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# Stochastic approximations of set-valued dynamical systems: Convergence with positive probability to an attractor \*

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## Abstract

A successful method to describe the asymptotic behavior of a discrete time stochastic process governed by some recursive formula is to relate it to the limit sets of a well chosen mean differential equation. Under an attainability condition, convergence to a given attractor of the flow induced by this dynamical system was proved to occur with positive probability (Benaïm, 1999) for a class of Robbins Monro algorithms. Benaïm et al. (2005) generalised this approach for stochastic approximation algorithms whose average behavior is related to a differential inclusion instead. We pursue the analogy by extending to this setting the result of convergence with positive probability to an attractor.

**Key words:** Stochastic approximations, set-valued dynamical systems, attractor, game theory, Markovian fictitious play

**MSC2000 Subject classification:** 62L20, 34A60, 34B40, 34B41, 91A25, 91A26

**OR/MS subject classification:** stochastic model applications

## 1 Introduction

### 1.1 Settings and bibliography

Stochastic approximation algorithms were born in the early 50s through the work of Robbins and Monro [20] and Kiefer and Wolfowitz [16]. Consider a discrete time stochastic process  $(x_n)_{n \geq 0}$  defined by the following recursive formula:

$$x_{n+1} - x_n = \gamma_{n+1} (F(x_n) + U_{n+1}), \quad (1)$$

where  $F : \mathbb{R}^m \rightarrow \mathbb{R}^m$  is a Lipschitz function,  $(\gamma_n)_n$  is a positive decreasing sequence and  $(U_n)_n$  a sequence of  $\mathbb{R}^m$ -valued random variables defined on a probability space  $(\Omega, \mathcal{F}, P)$ . In order to

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describe the limit behavior of the sample paths  $(x_n(\omega))_n$ , a natural idea is to compare them to the solution curves of the dynamical system induced by the ordinary differential equation

$$\frac{dx}{dt} = F(x). \quad (2)$$

This is the celebrated method of ordinary differential equation (ODE) which was introduced by Ljung in [19]. Heuristically, one can think of (1) as a kind of Cauchy-Euler approximation scheme for numerically solving (2) with step size  $(\gamma_n)_n$  and an added noise  $(U_n)_n$ . We could reasonably expect that, under appropriate assumptions on  $(\gamma_n)_n$  and if the noise  $(U_n)_n$  vanishes, the asymptotic behaviors of  $(x_n)_n$  and the ODE are closely related.

Thereafter, the method was studied and developed by many people (see Kushner and Clark [17], Benveniste et al [9], Duflo [12] or Kushner and Yin [18]). Originally, only simple dynamics were considered, for example the negative of the gradient of a cost function. However, it appears in several situations, for example, learning models or game theory, that the corresponding vector field may be more complex.

Benaïm and Hirsch have conducted, in a series of papers (essentially [5] and [3]), a thorough study of this method. They proved that the asymptotic behavior of stochastic approximation process can be described with a great deal of generality through the study of the asymptotics of the ODE. One of the main results is the characterization of limit sets  $(x_n(\omega))_n$  via the flow induced by  $F$ , in the sense that, almost surely, these sets are compact, invariant and contain no proper attractor for the deterministic flow (this is the notion of *internally chain recurrence* in the sense of Conley [11], see also Bowen [10]).

Now, let  $F : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$  be a sufficiently regular set-valued map and consider some discrete time stochastic processes  $(x_n)_{n \geq 0}$  defined by the following recursive formula:

$$x_{n+1} - x_n - \gamma_{n+1}U_{n+1} \in \gamma_{n+1}F(x_n), \quad (3)$$

where  $(\gamma_n)_n$  is a positive decreasing sequence and  $(U_n)_n$  a sequence of  $\mathbb{R}^m$ -valued random variables defined on a probability space  $(\Omega, \mathcal{F}, P)$ .

In [6], Benaïm, Hofbauer and Sorin have generalized the ODE method to the algorithms given by (3) and extended the characterization of limits set in the sense that they are again, under certain assumptions on the step size and the noise, connected and attractor free for the set-valued dynamic induced by the differential inclusion

$$\frac{dx}{dt} \in F(x). \quad (4)$$

This extension allows us to extend this technic to much wider class of problems arising, for exemple, in economics or game theory (see Benaïm, Hofbauer and Sorin [7]).

In this paper, we pursue the analogy between the ODE method and the differential inclusion method. The aim is to extend to the case of differential inclusions, the result of Benaïm (see theorem 7.3 in [4]) which guarantees that, under certain assumptions on the step size and the noise, the stochastic approximation process converges with positive probability to a given attractor of the set-valued dynamical system induced by  $F$ .

The organization of the paper is as follows. In section 1.2, we define a standard set-valued map and introduce the crucial notion of attainability so as to state a simple version of the main result. In part 2, we introduce the different notions of internally chain transitivity, asymptotic pseudotrajectories and perturbed solutions. Our main assumption (hypothesis 2.6) is given, the convergence result is stated in full generality and we define a generalised stochastic approximation process which satisfies the above assumption. An example of adaptive learning process to which our results may be applied is given in section 3. Finally, the proof of a crucial result needed in our study is postponed to part 4.

## 1.2 The main result, a simple version

In the following,  $M \subset \mathbb{R}^m$  is a compact set.

**Definition 1.1.** [Standard set-valued map] A correspondance  $F : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$  is said to be standard if it satisfies the following assumptions:

- for any  $x \in \mathbb{R}^m$ ,  $F(x)$  is a non empty, compact and convex set of  $\mathbb{R}^m$ ,
- $F$  is closed, which means that its graph

$$\text{Gr}(F) := \{(x, y) \in \mathbb{R}^m \times \mathbb{R}^m \mid y \in F(x)\}$$

is closed,

- there exists  $c > 0$  such that

$$\sup_{z \in F(x)} \|z\| \leq c(1 + \|x\|).$$

Under the above assumptions, it is well known (see Aubin and Cellina [1]) that (4) admits at least one solution (i.e. an absolutely continuous mapping  $\mathbf{x} : \mathbb{R} \rightarrow \mathbb{R}^m$  such that  $\dot{\mathbf{x}}(t) \in F(\mathbf{x}(t))$  for almost every  $t$ ) through every initial point.

We call  $S_x$  the set of solutions with initial condition  $\mathbf{x}(0) = x$ . The set-valued dynamical system induced by the differential inclusion will be denoted  $\Phi = (\Phi_t)_{t \in \mathbb{R}}$ . To any  $x \in \mathbb{R}^m$ , it associates the non empty set

$$\Phi_t(x) := \{\mathbf{x}(t) \mid \mathbf{x} \in S_x\}.$$

Finally,  $S_\Phi$  is the set of every solution curves. In order to understand the main result, recall some classical definitions about the set-valued dynamic.

**Definition 1.2.** A non empty compact set  $A \subset \mathbb{R}^m$  is called an attractor for  $\Phi$ , provided it is invariant (i.e. for all  $x \in A$ , there exists a solution  $\mathbf{x}$  to (5) with  $\mathbf{x}(0) = x$  and such that  $\mathbf{x}(\mathbb{R}) \subset A$ ) and that there is a neighborhood  $U$  of  $A$  with the property that, for every  $\epsilon > 0$ , there exists  $t_\epsilon > 0$  such that

$$\Phi_t(U) \subset N^\epsilon(A)$$

for all  $t \geq t_\epsilon$ .

**Definition 1.3.** Let  $A \subset \mathbb{R}^m$  be an attractor for the set-valued dynamical system. The basin of attraction of  $A$  is the set

$$\mathcal{B}(A) := \{x \in \mathbb{R}^m : \omega_\Phi(x) \subset A\},$$

where  $\omega_\Phi(x) = \bigcap_{t \geq 0} \overline{\Phi_{[t, \infty)}(x)}$  is the omega limit set of the point  $x$ .

Now consider a discrete time stochastic process  $(x_n)_n$  in  $M$  defined by (3), and satisfying the following assumptions :

- (i) For all  $c > 0$ ,

$$\sum_n e^{-c/\gamma_n} < \infty,$$

- (ii)  $(U_n)_n$  is an uniformly bounded and

$$\mathbb{E}(U_{n+1} \mid \mathcal{F}_n) = 0,$$

- (iii)  $F$  is a standard set-valued map.

