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SUMMING A POLYNOMIAL FUNCTION OVER INTEGRAL POINTS OF A POLYGON. USER'S GUIDE.

VELLEDA BALDONI, NICOLE BERLINE, AND MICHÈLE VERGNE

ABSTRACT. This document is a companion for the Maple program **Summing a polynomial function over integral points of a polygon**. It contains two parts. First, we see what this program does. In the second part, we briefly recall the mathematical background.

1. INTRODUCTION

The present article is a user's guide for the Maple program **Summing a polynomial function over integral points of a polygon**, available at <http://www.math.polytechnique.fr/~berline/maple.html>. The Maple program contains two types of computation. The first computation does just what the title says. The input consists of a finite set of rational points in \mathbb{Q}^2 , whose convex hull is a polygon \mathfrak{p} , and a polynomial $h(x, y)$ with rational coefficients. The output is the sum

$$\sum_{(x,y) \in \mathfrak{p} \cap \mathbb{Z}^2} h(x, y).$$

The second computation returns the function of $t \in \mathbb{N}$ which arises when the polytope \mathfrak{p} is dilated by t .

$$E(t) := \sum_{(x,y) \in t\mathfrak{p} \cap \mathbb{Z}^2} h(x, y).$$

This function is a *quasi-polynomial*, meaning that it has the form

$$E(t) = \sum_{i=0}^{\deg h+2} E_i(t) t^i,$$

where the coefficients depend only on $t \bmod q$, where q is the smallest integer such that $q\mathfrak{p}$ has integral vertices. The function $E(t)$ is called the *weighted Ehrhart quasi-polynomial* of \mathfrak{p} with respect to the weight $h(x, y)$.

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We apply two methods, the first one for a fixed polygon, the second one for the computation of the weighted Ehrhart quasi-polynomial. The first method is based directly on Brion's formula (2), [3], while the second method is based on the local Euler-Maclaurin formula of [2]. Both methods use Barvinok's decomposition into unimodular cones [1]. Although they are very similar, the first method is faster when we deal with a fixed polygon, while the second is faster when we want the Ehrhart quasi-polynomial.

The software libraries *LattE* [4] (improved version in [5]) and *Barvinok* [6] include the computation of the number of points of a rational polytope in any dimension, together with many other applications. Moreover, the weighted Ehrhart polynomials in any dimension are computed in *Barvinok*. The present program, in dimension two, is based on the same principles: Brion's formula and Barvinok's decomposition of cones. We use however some new ideas on "renormalisation" of Laurent series from [2] to speed up the computation. In the future, we will generalize it to higher dimensions.

2. MAIN COMMANDS

2.1. Summing a polynomial function over the set of integral points of a polygon. Let $P \subset \mathbb{Q}^2$ be a finite set of points. Let $\mathfrak{p} \subset \mathbb{R}^2$ be the polygon obtained as the convex hull of the set P . The program computes the sum

$$\sum_{(x,y) \in \mathfrak{p} \cap \mathbb{Z}^2} h(x,y)$$

of the values of a polynomial $h(x,y)$ over the set of integral points contained in \mathfrak{p} . In particular, when $h = 1$, it computes the number of integral points in \mathfrak{p} .

For a single monomial $h(x,y) = x^{m_1}y^{m_2}$, the command is

```
>sum_monomial_polygon(P,m);
```

Here P is a set of pairs of rational numbers, and $m = [m_1, m_2]$ is a pair of non negative integers, the multidegree of the monomial $x^{m_1}y^{m_2}$.

If we want just the number of integral of integral points, we can use the command

```
>number_points_polygon(P);
```

This number can be also obtained by the command

```
>sum_monomial_polygon(polygon,[0,0]);
```

We compute the sum of a polynomial $h(x,y)$ by the command

```
>sum_polynomial_polygon(P,h);
```

Here P is a set of pairs of rational numbers, and $h = \sum_m h_m x^{m_1} y^{m_2}$ is entered as an expression in x, y .

