}$. Thus ψ is constant along the \mathcal{A}_o -orbits in $\Omega \times \mathbb{R}$ and ψ factorizes to a map $\bar{\psi}$ from the suspension $(\Omega \times \mathbb{R})/\mathcal{A}_o$ to $X_{\mathcal{P}}^{\mathcal{N}}$. It is plain to check that $\bar{\psi} \circ \phi = Id$ on the dense set $\mathcal{N} \cdot \mathcal{P}$ and that $\phi \circ \bar{\psi} = Id$ on the dense set $\pi(\mathbb{N} \times \mathbb{R})$ where $\pi : \Omega \times \mathbb{R} \rightarrow (\Omega \times \mathbb{R})/\mathcal{A}_o$ denotes the canonical projection. Hence ϕ is an homeomorphism from $X_{\mathcal{P}}^{\mathcal{N}}$ onto $(\Omega \times \mathbb{R})/\mathcal{A}_o$. It is obvious that ϕ commutes with the \mathbb{R} -actions. \square

4. EXAMPLES

We construct in this section a family of tilings of \mathbb{H}_2 of finite affine type, indexed by sequences on a finite alphabet. For uncountably many of them, the tilings will be aperiodic and repetitive, the action on the associated hull will be free and minimal. A description of these actions in terms of double-suspension is given.

4.1. Construction of the examples. To construct such tilings we will use the hyperbolic Penrose's tiling described in section 2.1, so we will keep the notations of this section. Recall that its stabilizer group under the action of G , is the group $\langle R \rangle$ generated by the affine map R . The main idea is to "decorate" this tiling in order to break its symmetry, the decoration will be coded by a sequence on a finite alphabet. By a decoration, we mean that we will substitute to each tile t the same

polygon t equipped with a color. We take the convention that two colored polygons are the same if and only if the polygons are the same up to an affine map and they share the same color. By substituting each tile by a colored tile, we obtain a new tiling of finite affine type with a bigger number of prototiles.

Notice that the coloration is not canonical. It is also possible to do the same by substituting to a tile t , a unique finite family of convex tiles $\{t_i\}_i$, like triangles, such that the union of the t_i is t and the tiles t_i overlaps only on their borders. We choose the coloration only for presentation reasons.

Let r be an integer bigger than 1. We associate to each element of $\{1, \dots, r\}$ an unique color. Let P be the polygon defined in Section 2.1 to construct the Penrose's tiling. For an element i of $\{1, \dots, r\}$, we denote by P_i the prototile P colored in the color i . To a sequence $w = (w_k)_k \in \{1, \dots, r\}^{\mathbb{Z}}$, we associate the G -finite-type tiling $\mathcal{P}(w)$ built with the prototiles P_i for i in $\{1, \dots, r\}$ and defined by:

$$\mathcal{P}(w) = \{R^q \circ S^n(P_{w_{-q}}), n, q \in \mathbb{Z}\}.$$

Notice that the stabilizer of this tiling is a subgroup of $\langle R \rangle$.

The set of sequences on $\{1, \dots, r\}$ is the product space $\{1, \dots, r\}^{\mathbb{Z}}$ which is a Cantor set for the product topology. There exists a natural homeomorphism on it called the *shift*. To a sequence $(w_n)_{n \in \mathbb{Z}}$ the shift σ associates the sequence $(w_{n+1})_{n \in \mathbb{Z}}$.

Remark 4.1. *The map*

$$Z_w \rightarrow X_{\mathcal{P}(w)}^G; \omega' \mapsto \mathcal{P}(w')$$

is continuous.

Let Z_w denote the closure of the orbit of w by the action of the shift σ : $Z_w = \overline{\{\sigma^n(w), n \in \mathbb{Z}\}}$. The set Z_w is a compact metric space stable under the action of σ . Since $R \cdot \mathcal{P}(w)$ denotes the tiling image of $\mathcal{P}(w)$ by R , we get the relation

$$(4.1) \quad R \cdot \mathcal{P}(w) = \mathcal{P}(\sigma(w)).$$

Thanks this, we obtain the following property:

Lemma 4.2.

- *The sequence w is aperiodic for the shift-action, if and only if $\mathcal{P}(w)$ is stable under no non-trivial affine map.*
- *The dynamical system (Z_w, σ) is minimal, if and only if $(X_{\mathcal{P}(w)}^G, G)$ is minimal.*

Proof. The first point comes from the relation 4.1 and from the fact that the stabilizer of $\mathcal{P}(w)$ is a subgroup of $\langle R \rangle$. The last point comes from the characterization of minimal sequences: (Z_w, σ) is minimal if and only if each words in w appears infinitely many times with uniformly bounded gap [18]. This condition is equivalent to the repetitivity of $\mathcal{P}(w)$. \square

Recall that we have defined the group $\mathcal{N} = \{z \mapsto z + t, t \in \mathbb{R}\}$ and that $X_{\mathcal{P}}^{\mathcal{N}}$ stands for the closure (for the tiling topology) of the \mathcal{N} orbit $\mathcal{N} \cdot \mathcal{P}$ of the uncolored tiling \mathcal{P} . Notice that the continuous action of R on $X_{\mathcal{P}}^G$ preserves the orbit $\mathcal{N} \cdot \mathcal{P}$ so we get an homeomorphism of $X_{\mathcal{P}}^{\mathcal{N}}$ that we denote also by R . We consider on the space $X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times \mathbb{R}_+^*$ equipped with the product topology, the homeomorphism \mathcal{R} defined by $\mathcal{R}(\mathcal{T}, w', t) = (R \cdot \mathcal{T}, \sigma(w'), t/2)$. The quotient space $(X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times \mathbb{R}_+^*)/\mathcal{R}$, where the points in the same \mathcal{R} orbit are identified, is a compact space.

The affine group G also acts on the left on $X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times \mathbb{R}_+^*$; where the action of an element $g : z \mapsto az + b$ is given by the homeomorphism

$$(\mathcal{T}, w', t) \mapsto \left(\mathcal{T} + \frac{b}{at}, w', at\right) = g \cdot (\mathcal{T}, w', t).$$

It is straightforward to check that the application \mathcal{R} commutes with this action, so this defines a G continuous action on the quotient space $(X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times \mathbb{R}_+^*)/\mathcal{R}$.

Proposition 4.3. *Let w be an element of $\{1, \dots, r\}^{\mathbb{Z}}$ and let $X_{\mathcal{P}}^{\mathcal{N}}$ and $X_{\mathcal{P}(w)}^G$ be respectively the closures (for the tiling topology) of the orbits $\mathcal{N} \cdot \mathcal{P}$ and $G \cdot \mathcal{P}(w)$. Then the map*

$$\begin{aligned} \Psi : G \cdot \mathcal{P}(w) &\rightarrow (X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times \mathbb{R}_+^*)/\mathcal{R} \\ g \cdot \mathcal{P}(w) &\mapsto [g \cdot (\mathcal{P}, w, 1)] \end{aligned}$$

where $[x]$ denotes the \mathcal{R} -class of x , extends to a conjugacy map between $(X_{\mathcal{P}(w)}^G, G)$ and $((X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times \mathbb{R}_+^*)/\mathcal{R}, G)$.

Proof. Let Φ be the map $\mathcal{N} \cdot \mathcal{P} \times Z_w \times \mathbb{R}_+^* \rightarrow X_{\mathcal{P}(w)}^G$ defined by

$$\Phi(\mathcal{P} + \tau, w', t) = R_t \cdot (\mathcal{P}(w') + \tau)$$

where R_t denotes the map $z \mapsto tz$. According to remark 4.1, the application Φ is continuous for the tiling topology on $\mathcal{N} \cdot \mathcal{P}$, so it extends by continuity to a continuous map from $X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times \mathbb{R}_+^*$ to $X_{\mathcal{P}(w)}^G$. Thanks to relation (4.1), we get $\Phi \circ \mathcal{R} = \Phi$ on the dense subset $\mathcal{N} \cdot \mathcal{P} \times Z_w \times \mathbb{R}_+^*$. Therefore the map Φ factorizes through a continuous map $\bar{\Phi} : (X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times \mathbb{R}_+^*)/\mathcal{R} \rightarrow X_{\mathcal{P}(w)}^G$. Since the stabilizer of the tiling $\mathcal{P}(w)$ is a subgroup of the one generated by the map R , and by relation (4.1), the map

$$\Psi : G \cdot \mathcal{P}(w) \rightarrow (X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times \mathbb{R}_+^*)/\mathcal{R}; R_t \cdot \mathcal{P}(w) + \tau \mapsto [\mathcal{P} + \tau/t, w, t]$$

is well defined. It is straightforward to check that Ψ is continuous, so it extends to a continuous map from $X_{\mathcal{P}(w)}^G$ to $(X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times \mathbb{R}_+^*)/\mathcal{R}$ that we denote again Ψ . Furthermore we have $\bar{\Phi} \circ \Psi = Id$ on $G \cdot \mathcal{P}(w)$ and $\Psi \circ \bar{\Phi} = Id$ on the dense set $\pi(\mathcal{N} \cdot \mathcal{P} \times Z_w \times \mathbb{R}_+^*)$ where π denotes the canonical projection $X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times \mathbb{R}_+^* \rightarrow (X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times \mathbb{R}_+^*)/\mathcal{R}$. Hence the map $\bar{\Phi}$ is an homeomorphism. The homeomorphism Ψ obviously commutes with the action. \square

Notice that, $X_{\mathcal{P}}^{\mathcal{N}}$ is locally the Cartesian product of a Cantor set by an interval of \mathbb{R} . For $w \in \{1, \dots, r\}^{\mathbb{Z}}$, $X_{\mathcal{P}(w)}^G$ is locally homeomorphic the product of a Cantor set times an open subset (a slice) of $\mathbb{R}_+^* \times \mathbb{R}$ since the Cartesian product of two Cantor sets is again a Cantor set. The G -action maps slices onto slices.

4.2. Ergodic properties of Penrose's tilings. For a metric space X and a continuous action of a group Γ on it, a Γ -invariant measure is a measure μ defined on the Borel σ -algebra of X which is invariant under the action of Γ i.e.: For any measurable set $B \subset X$ and any $g \in \Gamma$, $\mu(B.g) = \mu(B)$. For instance, any group Γ acts on itself by right multiplication, there exists (up to a scalar) only one measure invariant for this action: it is called the *Haar* measure.

Any action of an amenable group Γ (like \mathbb{Z} , \mathbb{R} and all their extensions) on a compact metric space X admits a finite invariant measure and in particular, any homeomorphism f of X preserves a probability measure. An *ergodic* invariant

measure μ is such that every measurable functions constant along the orbits are μ almost surely constant. Every invariant measure is the sum of ergodic invariant measures [18]. A conjugacy map sends the invariant measure to invariant measure and the ergodic measures to the ergodic measures.

In our case, the group of affine maps G , is the extension of two groups isomorphic to \mathbb{R} , hence is amenable. It is well known that the only invariant measures for the suspension action $((X \times \mathbb{R})/\mathcal{A}_f, \mathbb{R})$ are locally the images through the canonical projection $\pi : X \times \mathbb{R} \rightarrow (X \times \mathbb{R})/\mathcal{A}_f$ of the measures $\mu \otimes \lambda$ where μ is a f -invariant measure on X and λ denotes the Lebesgue measure of \mathbb{R} . The proof is actually contained in Property 4.4.

It is well known also that the map $o : x \mapsto x + 1$ on the dyadic set of integers Ω , admits only one invariant probability measure: the Haar probability measure on Ω . Hence the suspension of this action $((\Omega \times \mathbb{R})/\mathcal{A}_o, \mathbb{R})$ admits only one invariant probability measure. By Proposition 3.3, $X_{\mathcal{P}}^{\mathcal{N}}$ has only one invariant probability measure ν . Notice that the map R preserves $X_{\mathcal{P}}^{\mathcal{N}}$, and since $R\mathcal{N}R^{-1} = \mathcal{N}$, the probability $R_*\nu$ is \mathcal{N} -invariant and hence R preserves ν . Nevertheless, the local product decomposition of ν is not invariant by R , because R divides by 2 the length of the intervals of the \mathcal{N} -orbit. So R has to inflate the Haar measure on Ω by a factor 2.

Proposition 4.4. *Let $w \in \{1, \dots, r\}^{\mathbb{Z}}$, and let $X_{\mathcal{P}}^{\mathcal{N}}$ be the closure of the orbit $\mathcal{N} \cdot \mathcal{P}$. A finite invariant measure of $((X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times \mathbb{R}_+^*)/\mathcal{R}, G)$ is locally the image through the projection $X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times \mathbb{R}_+^* \rightarrow (X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times \mathbb{R}_+^*)/\mathcal{R}$ of a measure $\nu \otimes \mu \otimes m$ where*

- ν is the only invariant probability measure of $(X_{\mathcal{P}}^{\mathcal{N}}, \mathbb{R})$;
- m is the Haar measure of (\mathbb{R}_+^*, \cdot) ;
- μ is a finite invariant measure of (Z_w, σ) .

Proof. It is enough to prove this for an ergodic finite G -invariant measure $\bar{\theta}$ on the suspension $(X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times \mathbb{R}_+^*)/\mathcal{R}$. Since \mathcal{R} acts cocompactly on $X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times \mathbb{R}_+^*$, $\bar{\theta}$ defines a finite measure on a fundamental domain of \mathcal{R} , and the sum of all the images of this measure by \mathcal{R} defines a σ -finite measure θ on $X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times \mathbb{R}_+^*$ which is G and \mathcal{R} -invariant.