Set  $\tau_n := \sum_{i=1}^n \gamma_i$  and  $m(t) := \sup\{j \mid \tau_j \leq t\}$ . We call  $X$  the continuous time affine interpolated process induced by  $(x_n)_n$  and  $\bar{\gamma}$  the piecewise constant deterministic process induced by  $(\gamma_n)_n$ :

$$X(\tau_i + s) = x_i + s \frac{x_{i+1} - x_i}{\gamma_{i+1}}, \text{ for } s \in [0, \gamma_{i+1}] \text{ and } \bar{\gamma}(\tau_i + s) := \gamma_{i+1} \text{ for } s \in [0, \gamma_{i+1}[,$$

and consider its limit set

$$\mathcal{L}(X) := \bigcap_{t \geq 0} \overline{\{X(s) : s \geq t\}}.$$

The attainability condition is crucial to show that  $X$  converges with positive probability to a given attractor

**Definition 1.4.** A point  $p \in M$  is attainable if, for any  $t > 0$  and any neighborhood  $U$  of  $p$ ,

$$\mathbb{P}(\exists s \geq t : X(s) \in U) > 0.$$

We call  $Att(X)$  the set of attainable points by  $X$ . The following statement is a special case of our main result, Theorem 2.14.

**Theorem 1.5.** Let  $A \subset M$  be an attractor for  $\Phi$  with basin of attraction  $\mathcal{B}(A)$ . If  $Att(X) \cap \mathcal{B}(A) \neq \emptyset$  then

$$P(\mathcal{L}(X) \subset A) > 0.$$

## 2 Convergence with positive probability

### 2.1 Set-valued dynamical systems relative to a differential inclusion.

We recall here some definitions and results due to Benaïm et al (see [6]).

Let  $F : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$  be a standard set-valued map and  $\Phi$  be the set-valued dynamical system associated to the differential inclusion

$$\frac{d\mathbf{x}}{dt} \in F(\mathbf{x}) \tag{5}$$

The notion of *internally chain transitive set* (ICT set) was introduced by Benaïm and Hirsch in [5] to analyse certain perturbations of flow relative to an ODE. This is an extension of the notion of chain recurrence due to Conley [11]. The concept of ICT sets was extended to differential inclusions by Benaïm et al. in [6].

We refer to this last reference for an accurate description of ICT sets. However, we only need the equivalent definition of these objects here and therefore we will say that  $L$  is internally chain transitive for  $\Phi$  if and only if it is invariant, compact and the restricted set-valued dynamical system  $\Phi|_L$  admits no proper attractor (i.e. no attractor distinct from  $L$ ).

**Theorem 2.1** (Benaïm et al, 2005). Let  $L$  be an internally chain transitive set and  $A$  be an attractor for  $\Phi$  with basin of attraction  $\mathcal{B}(A)$ . Then

$$\mathcal{B}(A) \cap L \neq \emptyset \Rightarrow L \subset A.$$

The space  $\mathcal{C}(\mathbb{R}_+, \mathbb{R}^m)$  of continuous paths, endowed with the metric

$$\mathbf{D}(\mathbf{x}, \mathbf{x}') := \sum_{k=1}^{\infty} \frac{1}{2^k} \min \left( \sup_{u \in [0, n]} \|\mathbf{x}(u) - \mathbf{x}'(u)\|, 1 \right)$$

is complete. A continuous map  $X : \mathbb{R}_+ \rightarrow \mathbb{R}^m$  is an *asymptotic pseudotrajectory* (APT) of the set-valued dynamical system  $(\Phi_t)_{t \geq 0}$  if  $\lim_{t \rightarrow +\infty} \mathbf{D}(X(t + \cdot), S_{\Phi}) = 0$ . In other terms, for any  $T > 0$ ,

$$\lim_{t \rightarrow +\infty} \inf_{\mathbf{z} \in S_{\Phi}} \|X(t + \cdot) - \mathbf{z}(\cdot)\|_{[0, T]} = 0,$$

where  $\|\cdot\|_{[0, T]}$  denotes the uniform norm on  $[0, T]$ . Heuristically this means that, for any  $T > 0$ , the curve joining  $X(t)$  to  $X(t+T)$  shadows the trajectory of some solution with arbitrary accuracy, provided  $t$  is large enough.

A fundamental property of asymptotic pseudotrajectories is that, if  $X$  is a bounded APT, then its limit set  $\mathcal{L}(X)$  is internally chain transitive. Consequently, as a consequence of Theorem 2.1, we have

**Corollary 2.2.** *Let  $X$  be an asymptotic pseudotrajectory of the set-valued dynamical system and  $A$  an attractor for  $\Phi$ . If  $\mathcal{L}(X)$  meets the basin of attraction of  $A$ , then it belongs to  $A$ .*

Let  $\delta$  be a positive real number. Then  $F^\delta$  is the set-valued map defined by

$$F^\delta(x) := \{y \mid \exists z \in B(x, \delta) \text{ such that } d(y, F(z)) < \delta\}. \quad (6)$$

**Definition 2.3.** *A continuous function  $\mathbf{y} : \mathbb{R}_+ \rightarrow \mathbb{R}^m$  is a  $(\delta(\cdot), \overline{U}(\cdot))$ -perturbed solution of the differential inclusion (5) if*

(i)  $\mathbf{y}$  is absolutely continuous,

(ii) there exists a function  $\delta : ]0, +\infty[ \rightarrow \mathbf{R}_+^*$  such that  $\delta(t) \downarrow_{t \rightarrow +\infty} 0$  and, for almost every  $t > 0$ ,

$$\frac{d\mathbf{y}(t)}{dt} - \overline{U}(t) \in F^{\delta(t)}(\mathbf{y}(t)),$$

(iii) the function  $\overline{U} : \mathbb{R}_+ \rightarrow \mathbb{R}^m$  is locally integrable and such that, for any  $T > 0$ ,

$$\lim_{t \rightarrow +\infty} \left\| \int_t^{t+T} \overline{U}(u) du \right\|_{[0, T]} = 0.$$

We recall the following theorem due to Benaïm et al (see [6] Theorem 4.2).

**Theorem 2.4** (Benaïm et al, 2005). *Any bounded perturbed solution of the differential inclusion (5) is an asymptotic pseudotrajectory of the set-valued dynamical system  $\Phi$ .*

## 2.2 A deterministic result

Let  $X$  be an APT of the set-valued dynamical system  $(\Phi_t)_{t \geq 0}$ . For any  $T > 0$ , we define the quantity

$$d_X(T) := \sup_{k \in \mathbb{N}} \inf_{\mathbf{z} \in S_{\Phi}} \|\mathbf{z}(\cdot) - X(kT + \cdot)\|_{[0, T]}.$$

The following lemma is the extension to differential inclusions of Lemma 6.8 in Benaïm[4].

**Lemma 2.5.** *Let  $A \subset \mathbb{R}^m$  be an attractor for the set-valued dynamical system  $\Phi$ , with basin of attraction  $\mathcal{B}(A)$ . Then, for any compact set  $K \subset \mathcal{B}(A)$ , there exists positive real numbers  $\alpha(K)$  and  $T(K)$  such that*

$$(X(0) \in K \text{ and } d_X(T(K)) < \alpha(K)) \Rightarrow \mathcal{L}(X) \subset A.$$

**Proof.** Let  $W$  be an open set such that

$$A \cup K \subset W \subset \overline{W} \subset \mathcal{B}(A).$$

There exists  $\alpha > 0$  such that  $N_{3\alpha}(A) \subset W$  and  $N_\alpha(K) \subset W$ . By definition of an attractor, there exists then a positive number  $T$  (which depends on  $\alpha$  and  $W$ ) such that

$$\Phi_{[T, +\infty[}(W) \subset N_\alpha(A).$$

Assume now that  $X(0) \in K$  and  $d_X(T) < \alpha$ . There exists a solution  $\mathbf{z}^1$  which shadows  $X$  on  $[0, T]$ ; in particular,

$$\mathbf{z}^1(0) \in N_\alpha(X(0)) \subset W \text{ and } X(T) \in N_\alpha(\mathbf{z}^1(T)).$$

By definition of  $T$ ,  $\mathbf{z}^1(T) \in N_\alpha(A)$ , which means that  $X(T) \in N_{2\alpha}(A) \subset W$ .