Example 1. P is the square $\{[0, 0], [1, 0], [1, 1], [0, 1]\}$.

```
>square:= {[0,0],[1,0],[1,1],[0,1]};
```

```
>number_points_polygon(square);
4
```

The sum of values $x^5 y^5$ over the 4 integral points in the square is

```
>sum_monomial_polygon(square,[5,5]);
1
```

Example 2. Here P is a randomly chosen set of 15 points.

```
> P := {[77/8,97/59],[93/44,70/29],[0,25/12],[25/32,29/48],
[92/41,57/91],[9/4,1/7],[64/43,31/75],[91/17,33/86],[12/37,77/8],
[8/5,41/27],[80/67,11/9],[16/73,11/89],[41/20,43/88],
[32/49,59/23],[77/94,65/46]};
```

The number of integral points in the convex hull is 45.

```
>number_points_polygon(P);
45
```

The vertices of the convex hull \mathbf{p} of P (listed in counter-clockwise order) are obtained with the command:

```
>vertices_in_counter_clock_order:=proc(polygon)
>vertices_in_counter_clock_order(P);
[[0,25/12],[16/73,11/89],[9/4,1/7],[91/17,33/86],[77/8,97/59],
[12/37,77/8]]
```

We compute the sum of $x^{32} y^{32}$ over the set of integral points (x, y) of the convex hull \mathbf{p} of P .

```
>sum_monomial_polygon(P,[32,32]);
987532646688766560932727042325214847653263886
```

We compute the sum of $x^{32} y^{32} + 7$ over all the integral points (x, y) of the polygon \mathbf{p} . (the preceding number $+7$ times 45)

```
>h:= x^{32}y^{32}+7;
>sum_polynomial_polygon(P,h);
987532646688766560932727042325214847653264201
```

2.2. Weighted Ehrhart polynomial of a polygon. Our program computes also the weighted Ehrhart quasi-polynomial of a polygon. For brevity, we treat only the case where the weight is a monomial $h(x, y) = x^{m_1} y^{m_2}$. When the polygon is dilated by a non negative

integer t , and if q is a positive integer such that $q\mathbf{p}$ has integral vertices, the function of t given by

$$t \mapsto \sum_{(x,y) \in t\mathbf{p} \cap \mathbb{Z}^2} x^{m_1} y^{m_2}$$

is a quasi-polynomial $S(t) = \sum_{i=0}^{m_1+m_2+2} E_i(t)t^i$ of degree $m_1 + m_2 + 2$. The coefficients $E_i(t)$ are functions of t modulo q . This program computes these coefficients $E_i(t)$ in terms of the symbolic function $fmod(p * t, q)$ which stands for $(t \mapsto pt \bmod q)$. We can either obtain each individual coefficient $E_i(t)$ or the full weighted Ehrhart polynomial $S(t)$.

Here are the commands:

```
> coeff_t_Ehrhart_polygon(i,t,P,m);
```

The input consists of i an integer, t a letter, P a set of points and $m = [m_1, m_2]$ a pair of integers which represents the weight; the output is the coefficient $E_i(t)$.

```
>Ehrhart_polynomial_polygon(t,P,m);
```

Input is as in the previous command, except i is not needed. The output is the full Ehrhart polynomial $S(t)$.

Examples

```
>transsquare:={[-1/2,-1/2]{[1/2,-1/2],[1/2,1/2],[-1/2,1/2]};
```

```
> coeff_t_Ehrhart_polygon(0,t,square,[0,0]);
```

1

```
> coeff_t_Ehrhart_polygon(0,t,transsquare,[0,0]);
```

```
-2*fmod(t, 2)+3/2+1/2*fmod(t, 2)^2
```

```
> Ehrhart_polynomial_polygon(t,square,[0,0]);
```