Let $\pi_2 : X_{\mathcal{P}}^{\mathcal{N}} \times Z_w \times \mathbb{R}_+^* \rightarrow Z_w$ be the projection to the second coordinate, then $\pi_{2*}\theta$ is a shift invariant measure on Z_w . The measure θ can be disintegrated over $\pi_{2*}\theta = \mu$ by a family of measures $(\lambda_{w'})_{w' \in Z_w}$ defined for μ -almost every $w' \in Z_w$ on $X_{\mathcal{P}}^{\mathcal{N}} \times \{w'\} \times \mathbb{R}_+^*$ such that

$$\theta(B \times C) = \int_C \lambda_{w'}(B) d\mu(w'),$$

for any Borel sets $B \subset X_{\mathcal{P}}^{\mathcal{N}} \times \mathbb{R}_+^*$ and $C \subset Z_w$.

The G -invariance of θ implies that the measures $\lambda_{w'}$ are G -invariant for almost all w' . The projection to the first coordinate $\pi_1 : X_{\mathcal{P}}^{\mathcal{N}} \times \{w'\} \times \mathbb{R}_+^* \rightarrow X_{\mathcal{P}}^{\mathcal{N}}$ is \mathcal{N} -equivariant. The measures $\pi_{1*}\lambda_{w'}$ are then \mathcal{N} -invariant measures, hence are proportional to ν . Each measure $\lambda_{w'}$ can be disintegrated over ν by a family of measures $(m_{x,w'})_{x \in \mathbb{R}, w' \in Z_w}$ on \mathbb{R}_+^* defined for ν almost every $x \in X_{\mathcal{P}}^{\mathcal{N}}$, so that

$$\lambda_{w'}(B \times \{w'\} \times I) = \int_B \int_I m_{x,w'} d\nu(x),$$

for any Borel sets $B \subset X_{\mathcal{P}}^{\mathcal{N}}$ and $I \subset \mathbb{R}_+^*$. Each measure $\lambda_{w'}$ is invariant under the action of maps of the kind $z \mapsto az$ for $a \in \mathbb{R}_+^*$. It is then straightforward to check

that the measures $m_{x,w'}$ are multiplication-invariant for almost every x . By unicity of the Haar measure, there exists a measurable positive function $(x, w') \mapsto h(x, w')$ defined almost everywhere so that $m_{x,w'} = h(x, w')m$. The \mathcal{N} -invariance of the measures $\lambda_{w'}$ implies that the map h is almost surely constant along the \mathcal{N} -orbits, and the \mathcal{R} -invariance of θ implies that h is almost surely constant along the \mathcal{R} -orbits. This defines then a measurable map on the quotient space by \mathcal{R} which is G -invariant, the ergodicity of $\bar{\theta}$ implies this map is almost surely constant. \square

Notice that an invariant measure on $X_{\mathcal{P}(w)}^G$ can be decomposed locally into the product of a measure on a Cantor set by a measure along the leaves. Since the map R does not preserve the transversal measure on Ω in $X_{\mathcal{P}}^{\mathcal{N}}$, the holonomy groupoid of $X_{\mathcal{P}(w)}^G$ does not preserve the transversal measure on the Cantor set.

The G -action is locally free and acts transitively on each leaf, so each orbits inherits a hyperbolic 2-manifolds structure. Actually $X_{\mathcal{P}(w)}^G$, can be equipped with a continuous metric with a constant curvature -1 in restriction to the leaves. The invariant measures of $X_{\mathcal{P}(w)}^G$ have then also a geometric interpretation in terms of *harmonic* measures, a notion introduced by L. Garnett in [7].

Definition 4.5. *A probability measure μ on M is harmonic if*

$$\int_M \Delta f d\mu = 0$$

for any continuous function $f \in C^2$ in restriction on the leaves, where Δ denotes the Laplace-Beltrami operator in restriction on each leaf.

Actually, it is shown in [13], that on $X_{\mathcal{P}(w)}^G$ the notions of harmonic and invariant measures are the same and such measures can be described in terms of inverse limit of vectoriel cones.

5. TRANSFORMATION GROUPOIDS

We gather this section with results on groupoids and their C^* -algebras. Good material on this topic can be found in [19]. Let us fix first some notations. Let \mathcal{G} be a locally compact groupoid, with base space X , range and source maps respectively $r : \mathcal{G} \rightarrow X$ and $s : \mathcal{G} \rightarrow X$. Recall that X can be viewed as a closed subset of \mathcal{G} (the set of units). For any element x of X , we set

$$\mathcal{G}^x = \{\gamma \in \mathcal{G} \text{ such that } r(\gamma) = x\}$$

and

$$\mathcal{G}_x = \{\gamma \in \mathcal{G} \text{ such that } s(\gamma) = x\}.$$

Let us denote for any γ in \mathcal{G} by $L_\gamma : \mathcal{G}^{s(\gamma)} \rightarrow \mathcal{G}^{r(\gamma)}$ the left translation by γ . Thourought this section, all the groupoids will be assumed locally compact and second countable. Recall that a Haar system λ for \mathcal{G} is a family $(\lambda^x)_{x \in X}$ of borelian measures on \mathcal{G} such that

- (1) the support of λ^x is \mathcal{G}^x ;
- (2) for any f in $C_c(\mathcal{G})$, the map $X \rightarrow \mathbb{C}; x \rightarrow \int_{\mathcal{G}^x} f d\lambda^x$ is continuous;
- (3) $L_{\gamma_*} \lambda^{s(\gamma)} = \lambda^{r(\gamma)}$ for all γ in Γ .

Our prominent examples of groupoid will be *semi-direct product groupoid*: let H be a locally compact group acting on a locally compact space X . The semi-direct product groupoid $X \rtimes H$ of X by H is defined by

- $X \times H$ as a topological space ;
- the base space is X and the structure maps are $r : X \rtimes H \rightarrow X; (x, h) \mapsto x$ and $s : X \rtimes H \rightarrow X; (x, h) \mapsto h^{-1}x$;
- the product is $(x, h) \cdot (h^{-1}x, h') = (x, hh')$ for x in X and h and h' in H .

Let μ be a left Haar measure on H . Then the groupoid $X \rtimes H$ is equipped with a Haar system $\lambda^\mu = (\lambda_x^\mu)_{x \in X}$ given for any f in $C_c(X \times H)$ and any x in X by $\lambda_x^\mu(f) = \int_H f(x, h) d\mu(h)$.

5.1. Suspension of a groupoid. Recall that any automorphism α of a groupoid \mathcal{G} induces an homeomorphism of its base space X that we shall denote by α_X .

Definition 5.1. Let \mathcal{G} be a groupoid with base space X equipped with a Haar system $\lambda = (\lambda^x)_{x \in X}$. A groupoid automorphism $\alpha : \mathcal{G} \rightarrow \mathcal{G}$ is said to preserve the Haar system λ if there exists a continuous function $\rho_\alpha : \mathcal{G} \rightarrow \mathbb{R}^+$ such that for any x in X the measures $\alpha_* \lambda^x$ and $\lambda^{\alpha(x)}$ on $\mathcal{G}^{\alpha(x)}$ are in the same class and ρ_α restricted to $\mathcal{G}^{\alpha(x)}$ is $\frac{d\alpha_* \lambda^x}{d\lambda^{\alpha(x)}}$. The map ρ_α is called the density of α .

Remark 5.2. Let \mathcal{G} be a groupoid with base space X and Haar system $\lambda = (\lambda^x)_{x \in X}$ and let $\alpha : \mathcal{G} \rightarrow \mathcal{G}$ be an automorphism of groupoid preserving the Haar system λ .

- (1) Since $L_\gamma \circ \alpha = \alpha \circ L_{\alpha^{-1}(\gamma)}$ for any γ in \mathcal{G} , we get that

$$L_{\gamma,*} \alpha_* \lambda^{\alpha^{-1}(s(\gamma))} = \alpha_* L_{\alpha^{-1}(\gamma),*} \lambda^{s(\alpha^{-1}(\gamma))} = \alpha_* \lambda^{r(\alpha^{-1}(\gamma))}.$$

Since $L_{\gamma,*} \alpha_* \lambda^{\alpha^{-1}(s(\gamma))}$ is a measure on $\mathcal{G}^{r(\gamma)}$ absolutely continuous with respect to $L_{\gamma,*} \lambda^{s(\gamma)} = \lambda^{r(\gamma)}$ with density $\rho_\alpha \circ L_{\gamma^{-1}}$ we see that $\rho_\alpha \circ L_{\gamma^{-1}}$ and ρ_α coincide on $\mathcal{G}^{r(\gamma)}$. In particular ρ_α is constant on \mathcal{G}_x for any x in X .

- (2) The automorphism of groupoid $\alpha^{-1} : \mathcal{G} \rightarrow \mathcal{G}$ also preserves the Haar system λ and $\rho_{\alpha^{-1}} = 1/\rho_\alpha \circ \alpha$.

Definition 5.3. Let \mathcal{G} be a groupoid with base space X , range and source map r and s and let $\alpha : \mathcal{G} \rightarrow \mathcal{G}$ be a groupoid automorphism. Using the notations of section 3.2 the suspension of the groupoid \mathcal{G} respectively to α is the groupoid $\mathcal{G}_\alpha \stackrel{\text{def}}{=} (\mathcal{G} \times \mathbb{R}) / \mathcal{A}_\alpha$ with base space $X_\alpha \stackrel{\text{def}}{=} (X \times \mathbb{R}) / \mathcal{A}_{\alpha_X}$. For any γ in \mathcal{G} and t in \mathbb{R} , let us denote by $[\gamma, t]$ the class of (γ, t) in \mathcal{G}_α .

- The range map r_α and the source map s_α are defined in the following way:
 - $r_\alpha([\gamma, t]) = [r(\gamma), t]$ for every γ in \mathcal{G} and t in \mathbb{R} ;
 - $s_\alpha([\gamma, t]) = [s(\gamma), t]$ for every γ in \mathcal{G} and t in \mathbb{R} ;
- Let γ and γ' be elements of \mathcal{G} such that $s(\gamma) = r(\gamma')$ and let t be in \mathbb{R} , then $[\gamma, t] \circ [\gamma', t] = [\gamma \circ \gamma', t]$;
- $[\gamma, t]^{-1} = [\gamma^{-1}, t]$.

There is an action of \mathbb{R} on \mathcal{G}_α by automorphisms given for s in \mathbb{R} and $[\gamma, t]$ in \mathcal{G}_α by $s \cdot [\gamma, t] = [\gamma, t + s]$.

Lemma 5.4. Let \mathcal{G} be a groupoid with base space X equipped with a Haar system $\lambda = (\lambda^x)_{x \in X}$ and let $\alpha : \mathcal{G} \rightarrow \mathcal{G}$ be an automorphism preserving the Haar System λ . Let us assume that $\rho_\alpha = \rho_\alpha \circ \alpha$. Then \mathcal{G}_α admits a Haar system $\lambda_\alpha = \left(\lambda_\alpha^{[x, t]} \right)_{[x, t] \in X_\alpha}$ given for any $[x, t]$ in X_α and any continuous fonction f in $C_c \left(\mathcal{G}_\alpha^{[x, t]} \right)$ by

$$\lambda_\alpha^{[x, t]}(f) = \int_{\mathcal{G}^x} \rho_\alpha(\gamma)^{-t} f([\gamma, t]) d\lambda^x(\gamma).$$

Proof.

- Let us prove first that the definition of $\lambda_\alpha^{[x,t]}(f)$ for $[x,t]$ in X_α and f in $C_c(\mathcal{G}_\alpha^{[x,t]})$ makes sense.

$$\begin{aligned} \int_{\mathcal{G}^x} \rho_\alpha(\gamma)^{-t} f([\gamma, t]) d\lambda^x(\gamma) &= \int_{\mathcal{G}^x} \rho_\alpha(\alpha(\gamma))^{-t} f([\alpha^{-1}(\alpha(\gamma), t]) d\lambda^x(\gamma) \\ &= \int_{\mathcal{G}^{\alpha(x)}} \rho_\alpha(\gamma)^{-t+1} f([\alpha^{-1}(\gamma), t]) d\lambda^{\alpha(x)}(\gamma) \\ &= \int_{\mathcal{G}^{\alpha(x)}} \rho_\alpha(\gamma)^{-t+1} f([\gamma, t-1]) d\lambda^{\alpha(x)}(\gamma) \end{aligned}$$