By a recursive argument, we show that the sequence  $(X(kT))_{k \geq 1}$  belongs to the set  $N_{2\alpha}(A)$ . Assume that  $X(kT) \in N_{2\alpha}(A)$ . Then, there exists a solution  $\mathbf{z}^{k+1}(\cdot)$  which is  $\alpha$ -close of  $X(kT + \cdot)$  on  $[0, T]$ : in particular,

$$\mathbf{z}^{k+1}(0) \in N_\alpha(X(kT)) \subset N_{3\alpha}(A) \subset W \text{ and } X(kT + T) \in N_\alpha(\mathbf{z}^{k+1}(T)) \subset N_{2\alpha}(A).$$

Consequently, the limit set  $\mathcal{L}(X)$  is contained in  $W \subset \mathcal{B}(A)$ . Hence,  $\mathcal{L}(X) \subset A$  by Corollary 2.2.  $\blacksquare$

### 2.3 Stochastic processes

In the following,  $(X(t))_{t \geq 0}$  will be a continuous time  $\mathbb{R}^m$ -valued stochastic process, adapted to some non decreasing sequence of  $\sigma$ -algebras  $(\mathcal{F}_t)_t$ .

**Hypothesis 2.6.** *There exists a map  $\omega : \mathbb{R}_+^3 \rightarrow \mathbb{R}_+$  such that, for any  $\alpha > 0$  and  $T > 0$ ,*

$$\mathbb{P} \left( \sup_{s \geq t} \inf_{\mathbf{z} \in S_\Phi} \|\mathbf{z}(\cdot) - X(s + \cdot)\|_{[0, T]} \geq \alpha \mid \mathcal{F}_t \right) \leq \omega(t, \alpha, T) \downarrow_{t \rightarrow +\infty} 0. \quad (7)$$

Recall that  $\text{Att}(X)$  is the set of attainable points by  $X$ .

**Theorem 2.7.** *Let  $A$  be an attractor and  $(X(t))_{t \geq 0}$  be an adapted process satisfying hypothesis 2.6. Then, if  $\text{Att}(X) \cap \mathcal{B}(A) \neq \emptyset$ , we have*

$$\mathbb{P}(\mathcal{L}(X) \subset A) > 0.$$

**Proof.** We adapt the proof of Theorem 7.3 in Benaïm [4]. Let  $U$  be an open set included in  $\mathcal{B}(A)$  and call  $K = \overline{U}$ . There exists  $\alpha(K)$  and  $T(K)$  such that

$$X(0) \in K \text{ and } d_X(T(K)) < \alpha(K) \Rightarrow \mathcal{L}(X) \subset A.$$

Let  $t > 0$  such that  $\omega(t, \alpha, T) < 1$  and denote  $t_n(k) = \frac{k}{2^n}$  for  $n$  and  $k$  in  $\mathbb{N}$ . We define the stopping time

$$\tau_n := \inf_{k \in \mathbb{N}} \{t_n(k) \mid X(t_n(k)) \in U, t_n(k) \geq t\}.$$

On the intersection of the events  $\{\tau_n < \infty\}$  and  $\{\sup_{s \geq \tau_n} \inf_{\mathbf{z} \in S_\Phi} \|\mathbf{z}(\cdot) - X(s + \cdot)\|_{[0, T]} \leq \alpha\}$  the set  $\mathcal{L}(X)$  is included in  $A$ . Consequently, we have

$$\begin{aligned} \mathbb{P}(\mathcal{L}(X) \subset A) &\geq \sum_{k \geq [2^n t] + 1} \mathbb{E} \left( \mathbb{P} \left( \sup_{s \geq \tau_n} \inf_{\mathbf{z} \in S_\Phi} \|\mathbf{z}(\cdot) - X(s + \cdot)\|_{[0, T]} \leq \alpha \mid \mathcal{F}_{t_n(k)} \right) \mathbb{I}_{\tau_n = t_n(k)} \right) \\ &\geq \sum_{k \geq [2^n t] + 1} (1 - \omega(t_n(k), \alpha, T)) \mathbb{P}(\tau_n = t_n(k)) \geq (1 - \omega(t, \alpha, T)) \mathbb{P}(\tau_n < +\infty), \end{aligned}$$

since  $\omega(t_n(k), \alpha, T) \leq \omega(t, \alpha, T)$ ,  $\forall k \geq [2^n t] + 1$ . On the other hand, the sequence of events  $\{\tau_n < +\infty\}$  is increasing and we have

$$\lim \uparrow \{\tau_n < +\infty\} = \{\exists s \geq t \mid X(s) \in U\}.$$

Now take an attainable point  $p \in \mathcal{B}(A)$  and  $U$  a neighborhood of  $p$ , such that  $\overline{U} \subset \mathcal{B}(A)$ . We have

$$\mathbb{P}(\mathcal{L}(X) \subset A) \geq (1 - \omega(t, \alpha, T)) \mathbb{P}(\exists s \geq t \mid X(s) \in U) > 0,$$

and the proof is complete.  $\blacksquare$

Now, we consider a compact set  $M \subset \mathbb{R}^m$  and a standard set-valued map  $F : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ . Let  $T > 0$ . Denote  $\Phi^T(M) := \bigcup_{s \in [0, T]} \overline{\Phi_s(M)}$ ,  $\|F\| = \sup_{x \in \Phi^T(M)} \sup_{y \in F(x)} \|y\|$  and let us define the compact set

$$K_C := K_{(\|F\|, C)} = \{\mathbf{y} \in \text{Lip}([0, T], \mathbb{R}^m) \mid \text{Lip}(\mathbf{y}) \leq \|F\| + C + 1, \mathbf{y}(0) \in M\},$$

where  $C$  is a positive constant.

**Remark 2.8.**  $K_C$  is appropriate to our situation since it contains every solution curve, restricted to an interval of length  $T$  and any  $(\delta(\cdot), \overline{U}(\cdot))$ -perturbed solution of the differential inclusion, with  $\sup_{t \in [0, T]} \overline{U}(t) \leq C$  and  $\delta \leq 1$ .

For  $\delta \in [0, 1]$ , let us define the set-valued application (with the convention  $\Lambda^0 = \Lambda$ ):

$$\Lambda^\delta : K_C \rightrightarrows K_C, \mathbf{z} \mapsto \Lambda^\delta(\mathbf{z}), \quad (8)$$

where  $\mathbf{y} \in \Lambda^\delta(\mathbf{z})$  if and only if there exists an integrable  $h : [0, T] \rightarrow \mathbb{R}^m$  such that  $h(u) \in F^\delta(\mathbf{z}(u)) \forall u \in [0, T]$  and

$$\mathbf{y}(\tau) = \mathbf{z}(0) + \int_0^\tau h(u) du, \quad \forall \tau \in [0, T].$$

Remark that  $\text{Fix}(\Lambda) := \{\mathbf{z} \in K_C \mid \mathbf{z} \in \Lambda(\mathbf{z})\}$  is the set of the restrictions on  $[0, T]$  of the solutions curves starting from  $M$ , which we denote  $S_{[0, T]}$  from now on. Additionally, we call  $d_{[0, T]}$  the distance associated to the uniform norm on  $[0, T]$ . The following lemma is an immediate consequence of Corollary 4.11, proved in the last section.

**Lemma 2.9.** *Let  $C > 0$  and  $\alpha > 0$ . There exists  $\varepsilon > 0$  and  $\delta_0 > 0$  such that, for any  $\delta < \delta_0$*

$$d_{[0, T]}(\mathbf{z}, \Lambda^\delta(\mathbf{z})) < \varepsilon \Rightarrow d_{[0, T]}(\mathbf{z}, S_{[0, T]}) < \alpha.$$

As a consequence, we obtain the following crucial result.



**Proposition 2.10.** *Assume that there exists a function  $\delta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  converging to zero and an uniformly bounded random process  $(\bar{U}(t))_{t \geq 0}$  such that  $(X(t))_{t \geq 0}$  is almost surely a bounded  $(\delta, \bar{U})$ -perturbed solution of the differential inclusion (5) and such that  $X(0) \in M$ . Then, if  $\bar{U}$  satisfies the following property*

$$\mathbb{P} \left( \sup_{s \geq t} \left\| \int_s^{s^+} \bar{U}(u) du \right\|_{[0, T]} \geq \varepsilon \mid \mathcal{F}_t \right) \leq \omega(t, \varepsilon, T) \downarrow_{t \rightarrow +\infty} 0, \quad (9)$$

$X(\cdot)$  satisfies hypothesis 2.6 and Theorem 2.7 may be applied.