1 + 2 t + t^2

```
> Ehrhart_polynomial_polygon(t,transsquare,[0,0]);
```

(fmod(t, 2)-1)^2+(-2*fmod(t, 2)+2)*t+t^2

2.3. Experiments. The following experiments were done with a laptop, processor 1,86 GHz, RAM 782 MHz,0,99 Go.

```
>A:={ [(-567337)/102495,-1414975/95662],[1/3,1/5],[-88141/20499,12732/47831]}
```

```
>largeA:={ [1000*(-567337)/102495,1000*(-1414975/95662)],
```

```
[1000*1/3,1000*1/5],[-1000*88141/20499,1000*12732/47831]};
```

```
> number_points_polygon(A);
```

```
> number_points_polygon(largeA);
      34922612
```

In the next experiments, we indicate the time of computation T in seconds. The number of integral points in the rational triangle with vertices A is 36. If we dilate A by the factor 1000, we obtain the triangle $largeA$ where the number of points (34922612) is approximatively 10^6 times larger. Observe that we compute in 14 seconds the sum of the large degree monomial $h(x, y) = x^{64}y^{64}$ over the set of integral points of A , and that we compute in 16 seconds the sum of the same monomial over the integral points of $largeA$. The computation time is almost the same, although any computation by enumeration would be 10^6 times longer.

```
> T:=time(): sum_monomial_polygon(A, [32,32]);Time:=time()-T;
      11156693714080121436809683716369682546812787494001398139657
      Time := 1.766
```

```
> T:=time():sum_monomial_polygon(A, [64,64]);Time:=time()-T;
      10691662746975383171690687952963005219723639375189814
      217756070191566530558879\
      3836513555847334896253718879462978590217
      Time := 13.640
```

```
> T:=time(): sum_monomial_polygon(largeA, [64,64]): Time:=time()-T;
      1783103591372206604358967784049666198798919356345005767183297976710270822606
      90519522342895765988221612337480372436229072894493363579270305297678267123
      40160119137597718403779959778986161713238013119891186401529359213636522185
      95244921491613319792892241946298996049559369929767070065285383458443917290
      85791611969462010599632957347801451344938373887397255088905193762020134177
      82911075684183735887058845417207962477000059284528111310251783657942905087
      20099703621578931359063825440122383120351301766010118556183
      Time:= 15.516
```

Finally, we computed the weighted Ehrhart polynomial with weight $x^{32}y^{32}$ over the triangle with vertices $[[-567337/102495, -1414975/95662], [88141, 292844676/6833], [-88141/20499, 12732/47831]]$. We compute the coefficient of t^2 for example. The time of computation is 268 seconds. The result is too big to be printed here, as it involves many functions $fmod(c*t, D)$ where D runs through the denominators of the coordinates of the vertices of A (large numbers).

```
> T:=time():coeff_t_Ehrhart_polygon(2,t, [[-567337/102495,
```

```
-1414975/95662], [88141, 292844676/6833], [-88141/20499,
12732/47831]], [32,32]): Time:=time()-T;
Time := 268.266
```

3. MATHEMATICAL BACKGROUND

The first method is for a fixed polygon, the second one for the computation of the weighted Ehrhart quasi-polynomial.

3.1. First method: Brion's formula, Barvinok's decomposition into unimodular cones and iterated Laurent series. Let \mathfrak{p} be a convex polygon in \mathbb{R}^2 with rational vertices s_i , $1 \leq i \leq n+1$. We want to compute the sum

$$(1) \quad \sum_{x \in \mathfrak{p} \cap \mathbb{Z}^2} x^{m_1} y^{m_2}.$$

We start by observing that (1) is equal to the coefficient of $\frac{\xi_1^{m_1} \xi_2^{m_2}}{m_1! m_2!}$ in

$$\sum_{x \in \mathfrak{p} \cap \mathbb{Z}^2} e^{\langle \xi, x \rangle}.$$

Our method is based on Brion's formula (2). Brion's formula is the generalization of the following formula for the sum of geometric progressions over the interval $[A, B]$ (with $A \leq B$ integers):

$$\sum_A^B e^{n\xi} = \frac{e^{A\xi}}{1 - e^\xi} + \frac{e^{B\xi}}{1 - e^{-\xi}}.$$