- It is clear that the continuity condition is fulfilled. Let us show then that $(\lambda^{[x,t]})_{[x,t] \in X_\alpha}$ is a Haar system. Let γ' be an element of \mathcal{G} , let t be a real number and let f be a function in $C_c(\mathcal{G}_\alpha^{[r(\gamma'), t]})$. Then

$$\begin{aligned} \lambda_\alpha^{[r(\gamma'), t]}(f) &= \int_{\mathcal{G}^{r(\gamma')}} \rho_\alpha(\gamma)^{-t} f([\gamma, t]) d\lambda^{r(\gamma')}(\gamma) \\ &= \int_{\mathcal{G}^{s(\gamma')}} \rho_\alpha(\gamma' \cdot \gamma)^{-t} f([\gamma' \cdot \gamma, t]) d\lambda^{s(\gamma')}(\gamma) \\ &= \int_{\mathcal{G}^{s(\gamma')}} \rho_\alpha(\gamma)^{-t} f([\gamma' \cdot \gamma, t]) d\lambda^{s(\gamma')}(\gamma) \\ &= \lambda_\alpha^{[s(\gamma'), t]}(f \circ L_{[\gamma', t]}), \end{aligned}$$

where the third equality holds in view of remark 5.2. □

5.2. C^* -algebra of a suspension groupoid. Let us recall first the construction of the reduced C^* -algebra $C_r^*(\mathcal{G}, \lambda)$ associated to a groupoid \mathcal{G} with base X and Haar system $\lambda = (\lambda^x)_{x \in X}$. Let $\mathcal{L}^2(\mathcal{G}, \lambda)$ be the $C_0(X)$ -Hilbert completion of $C_c(\mathcal{G})$ equipped with the $C_0(X)$ -valued scalar product

$$\langle \phi, \phi' \rangle(x) = \int_{\mathcal{G}^x} \bar{\phi}(\gamma^{-1}) \phi'(\gamma^{-1}) d\lambda^x(\gamma)$$

for ϕ and ϕ' in $C_c(\mathcal{G})$ and x in X , i.e the completion of $C_c(\mathcal{G})$ with respect to the norm $\|\phi\| = \sup_{x \in X} \langle \phi, \phi \rangle^{1/2}$. The $C_0(X)$ -module structure on $C_c(\mathcal{G})$ extends to $\mathcal{L}^2(\mathcal{G}, \lambda)$ and $\langle \bullet, \bullet \rangle$ extends to a $C_0(X)$ -valued scalar product on $\mathcal{L}^2(\mathcal{G}, \lambda)$. Recall that an operator $T : \mathcal{L}^2(\mathcal{G}, \lambda) \rightarrow \mathcal{L}^2(\mathcal{G}, \lambda)$ is called adjointable if there exists an operator $T^* : \mathcal{L}^2(\mathcal{G}, \lambda) \rightarrow \mathcal{L}^2(\mathcal{G}, \lambda)$, called the adjoint of T such that

$$\langle T^* \phi, \phi' \rangle = \langle \phi, T \phi' \rangle$$

for all ϕ and ϕ' in $\mathcal{L}^2(\mathcal{G}, \lambda)$. Notice that the adjoint, when it exists is unique and that operator that admits an adjoint are automatically $C_0(X)$ -linear and continuous. The set of adjointable operators on $\mathcal{L}^2(\mathcal{G}, \lambda)$ is then a C^* -algebra with respect to the operator norm. Then any f in $C_c(\mathcal{G})$ acts as an adjointable operator on $\mathcal{L}^2(\mathcal{G}, \lambda)$ by convolution

$$f \cdot \phi(\gamma) = \int_{\mathcal{G}^{r(\gamma)}} f(\gamma') \phi(\gamma'^{-1} \gamma) d\lambda^{r(\gamma)}(\gamma')$$

where ϕ is in $C_c(\mathcal{G})$, the adjoint of this operator being given by the action of $f^* : \gamma \mapsto \bar{f}(\gamma^{-1})$. The convolution product provides a structure of involutive algebra on $C_c(\mathcal{G})$ and using the action defined above, this algebra can be viewed as a subalgebra of the C^* -algebra of adjointable operators of $\mathcal{L}^2(\mathcal{G}, \lambda)$. The reduced C^* -algebra $C_r^*(\mathcal{G}, \lambda)$ is then the closure of $C_c(\mathcal{G})$ in the C^* -algebra of adjointable operators of $\mathcal{L}^2(\mathcal{G}, \lambda)$. Namely, if we define for x in X the measure on \mathcal{G}_x by $\lambda_x(\phi) = \int_{\mathcal{G}_x} \phi(\gamma^{-1}) d\lambda^x(\gamma)$ for any ϕ in $C_c(\mathcal{G}_x)$, then $\mathcal{L}^2(\mathcal{G}, \lambda)$ is a continuous field of Hilbert spaces over X with fiber $\mathcal{L}^2(\mathcal{G}_x, \lambda_x)$ at x in X . The corresponding $C_0(X)$ -structure on $C_r^*(\mathcal{G}, \lambda)$ is then given for h in $C_0(X)$ by the multiplication by $h \circ s$.

Example 5.5. *Let H be a locally compact group acting on a locally compact space X , and consider the semi-direct product groupoid $X \rtimes H$ equipped with a Haar system arising from the Haar measure on H . Then $C^*(X \rtimes H, \lambda^\mu)$ is the usual reduced cross-product $C_0(X) \rtimes H$.*

Let us denote for any x in X by ν_x the representation of $C_r^*(\mathcal{G}, \lambda)$ on the fiber $\mathcal{L}^2(\mathcal{G}_x, \lambda_x)$. Then for any f in $C_r^*(\mathcal{G}, \lambda)$, we get that $\|f\|_{C_r^*(\mathcal{G}, \lambda)} = \sup_{x \in X} \|\nu_x(f)\|$.

Lemma 5.6. *Let \mathcal{G} be a locally compact groupoid with base space X equipped with a Haar system $\lambda = (\lambda^x)_{x \in X}$ and let $\alpha : \mathcal{G} \rightarrow \mathcal{G}$ be an automorphism preserving the Haar System λ . Let us define the continuous map $\rho'_\alpha : \mathcal{G} \rightarrow \mathbb{R}; \gamma \mapsto \rho_\alpha(\gamma^{-1})$. Then there exists a unique automorphism $\tilde{\alpha}$ of the C^* -algebra $C_r^*(\mathcal{G}, \lambda)$ such that for every f in $C_c(\mathcal{G})$ we have $\tilde{\alpha}(f) = (\rho'_\alpha \rho_\alpha)^{1/2} f \circ \alpha^{-1}$.*

Proof. The map $C_c(\mathcal{G}) \rightarrow C_c(\mathcal{G}); \phi \mapsto \rho_\alpha^{1/2} \phi \circ \alpha^{-1}$ extends to a continuous linear and invertible map $W : \mathcal{L}^2(\mathcal{G}, \lambda) \rightarrow \mathcal{L}^2(\mathcal{G}, \lambda)$ such that

$$\langle W \cdot \phi, W \cdot \phi \rangle(x) = \langle \phi, \phi \rangle(\alpha^{-1}(x)),$$

for all x in X . Its inverse W^{-1} is defined by $W^{-1}(\phi) = (\rho'_\alpha \circ \alpha)^{-1/2} \phi \circ \alpha$ for all ϕ in $C_c(\mathcal{G})$. Let us define

$$\tilde{\alpha} : C_r^*(\mathcal{G}, \lambda) \rightarrow C_r^*(\mathcal{G}, \lambda); x \mapsto W \cdot x \cdot W^{-1}.$$

Then $W \cdot f \cdot W^{-1} = (\rho'_\alpha \rho_\alpha)^{1/2} f \circ \alpha^{-1}$ for all f in $C_c(\mathcal{G})$. \square

Recall that if A is a C^* -algebra and if β is an automorphism of A then the mapping torus of A is the C^* -algebra

$$A_\beta = \{f \in C([0, 1], A) \text{ such that } \beta(f(1)) = f(0)\}.$$

Namely, the mapping torus A_β can be viewed as the algebra of continuous function $h : \mathbb{R} \rightarrow A$ such that $h(t) = \beta(h(t+1))$ for all t in \mathbb{R} . In this picture, there is an action of \mathbb{R} on A_β by translations defined for t in \mathbb{R} and f in A_β by

$$t \cdot f(s) = f(s - t)$$

for any s in \mathbb{R} . Translations then define a strongly continuous action by automorphisms $\hat{\beta}$ of \mathbb{R} on A_β . By the mapping torus isomorphism, we have a natural Morita equivalence between $A \rtimes_\beta \mathbb{Z}$ and $A \rtimes_{\hat{\beta}} \mathbb{R}$.

Let α be an automorphism of a groupoid \mathcal{G} preserving a Haar system λ and with density ρ_α . For a function f in $C_c(\mathcal{G}_\alpha)$, we define \hat{f} in $C_c([0, 1] \times \mathcal{G}) \subset C([0, 1], C_r^*(\mathcal{G}, \lambda))$ by $\hat{f}(t, \gamma) = \rho_\alpha^{-t/2}(\gamma) \rho_\alpha^{t/2}(\gamma) f([\gamma, t])$. We can check easily that \hat{f} belongs to the mapping torus $C_r^*(\mathcal{G}, \lambda)_{\tilde{\alpha}}$.

Proposition 5.7. *Let \mathcal{G} be a locally compact groupoid with base space X equipped with a Haar system $\lambda = (\lambda^x)_{x \in X}$ and let $\alpha : \mathcal{G} \rightarrow \mathcal{G}$ be an automorphism preserving the Haar System λ such that $\rho_\alpha \circ \alpha = \rho_\alpha$. Then there is a unique automorphism of C^* -algebras*

$$\Lambda_\alpha : C_r^*(\mathcal{G}_\alpha, \lambda_\alpha) \rightarrow C_r^*(\mathcal{G}, \lambda)_{\tilde{\alpha}}$$

such that $\Lambda_\alpha(f) = \hat{f}$ for any f in $C_c(\mathcal{G}_\alpha)$.

Proof. Let f be a function of $C_c(\mathcal{G}_\alpha)$. Then

$$\begin{aligned} \|\hat{f}\|_{C_r^*(\mathcal{G}, \lambda)_{\tilde{\alpha}}} &= \sup_{t \in [0, 1]} \|\hat{f}(t, \bullet)\|_{C_r^*(\mathcal{G}, \lambda)} \\ &= \sup_{t \in [0, 1], x \in X} \|\nu_x(\hat{f}(t, \bullet))\| \end{aligned}$$

On the other hand,

$$\|f\|_{C_r^*(\mathcal{G}_\alpha, \lambda_\alpha)} = \sup_{t \in [0, 1], x \in X} \|\nu_{[x, t]}(f)\|,$$

where $\nu_{[x, t]}$ is the representation of $C_r^*(\mathcal{G}_\alpha, \lambda_\alpha)$ on the fiber $\mathcal{L}^2(\mathcal{G}_{\alpha, [x, t]}, \lambda_{\alpha, [x, t]})$ at $[x, t] \in (X \times \mathbb{R})/\mathcal{A}_\alpha$. If we define for t in $[0, 1]$ the map $\pi_t : \mathcal{G} \rightarrow \mathcal{G}_\alpha : \gamma \mapsto [\gamma, t]$, then

$$C_c(\mathcal{G}_{[x, t]}) \rightarrow C_c(\mathcal{G}_x) : \phi \mapsto \rho_\alpha^{-t/2} \phi \circ \pi_t$$

extends to an isometry $W_t : \mathcal{L}^2(\mathcal{G}_{\alpha, [x, t]}, \lambda_{\alpha, [x, t]}) \rightarrow \mathcal{L}^2(\mathcal{G}_x, \lambda_x)$ and W_t conjugate $\nu_{[x, t]}(f)$ and $\nu_x(\hat{f}(t, \bullet))$. Thus $\|\hat{f}\|_{C_r^*(\mathcal{G}, \lambda)_{\tilde{\alpha}}} = \|f\|_{C_r^*(\mathcal{G}_\alpha, \lambda_\alpha)}$ and

$$C_c(\mathcal{G}_\alpha) \rightarrow C_r^*(\mathcal{G}, \lambda)_{\tilde{\alpha}}; f \mapsto \hat{f}$$

extends to a monomorphism $\Lambda_\alpha : C_r^*(\mathcal{G}_\alpha, \lambda_\alpha) \rightarrow C_r^*(\mathcal{G}, \lambda)_{\tilde{\alpha}}$. The set

$$\mathcal{A}_\alpha = \{h \in C_c([0, 1] \times \mathcal{G}) \text{ such that } h(1, \alpha(\gamma)) = \rho_\alpha^{t/2} \rho_\alpha^{-t/2} h(0, \gamma) \text{ for all } \gamma \in \Gamma\}$$

is dense in $C_r^*(\mathcal{G}, \lambda)_{\tilde{\alpha}}$. Let us define for an element h of \mathcal{A}_α the map $\tilde{h} : \mathcal{G} \times \mathbb{R} \rightarrow \mathbb{C}$ as the unique map such that

- $\tilde{h}(\gamma, t) = \rho_\alpha^{-t/2} \rho_\alpha^{-t/2} h(t, \gamma)$ for all γ in \mathcal{G} and t in $[0, 1]$;
- $h(\alpha(\gamma), t) = h(\gamma, t + 1)$ for all γ in \mathcal{G} and t in \mathbb{R} .

Then \tilde{h} defines a continuous map of $C_c(\mathcal{G}_\alpha)$ whose image under Λ_α is h . Hence Λ_α has dense range in $C_r^*(\mathcal{G}, \lambda)_{\tilde{\alpha}}$ and thus is surjective. \square

Remark 5.8. *With the notations of above proposition, let us define for a real s the automorphism of groupoid $\theta_s : \mathcal{G}_\alpha \rightarrow \mathcal{G}_\alpha; [\gamma, t] \mapsto [\gamma, s + t]$. Then θ_s is preserving the Haar system $\lambda_\alpha = (\lambda^{[x, t]})_{[x, t] \in X_\alpha}$ with density*

$$\mathcal{G}_\alpha \rightarrow \mathbb{R}; [\gamma, t] \mapsto \rho_\alpha(\gamma)^s.$$

We obtain from lemma 5.6 an automorphism $\tilde{\theta}_s$ of $C_r^*(\mathcal{G}_\alpha, \lambda_\alpha)$ which gives rise to a strongly continuous action of \mathbb{R} on $C_r^*(\mathcal{G}_\alpha, \lambda_\alpha)$ by automorphism. The isomorphism

$$\Lambda_\alpha : C_r^*(\mathcal{G}_\alpha, \lambda_\alpha) \rightarrow C_r^*(\mathcal{G}, \lambda)_{\tilde{\alpha}}$$

of proposition 5.7 is then \mathbb{R} -equivariant, where the action of \mathbb{R} on $C_r^*(\mathcal{G}, \lambda)_{\tilde{\alpha}}$ is the action $\hat{\alpha}$ associated to a mapping torus.