**Proof.** First,  $X(\cdot)$  is almost surely an asymptotic pseudotrajectory of the set-valued dynamical system by Theorem 2.4. For Lipschitz (classical) dynamical systems, the fact that hypothesis 2.6 is checked follows from an application of Gronwall lemma (see Benaïm [4], section 7). However, this is no longer possible in our settings and this is the reason why we need Lemma 2.9. By assumption, we have almost surely

$$\frac{dX(t)}{dt} - \bar{U}(t) \in F^{\delta(t)}(X(t)), \quad \text{for almost every } t > 0.$$

Let  $T > 0$ . For any  $\tau \in [0, T]$ ,

$$X(s + \tau) - \int_s^{s+\tau} \bar{U}(u) du \in X(s) + \int_0^\tau F^{\delta(s)}(X(s + u)) du,$$

Hence,  $d_{[0, T]}(X(s + \cdot), \Lambda^{\delta(s)}(X(s + \cdot))) \leq \left\| \int_s^{s^+} \bar{U}(u) du \right\|_{[0, T]}$  and

$$\begin{aligned} \mathbb{P} \left( \sup_{s \geq t} d_{[0, T]}(X(s + \cdot), \Lambda^{\delta(t)}(X(s + \cdot))) \geq \varepsilon \mid \mathcal{F}_t \right) &\leq \mathbb{P} \left( \sup_{s \geq t} \left\| \int_s^{s^+} \bar{U}(u) ds \right\|_{[0, T]} \geq \varepsilon \mid \mathcal{F}_t \right) \\ &\leq \omega(t, \varepsilon, T). \end{aligned}$$

Now let  $\alpha > 0$ . By Lemma 2.9 there exists  $\varepsilon > 0$  (which depends on  $T$  and  $\alpha$ ) such that, for  $t$  large enough and  $s \geq t$ ,

$$d_{[0, T]}(X(s + \cdot), S_{[0, T]}) \geq \alpha \Rightarrow d_{[0, T]}(X(s + \cdot), \Lambda^{\delta(t)}(X(s + \cdot))) \geq \varepsilon.$$

Consequently, for these choices of  $t$  and  $\varepsilon$ ,

$$\begin{aligned} \mathbb{P} \left( \sup_{s \geq t} d_{[0, T]}(X(s + \cdot), S_{[0, T]}) \geq \alpha \mid \mathcal{F}_t \right) &\leq \mathbb{P} \left( \sup_{s \geq t} d_{[0, T]}(X(s + \cdot), \Lambda_{\delta(t)}(X(s + \cdot))) \geq \varepsilon \mid \mathcal{F}_t \right) \\ &\leq \mathbb{P} \left( \sup_{s \geq t} \left\| \int_s^{s^+} \bar{U}(u) ds \right\|_{[0, T]} \geq \varepsilon \mid \mathcal{F}_t \right) \\ &\leq \omega(t, \varepsilon(\alpha, T), T), \end{aligned}$$

and the proof is complete. ■

## 2.4 Convergence of Stochastic approximation algorithms

We introduce here a class of stochastic approximation processes which generalize the Robbins-Monro algorithms. Under some assumptions on the step size and the noise, we prove that hypothesis 2.6 is verified and that the conclusion of Theorem 2.7 holds.

**Definition 2.11** (Generalised stochastic approximation process). *Let  $(U_n)_n$  be an uniformy bounded  $\mathbb{R}^m$ -valued random process,  $(\gamma_n)_n$  a deterministic positive real sequence and  $(F_n)_n$  a sequence of set-valued maps on  $\mathbb{R}^m$ . We say that  $(x_n)_n$  is a generalised stochastic approximation process relative to the standard set-valued map  $F$  if the following assumptions are satisfied:*

(i) *we have the recursive formula*

$$x_{n+1} - x_n - \gamma_{n+1}U_{n+1} \in \gamma_{n+1}F_n(x_n),$$

(ii) *the step size satisfies*

$$\sum_n \gamma_n = +\infty, \quad \lim_n \gamma_n = 0,$$

(iii) *for all  $T > 0$ , we have almost surely*

$$\lim_{n \rightarrow +\infty} \sup \left\{ \left\| \sum_{i=n}^{k-1} \gamma_{i+1} U_{i+1} \right\| \mid k \text{ such that } \sum_{i=n}^{k-1} \gamma_i \leq T \right\} = 0.$$

(iv) *For all  $n \geq 0$ ,  $x_n \in M$ .*

(v) *For any  $\delta > 0$ , there exists  $n_0 \in \mathbb{N}$  such that*

$$\forall n \geq n_0, \quad F_n(x_n) \subset F^\delta(x_n).$$

In the following we will call  $X = (X(t))_t$  the continuous time affine interpolated process induced by a given generalised stochastic approximation process  $(x_n)_n$ .

**Proposition 2.12.** *The process  $X$  is almost surely a  $(\delta, \bar{U})$ -perturbed solution, for some deterministic function  $\delta$ , and  $\bar{U}$  the piecewise constant continuous time process associated to  $(U_n)_n$ :*

$$\bar{U}(t) := U_{n+1}, \quad \forall t \in [\tau_n, \tau_{n+1}[.$$

**Proof.** By straightforward computations (see the proof of proposition 1.3 in Benaïm et al. [6]), it is not difficult to see that almost surely,  $(X(t))_t$  is a perturbed solution associated to  $\bar{U}$  and

$$\delta(t) := \inf \left\{ \delta > 0 \mid \tau_n \geq t \Rightarrow F_n(x_n) \subset F^\delta(x_n) \right\} + \bar{\gamma}(t) \left( \bar{U}(t) + c \left( 1 + \sup_{x \in M} F(x) \right) \right),$$

which obviously converges to 0. ■

**Remark 2.13.** *The condition (9) is equivalent to*

$$\mathbb{P} \left( \sup_{m \geq n} \sup_{m < k \leq m(\tau_m + T)} \left\| \sum_{i=m}^{k-1} \gamma_{i+1} U_{i+1} \right\| \geq \varepsilon \mid \mathcal{F}_n \right) \leq \omega(n, \varepsilon, T) \downarrow_{n \rightarrow +\infty} 0. \quad (10)$$

*and we use the notation  $\Delta(n, T) := \sup_{n < k \leq m(\tau_m + T)} \left\| \sum_{i=n}^{k-1} \gamma_{i+1} U_{i+1} \right\|$  in the sequel.*

Our main result is now stated in full generality.

**Theorem 2.14.** *Let  $(x_n)_n$  be a stochastic approximation algorithm such that  $(U_n)_n$  satisfies (10). Then, if  $A$  is an attractor relatively to  $F$ , we have*

$$\text{Att}(X) \cap \mathcal{B}(A) \neq \emptyset \Rightarrow \mathbb{P}(\mathcal{L}(X) \subset A) > 0.$$

**Proof.** By Proposition 2.12, the conditions requested to apply Proposition 2.10 are satisfied. Hence, hypothesis 2.6 is checked and the result follows directly from Theorem 2.7. ■

In the particular case where  $(U_n)_n$  is a martingale difference:  $\mathbb{E}(U_{n+1} | \mathcal{F}_n) = 0$ , (10) is checked under simple assumptions on the noise and step size.

**Proposition 2.15.** *Let  $(U_n^0)_n$  be a martingale difference noise (not necessarily bounded) and assume that one of the following assumptions is satisfied:*

1) *There exists some  $q \geq 2$  such that*

$$\sum \gamma_n^{1+q/2} < +\infty \text{ and } \sup_n \mathbb{E}(\|U_n^0\|^q) < +\infty.$$

2) *There exists a deterministic sequence  $(M_n)_n$  such that  $M_n^2 = o((\gamma_n \log n)^{-1})$  and, for any  $n \in \mathbb{N}$ ,*

$$\forall \theta \in \mathbb{R}^m, \mathbb{E}(\exp(\langle \theta, U_{n+1}^0 \rangle) | \mathcal{F}_n) \leq \exp\left(\frac{M_n^2}{2} \|\theta\|^2\right),$$

then (10) is checked.