For any rational polygon $\mathfrak{q} \subset \mathbb{R}^2$ define

$$S(\mathfrak{q})(\xi) = \sum_{x \in \mathfrak{q} \cap \mathbb{Z}^2} e^{\langle \xi, x \rangle}.$$

This a meromorphic function near $\xi = 0$. Moreover the map $\mathfrak{q} \mapsto S(\mathfrak{q})(\xi)$ is a valuation on the set of rational polyhedra, and $S(\mathfrak{q}) = 0$ if \mathfrak{q} contains a line. Brion's formula is the following. Let \mathfrak{c}_i be the cone at vertex S_i of the polygon.

$$(2) \quad S(\mathfrak{p}) = \sum_{i=1}^{n+1} S(\mathfrak{c}_i).$$

Each term $S(\mathfrak{c}_i)(\xi) = \sum_{x \in \mathfrak{c}_i \cap \mathbb{Z}^2} e^{\langle \xi, x \rangle}$ in (2) is a meromorphic function near $\xi = 0$. The poles cancel and the sum is a holomorphic function of ξ . Thus we compute (1) as the coefficient of $\frac{\xi_1^{m_1} \xi_2^{m_2}}{m_1! m_2!}$ in the right-hand-side of (2). We actually compute the individual contribution of each cone \mathfrak{c}_i (associated to the vertex s_i) to the sum. The coefficient of

$\frac{\xi_1^{m_1} \xi_2^{m_2}}{m_1! m_2!}$ in the meromorphic function $S(\mathbf{c}_i)$ of two variables ξ_1, ξ_2 has no intrinsic meaning. Our method consists in applying **iterated Laurent series** expansions to $S(\mathbf{c}_i)(\xi)$ with respect to the variables ξ_1 then ξ_2 . We obtain a Laurent series $L(\mathbf{c}_i)$ in the ring $\mathbb{Q}[\xi_1, \xi_1^{-1}, \xi_2, \xi_2^{-1}]$ and we compute the coefficient $\frac{\xi_1^{m_1} \xi_2^{m_2}}{m_1! m_2!}$ in $L(\mathbf{c}_i)$.

Thus, in order to compute the contribution of a vertex s to the sum (2), we need to compute $S(\mathbf{c})(\xi)$ for the supporting cone \mathbf{c} . The crucial tool here is Barvinok's decomposition into unimodular cones. Actually, we use the following variant of Barvinok's decomposition, (procedure `signed_decomp`).

Let \mathbf{c} be a simplicial cone in \mathbb{R}^d . Let V_i , for $i = 1, \dots, d$, be the generators of \mathbf{c} . Let V be a vector in \mathbb{R}^d . We write $V = \sum_i u_i V_i$. We split $[V_1, \dots, V_d]$ into three parts, as follows.

$$L_+ := [X_1, \dots, X_k]$$

formed by the V_i such that $u_i > 0$,

$$L_- := [Y_1, \dots, Y_m]$$

formed by the V_i such that $u_i < 0$,

$$L_0 := \{Z_1, \dots, Z_b\}$$

formed by the V_i such that $u_i = 0$.

Then we have the equality of characteristic functions modulo characteristic functions of cones containing lines.

$$\begin{aligned}
 (-1)^{(k+1)}[\mathbf{c}] &= \sum_{i=1}^k (-1)^{i+1} [\mathbf{c}(X_1, \dots, X_{i-1}, -X_{i+1}, \dots, -X_k, V, L_-, L_0)] + \\
 &\sum_{j=1}^m (-1)^{j+k} [\mathbf{c}(L_+, -V, -Y_1, \dots, -Y_{j-1}, Y_{j+1}, \dots, Y_m, L_0)].
 \end{aligned}$$

Remark. This decomposition is not the stellar decomposition. It involves only cones of maximal dimension d . It avoids the dualizing trick of Brion.