6. THE DYNAMIC OF THE UNCOLORED PENROSE TILING UNDER TRANSLATIONS

As we have seen before, the continuous hull $X_{\mathcal{P}}^{\mathbb{R}}$ of the action of the uncolored Penrose tiling \mathcal{P} under the actions of \mathbb{R} by translations is the suspension $\frac{\Omega \times \mathbb{R}}{\mathcal{A}_o}$ of the odometer homeomorphism $o : \Omega \rightarrow \Omega; x \mapsto x + 1$, where Ω is the dyadic completion of the integers. The \mathbb{R} -algebra $C(\frac{\Omega \times \mathbb{R}}{\mathcal{A}_o})$ is then the mapping torus algebra of $C(\Omega)$ with respect to automorphism induced by o . In consequence, the crossed-product algebras $C(X_{\mathcal{P}}^{\mathbb{R}}) \rtimes \mathbb{R}$ and $C(\Omega) \rtimes \mathbb{Z}$ are Morita equivalent. The purpose of this section is to recall the explicit description of the isomorphism $C(\Omega) \rtimes \mathbb{Z} \xrightarrow{\cong} C(X_{\mathcal{P}}^{\mathbb{R}}) \rtimes \mathbb{R}$ arising from this Morita equivalence.

For this, let us define on $C_c(\Omega \times \mathbb{R})$ the $C(\Omega) \rtimes \mathbb{Z}$ -valued inner product

$$\langle \xi, \xi' \rangle(\omega, k) = \int_{\mathbb{R}} \bar{\xi}(\omega, s) \xi'(\omega - k, s + k) ds$$

for ξ and ξ' in $C_c(\Omega \times \mathbb{R})$ and (ω, k) in $(\Omega \times \mathbb{R})$. This inner product is positive and gives rise to a right $C(\Omega) \rtimes \mathbb{Z}$ -Hilbert module \mathcal{E} , the action of $C(\Omega) \rtimes \mathbb{Z}$ being given for h in $C_c(\Omega \times \mathbb{Z})$ and ξ in $C_c(\Omega \times \mathbb{R})$ by

$$\xi \cdot h(\omega, t) = \sum_{n \in \mathbb{Z}} \xi(n + \omega, t - n) h(n + \omega, n)$$

for (ω, k) in $\Omega \times \mathbb{R}$. The right $C(\Omega) \rtimes \mathbb{Z}$ -Hilbert module \mathcal{E} is also equipped with a left action of $C(\frac{\Omega \times \mathbb{R}}{\mathcal{A}_o}) \rtimes \mathbb{R}$ given for f in $C_c(\frac{\Omega \times \mathbb{R}}{\mathcal{A}_o} \times \mathbb{R})$ and ξ in $C_c(\Omega \times \mathbb{R})$ by

$$f \cdot \xi(\omega, t) = \int_{\mathbb{R}} f([\omega, t], s) \xi(\omega, t - s) ds$$

for (ω, k) in $\Omega \times \mathbb{R}$. We get in this way a $C(\frac{\Omega \times \mathbb{R}}{\mathcal{A}_o}) \rtimes \mathbb{R} - C(\Omega) \rtimes \mathbb{Z}$ imprimitivity bimodule which implements the Morita equivalence we are looking for. Actually, there is an isomorphism of right $C(\Omega) \rtimes \mathbb{Z}$ -Hilbert module

$$\Psi : \mathcal{E} \rightarrow L^2([0, 1]) \otimes C(\Omega) \rtimes \mathbb{Z}$$

defined in a unique way by $\Psi(g) = g \otimes \delta_0$ for g in $C_c(\mathbb{R})$ supported in $(0, 1)$ (here δ_0 is the Dirac function at $0 \in \mathbb{Z}$). Using the right $C(\frac{\Omega \times \mathbb{R}}{\mathcal{A}_o}) \rtimes \mathbb{R}$ -module structure of the $C(\frac{\Omega \times \mathbb{R}}{\mathcal{A}_o}) \rtimes \mathbb{R} - C(\Omega) \rtimes \mathbb{Z}$ imprimitivity bimodule \mathcal{E} and the isomorphism Ψ , we get an isomorphism

$$(6.1) \quad C\left(\frac{\Omega \times \mathbb{R}}{\mathcal{A}_o}\right) \rtimes \mathbb{R} \xrightarrow{\cong} \mathcal{K}(L^2([0, 1])) \otimes C(\Omega) \rtimes \mathbb{Z}.$$

This isomorphism can be described as follows. Let us define for f and g in $L^2([0, 1])$ the rank one operator

$$\Theta_{f,g} : L^2([0, 1]) \rightarrow L^2([0, 1]); h \mapsto f \langle g, h \rangle.$$

We define for ξ and ξ' in $C_c(\Omega \times \mathbb{R})$ the continuous function of $C_c(\frac{\Omega \times \mathbb{R}}{\mathcal{A}_o} \times \mathbb{R})$

$$\Theta_{\xi, \xi'}^{\Omega}([\omega, s], t) = \sum_{k \in \mathbb{Z}} \xi(\omega + k, s - k) \bar{\xi}'(\omega + k, s - t - k)$$

for all ω in Ω and s and t in \mathbb{R} . It is straightforward to check that $\Theta_{\xi, \xi'}^{\Omega}$ is well defined and that

$$\Theta_{\xi, \xi'}^{\Omega} \cdot \eta = \xi \langle \xi', \eta \rangle$$

for all η in $C_c(\Omega \times \mathbb{R})$. If we set for f and g in $C_c(\mathbb{R})$ with support in $(0, 1)$ and for ϕ in $C(\Omega)$, $\xi = 1 \otimes f$, $\xi' = \phi \otimes g$ and $\xi'' : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$; $(\omega, t) \mapsto g(t+1)$, then the image of $\Theta_{\xi, \xi'}^\Omega$ under the isomorphism of equation (6.1) is $\Theta_{f, g} \otimes \phi \in \mathcal{K}(L^2([0, 1])) \otimes C(\Omega) \rtimes \mathbb{Z}$ and moreover,

$$(6.2) \quad \Theta_{\xi, \xi'}^\Omega([\omega, s], t) = \sum_{k \in \mathbb{Z}} f(s-k) \bar{\phi}(\omega+k) \bar{g}(s-t-k).$$

Let us denote by u the unitary of $C(\Omega) \rtimes \mathbb{Z}$ corresponding to the positive generator of \mathbb{Z} . Then the image of $\Theta_{\xi, \xi''}^\Omega$ under the isomorphism of equation (6.1) is $\Theta_{f, g} \otimes u \in \mathcal{K}(L^2([0, 1])) \otimes C(\Omega) \rtimes \mathbb{Z}$ and moreover,

$$(6.3) \quad \Theta_{\xi, \xi''}^\Omega([\omega, s], t) = \sum_{k \in \mathbb{Z}} f(s-k) \bar{g}(s+1-t-k).$$

Let us define the automorphism α of the groupoid $\frac{\Omega \times \mathbb{R}}{\mathbb{Z}} \rtimes \mathbb{R}$ in the following way

- $\alpha([\omega, s], t) = ([\omega/2, s/2], t/2)$ if ω is even;
- $\alpha([\omega, s], t) = ([(\omega+1)/2, (s+1)/2], t/2)$ if ω is odd.

Notice that $\alpha^{-1}([\omega, s], t) = ([2\omega, 2s], 2t)$ for all ω in Ω and s and t in \mathbb{R} . Then α preserves the Haar system of $\frac{\Omega \times \mathbb{R}}{\mathbb{Z}} \rtimes \mathbb{R}$ arising from the Haar measure on \mathbb{R} and has constant density $\rho_\alpha = 2$. Hence according to lemma 5.6, the automorphism of groupoid α induces an automorphism $\tilde{\alpha}$ of C^* -algebra $C(\frac{\Omega \times \mathbb{R}}{\mathbb{Z}}) \rtimes \mathbb{R}$ such that $\tilde{\alpha}(h) = 2h \circ \alpha^{-1}$ for all h in $C(\frac{\Omega \times \mathbb{R}}{\mathbb{Z}} \times \mathbb{R})$. We are now in position to describe how $\tilde{\alpha}$ is transported under the isomorphism of equation (6.1) to an automorphism Υ of $\mathcal{K}(L^2([0, 1])) \otimes C(\Omega) \rtimes \mathbb{Z}$. With ξ , ξ' and ξ'' as defined above,

$$(6.4) \quad \begin{aligned} \tilde{\alpha}(\Theta_{\xi, \xi'}^\Omega)([\omega, s], t) &= 2\Theta_{\xi, \xi'}^\Omega([2\omega, 2s], 2t) \\ &= 2 \sum_{k \in \mathbb{Z}} f(2s-k) \bar{\phi}(2\omega+k) \bar{g}(2s-2t-k) \\ &= 2 \sum_{k \in \mathbb{Z}} f(2s-2k) \bar{\phi}(2\omega+2k) \bar{g}(2s-2t-2k) + \\ &\quad 2 \sum_{k \in \mathbb{Z}} f(2s-2k-1) \bar{\phi}(2\omega+2k-1) \bar{g}(2s-2t-2k-1) \end{aligned}$$

and

$$(6.5) \quad \begin{aligned} \tilde{\alpha}(\Theta_{\xi, \xi''}^\Omega)([\omega, s], t) &= 2\Theta_{\xi, \xi''}^\Omega([2\omega, 2s], 2t) \\ &= 2 \sum_{k \in \mathbb{Z}} f(2s-k) \bar{g}(2s+1-2t-k) \\ &= 2 \sum_{k \in \mathbb{Z}} f(2s-2k) \bar{g}(2s+1-2t-2k) + \\ &\quad 2 \sum_{k \in \mathbb{Z}} f(2s-2k-1) \bar{g}(2s-2t-2k). \end{aligned}$$

To complete the description of the automorphism Υ of $\mathcal{K}(L^2([0, 1])) \otimes C(\Omega) \rtimes \mathbb{Z}$ corresponding to $\tilde{\alpha}$, we need to introduce some further notations. We define the partial isometries U_0, U_1 and V of $L^2([0, 1])$ by

- $U_0 f(t) = \sqrt{2} f(2t)$ if $t \in [0, 1/2]$ and $U_0 f(t) = 0$ otherwise;
- $U_1 f(t) = \sqrt{2} f(2t-1)$ if $t \in [1/2, 1]$ and $U_1 f(t) = 0$ otherwise;

- $Vf(t) = f(t + 1/2)$ if $t \in [0, 1/2]$ and $Vf(t) = 0$ otherwise,

for f in $C([0, 1])$. Let us define also the endomorphisms W_0 and W_1 of the C^* -algebra $C(\Omega)$ by $W_0\phi(\omega) = \phi(2\omega)$ and $W_1\phi(\omega) = \phi(2\omega + 1)$, for ϕ in $C(\Omega)$ and ω in Ω . Using this notations, equations (6.4) and (6.5) can be rewritten

$$\tilde{\alpha}(\Theta_{\xi, \xi'}^\Omega)([\omega, s], t) = \sum_{k \in \mathbb{Z}} U_0 f(s-k) W_0 \bar{\phi}(\omega+k) U_0 \bar{g}(s-t-k) + \sum_{k \in \mathbb{Z}} U_1 f(s-k) W_1 \bar{\phi}(\omega+k) U_1 \bar{g}(s-t-k)$$

and

$$\tilde{\alpha}(\Theta_{\xi, \xi''}^\Omega)([\omega, s], t) = \sum_{k \in \mathbb{Z}} U_0 f(s-k) U_1 \bar{g}(s-t-k+1) + \sum_{k \in \mathbb{Z}} U_1 f(s-k) U_0 \bar{g}(s-t-k).$$

Thus, in view of equations (6.2) and (6.3), we get that

$$\Upsilon(\Theta_{f,g} \otimes \phi) = \Theta_{U_0 f, U_0 g} \otimes W_0 \phi + \Theta_{U_1 f, U_1 g} \otimes W_1 \phi$$

and

$$\Upsilon(\Theta_{f,g} \otimes u) = \Theta_{U_0 f, U_1 g} \otimes u + \Theta_{U_1 f, U_0 g} \otimes 1.$$

From this we deduce

$$\Upsilon(k \otimes \phi) = U_0 \cdot k \cdot U_0^* \otimes W_0 \phi + U_1 \cdot k \cdot U_1^* \otimes W_1 \phi$$

and

$$\begin{aligned} \Upsilon(k \otimes u) &= U_0 \cdot k \cdot U_1^* \otimes u + U_1 \cdot k \cdot U_0^* \otimes 1 \\ &= U_0 \cdot k \cdot U_0^* \cdot V \otimes u + U_1 \cdot k \cdot U_1^* \cdot V^* \otimes 1 \\ &= (U_0 \cdot k \cdot U_0^* + U_1 \cdot k \cdot U_1^*) \cdot (V \otimes u + V^* \otimes 1) \end{aligned}$$

where the second equality holds since $V^* \cdot U_0 = U_1$ and $V \cdot U_1 = U_0$ and the third holds since $V^* U_1 = V U_0 = 0$. In consequence, if we extend Υ to the multiplier algebra of $\mathcal{K}(L^2([0, 1])) \otimes C(\Omega) \rtimes \mathbb{Z}$, we finally obtain that the automorphism Υ is the unique homomorphism of C^* -algebra such that

$$\Upsilon(k \otimes \phi) = U_0^* \cdot k \cdot U_0 \otimes W_0 \phi + U_1^* \cdot k \cdot U_1 \otimes W_1 \phi$$

and

$$(6.6) \quad \Upsilon(1 \otimes u) = V \otimes u + V^* \otimes 1,$$

where k is in $\mathcal{K}(L^2([0, 1]))$, ϕ is in $C(\Omega)$ and $1 \otimes u$ and $V \otimes u + V^* \otimes 1$ are viewed as multipliers of $\mathcal{K}(L^2([0, 1])) \otimes C(\Omega) \rtimes \mathbb{Z}$.