**Proof.** For the first point, we refer the reader to Benaïm [4]. Now for the second, let  $\theta \in \mathbb{R}^m$  and consider the process  $(Z_n(\theta))_n$  defined by

$$Z_n(\theta) := \exp\left(\sum_{i=1}^n \langle \theta, \gamma_i U_i^0 \rangle - \frac{\|\theta\|^2}{2} \sum_{i=1}^n \gamma_i^2 M_i^2\right).$$

$(Z_n(\theta))_n$  is a supermartingale by assumption. Hence, if we denote  $S_n := \sum_{i=n}^{k-1} \gamma_{i+1}^2 M_{i+1}^2$  and  $m_n := m(\tau_n + T)$ , for any  $\beta > 0$ ,

$$\begin{aligned} \mathbb{P}\left(\sup_{n < k \leq m_n} \left\langle \theta, \sum_{i=n}^{k-1} \gamma_{i+1} U_{i+1} \right\rangle \geq \beta \mid \mathcal{F}_n\right) &= \mathbb{P}\left(\sup_{n < k \leq m_n} \frac{Z_k(\theta)}{Z_n(\theta)} \exp\left(\frac{\|\theta\|^2}{2} \sum_{i=1}^k \gamma_i^2 M_i^2\right) \geq e^\beta \mid \mathcal{F}_n\right) \\ &\leq \mathbb{P}\left(\sup_{n < k \leq m_n} Z_k(\theta) \geq Z_n(\theta) \exp\left(\beta - \frac{\|\theta\|^2}{2} S_n\right) \mid \mathcal{F}_n\right) \\ &\leq \exp\left(\frac{\|\theta\|^2}{2} S_n - \beta\right). \end{aligned}$$

Let  $e \in \{e_1, \dots, e_m, -e_1, \dots, -e_m\}$ . We have

$$\begin{aligned} \mathbb{P}\left(\sup_{n < k \leq m_n} \left\langle e, \sum_{i=n}^{k-1} \gamma_{i+1} U_{i+1} \right\rangle \geq \varepsilon \mid \mathcal{F}_n\right) &= \mathbb{P}\left(\sup_{n < k \leq m_n} \left\langle \frac{\delta e}{S_n}, \sum_{i=n}^{k-1} \gamma_{i+1} U_{i+1} \right\rangle \geq \frac{\varepsilon^2}{S_n} \mid \mathcal{F}_n\right) \\ &\leq \exp\left(\frac{-\varepsilon^2}{2S_n}\right). \end{aligned}$$

$$\mathbb{P} \left( \sup_{n < k \leq m_n} \left\| \sum_{i=n}^{k-1} \gamma_{i+1} U_{i+1} \right\| \geq \varepsilon \mid \mathcal{F}_n \right) \leq 2m \exp \left( \frac{-\varepsilon^2}{2S_n} \right).$$

Let us introduce  $\varepsilon_n := \gamma_n M_n^2 \log n$ . Then, since  $\sum_{i=n}^{k-1} \gamma_{i+1} \leq T$ , we have

$$\mathbb{P} \left( \sup_{n < k \leq m_n} \left\| \sum_{i=n}^{k-1} \gamma_{i+1} U_{i+1} \right\| \geq \varepsilon \mid \mathcal{F}_n \right) \leq 2m \exp \left( \frac{-\varepsilon^2 \log n}{2T \sup_{k \geq n} \varepsilon_k} \right).$$

Since  $\sup_{k \geq n} \varepsilon_k \rightarrow 0$ , the application  $(n, \varepsilon, T) \mapsto \omega(n, \varepsilon, T) := 2d \sum_{m \geq n} \exp \left( \frac{-\varepsilon^2 \log n}{2T \sup_{k \geq n} \varepsilon_k} \right)$  converges to 0 as  $n$  tends to infinity and the proof is complete. ■

The following is a useful consequence of this statement.

**Corollary 2.16.** *Assume that  $U_n$  can be written  $U_n = U_n^0 + U_n^1$ , where*

- $(U_n^0)_n$  a martingale difference noise satisfying one of the assumptions in the Proposition 2.15,
- (10) is satisfied for  $(U_n^1)_n$ .

Then (10) is satisfied for  $(U_n)_n$

### 3 Application to the Markovian fictitious play learning model

We discuss here a Markovian strategy in a two-person game and study the induced dynamic. The model is studied by Benaïm and Raimond in [8] and was inspired by a so-called *pairwise comparison dynamic* introduced in Benaïm et al. [7].

#### 3.1 The model

The motivation is the following. We assume, in the initial model, that the *information situation* is the same as in the smooth fictitious play developed by Fudenberg and Levine (see [14] and [15]) where the considered player uses a best response strategy against the empirical moves of his opponent, with respect to a smooth perturbation of the payoff function. A player adopting a smooth fictitious play strategy needs to be informed of his payoff function as well as the moves of his opponents up to this stage.

For some reason (computational limits, restricted available strategy...), we consider here that the set of moves he can play at some stage is a subset of his action set, which depends of the last action taken.

More formally, we consider a two players game in normal form.  $I$  and  $L$  are the (finite) set of moves of respectively player 1 and player 2. These sets are of the form

$$I = \{1, \dots, m^1\}, \quad L = \{1, \dots, m^2\};$$

The maps  $(U^1, U^2) : I \times L \rightarrow \mathbb{R} \times \mathbb{R}$  denote the payoff (or utility) functions of players. The sets of mixed strategies available to players are denoted  $\mathcal{X} = \Delta(I)$  and  $\mathcal{Y} = \Delta(L)$ , where

$$\Delta(I) := \left\{ x = (x_1, \dots, x_{m^1}) \in \mathbb{R}_+^{m^1} \mid \sum_{i=1, \dots, m^1} x_i = 1 \right\},$$

and analogously for  $\Delta(L)$ . We will use the classical abuse of language for  $y \in Y$ :

$$U^1(i, y) = \sum_{l \in L} U^1(i, l) y_l.$$

For  $x \in \mathcal{X}$ ,  $y \in \mathcal{Y}$ , we call  $br^1(y) := \text{Argmax}_{x \in \mathcal{X}} U^1(x, y)$  and  $br^2(x) = \text{Argmax}_{y \in \mathcal{Y}} U^2(x, y)$ . We assume that a given game is played repeatedly and call  $X_n$  (resp.  $Y_n$ ) the move of player 1 (resp. player 2) at stage  $n$ . The empirical distribution of moves up to stage  $n$  is denoted  $\bar{x}_n$  (resp.  $\bar{y}_n$ )

Let  $M_0^1$  be an irreducible matrix, reversible with respect to its invariant probability distribution  $\pi_0^1$ .  $M_0^1$  represents the possibility or not to play an action depending on the last move: player 1 will be able to play action  $j$  after having played  $i$  if and only if  $M_0^1(i, j) > 0$ . For  $n \in \mathbb{N}$  and  $y \in Y$ , let us define the Markov matrix

$$M_n^1(i, j; y) = \begin{cases} M_0^1(i, j) \exp\left(-\beta_n^1 (U^1(i, y) - U^1(j, y))^+\right) & \text{if } i \neq j, \\ 1 - \sum_{k \neq i} M_n^1(i, k; y) & \text{if } i = j, \end{cases}$$

where  $(\beta_n^1)_n$  is some positive deterministic sequence.

**Definition 3.1.** A Markovian fictitious play (MFP) strategy for player 1, associated with  $(\beta_n^1)_n$  and  $(M_0^1, \pi_0^1)$  is a strategy  $\sigma$  such that, for any  $n \in \mathbb{N}$ ,

$$\mathbb{P}_\sigma(X_{n+1} = j \mid \mathcal{F}_n) = M_n^1(X_n, j; \bar{y}_n).$$

From now, we assume that both players use a Markovian fictitious play strategy, associated to  $M_0^p$  and  $(\beta_n^p)_n$  ( $p = 1, 2$ ). Let us introduce the random sequences

$$V_n := (\delta_{X_n}, \delta_{Y_n}) \quad \text{and} \quad v_n := \frac{1}{n} \sum_{i=1}^n V_i = (\bar{x}_n, \bar{y}_n).$$

Benaïm and Raimond proved that there exists positive constants  $\tilde{A}^p$ ,  $p = 1, 2$  (see section 3.2) such that, if  $\beta_n^p = A^p \log n$  with  $A^p < \tilde{A}^p$ , then  $(v_n)_n$  is a generalised stochastic approximation process, taking values in  $\Delta = \mathcal{X} \times \mathcal{Y}$ , with step size  $\gamma_n = 1/n$ , relatively to the map  $F(x, y) = \{( \alpha, \beta, \mid \alpha \in br^1(y) - x, \beta \in br^2(x) - y \}$ . Note that the corresponding differential inclusion is the best response dynamic:

$$(\dot{x}, \dot{y}) \in (br^1(y), br^2(x)) - (x, y)$$

We call  $Att(v)$  the attainability set of the discrete process  $(v_n)_n$ . Recall that  $p \in Att(v)$  if and only if, for any neighborhood  $N$  of  $p$  and any  $n_0 \in \mathbb{N}$ ,

$$\mathbb{P}(\exists n \geq n_0 \mid v_n \in N) > 0.$$

**Proposition 3.2.** Assume that the exploration matrices have positive diagonal values. Then  $Att(v)$  is equal to the whole state space  $\Delta$