Example. $\mathbf{c} = \mathbb{R}^+ e_1 \oplus \mathbb{R}^+ e_2$, $V = e_1 + e_2$, so that L_- and L_0 are empty and $k = 2$. Then

$$-[\mathbf{c}] = \mathbf{c}(V, -e_2) - \mathbf{c}(e_1, V) - \mathbf{c}[e_2, -e_2, e_1].$$

Indeed $[\mathbf{c}(e_2, -e_2, e_1)] - [\mathbf{c}]$ is equal to the characteristic function of the quadrant $(e_1, -e_2)$ minus that of the half-line $\mathbb{R}^+ e_1$. This is also the case for $[\mathbf{c}(V, -e_2)] - [\mathbf{c}(e_1, V)]$.

If we use a lattice vector V with sufficiently small coordinates in the basis (V_i) , the cones appearing in this decomposition have indices smaller than \mathbf{c} . One obtains such a *short* vector V by the Lenstra-Lenstra-Lovasz algorithm. By a repeated application of this decomposition, one obtains a decomposition of \mathbf{c} in a signed sum of unimodular cones \mathbf{c}_z (modulo cones containing lines). As $S(\mathbf{a}) = 0$ for a cone \mathbf{a} which contains a line, we can use this decomposition to compute $S(\mathbf{c})$.

For a unimodular cone \mathbf{c} , the sum $S(\mathbf{c})$ has a simple closed expression. Let (V_1, V_2) be primitive generators of the edges of \mathbf{c} and let s be its vertex. Let \tilde{s} be the unique integral point contained in the *semi-closed box*

$$\{s + t_1V_1 + t_2V_2, 0 \leq t_i < 1\}$$

If $s = s_1V_1 + s_2V_2$, then $\tilde{s} = \tilde{s}_1V_1 + \tilde{s}_2V_2$ with $\tilde{s}_i = \text{ceil}(s_i)$. Then

$$(3) \quad S(\mathbf{c})(\xi) = \frac{e^{\langle \xi, \tilde{s} \rangle}}{(1 - e^{\langle \xi, V_1 \rangle})(1 - e^{\langle \xi, V_2 \rangle})}.$$

In order to simplify the computation of iterated Laurent series, we introduce the analytic function

$$B(X, u) = \frac{e^{uX}}{1 - e^X} + \frac{1}{X} = - \sum_{n=0}^{\infty} \frac{b(n+1, u)}{(n+1)!} X^n$$

where $b(n, u)$ are the Bernoulli polynomials. Writing

$$(4) \quad \frac{e^{uX}}{1 - e^X} = B(X, u) - \frac{1}{X},$$

we obtain

$$S(\mathbf{c})(\xi) = A + G + R,$$

where

$$A = B(\langle \xi, V_1 \rangle, \tilde{s}_1)B(\langle \xi, V_2 \rangle, \tilde{s}_2)$$

is an analytic function of ξ ,

$$G := -\frac{1}{\langle \xi, V_1 \rangle} B(\langle \xi, V_2 \rangle, \tilde{s}_2) - \frac{1}{\langle \xi, V_2 \rangle} B(\langle \xi, V_1 \rangle, \tilde{s}_1),$$

$$R := \frac{1}{\langle \xi, V_1 \rangle \langle \xi, V_2 \rangle}.$$

We replace $\frac{1}{\langle \xi, V_1 \rangle}$ and $\frac{1}{\langle \xi, V_2 \rangle}$ by their iterated Laurent series expansion in the ring $R[\xi_1, \xi_1^{-1}, \xi_2, \xi_2^{-1}]$. For example, if $V_1 = [2, 1]$, we write

$$\frac{1}{2\xi_1 + \xi_2} = \frac{1}{\xi_2} \frac{1}{(1 + 2\xi_1/\xi_2)} = \frac{1}{\xi_2} \sum_{k=0}^{\infty} (-1)^k 2^k (\xi_1/\xi_2)^k.$$

We then replace $S(\mathbf{c})$ by the corresponding element in $\mathbb{Q}[[\xi_1, \xi_2, \xi^{-1}, \xi^{-2}]]$ and we take the coefficient of $\xi_1^{m_1} \xi_2^{m_2}$.