The following lemma will be helpful to compute the K -theory of the C^* -algebra of the Penrose hyperbolic tiling. For short, we will denote from now $\mathcal{K}(L^2([0, 1]))$ by \mathcal{K} .

Lemma 6.1. *Let A be the unitarisation of $\mathcal{K} \otimes C(\Omega) \rtimes \mathbb{Z}$ and let f be a norm one function of $L^2([0, 1])$. Then the unitaries*

$$(1 - \Theta_{f,f} \otimes 1) + \Theta_{f,f} \otimes u$$

and

$$(6.7) \quad \Theta_{U_0 f, U_1 f} \otimes u + \Theta_{U_1 f, U_0 f} \otimes 1 + 1 - \Theta_{U_0 f, U_0 f} \otimes 1 - \Theta_{U_1 f, U_1 f} \otimes 1$$

of A are homotopic.

Proof. If we set $f_0 = f$ and complete to a Hilbertian base f_0, \dots, f_n, \dots of $L^2([0, 1])$, then $U_0 f_0, \dots, U_0 f_n, \dots; U_1 f_0, \dots, U_1 f_n, \dots$ is a Hilbertian basis of $L^2([0, 1])$. In this basethe unitary of equation 6.7 can be written down

$$\left(\begin{array}{c|c} 0 & u \\ \hline 1 & 0 \\ \vdots & \vdots \\ 0 & 1 \\ \vdots & \vdots \end{array} \right)$$

which is homotopic to

$$\left(\begin{array}{c|c} u & 0 \\ \hline 0 & 1 \\ \vdots & \vdots \\ 0 & 1 \\ \vdots & \vdots \end{array} \right).$$

All unitaries that can be written down in such way in some hilbertian basis of $L^2([0, 1])$ are homotopic and since this is the case for $1 - \Theta_{f,f} \otimes 1 + \Theta_{f,f} \otimes u$, we get the result. \square

7. THE C^* -ALGEBRA OF A PENROSE HYPERBOLIC TILING

Let us consider the semi-direct product groupoid $\mathcal{G} = (X_{\mathcal{P}}^{\mathbb{R}} \times Z_w) \rtimes \mathbb{R}$ corresponding to the diagonal action of \mathbb{R} on $X_{\mathcal{P}}^{\mathbb{R}} \times Z_w$, by translation on $X_{\mathcal{P}}^{\mathbb{R}}$ and trivial on Z_w . Let us denote by $\lambda = (\lambda_{(\mathcal{P}', \omega)})_{(\mathcal{P}', \omega) \in X_{\mathcal{P}}^{\mathbb{R}} \times Z_w}$ the Haar system provided by the left Haar mesure on \mathbb{R} . Let us define the groupoid automorphism $\alpha_w : \mathcal{G} \rightarrow \mathcal{G}; (\mathcal{P}', w', t) \mapsto (R \cdot \mathcal{P}', \sigma(w'), 2t)$. Then α_w preserves the Haar system λ with constant density $\rho_{\alpha_w} = 1/2$ and thus according to lemma 5.4 the mapping torus groupoid \mathcal{G}_{α_w} admits a Haar system λ_{α_w} . The semi-direct product groupoid $X_{\mathcal{P}(w)}^G \rtimes \mathbb{R}$, where \mathbb{R} acts on $X_{\mathcal{P}(w)}^G$ by translations, is equipped with an action of \mathbb{R} by automorphisms $\beta_t : X_{\mathcal{P}(w)}^G \rtimes \mathbb{R} \rightarrow X_{\mathcal{P}(w)}^G \rtimes \mathbb{R}; (\mathcal{T}, s) \mapsto (2^t \cdot \mathcal{T}, 2^t s)$ for any t in \mathbb{R} . The automorphism β_t preserves the Haar system with constant density 2^{-t} and thus in view of proposition 5.7 induced a strongly continuous action of \mathbb{R} on the C^* -algebra of the semi-direct groupoid \mathcal{G} .

Lemma 7.1. *Let w be an element of $\{1, \dots, r\}^{\mathbb{Z}}$. Then there is a unique isomorphism of groupoids $\Phi_w : \mathcal{G}_{\alpha_w} \rightarrow X_{\mathcal{P}(w)}^G \rtimes \mathbb{R}$ such that:*

- (1) $\Phi_w([\mathcal{P} + x, w, y, 0]) = (\mathcal{P}(w) + x, y)$ for all x and y in \mathbb{R} ;
- (2) Φ_w is equivariant with respect to the actions of \mathbb{R} ;
- (3) $\Phi_{w,*} \lambda_{\alpha_w}$ is the Haar system on $X_{\mathcal{P}(w)}^G \rtimes \mathbb{R}$ provided by the Haar measure on \mathbb{R} .

Proof. With notations of the proof of proposition 4.3, let us define $\mathcal{T}(w') = \overline{\Phi}([T, w, 1])$, were \mathcal{T} is in $X_{\mathcal{P}}^{\mathbb{R}}$ and w' is in $\{1, \dots, r\}^{\mathbb{Z}}$. Then the map

$$X_{\mathcal{P}}^{\mathbb{R}} \times Z_w \rightarrow X_{\mathcal{P}(w)}^G; (\mathcal{T}, w') \mapsto \mathcal{T}(w')$$

is continuous and since $(R \cdot \mathcal{T})(\sigma(w')) = R \cdot \mathcal{T}(w')$, the continuous map

$$\mathcal{G} \times \mathbb{R} \rightarrow X_{\mathcal{P}(w)}^G \rtimes \mathbb{R}; (\mathcal{T}, w', x, y) \mapsto (R_{2^y} \mathcal{T}(w'), 2^y x)$$

induces a continuous homomorphism of groupoids

$$\Phi_w : \mathcal{G}_{\alpha_w} \rightarrow X_{\mathcal{P}(w)}^G \rtimes \mathbb{R}.$$

This map is clearly one-to-one since the equality $R_{2^t} \mathcal{T}(w') = \mathcal{T}'(w'')$ for t in \mathbb{R} , \mathcal{T} and \mathcal{T}' in $X_{\mathcal{P}}^{\mathbb{R}}$ and w'' and w' in Z_w holds if and only if t is integer, $w'' = \sigma^t(w')$ and $R_{2^t} \mathcal{T} = \mathcal{T}'$. To prove that Φ_w is onto, let us remark that any element of $X_{\mathcal{P}(w)}^G$ can be written as $R_a \mathcal{T}(w')$, with a in \mathbb{R} and \mathcal{T} in $X_{\mathcal{P}}^{\mathbb{R}}$. We get then

$$\Phi_w([\mathcal{T}, w', 2^a t, a]) = (R_{2^a} \mathcal{T}(w'), t)$$

for all t in \mathbb{R} .

It is then straightforward to check that condition (3) of the lemma is satisfied. The uniqueness of Φ_w is a consequence on one hand of its equivariance and on the other hand of the density of the \mathbb{R} -orbit of \mathcal{P} in $X_{\mathcal{P}}^{\mathbb{R}}$. \square

As a consequence of lemma 7.1, we get

Corollary 7.2. *The map*

$$C_c(X_{\mathcal{P}(w)}^G \rtimes \mathbb{R}) \rightarrow C_c(\mathcal{G}_{\alpha_w}); f \mapsto f \circ \Phi_w$$

induced a \mathbb{R} -equivariant isomorphism

$$\tilde{\Phi}_w : C_0(X_{\mathcal{P}(w)}^G) \rtimes \mathbb{R} \rightarrow C^*(\mathcal{G}_{\alpha_w}, \lambda_{\alpha_w}).$$

Proposition 7.3. *Using the notations of lemmas 5.6 and 7.1, the C^* -algebras $C(X_{\mathcal{P}(w)}^G) \rtimes G$ and $C_r^*(\mathcal{G}, \lambda) \rtimes_{\tilde{\alpha}_w} \mathbb{Z}$ are Morita equivalent.*

Proof. Recall that $G = \mathbb{R} \rtimes \mathbb{R}_+^*$, where the group (\mathbb{R}_+^*, \cdot) acts on $(\mathbb{R}, +)$ by multiplication. Iterate crossed-products leads to an isomorphism

$$C(X_{\mathcal{P}(w)}^G) \rtimes G \cong (C(X_{\mathcal{P}(w)}^G) \rtimes \mathbb{R}) \rtimes \mathbb{R}_+^*.$$

If we identify the groups $(\mathbb{R}, +)$ and (\mathbb{R}_+^*, \cdot) using the isomorphism

$$\mathbb{R} \rightarrow \mathbb{R}_+^*; t \mapsto 2^t,$$

this provides the action under consideration in lemma 7.1 of \mathbb{R} on $C(X_{\mathcal{P}(w)}^G) \rtimes \mathbb{R}$ and hence, the algebras $C(X_{\mathcal{P}(w)}^G) \rtimes G$ and $C^*(\mathcal{G}_{\alpha_w}, \lambda_{\alpha_w}) \rtimes \mathbb{R}$ are isomorphic. In view of lemma 5.7 and of remark 5.8, the C^* -algebra $C(X_{\mathcal{P}(w)}^G) \rtimes G$ is isomorphic to $C_r^*(\mathcal{G}, \lambda)_{\tilde{\alpha}_w} \rtimes \mathbb{R}$. But since $C_r^*(\mathcal{G}, \lambda)_{\tilde{\alpha}_w}$ is the mapping torus algebra with respect to the automorphism $\tilde{\alpha}_w : C_r^*(\mathcal{G}, \lambda) \rightarrow C_r^*(\mathcal{G}, \lambda)$, the crossed product C^* -algebra $C_r^*(\mathcal{G}, \lambda)_{\tilde{\alpha}_w} \rtimes \mathbb{R}$ is Morita equivalent to $C_r^*(\mathcal{G}, \lambda) \rtimes_{\tilde{\alpha}_w} \mathbb{Z}$ and hence we get the result. \square

8. THE K -THEORY OF THE C^* -ALGEBRA OF A PENROSE HYPERBOLIC TILING

Let us consider the semi-direct groupoid $\mathcal{G} = (X_{\mathcal{P}}^{\mathbb{R}} \times Z_w) \rtimes \mathbb{R}$ corresponding to the diagonal action of \mathbb{R} on $X_{\mathcal{P}}^{\mathbb{R}} \times Z_w$, by translations on $X_{\mathcal{P}}^{\mathbb{R}}$ and trivial on Z_w . According to proposition 7.3 we have an isomorphism

$$K_*(C(X_{\mathcal{P}(w)}^G) \rtimes G) \xrightarrow{\cong} K_*(C_r^*(\mathcal{G}, \lambda) \rtimes_{\tilde{\alpha}_w} \mathbb{Z})$$

induced by the Morita equivalence. In order to compute this K -theory group, need to recall some basic facts concerning the K -theory group of a crossed product of

a C^* -algebra A by an action of \mathbb{Z} provided by an automorphism θ of A . This K -theory can be computed by using the Pimsner-Voiculescu exact sequence [17]

$$\begin{array}{ccccc} K_0(A) & \xrightarrow{\theta_* - Id} & K_0(A) & \xrightarrow{\iota_*} & K_0(A \rtimes_{\theta} \mathbb{Z}) \\ \uparrow & & & & \downarrow \\ K_1(A \rtimes_{\theta} \mathbb{Z}) & \xleftarrow{\iota_*} & K_1(A) & \xleftarrow{\theta_* - Id} & K_1(A) \end{array},$$

where ι_* is the homomorphism induced in K -theory by the inclusion $\iota : A \hookrightarrow A \rtimes_{\theta} \mathbb{Z}$ and θ_* is the homomorphism in K -theory induced by θ . The vertical maps are given by the composition

$$K_*(A \rtimes_{\theta} \mathbb{Z}) \xrightarrow{\cong} K_*(A_{\theta} \rtimes_{\widehat{\theta}} \mathbb{R}) \xrightarrow{\cong} K_{*+1}(A_{\theta}) \xrightarrow{\text{ev}_*} K_{*+1}(A),$$

where

- A_{θ} is the mapping torus of A with respect to the action θ endowed, with its associated action $\widehat{\theta}$ of \mathbb{R} ;
- the first map is induced by the Morita equivalence between $A \rtimes_{\theta} \mathbb{Z}$ and $A_{\theta} \rtimes_{\widehat{\theta}} \mathbb{R}$;
- the second map is the Thom-Connes isomorphism;
- the third map is induced in K -theory by the evaluation map

$$\text{ev} : A_{\theta} \rightarrow A; f \mapsto f(0).$$

For a automorphism Ψ of an abelian group M , let us define $\text{Inv } M$ as the set of invariant elements of M and by $\text{Coinv } M = M / \{x - \Psi(x), x \in M\}$ the set of coinvariants elements. We then get a short exact sequences

$$(8.1) \quad 0 \rightarrow \text{Coinv } K_0(A) \rightarrow K_0(A \rtimes_{\theta} \mathbb{Z}) \rightarrow \text{Inv } K_1(A) \rightarrow 0$$

and

$$(8.2) \quad 0 \rightarrow \text{Coinv } K_1(A) \rightarrow K_1(A \rtimes_{\theta} \mathbb{Z}) \rightarrow \text{Inv } K_0(A) \rightarrow 0.$$

Moreover the inclusions in these exact sequence are induced by ι_* . The first step in the computation of $K_*(C_r^*(\mathcal{G}, \lambda) \rtimes_{\alpha_w} \mathbb{Z})$ is provided by next lemma, which is straightforward to prove.