**Proof.** This is a consequence of the definition of an exploration matrix. From any instant  $n$ , any player  $i$ , any move  $a_j^i$  and any positive integer  $p$ , player  $i$  will play action  $a_j^i$   $p$  times in a row with positive probability. ■

**Theorem 3.3.** There exists positive values  $\tilde{A}^p$ ,  $p = 1, 2$  (which depend only on the payoff functions and the exploration matrices) such that, if agent  $p$  plays accordingly to a MFP strategy with  $\beta_n^p = A^p \log n$  and  $A^p < \tilde{A}^p$ , then

$$\mathbb{P}(\mathcal{L}((v_n)_n) \subset A) > 0,$$

for any attractor  $A$  for the set-valued dynamical system induced by  $F$ .

In particular, a strict Nash equilibrium is always an attractor for the best response dynamic. Hence

**Corollary 3.4.** *Let  $\hat{v} = (\hat{x}, \hat{y})$  be a strict Nash equilibrium. Then, under the assumptions of Theorem 3.3,*

$$\mathbb{P}(v_n \rightarrow \hat{v}) > 0.$$

### 3.2 Proof of Theorem 3.3

Note that the Markov matrix  $M_n^1(\cdot, \cdot; y)$  defined in the previous section is reversible with respect to its invariant distribution  $\pi_n^1[y]$ :

$$(\pi_n^1[y])_i = (\pi_0^1)_i \exp(\beta U^1(i, y)).$$

Also, considering an irreducible Markovian matrix  $M$  and its invariant probability measure  $\pi$ , one can define the pseudo inverse  $Q$  of  $M$ , characterized by

$$Q(I - M) = (I - M)Q = I - \Pi, \quad Q\mathbf{1} = 0,$$

where  $\Pi$  is the matrix defined by  $\Pi(i, j) = \Pi(j)$ . Let us call  $\pi_n^1$  and  $Q_n^1$  (respectively  $\pi_n^2$  and  $Q_n^2$ ) the invariant probability and the pseudo inverse of the matrix  $M_n^1 := M_n^1(\cdot, \cdot; \bar{y}_n)$  (resp.  $M_n^2 := M_n^2(\cdot, \cdot; \bar{x}_n)$ ). We now define the *energy barrier* of the exploration matrix  $M_0^1$  with respect to the payoff function  $U^1$ . Let  $\Gamma_{i,j}$  be the set of admissible paths from  $i$  to  $j$  in the graph associated to  $M_0^1$ :  $\gamma = (i = i_0, i_1, \dots, i_n = j)$  is admissible if  $M_0^1(i_k, i_{k+1}) > 0$ ,  $k = 0, \dots, n-1$ . Then, denoting for  $y \in Y$ ,

$$Elev(i, j; y) := \min \{ \max \{ -U^1(k, y) \mid k \in \gamma \}, \gamma \in \Gamma_{i,j} \}.$$

we call

$$U^{1,\#}(y) := \max \{ Elev(i, j; y) + U^1(i, y) + U^1(j, y) - \max U^1(\cdot, y) \}, \quad U^{1,\#} := \max_{y \in Y} U^{1,\#}(y).$$

Obviously, the quantity  $U^{2,\#}$  is defined analogously. Let  $\tilde{A}^i := 1/2U^{i,\#}$ . The following proposition can be easily derived from the proof of Proposition 4.4 in Benaïm and Raimond [8].

**Proposition 3.5.** *Assume that the sequence  $\beta_n^1$  satisfies*

$$\frac{\beta_n^1}{A^1 \log n} \rightarrow_n 0, \quad \text{for some } 0 < A^1 < \tilde{A}^1.$$

*Then there exists a positive deterministic sequence  $(u_n)_n$  such that*

- a)  $\frac{|Q_n^1|^2 \log n}{n} \leq u_n \rightarrow_n 0$ ,
- b)  $\lim_n |\Pi_{n+1}^1 - \Pi_n^1| \leq u_n \rightarrow_n 0$ ,
- c)  $\lim_n |Q_{n+1}^1 - Q_n^1| \leq u_n \rightarrow_n 0$ .

Let  $\theta_n$  be the random variable

$$\theta_n := (\pi_n^1[\bar{y}_n], \pi_n^2[\bar{x}_n]),$$

we have the following

$$v_{n+1} - v_n = \frac{1}{n+1} (-v_n + V_{n+1}) = \frac{1}{n+1} (-v_n + \theta_n + U_{n+1}),$$

with

$$U_{n+1} = V_{n+1} - \theta_n = (\delta_{X_{n+1}} - \pi_n^1[\bar{y}_n], \delta_{Y_{n+1}} - \pi_n^2[\bar{x}_n]).$$

There remains to prove that the noise  $U_n$  checks the property (10).

**Lemma 3.6.** *Assume that the sequences  $(\beta_n^p)_n$  ( $p = 1, 2$ ) satisfy the assumption of Proposition 3.5. Then the random sequence  $\Delta(n, T)$  relative to  $(1/n)_n$  and  $(U_n)_n$  satisfies the identity (10).*

**Proof.** We call  $\zeta_{n+1}$  the term  $\delta_{X_{n+1}} - \pi_n^1[\bar{y}_n]$ , and we aim to prove that the inequality (10) is checked for  $(\zeta_n)_n$ . Clearly, this will imply that (10) is also satisfied for  $(U_n)_n$ . Without loss of generality, we therefore denote  $\Delta(n, T) := \sup_{n < k < m(\tau_n + T)} \left\| \sum_{i=n}^{k-1} \frac{1}{i+1} \zeta_{i+1} \right\|$ ,

First of all,  $\zeta_{n+1}$  can be written

$$\zeta_{n+1} = \delta_{X_{n+1}} (Id - \Pi_n) = \delta_{X_{n+1}} (Q_n - M_n Q_n).$$

There is then a natural decomposition:

$$\zeta_{n+1} = (\delta_{X_{n+1}} Q_n - \delta_{X_n} M_n Q_n) + (\delta_{X_n} M_n Q_n - \delta_{X_{n+1}} M_n Q_n).$$

The first term is a martingale difference, bounded by  $|Q_n|$  (up to a constant). Hence it satisfies the assumption 2) of Proposition 2.15, with  $M_n = \frac{u_n \sqrt{n}}{\log n}$ .

Now, for the second term, we have

$$\begin{aligned} \sum_{i=n}^{k-1} \frac{1}{i+1} (\delta_{X_i} M_i Q_i - \delta_{X_{i+1}} M_i Q_i) &\leq \sum_{i=n}^{k-1} \frac{1}{i+1} (\delta_{X_{i+1}} M_{i+1} Q_{i+1} - \delta_{X_{i+1}} M_i Q_i) \\ &+ \sum_{i=n}^{k-1} \left( \frac{1}{i} \delta_{X_i} M_i Q_i - \frac{1}{i+1} \delta_{X_{i+1}} M_{i+1} Q_{i+1} \right) + T \sup_{n \leq i \leq k-1} \frac{|Q_i|}{i}, \end{aligned}$$

since  $\left\| \sum_{i=n}^k \frac{1}{i(i+1)} \delta_{X_i} M_i Q_i \right\| \leq \max \{ |Q_i|/i \mid n \leq i \leq k-1 \} \sum_{i=n, \dots, k-1} 1/i$ .

The first term on the right side can be written

$$\sum_{i=n}^{k-1} \frac{1}{i+1} \delta_{X_{i+1}} (Q_{n+1} - Q_n + \Pi_{n+1} - \Pi_n),$$

and is bounded by the quantity  $T \max \{ |Q_{i+1} - Q_i| + |\Pi_{i+1} - \Pi_i| \mid i = n, \dots, k-1 \}$ .

Finally, we have

$$\left\| \sum_{i=n}^{k-1} \frac{1}{i+1} (\delta_{X_i} M_i Q_i - \delta_{X_{i+1}} M_i Q_i) \right\| \leq K(T) \sup_{i \geq n} \left\{ \frac{|Q_i|}{i} + |Q_{i+1} - Q_i| + |\Pi_{i+1} - \Pi_i| \right\}.$$

By Proposition 3.5, the term on the right is decreasing to zero and

$$\mathbb{P} \left( \sup_{m \geq n} \Delta(n, T) \geq \varepsilon \mid \mathcal{F}_n \right) \leq \omega(n, \varepsilon, T) \downarrow_n 0.$$

This concludes the proof. ■

## 4 Proof of Lemma 2.9

In the following,  $L^1([0, T])$  the set of all Lebesgue-integrable functions from  $[0, T]$  to  $\mathbb{R}^m$ . Let  $H : [0, T] \rightrightarrows \mathbb{R}^m$  be a set-valued map, such that, for any  $u \in [0, T]$ ,  $H(u)$  is a nonempty subset of  $\mathbb{R}^m$ .

**Definition 4.1.** We call  $\mathcal{S}(H)$  the set of integrable selections from  $[0, T]$  to  $\mathbb{R}^m$ :

$$\mathcal{S}(H) := \{h \in L^1([0, T]) \text{ such that } \forall u \in [0, T], h(u) \in H(u)\}.$$

With such a definition, we introduce the set-valued integral of  $H$  on  $[0, T]$ :

$$\int_{[0, T]} H(u) du := \left\{ \int h(u) du \mid h \in \mathcal{S}(H) \right\}.$$

$H$  is said to be *measurable* if its graph  $\{(t, x) \mid x \in H(t)\}$  is measurable and *integrally bounded* if there exists an integrable function  $h : [0, T] \rightarrow \mathbb{R}_+$  such that

$$\sup_{x \in H(t)} \|x\| \leq h(t), \quad \forall t \in [0, T].$$

Let  $h \in \mathcal{S}(H)$ . We call  $\Psi_h$  the map defined by  $\tau \in [0, T] \mapsto \int_0^\tau h(u) du$ .

The following theorem is due to Aumann [2]

**Theorem 4.2.** Let  $H$  be a set-valued map on  $[0, T]$  with nonempty images. Then

- \*  $\int_{[0, T]} H(u) du$  is convex,
- \* If  $H$  is measurable and integrally bounded then  $\int_{[0, T]} H(u) du$  is nonempty.
- \* If  $H$  has closed images then  $\int_{[0, T]} H(u) du$  is compact.
- \* If  $(H_k)_k$  is a sequence of set-valued functions uniformly integrally bounded then

$$\limsup_k \int_{[0, T]} H_k(u) du \subset \int_{[0, T]} \limsup_k H_k(u) du,$$

where  $x \in \limsup_k A_k$  if and only if every neighborhood of  $x$  intersects infinitely many  $A_k$ .

The next proposition is not a direct consequence of these results. However, the proofs of third and fourth points can be adapted to derive it:

**Proposition 4.3.** Let  $(H_n)$  be a sequence of set-valued map from  $[0, T]$  to  $\mathbb{R}^m$  uniformly integrally bounded and  $H$  be a set-valued map with non empty images. We assume that, for any  $\tau \in [0, T]$ ,  $\limsup_n H_n(\tau) \subset H(\tau)$ . For any  $n \in \mathbb{N}$ , let  $h_n \in \mathcal{S}(H_n)$ . Then,

- (i) If, for any  $u$ ,  $H(u)$  is convex, there exists  $h \in \mathcal{S}(H)$  such that  $h_n$  converges weakly in  $L^1([0, T])$  to  $h$  (up to a subsequence). In particular,  $\Psi_{h_n}$  converges simply to  $\Psi_h$ .
- (ii) Without the convexity assumption, there exists a function  $h$  on  $[0, T]$  with the property that, for any  $u \in [0, T]$ ,  $h(u) \in \text{co}(H(u))$  and such that  $h_n$  converges weakly in  $L^1([0, T])$  to  $h$ . In particular,  $\Psi_{h_n}$  converges simply to  $\Psi_h$ .

**Proof.** Since the sequence  $(H_n)$  is uniformly integrally bounded, the  $h_n$  are all bounded by an integrable function  $g : [0, T] \rightarrow \mathbb{R}_+$ . Then there is a subsequence of  $(h_n)$  with a weak limit  $h \in L^1([0, T])$  (See Dunford and Schwartz [13], Theorem IV.8.9). We may assume without loss of generality that  $(h_n)$  actually converges weakly to  $h$ . There remains to check that  $h$  belongs to the set  $\mathcal{S}(H)$ .

For  $A \subset L^1([0, T])$ , we note  $\text{co}(A)$  the *convex hull* (i.e. the smallest convex set containing  $A$ ) and  $\overline{\text{co}}(A)$  the smallest closed (for the  $L^1$  norm) convex set containing  $A$ .



Recall that, by Mazur's theorem, a convex subset of a  $L^1([0, T])$  is closed if and only if it is weakly closed. Consequently, let  $k \in \mathbb{N}$ . the set  $\overline{co}((h_n)_{n \geq k})$  is closed and convex and therefore weakly closed. Hence it contains  $h$ , which belongs to the weakly closed convex hull of  $(h_n)_{n \geq k}$ . Finally,

$$h \in \overline{co}((h_n)_{n \geq k}) = \overline{co((h_n)_{n \geq k})},$$

which means that there exists  $g_k \in co((h_n)_{n \geq k})$  such that  $\|h - g_k\|_{L^1} \leq 1/k$ . Finally,  $(g_k)$  converges to  $h$  in  $L^1$  and we may assume without loss of generality that  $(g_k)_k$  converges to  $h$  almost everywhere on  $[0, T]$ .

From Caratheodory's theorem, the convex hull of a set  $A$  is the set of all barycenters of families of  $m + 1$  elements of  $A$ . Consequently, for any  $k$  and  $u \in [0, T]$ , since  $(h_n(u))_{n \geq k}$  is a set of points in  $\mathbb{R}^m$ , we have

$$g_k(u) = \sum_{j=0}^m \lambda_k^j(u) e_k^j(u),$$

where  $\lambda_k^j(u) \geq 0$ ,  $\sum_{j=0}^m \lambda_k^j(u) = 1$  and  $e_k^j(u) \in \{h_n(u) \mid n \geq k\}$ . By compactity, we may assume that, for any  $j$ ,  $(e_k^j(u))_k$  converges to some  $e^j(u)$  and  $(\lambda_k^j(u))_k$  converges to  $\lambda^j(u)$ , such that  $\lambda_j(u) \geq 0$  and  $\sum_{j=0}^m \lambda^j(u) = 1$ . Finally,

$$h(u) = \lim_k g_k(u) = \sum_{j=0}^m \lambda^j(u) e^j(u).$$

For any  $j$ , since  $e^j(u)$  belongs to the limit set of the sequence  $(h_n(u))_n$ , it belongs to  $H(u)$ . Hence,  $h(u)$  belongs to  $co(H(u))$  and, if  $H(u)$  is convex,  $h(u) \in H(u)$ . The proof is complete. ■

**Lemma 4.4.** *Let  $(K, d)$  be a compact metric space and  $(\Lambda_n)_n, \Lambda : K \rightrightarrows K$  be set-valued maps such that  $\Lambda$  is standard and  $Fix(\Lambda) = \{x \in K \mid x \in \Lambda(x)\} \neq \emptyset$ . Assume that, for any  $x \in K$ ,*

- \*  $\Lambda_{n+1}(x) \subset \Lambda_n(x)$ ,
- \*  $\lim_n x_n = x \Rightarrow \limsup_n \Lambda_n(x_n) \subset \Lambda(x)$ .