Lemma 8.1. *Let Z be a Cantor set and let us denote by $C(Z, \mathbb{Z})$ the algebra of continuous and integer valued functions on Z .*

- (1) *we have an isomorphism $C(Z, \mathbb{Z}) \rightarrow K_0(C(Z)); \chi_E \mapsto [\chi_E]$.*
- (2) *$K_1(C(Z)) = \{0\}$,*

where for a compact open subset E of Z , then χ_E stands for the characteristic function of E .

Plugging $C_r^*(\mathcal{G}, \lambda) \rtimes_{\alpha_w} \mathbb{Z}$ into the short exact sequences (8.1) and (8.2), we get

$$(8.3) \quad 0 \rightarrow \text{Coinv } K_0(C_r^*(\mathcal{G}, \lambda)) \rightarrow K_0(C_r^*(\mathcal{G}, \lambda) \rtimes_{\alpha_w} \mathbb{Z}) \rightarrow \text{Inv } K_1(C_r^*(\mathcal{G}, \lambda)) \rightarrow 0$$

and

$$(8.4) \quad 0 \rightarrow \text{Coinv } K_1(C_r^*(\mathcal{G}, \lambda)) \rightarrow K_1(C_r^*(\mathcal{G}, \lambda) \rtimes_{\alpha_w} \mathbb{Z}) \rightarrow \text{Inv } K_0(C_r^*(\mathcal{G}, \lambda)) \rightarrow 0.$$

According to equation (6.1), the C^* -algebra $C_r^*(\mathcal{G}, \lambda)$ is isomorphic to $C(Z_w) \otimes \mathcal{K} \otimes C(\Omega) \rtimes \mathbb{Z}$. The K -theory of $C_r^*(\mathcal{G}, \lambda)$ can be computed by using the

Künneth formula: in view of lemma 8.1, $K_0(C(Z_w)) \cong C(Z_w, \mathbb{Z})$ is torsion free and $K_1(C(Z_w)) = \{0\}$ and by Morita equivalence, we get that

$$K_0(C_r^*(\mathcal{G}, \lambda)) \cong C(Z_w, \mathbb{Z}) \otimes K_0(C(\Omega) \rtimes \mathbb{Z})$$

and

$$K_1(C_r^*(\mathcal{G}, \lambda)) \cong C(Z_w, \mathbb{Z}) \otimes K_1(C(\Omega) \rtimes \mathbb{Z}).$$

This isomorphism, up to the Morita equivalence and to the isomorphism of equation (6.1) are implemented by the external product in K -theory and will be precisely described later on. Once again, $K_*(C(\Omega) \rtimes \mathbb{Z})$ can be computed from the short exact sequences (8.1) and (8.2), and we get, using lemma 8.1 that

$$(8.5) \quad K_0(C(\Omega) \rtimes \mathbb{Z}) \cong \text{Coinv } C(\Omega, \mathbb{Z})$$

and

$$(8.6) \quad K_1(C(\Omega) \rtimes \mathbb{Z}) \cong \text{Inv } C(\Omega, \mathbb{Z}) \cong \mathbb{Z}.$$

The isomorphism of equation (8.5) is induced by the composition

$$C(\Omega, \mathbb{Z}) \xrightarrow{\cong} K_0(C(\Omega)) \rightarrow K_0(C(\Omega) \rtimes \mathbb{Z}),$$

which factorizes through $\text{Coinv } C(\Omega, \mathbb{Z})$, where the first map is described in lemma 8.1, and the second map is induced on K -theory by the inclusion $C(\Omega) \hookrightarrow C(\Omega) \rtimes \mathbb{Z}$. In the first isomorphism of equation (8.6) the class of $[u]$ in $K_1(C(\Omega) \rtimes \mathbb{Z})$ of the unitary u of $C(\Omega) \rtimes \mathbb{Z}$ corresponding to the positive generator of \mathbb{Z} is mapped to the constant function 1 of $C(\Omega, \mathbb{Z})$.

Lemma 8.2. *Let ν be the Haar measure on Ω . Then*

- (1) $\int f d\nu$ is in $\mathbb{Z}[1/2]$ for all f in $C(\Omega, \mathbb{Z})$;
- (2) $C(\Omega, \mathbb{Z}) \rightarrow \mathbb{Z}[1/2]; f \mapsto \int f d\nu$ factorizes through an isomorphism

$$\text{Coinv } C(\Omega, \mathbb{Z}) \xrightarrow{\cong} \mathbb{Z}[1/2].$$

Proof. It is enough to check the first point for characteristic function of compact-open subset of Ω . For an integer n and k in $\{0, \dots, 2^n - 1\}$, we set $F_{n,k} = 2^n \Omega + k$. Then $(F_{n,k})_{n \in \mathbb{N}, 0 \leq k \leq 2^n - 1}$ is a basis of compact open neighborhoods for Ω and thereby, every compact open subset of Ω is a finite disjoint union of some $F_{n,k}$. Since $\nu(F_{n,k}) = 2^{-n}$, we get the first point.

The measure μ being invariant by translation, the map

$$C(\Omega, \mathbb{Z}) \rightarrow \mathbb{Z}[1/2]; f \mapsto \int f d\nu$$

factorizes through a group homomorphism $\text{Coinv } C(\Omega, \mathbb{Z}) \rightarrow \mathbb{Z}[1/2]$. This homomorphism admits a cross-section

$$(8.7) \quad \mathbb{Z}[1/2] \rightarrow \text{Coinv } C(\Omega, \mathbb{Z}); 2^{-n} \mapsto [\chi_{F_{n,0}}].$$

This map is well defined since $F_{n,0} = F_{n+1,0} \coprod ((2^n + F_{n+1,0}))$ and thus

$$[\chi_{F_{n,0}}] = [\chi_{F_{n+1,0}}] + [\chi_{2^n + F_{n+1,0}}] = 2[\chi_{F_{n+1,0}}]$$

in $\text{Coinv } C(\Omega, \mathbb{Z})$. Since the $(\chi_{F_{n,k}})_{n \in \mathbb{N}, 0 \leq k \leq 2^n - 1}$ generates $C(\Omega, \mathbb{Z})$ as an abelian group, it is enough to check that the cross-section of equation (8.7) is a left inverse on $\chi_{F_{n,k}}$, which is true since $[\chi_{F_{n,k}}] = [\chi_{k + F_{n,0}}] = [\chi_{F_{n,0}}]$ in $\text{Coinv } C(\Omega, \mathbb{Z})$. \square

Proposition 8.3. *Let $C(Z_w, \mathbb{Z}[1/2]) \cong C(Z_w, \mathbb{Z}) \otimes \mathbb{Z}[1/2]$ be the algebra of continuous function on Z_w , valued in $\mathbb{Z}[1/2]$ (equipped with the discrete topology). Then with the notations of the proof of lemma 8.2, we have isomorphisms*

(1)

$$\begin{aligned} C(Z_w, \mathbb{Z}[1/2]) &\xrightarrow{\cong} K_0(C(Z_w) \otimes C(\Omega) \rtimes \mathbb{Z}) \\ \frac{\chi_E}{2^n} &\mapsto [\chi_E \otimes \chi_{F_{n,0}}], \end{aligned}$$

where E is a compact open subset of Z_w and χ_E is its characteristic function.

(2)

$$\begin{aligned} C(Z_w, \mathbb{Z}) &\xrightarrow{\cong} K_1(C(Z_w) \otimes C(\Omega) \rtimes \mathbb{Z}) \\ \chi_E &\mapsto [\chi_E \otimes u + (1 - \chi_E) \otimes 1], \end{aligned}$$

where u is the unitary of $C(\Omega) \rtimes \mathbb{Z}$ corresponding to the positive generator of \mathbb{Z} .

Proof. As we have already mentionned, $K_*(C(Z_w))$ is torsion free and the Künneth formula provides isomorphisms

$$\begin{aligned} K_0(C(Z_w)) \otimes K_0(C(\Omega) \rtimes \mathbb{Z}) &\xrightarrow{\cong} K_0(C(Z_w) \otimes C(\Omega) \rtimes \mathbb{Z}) \\ [p] \otimes [q] &\mapsto [p \otimes q], \end{aligned}$$

where p and q are some matrix projectors with coefficients respectively in $C(Z_w)$ and $C(\Omega) \rtimes \mathbb{Z}$, and

$$\begin{aligned} K_0(C(Z_w)) \otimes K_1(C(\Omega) \rtimes \mathbb{Z}) &\xrightarrow{\cong} K_1(C(Z_w) \otimes C(\Omega) \rtimes \mathbb{Z}) \\ [p] \otimes [v] &\mapsto [p \otimes v + (I_k - p) \otimes I_l], \end{aligned}$$

where p is a projector in $M_l(C(Z_w))$ and v is a unitary in $M_k(C(\Omega) \rtimes \mathbb{Z})$. The proposition is then consequence of lemmas 8.1, 8.2 and of the discussion related to equations (8.5) and (8.6). \square

In order to compute the invariants and the coinvariants of

$$K_*(C_r^*(\mathcal{G}, \lambda)) \cong K_*(C(Z_w) \otimes \mathcal{K} \otimes C(\Omega) \rtimes \mathbb{Z}),$$

we will need a carefull description of the action induced in K -theory by the automorphism $\sigma^* \otimes \Upsilon$ of $C(Z_w) \otimes \mathcal{K} \otimes C(\Omega) \rtimes \mathbb{Z}$, where σ^* is the automorphism of $C(\Omega)$ induced by the shift σ and where Υ was defined in section 6.

Lemma 8.4. *If we equip $C(Z_w) \otimes \mathcal{K} \otimes C(\Omega) \rtimes \mathbb{Z}$ with the \mathbb{Z} -action provided by $\sigma^* \otimes \Upsilon$ and under the \mathbb{Z} -equivariant isomorphism*

$$C_r^*(\mathcal{G}, \lambda) \cong C(Z_w) \otimes \mathcal{K} \otimes C(\Omega) \rtimes \mathbb{Z},$$

the action induced by α_w on $K_0(C_r^*(\mathcal{G}, \lambda)) \cong C(Z_w, \mathbb{Z}[1/2])$ and on $K_1(C_r^*(\mathcal{G}, \lambda)) \cong C(Z_w, \mathbb{Z})$ are given by the automorphisms of abelian groups

$$\begin{aligned} \Psi_0 : C(Z_w, \mathbb{Z}[1/2]) &\rightarrow C(Z_w, \mathbb{Z}[1/2]) \\ f &\mapsto 2f \circ \sigma^{-1} \end{aligned}$$

and

$$\begin{aligned}\Psi_1 : C(Z_w, \mathbb{Z}) &\rightarrow C(Z_w, \mathbb{Z}) \\ f &\mapsto f \circ \sigma^{-1}.\end{aligned}$$

Proof. According to proposition 8.3 and using the Morita equivalence between $C(Z_w) \otimes C(\Omega) \rtimes \mathbb{Z}$ and $C(Z_w) \otimes \mathcal{K} \otimes C(\Omega) \rtimes \mathbb{Z}$, in order to describe Ψ_0 , we have to compute the image under $(\sigma^* \otimes \Upsilon)_*$ of

$$[\chi_E \otimes \Theta_{f,f} \otimes \chi_{F_{n,0}}] \in K_0(C(Z_w) \otimes \mathcal{K} \otimes C(\Omega) \rtimes \mathbb{Z})$$

where,

- χ_E is the characteristic function of a compact open subset E of Z_w ;
- $\chi_{F_{n,0}}$ is the characteristic function of $F_{n,0} = 2^n \Omega$ for $n \geq 1$;
- $\Theta_{f,f}$ is the rank one projector associated to a norm 1 function f of $L^2([0, 1])$.

We have

$$\begin{aligned}\sigma^* \otimes \Upsilon(\chi_E \otimes \Theta_{f,f} \otimes \chi_{F_{n,0}}) &= \chi_{\sigma(E)} \otimes \Theta_{U_0 f, U_0 f} \otimes W_0 \chi_{F_{n,0}} + \chi_{\sigma(E)} \otimes \Theta_{U_1 f, U_1 f} \otimes W_1 \chi_{F_{n,0}} \\ &= \chi_{\sigma(E)} \otimes \Theta_{U_0 f, U_0 f} \otimes \chi_{F_{n-1,0}},\end{aligned}$$

where the last equality holds since $W_0 \chi_{F_{n,0}} = \chi_{F_{n-1,0}}$ and $W_1 \chi_{F_{n,0}} = 0$. Since $\Theta_{U_0 f, U_0 f}$ is again a rank one projector, then up to the Morita equivalence between $C(Z_w) \otimes C(\Omega) \rtimes \mathbb{Z}$ and $C(Z_w) \otimes \mathcal{K} \otimes C(\Omega) \rtimes \mathbb{Z}$, the image of $[\chi_E \otimes \chi_{F_{n,0}}] \in K_0(C(Z_w) \otimes C(\Omega) \rtimes \mathbb{Z})$ under $(\sigma^* \otimes \Upsilon)_*$ is $[\chi_{\sigma(E)} \otimes \chi_{F_{n-1,0}}] \in K_0(C(Z_w) \otimes C(\Omega) \rtimes \mathbb{Z})$. Using proposition 8.3, this completes the description of Ψ_0 . For Ψ_1 , notice first that up to the isomorphism

$$K_0(C(Z_w)) \otimes K_0(\mathcal{K} \otimes C(\Omega) \rtimes \mathbb{Z}) \xrightarrow{\cong} K_1(C(Z_w) \otimes \mathcal{K} \otimes C(\Omega) \rtimes \mathbb{Z})$$

provided by the Künneth formula, the action of $(\sigma^* \otimes \Upsilon)_*$ is $\sigma_*^* \otimes \Upsilon_*$ and then the result is a consequence of lemma 6.1 and of proposition 8.3. \square

Let us equip $C(Z_w, \mathbb{Z}[1/2])$ and $C(Z_w, \mathbb{Z})$ with the \mathbb{Z} -actions respectively provided by Ψ_0 and Ψ_1 . Then since $\|\Psi_0(h)\| = \|2h\|$ for any h in $C(Z_w, \mathbb{Z}[1/2])$, we get that $\text{Inv } C(Z_w, \mathbb{Z}[1/2]) = \{0\}$. We are now in position to get a complete description of the K -theory of $C(X_{\mathcal{P}(w)}^G) \rtimes G$. In view of the short exact sequences of equations (8.3) and (8.4), the two following theorems are then consequences of lemma 8.4 and of proposition 8.3.

Theorem 8.5. *We have a short exact sequence*

$$0 \rightarrow \text{Coinv } C(Z_w, \mathbb{Z}[1/2]) \xrightarrow{\iota_0} K_0(C(X_{\mathcal{P}(w)}^G) \rtimes G) \rightarrow \text{Inv } C(Z_w, \mathbb{Z}) \rightarrow 0,$$

where up to the Morita equivalence $C(X_{\mathcal{P}(w)}^G) \rtimes G \cong C_r^*(\mathcal{G}, \lambda) \rtimes_{\alpha_w} \mathbb{Z}$, the element $\iota_0[2^{-n} \chi_E]$ is the image of $[\chi_E \otimes \Theta_{f,f} \otimes \chi_{F_{n,0}}] \in K_0(C(Z_w) \otimes \mathcal{K} \otimes C(\Omega) \rtimes \mathbb{Z})$ under the homomorphism induced in K -theory by the inclusion

$$C(Z_w) \otimes \mathcal{K} \otimes C(\Omega) \rtimes \mathbb{Z} \cong C_r^*(\mathcal{G}, \lambda) \hookrightarrow C_r^*(\mathcal{G}, \lambda) \rtimes_{\alpha_w} \mathbb{Z},$$

where

- χ_E is the characteristic function of a compact open subset E of Z_w ;
- $\chi_{F_{n,0}}$ is the characteristic function of $F_{n,0} = 2^n \Omega$;
- $\Theta_{f,f}$ is the rank one projector associated to a norm 1 function f of $L^2([0, 1])$.

Theorem 8.6. *We have an isomorphism*

$$\text{Coinv } C(Z_w, \mathbb{Z}) \xrightarrow{\cong} K_1(C(X_{\mathcal{P}(w)}^G) \rtimes G)$$

induced on the coinvariants by the composition

$$C(Z_w, \mathbb{Z}) \cong K_0(C(Z_w)) \xrightarrow{\otimes[u]} K_1(C(Z_w) \otimes C(\Omega) \rtimes \mathbb{Z}) \cong K_1(C_r^*(\mathcal{G}, \lambda)) \rightarrow K_1(C_r^*(\mathcal{G}, \lambda) \rtimes_{\alpha_w} \mathbb{Z}),$$

where

- $\otimes[u]$ is the external product in K -theory by the class in $K_1(C(\Omega) \rtimes \mathbb{Z})$ of the unitary u of $C(\Omega) \rtimes \mathbb{Z}$ corresponding to the positive generator of \mathbb{Z} ;
- the last map in the composition is the homomorphism induced in K -theory by the inclusion $C_r^*(\mathcal{G}, \lambda) \hookrightarrow C_r^*(\mathcal{G}, \lambda) \rtimes_{\alpha_w} \mathbb{Z}$.

The short exact sequence of theorem 8.5, admits an explicit splitting which can be describe in the following way: assume first that (Z_w, σ) is minimal. In particular, $\text{Inv } C(Z_w, \mathbb{Z}) \cong \mathbb{Z}$ is generated by $1 \in C(Z_w, \mathbb{Z})$. Let us consider the following diagram, whose left square is commutative

$$\begin{array}{ccc} K_1(C^*(\mathbb{R})) & \longrightarrow & K_1(C^*(\mathcal{G}, \lambda)) \\ \downarrow & & \downarrow \\ \mathbb{Z} \cong K_0(\mathbb{C}) & \longrightarrow & K_0(C(X_{\mathcal{P}}^{\mathcal{N}} \times Z_w)) \xrightarrow{\text{ev}_*} K_0(C(\Omega \times Z_w)) \end{array},$$

where

- the horizontal maps of the left square are induced by the inclusion $\mathbb{C} \hookrightarrow C(X_{\mathcal{P}}^{\mathcal{N}} \times Z_w)$.
- vertical maps are the Thom-Connes isomorphisms.
- The map $\text{ev} : C(X_{\mathcal{P}}^{\mathcal{N}} \times Z_w) \rightarrow C(\Omega \times Z_w)$ is induced by the continuous map $\Omega \rightarrow X_{\mathcal{P}}^{\mathcal{N}} \cong (\Omega \times \mathbb{R})/\mathcal{A}_0$; $x \mapsto [x, 0]$;

Up to the Morita equivalence between $C^*(\mathcal{G}, \lambda)$ and $C(Z_w) \otimes C(\Omega) \rtimes \mathbb{Z}$, the left down stair case is the boundary of the Pimsner-Voiculescu six-term exact sequence that computes $K_*(C(Z_w) \otimes C(\Omega) \rtimes \mathbb{Z})$. From this, we see that $K_1(C^*(\mathcal{G}, \lambda)) \cong \mathbb{Z}$ is generated by the image of the generator ζ of $K_1(C^*(\mathbb{R}))$ corresponding under the canonical identification $K_1(C^*(\mathbb{R})) \cong K_1(C_0(\mathbb{R})) \cong K_0(\mathbb{C}) \cong \mathbb{Z}$ to the class of any rank one projector in some $M_n(\mathbb{C})$. On the other hand, we have a diagram with commutative square

$$\begin{array}{ccccccc} K_0(C^*(\mathbb{R}) \rtimes \mathbb{R}_+) & \longrightarrow & K_0(C(X_{\mathcal{P}(w)}^G) \rtimes G) & \xrightarrow{\cong} & K_0(C^*(\mathcal{G}_{\alpha_w}, \lambda_{\alpha_w}) \rtimes \mathbb{R}) & \xrightarrow{\cong} & K_0(C^*(\mathcal{G}, \lambda)_{\alpha_w} \rtimes \mathbb{R}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ K_1(C^*(\mathbb{R})) & \longrightarrow & K_1(C(X_{\mathcal{P}(w)}^G) \rtimes \mathbb{R}) & \xrightarrow{\cong} & K_1(C^*(\mathcal{G}_{\alpha_w}, \lambda_{\alpha_w})) & \xrightarrow{\cong} & K_1(C^*(\mathcal{G}, \lambda)_{\alpha_w}) \\ & & & & & & \downarrow \text{ev}_* \\ & & & & & & K_0(C^*(\mathcal{G}, \lambda)) \end{array}$$

where,

- the horizontal maps of the left square are induced by the inclusion $C^*(\mathbb{R}) \hookrightarrow C(X_{\mathcal{P}(w)}^G) \rtimes \mathbb{R}$;
- the horizontal maps of the middle square are induced by the isomorphism of lemma 7.1
- the horizontal maps of the right square are induced by the isomorphism of proposition 5.7

- the first row of vertical maps are Thom-Connes isomorphisms.

It is then straightforward to check that the downstaircase of the diagram is indeed induced by the inclusion $C^*(\mathbb{R}) \hookrightarrow C(X_{\mathcal{P}}^{\mathcal{N}} \times Z_w) \rtimes \mathbb{R} = C^*(\mathcal{G}, \lambda)$. Notice that $K_0(C^*(\mathbb{R}) \rtimes \mathbb{R}_+^*) \cong \mathbb{Z}$ (by Thom-Connes isomorphism). Moreover, under the inclusion $C_0(\mathbb{R}_+^*) \rtimes \mathbb{R}_+^* \hookrightarrow C_0(\mathbb{R}) \rtimes \mathbb{R}_+^* \cong C^*(\mathbb{R}) \rtimes \mathbb{R}_+^*$, any rank one projector e of $\mathcal{K}(L^2(\mathbb{R}_+^*)) \cong C_0(\mathbb{R}_+^*) \rtimes \mathbb{R}_+^*$ provides a generator for $K_0(C^*(\mathbb{R}) \rtimes \mathbb{R}_+^*)$ whose image under the left vertical map is the generator ζ for $K_1(C^*(\mathbb{R})) \cong K_0(\mathbb{C}) \cong \mathbb{Z}$. Using the description of the boundary map of Pimsner-Voiculescu six-term exact sequence, we see that e , viewed as an element of $C(X_{\mathcal{P}_w}^G) \rtimes G$ whose class in K-theory provides a lift for $1 \in C(Z_w, \mathbb{Z})$ in the short exact sequence of theorem 8.5.

In general, $\text{Inv } C(Z_w, \mathbb{Z})$ is generated by characteristic functions of \mathbb{Z} -invariant compact-open subsets of Z_w . According to proposition 4.3, any \mathbb{Z} -invariant compact-open subset E of Z_w provides a \mathbb{R} -invariant compact subset \tilde{E} of $X_{\mathcal{P}}^G$. Hence, with above notations, if $\chi_{\tilde{E}}$ is the characteristic function for \tilde{E} , then $\chi_{\tilde{E}}e$ can be viewed as an element of $C(X_{\mathcal{P}_w}^G) \rtimes G$. Let $v : \text{Inv } C(Z_w, \mathbb{Z}) \rightarrow K_0(C(X_{\mathcal{P}_w}^G) \rtimes G)$ be the group homomorphism uniquely defined by $v(\chi_E) = \chi_{\tilde{E}}e$ for E a \mathbb{Z} -invariant compact-open subset of Z_w . Then v is a section for the short exact sequence of theorem 8.5.

9. THE CYCLIC COCYCLE ASSOCIATED TO A HARMONIC PROBABILITY

Recall that according to the discussion ending section 4.2, a probability is harmonic if and only if it is G -invariant. In this section, we associate to an harmonic probability a 3-cyclic cocycle on the smooth cross product algebra of $X_{\mathcal{P}(\omega)}^G \rtimes G$. This cyclic cocycle is indeed builded out a 1-cyclic cocycle on the algebra of smooth (along the leaves) functions on $X_{\mathcal{P}(\omega)}^G$ by using the analogue in cyclic cohomology of the Thom-Connes isomorphism (see [6]). We give a description of this cocycle and we discuss an odd version of the gap-labelling.

9.1. Review on smooth cross products. We collect here results from [6] concerning smooth cross products by an action of \mathbb{R} that we will need later on.

Let \mathcal{A} be a Frechet algebra with respect to an increasing family of semi-norms $(\|\bullet\|_k)_{k \in \mathbb{N}}$.

Definition 9.1. *A smooth action on \mathcal{A} is a homomorphism $\alpha : \mathbb{R} \rightarrow \text{Aut } \mathcal{A}$ such that*

- (1) *For every t in \mathbb{R} and a in \mathcal{A} , the function $t \mapsto \alpha_t(a)$ is smooth.*
- (2) *For every integers k and m , there exist integers j and n and a real C such that $\left\| \frac{d^k}{dt^k} \alpha_t(a) \right\|_m \leq C(1+t^2)^{j/2} \|a\|_n$ for every a in \mathcal{A} .*

If α is a smooth action on \mathcal{A} , then the smooth cross product $\mathcal{A} \rtimes_{\alpha} \mathbb{R}$ is defined as the set of smooth functions $f : \mathbb{R} \rightarrow \mathcal{A}$ such that

$$\|f\|_{k,m,n} \stackrel{\text{def}}{=} \sup_{t \in \mathbb{R}} (1+t^2)^{k/2} \left\| \frac{d^m}{dt^m} f(t) \right\|_n < +\infty$$

for all integers k, m and n . The smooth cross product $\mathcal{A} \rtimes_{\alpha} \mathbb{R}$ provided with the family of semi-norm $\|\bullet\|_{k,m,n}$ for k, m and n integers together with the convolution product

$$f * g(t) = \int f(s) \alpha_s(g(t-s)) dt$$

is then a Frechet algebra. Notice that a smooth action α on a Frechet algebra \mathcal{A} gives rise to a bounded derivation Z_α of $\mathcal{A} \rtimes_\alpha \mathbb{R}$ defined by $Z_\alpha(f)(t) = tf(t)$ for all f in $\mathcal{A} \rtimes_\alpha \mathbb{R}$ and t in \mathbb{R} .