Then, for all  $\delta > 0$ , there exists  $\epsilon > 0$  and  $n_0 \in \mathbb{N}$  such that for all  $n > n_0$

$$d(x, \Lambda_n(x)) \leq \epsilon \Rightarrow d(x, Fix(\Lambda)) \leq \delta.$$

**Proof.** First notice that, since  $Fix(\Lambda)$  is non empty and  $\Lambda_{n+1}(x) \subset \Lambda_n(x)$ , there exists  $x \in K$  such that  $x \in \Lambda_n(x)$  for all  $n$ . Assume that there are  $\delta > 0$ , a strictly increasing sequence of integers  $(n_k)_{k \geq 1}$  and a sequence  $(x_k)_{k \geq 1}$  in  $K$  such that

$$d(x_k, \Lambda_{n_k}(x_k)) \leq \frac{1}{k} \text{ and } d(x_k, Fix(\Lambda)) > \delta.$$

Then, there exists a sequence  $(y_k)_{k \geq 1}$  such that  $y_k \in \Lambda_{n_k}(x_k)$  and

$$d(x_k, y_k) \leq \frac{1}{k} \text{ and } d(x_k, Fix(\Lambda)) > \delta.$$

Without loss of generality, we may assume that  $x_k \rightarrow x \in K$  and  $y_k \rightarrow y$ . Consequently,  $d(x, Fix(\Lambda)) > \delta > 0$  and  $x = y$ . On the other hand,

$$y \in \limsup_k \Lambda_{n_k}(x_k) \subset \limsup_k \Lambda_k(x_k) \subset \Lambda(x),$$

which means that  $x \in \Lambda(x)$ , a contradiction. ■

**Remark 4.5.** If the  $\Lambda_n$  are closed, then it is sufficient to assume that  $\bigcap_n \Lambda_n(x) = \Lambda(x)$ ,  $\forall x$ . Indeed, by monotonicity and the fact that  $\Lambda_{n_k}$  is closed,  $y \in \Lambda_{n_k}(x)$ ,  $\forall k$ .

From now, we consider a bounded standard set-valued map  $F : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$  and  $T > 0$ . Let  $\delta$  be a positive real number. Then  $F^\delta$  is the set-valued map defined by (6) and  $\Lambda^\delta$  is the set-valued map defined by (8), extended to  $\mathcal{C}([0, T], \mathbb{R}^m)$ . Note that, with our current notations, the definition of the set-valued map  $\Lambda^\delta$  can be written, for  $\delta \geq 0$ :

$$\Lambda^\delta : \mathcal{C}([0, T], \mathbb{R}^m) \rightrightarrows \mathcal{C}([0, T], \mathbb{R}^m), \quad \mathbf{z} \mapsto \{\mathbf{z}(0) + \Psi_h \mid h \in \mathcal{S}(F^\delta(\mathbf{z}))\}.$$

**Proposition 4.6.**  $\Lambda$  is a closed set-valued map with non empty images.

**Proof.** First,  $\Lambda$  has non empty images since  $\int_0^T F(\mathbf{z})$  is non empty, for any  $\mathbf{z}$  (see Theorem 4.2). Let  $(\mathbf{z}_n)_n$  be a sequence of  $\mathcal{C}([0, T], \mathbb{R}^m)$ , which converges to some  $\mathbf{z}$  in  $\mathcal{C}([0, T], \mathbb{R}^m)$  and let  $(\mathbf{y}_n)_n$  be a sequence converging to  $\mathbf{y}$  such that, for all  $n \in \mathbb{N}$ ,  $\mathbf{y}_n \in \Lambda(\mathbf{z}_n)$ . This implies that,

$$\forall n \in \mathbb{N}, \exists h_n \in \mathcal{S}(F(\mathbf{z}_n)) \text{ such that } \mathbf{y}_n(\tau) = \mathbf{z}_n(0) + \Psi_{h_n}(\tau).$$

We call  $H_n := F(\mathbf{z}_n)$  and  $H := F(\mathbf{z})$ . By assumptions we made on  $F$ ,  $H_n$  and  $H$  have compact, convex and nonempty values. For  $\tau \in [0, T]$ ,  $\mathbf{z}_n(\tau)$  converges to  $\mathbf{z}(\tau)$  and, since the graph of  $F$  is closed,

$$\limsup_n H_n(\tau) = \limsup_n F(\mathbf{z}_n(\tau)) \subset F(\mathbf{z}(\tau)) = H(\tau).$$

The assumptions of Proposition 4.3 are satisfied. Hence, there exists  $h \in \mathcal{S}(F(\mathbf{z}))$  such that  $\mathbf{y} = \mathbf{z}(0) + \Psi_h$ . ■

**Lemma 4.7.** For any  $\delta > 0$ ,  $F^\delta$  is a closed set-valued map with non empty images.

**Proof.** Let  $x \in \mathbb{R}^m$ .  $F(x)$  is contained in  $F^\delta$ . Hence, it is not empty. Let  $x_n \rightarrow x$  and  $(y_n)_n$  be a sequence of  $F^\delta(x_n)$  converging to some  $y$ . Then there exists a sequence  $(z_n)_n$  such that

$$d(z_n, x_n) < \delta \text{ and } d(y_n, F(z_n)) < \delta.$$

hence there exists a sequence  $(\alpha_n)_n$  such that  $\alpha_n \in F(z_n)$  and  $d(y_n, \alpha_n) < \delta$ . Without loss of generality, we may assume that  $z_n \rightarrow z$  and  $\alpha_n \rightarrow \alpha$ . By closeness of the graph of  $F$ , we obtain

$$d(z, x) < \delta, \quad \alpha \in F(z) \text{ and } d(y, \alpha) < \delta,$$

and  $F^\delta$  is closed (and, in particular, has closed images). ■

**Remark 4.8.** Note that the images are, a priori, not convex.

**Lemma 4.9.** Let  $(x_n)_n$  be a sequence of  $\mathbb{R}^m$ , converging to  $x$  and  $(\delta_n)_n$  be a positive, vanishing sequence. Then we have

$$\limsup_{n \rightarrow +\infty} F^{\delta_n}(x_n) \subset F(x).$$

**Proof.** Let  $y \in \limsup_n F^{\delta_n}(x_n)$ . By definition, there exists a sequence  $(y_n)$  which converges to  $y$  and such that  $y_n \in F^{\delta_n}(x_n)$  (actually it is a subsequence but there is no loss of generality to keep the initial indexation). Hence there exists a sequence  $(z_n)$  such that

$$d(z_n, x_n) < \delta_n, \quad d(y_n, F(z_n)) < \delta_n,$$

which means that  $d(x_n, \alpha_n) < \delta_n$  for some sequence  $(\alpha_n)_n$  satisfying  $\alpha_n \in F(z_n)$ . Without loss of generality we may assume that  $\alpha_n \rightarrow \alpha$  and  $z_n \rightarrow z$ . Hence we have  $y = \alpha \in F(z) = F(x)$ . ■

**Corollary 4.10.** Let  $(\mathbf{z}_n)_n$  be a sequence converging to  $\mathbf{z}$  in  $\mathcal{C}([0, T], \mathbb{R}^m)$ . Then we have, for any positive, vanishing sequence  $(\delta_n)_n$ ,

$$\limsup_{n \rightarrow +\infty} \Lambda^{\delta_n}(\mathbf{z}_n) \subset \Lambda(\mathbf{z}).$$

**Proof.** Let  $\mathbf{y} \in \limsup_n \Lambda^{\delta_n}(\mathbf{z}_n)$ . This means that there exists a sequence  $\mathbf{y}_n \in \Lambda^{\delta_n}(\mathbf{z}_n)$  which converges to  $\mathbf{y}$ . Hence, for all  $n \in \mathbb{N}$ , there exists  $h_n \in \mathcal{S}(F^{\delta_n}(\mathbf{z}_n))$  such that

$$\forall \tau \in [0, T], \mathbf{y}_n(\tau) = \mathbf{z}_n(0) + \int_0^\tau h_n(u) du$$

By Corollary 4.3 and Lemma 4.9, there exists a function  $h$  on  $[0, T]$  such that

$$\int_0^\tau h_n(u) du \rightarrow_n \int_0^\tau h(u) du, \quad \forall \tau \in [0, T]$$

and  $h \in \mathcal{S}(F(\mathbf{z}))$ , which completes the proof. ■

**Corollary 4.11.** Let  $\Lambda_n := \Lambda^{\delta_n}$ . Suppose there exists a compact  $K \subset \mathcal{C}([0, T], \mathbb{R}^m)$  such that  $\Lambda_n : K \rightrightarrows K$  for all  $n$ . Then, for all  $\delta > 0$ , there exists  $\epsilon > 0$  and  $n_0 \in \mathbb{N}$  such that, for all  $n > n_0$ ,

$$d(\mathbf{z}, \Lambda_n(\mathbf{z})) \leq \epsilon \Rightarrow d(\mathbf{z}, \text{Fix}(\Lambda)) \leq \delta.$$

**Proof.** This result follows from Lemma 4.4 and Corollary 4.10. ■

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