Let $\mathcal{A}_{\mathcal{P}(\omega)}^G$ be the algebra of continuous and smooth along the leaves functions on $X_{\mathcal{P}(\omega)}^G$, i.e functions whose restrictions to leaves admit at all order differential which are continuous as functions on $X_{\mathcal{P}(\omega)}^G$. Let β_0 and β_1 be the two actions of \mathbb{R} on $\mathcal{A}_{\mathcal{P}(\omega)}^G$ respectively induced by

$$\mathbb{R} \times X_{\mathcal{P}(\omega)}^G \rightarrow X_{\mathcal{P}(\omega)}^G; (t, \mathcal{T}) \mapsto \mathcal{T} + t$$

and

$$\mathbb{R} \times X_{\mathcal{P}(\omega)}^G \rightarrow X_{\mathcal{P}(\omega)}^G; (t, \mathcal{T}) \mapsto R_{2^{-t}} \cdot \mathcal{T}.$$

Let X and Y be respectively the vector fields associated to β^0 and β^1 . Then $\mathcal{A}_{\mathcal{P}(\omega)}^G$ is a Frechet algebra with respect to the family of semi-norms

$$f \mapsto \sup_{X_{\mathcal{P}(\omega)}^G} |X^k Y^l(f)|,$$

where k and l run through integers. It is clear that β^0 is a smooth action on $\mathcal{A}_{\mathcal{P}(\omega)}^G$. Moreover,

$$\mathbb{R} \times \mathcal{A}_{\mathcal{P}(\omega)}^G \rtimes_{\beta^0} \mathbb{R} \rightarrow \mathcal{A}_{\mathcal{P}(\omega)}^G \rtimes_{\beta^0} \mathbb{R}; (t, f) \mapsto [s \mapsto \beta^1(f(2^{-t}s))]$$

is an action of \mathbb{R} on $\mathcal{A}_{\mathcal{P}(\omega)}^G \rtimes_{\beta^0} \mathbb{R}$ by automorphisms. This action is not smooth in the previous sense. Nevertheless, the action β^1 satisfies conditions (1),(2) and (3) of [6, Section 7.2] with respect to the family of functions $\rho_n : \mathbb{R} \rightarrow \mathbb{R}; t \mapsto 2^{2n|t|}$, where n runs through integers. In this situation, we can define the smooth cross product $\mathcal{A}_{\mathcal{P}(\omega)}^G \rtimes_{\beta^0} \mathbb{R} \rtimes_{\beta^1}^{\rho} \mathbb{R}$ of $\mathcal{A}_{\mathcal{P}(\omega)}^G \rtimes_{\beta^0} \mathbb{R}$ by β^1 to be set of smooth functions $f : \mathbb{R} \rightarrow \mathcal{A}_{\mathcal{P}(\omega)}^G \rtimes_{\beta^0} \mathbb{R}$ such that

$$\|f\|_{k,l,m,n} \stackrel{\text{def}}{=} \sup_{t \in \mathbb{R}} \rho_k(t) \left\| \frac{d^l}{dt^l} f(t) \right\|_m < +\infty$$

for all integers k, l and m (we have reindexed for convenience the family of semi-norms on $\mathcal{A}_{\mathcal{P}(\omega)}^G \rtimes_{\beta^0} \mathbb{R}$ using integers). Then $\mathcal{A}_{\mathcal{P}(\omega)}^G \rtimes_{\beta^0} \mathbb{R} \rtimes_{\beta^1}^{\rho} \mathbb{R}$ provided with the family of semi-norm $\|\bullet\|_{k,l,m,n}$ for k, m and n integers together with the convolution product is a Frechet algebra. Moreover, this algebra can be viewed as a dense subalgebra of $C(X_{\mathcal{P}(\omega)}^G) \rtimes G$. As for smooth actions, the action β^1 gives rise to a derivation Z_{β^1} of $\mathcal{A}_{\mathcal{P}(\omega)}^G \rtimes_{\beta^0} \mathbb{R} \rtimes_{\beta^1}^{\rho} \mathbb{R}$ (defined by the same formula).

9.2. The 3-cyclic cocycle. Let η be a G -invariant probability on $X_{\mathcal{P}(\omega)}^G$. Define

$$\tau_{w,\eta} : \mathcal{A}_{\mathcal{P}(\omega)}^G \times \mathcal{A}_{\mathcal{P}(\omega)}^G \rightarrow \mathbb{C}; (f, g) \mapsto \int Y(f)gd\eta.$$

Using the Leibnitz rules and the invariance of G , it is straightforward to check that $\tau_{w,\eta}$ is 1-cyclic cocycle. In [6] was constructed for a smooth action α on a Frechet algebra \mathcal{A} a homomorphism $H_\lambda^n(\mathcal{A}) \rightarrow H_\lambda^{n+1}(\mathcal{A} \rtimes_\alpha \mathbb{R})$, where $H_\lambda^*(\bullet)$ stands for the cyclic cohomology. This homomorphism is indeed induced by a homomorphism at the level of cyclic cocycles $\#_\alpha : Z_\lambda^n(\mathcal{A}) \rightarrow Z_\lambda^{n+1}(\mathcal{A} \rtimes_\alpha \mathbb{R})$ and commutes with the periodisation operator S . Hence it gives rise to homomorphism in periodic cohomology $HP^*(\mathcal{A}) \rightarrow HP^{*+1}(\mathcal{A} \rtimes_\alpha \mathbb{R})$ which turns out to be an isomorphism.

This isomorphism is for periodic cohomology the analogue of the Thom-Connes isomorphism in K -theory.

We give now the description of $\#_{\beta^0} \tau_{w,\eta}$. Let us define first

$$X_{\beta^0} : \mathcal{A}_{\mathcal{P}(w)}^G \rtimes_{\beta^0} \mathbb{R} \rightarrow \mathcal{A}_{\mathcal{P}(w)}^G \rtimes_{\beta^0} \mathbb{R}$$

and

$$Y_{\beta^0} : \mathcal{A}_{\mathcal{P}(w)}^G \rtimes_{\beta^0} \mathbb{R} \rightarrow \mathcal{A}_{\mathcal{P}(w)}^G \rtimes_{\beta^0} \mathbb{R}$$

respectively by $X_{\beta^0} f(t) = X(f)(t)$ and $Y_{\beta^0} f(t) = Y(f)(t)$, for all f in $\mathcal{A}_{\mathcal{P}(w)}^G \rtimes_{\beta^0} \mathbb{R}$ and t in \mathbb{R} . Using the relation $Y \circ \beta_t^0 = \beta_t^0 \circ Y - t \ln 2 \beta_t^0 \circ X$ and applying the definition of $\#_{\beta^0}$ (see [6, section 3.3]), we get:

Proposition 9.2. *For any elements f, g and h in $\mathcal{A}_{\mathcal{P}(w)}^G \rtimes_{\beta^0} \mathbb{R}$, we have*

(1)

$$\begin{aligned} \#_{\beta^0} \tau_{w,\eta}(f, g, h) &= -2\pi i \eta(Y_{\beta^0} f * g * Z_{\beta^0} h(0) + Z_{\beta^0} f * g * Y_{\beta^0} h(0)) \\ &- 2\pi i \ln 2 (\eta(1/2 Z_{\beta^0}^2 f * g * X_{\beta^0} h(0) + Z_{\beta^0} f * Z_{\beta^0} g * X_{\beta^0} h(0)) - 1/2 X_{\beta^0} f * g * Z_{\beta^0}^2(0)) \end{aligned}$$

(2)

$$\#_{\beta^0} \tau_{w,\eta}(\beta_t^1 f, \beta_t^1 g, \beta_t^1 h) = \#_{\beta^0} \tau_{w,\eta}(f, g, h)$$

for all t in \mathbb{R} , i.e the cocycle $\#_{\beta^0} \tau_{w,\eta}$ is β^1 -invariant.

According to [6, Section 7.2], the action β^1 on $\mathcal{A}_{\mathcal{P}(w)}^G \rtimes_{\beta^0} \mathbb{R}$ also gives rise to a homomorphism $\#_{\beta^1} : Z_{\lambda}^n(\mathcal{A}_{\mathcal{P}(w)}^G \rtimes_{\beta^0} \mathbb{R}) \rightarrow Z_{\lambda}^{n+1}(\mathcal{A}_{\mathcal{P}(w)}^G \rtimes_{\beta^0} \mathbb{R} \rtimes_{\beta^1}^{\rho} \mathbb{R})$ which induces an isomorphism $HP^*(\mathcal{A}_{\mathcal{P}(w)}^G \rtimes_{\beta^0} \mathbb{R}) \xrightarrow{\cong} HP^{*+1}(\mathcal{A}_{\mathcal{P}(w)}^G \rtimes_{\beta^0} \mathbb{R} \rtimes_{\beta^1}^{\rho} \mathbb{R})$. A direct application of the definition of $\#_{\beta^1}$ leads to

Lemma 9.3. *Let ϕ be a β^1 -invariant 3-cyclic cocycle for $\mathcal{A}_{\mathcal{P}(w)}^G \rtimes_{\beta^0} \mathbb{R}$. Let us define for any f, g and h in $\mathcal{A}_{\mathcal{P}(w)}^G \rtimes_{\beta^0} \mathbb{R} \rtimes_{\beta^1}^{\rho} \mathbb{R}$.*

$$\tilde{\phi}(f, g, h) = 2\pi i \int_{t_0+t_1+t_2=0} f(t_0) \beta_{t_0}^1 g(t_1) \beta_{-t_2}^1(h).$$

Then

$$\begin{aligned} \#_{\beta^1} \phi(f_0, f_1, f_2, f_3) &= -\tilde{\phi}(f_0, f_1, f_2 * Z_{\beta^1} f_3) + \tilde{\phi}(Z_{\beta^1} f_0 * f_1, f_2, f_3) \\ &- \tilde{\phi}(f_0, Z_{\beta^1} f_1 * f_2, f_3) - \tilde{\phi}(Z_{\beta^1} f_0, f_1 * f_2, f_3) \end{aligned}$$

Definition 9.4. *With above notations, the 3-cyclic cocycle on $\mathcal{A}_{\mathcal{P}(w)}^G \rtimes_{\beta^0} \mathbb{R} \rtimes_{\beta^1}^{\rho} \mathbb{R}$ associated to the Penrose hyperbolic tiling coloured by w and to a G -invariant probability η on $X_{\mathcal{P}(w)}^G$ is*

$$\phi_{w,\eta} = \#_{\beta^1} \#_{\beta^0} \tau_{w,\eta}.$$

Notice that if we carry out this construction for a tiling \mathcal{T} of the Euclidian space with continuous hull $X_{\mathcal{T}}^{\mathbb{R}^2}$ with respect to the \mathbb{R}^2 -action by translations, we get taking twice the cross-product by \mathbb{R} a 3-cyclic cocycle which is indeed equivalent (via the periodisation operator) to the 1-cycle cocycle on $C(X_{\mathcal{T}}^{\mathbb{R}^2}) \rtimes \mathbb{R}^2 \cong (C(X_{\mathcal{T}}^{\mathbb{R}^2}) \rtimes \mathbb{R}) \rtimes \mathbb{R}$ arising from the trace on $C(X_{\mathcal{T}}^{\mathbb{R}^2}) \rtimes \mathbb{R}$ associated to an \mathbb{R} -invariant probability on $X_{\mathcal{T}}^{\mathbb{R}^2}$.

The class $[\phi_{w,\eta}]$ of $\phi_{w,\eta}$ in $HP^1(\mathcal{A}_{\mathcal{P}(w)}^G \rtimes_{\beta^0} \mathbb{R} \rtimes_{\beta^1}^{\rho} \mathbb{R})$ is the image of the class of $\tau_{w,\eta}$ under the composition of isomorphism

$$HP^1(\mathcal{A}_{\mathcal{P}(w)}^G) \xrightarrow{\cong} HP^0(\mathcal{A}_{\mathcal{P}(w)}^G \rtimes_{\beta^0} \mathbb{R}) \xrightarrow{\cong} HP^1(\mathcal{A}_{\mathcal{P}(w)}^G \rtimes_{\beta^0} \mathbb{R} \rtimes_{\beta^1}^{\rho} \mathbb{R}).$$

Since pairing with periodic cohomology provides linear forms for K -theory groups, the 3-cyclic cocycle $\phi_{w,\eta}$ provides a linear map

$$\phi_{w,\eta,*} : K^1(\mathcal{A}_{\mathcal{P}(w)}^G \rtimes_{\beta^0} \mathbb{R} \rtimes_{\beta^1}^{\rho} \mathbb{R}) \rightarrow \mathbb{C}; x \mapsto \langle [\phi_{w,\eta}], x \rangle.$$

The main issue in computing $\phi_{w,\eta,*}(K^1(\mathcal{A}_{\mathcal{P}(w)}^G \rtimes_{\beta^0} \mathbb{R} \rtimes_{\beta^1}^{\rho} \mathbb{R}))$ is that the Thom-Connes isomorphism a priori may not hold for $K^1(\mathcal{A}_{\mathcal{P}(w)}^G \rtimes_{\beta^0} \mathbb{R})$. If it were the case, the inclusion $\mathcal{A}_{\mathcal{P}(w)}^G \rtimes_{\beta^0} \mathbb{R} \rtimes_{\beta^1}^{\rho} \mathbb{R} \hookrightarrow C(X_{\mathcal{P}(w)}^G) \rtimes G$ would induce an isomorphism $K_1(\mathcal{A}_{\mathcal{P}(w)}^G \rtimes_{\beta^0} \mathbb{R} \rtimes_{\beta^1}^{\rho} \mathbb{R}) \xrightarrow{\cong} K_1(C(X_{\mathcal{P}(w)}^G) \rtimes G)$ and from this we could get that

$$\phi_{w,\eta,*}(K^1(\mathcal{A}_{\mathcal{P}(w)}^G \rtimes_{\beta^0} \mathbb{R} \rtimes_{\beta^1}^{\rho} \mathbb{R})) = \mathbb{Z}[\hat{\eta}] \stackrel{\text{def}}{=} \{\hat{\eta}(E), E \text{ compact open subset of } Z_w\},$$

where $\hat{\eta}$ is the probability on Z_w from proposition 4.4 in one-to-one correspondence with η . Since $\mathbb{Z}[\hat{\eta}]$ is indeed the one dimension gap-labelling for the subshift corresponding to w , this would be viewed as an odd version of the gap labelling. Nevertheless, the right setting to state this generalisation for this gap-labelling seems to be the Frechet algebra and natural question is whether we have

$$\{\langle [\phi_{w,\eta}], x \rangle; x \in K_1(\mathcal{A}_{\mathcal{P}(w)}^G \rtimes_{\beta^0} \mathbb{R} \rtimes_{\beta^1}^{\rho} \mathbb{R})\} = \mathbb{Z}[\hat{\eta}]$$

or if the pairing bring in new invariants.

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