Remark The weighted Ehrhart polynomial can also be computed by this method. We did not write the corresponding algorithm in the Maple file, because we observed that a faster algorithm is given by the second method which we describe in the next section. However, let us explain what one should do. When the polytope \mathbf{p} is dilated in $t\mathbf{p}$, its vertices are dilated by t , while the edges of the cones at vertices do not change. Thus we have to compute

$$(5) \quad S(\mathbf{c}_t)(\xi) = \frac{e^{\langle \xi, \bar{s}_t \rangle}}{(1 - e^{\langle \xi, V_1 \rangle})(1 - e^{\langle \xi, V_2 \rangle})}$$

where now s_t is the unique point with integral coordinates in the box

$$\{ts + u_1V_1 + u_2V_2, 0 \leq u_i < 1\}.$$

If $s = s_1V_1 + s_2V_2$, with $s_i = p_i/q_i$, we see that

$$s_t = [ts_1 + \text{mod}(-tp_1, q_1)/q_1, ts_2 - \text{mod}(-tp_2, q_2)/q_2].$$

The iterated Laurent series in ξ_1, ξ_2 has coefficients which are polynomials in t and the periodic functions $\text{mod}(tp_i, q_i)$. We extract the coefficient of $t^j \xi_1^{m_1} \xi_2^{m_2}$.

3.2. Second method. Weighted Ehrhart quasi-polynomial using local Euler-Maclaurin formula. We now recall the results of [2] and explain how they can be applied to the computation of the weighted Ehrhart quasi-polynomials. Let \mathbf{p} be a convex polytope in \mathbb{R}^d , with rational vertices. Let $h(x)$ be a polynomial function of degree r on \mathbb{R}^d . We want to compute the sum $\sum_{x \in \mathbf{p} \cap \Lambda} h(x)$ of values $h(x)$ over the set of integral points of the polytope \mathbf{p} .

The local Euler-Maclaurin formula has the following form.

$$(6) \quad \sum_{x \in \mathbf{p} \cap \Lambda} h(x) = \sum_{\mathbf{f} \in \mathcal{F}(\mathbf{p})} \int_{\mathbf{f}} D(\mathbf{p}, \mathbf{f}) \cdot h$$

where $\mathcal{F}(\mathbf{p})$ is the set of all faces of \mathbf{p} . For each face \mathbf{f} , $D(\mathbf{p}, \mathbf{f})$ is a differential operator of infinite degree with constant coefficients associated to \mathbf{f} . The operator $D(\mathbf{p}, \mathbf{f})$ is *local*, in the sense that it depends only on the intersection of \mathbf{p} with a neighborhood of any generic point of \mathbf{f} . The integral on the face \mathbf{f} is taken with respect to the Lebesgue measure on $\langle \mathbf{f} \rangle$ defined by the lattice $\mathbb{Z}^d \cap \text{lin}(\mathbf{f})$. Here $\langle \mathbf{f} \rangle$ is the affine span of the face \mathbf{f} and $\text{lin}(\mathbf{f})$ is the linear subspace parallel to $\langle \mathbf{f} \rangle$.

Let us recall the construction of the operators $D(\mathbf{p}, \mathbf{f})$. We denote by $\mathbf{t}(\mathbf{p}, \mathbf{f})$ the transverse cone to \mathbf{p} along \mathbf{f} . Using the standard scalar

product, $\mathfrak{t}(\mathfrak{p}, \mathfrak{f})$ is described as the following affine cone in \mathbb{R}^d . Let $\text{lin}(\mathfrak{f})^\perp$ be the vector subspace orthogonal to $\text{lin}(\mathfrak{f})$. Then $\mathfrak{t}(\mathfrak{p}, \mathfrak{f})$ is the orthogonal projection on $\text{lin}(\mathfrak{f})^\perp$ of the supporting cone of \mathfrak{p} along \mathfrak{f} . The operator $D(\mathfrak{p}, \mathfrak{f})$ is defined in terms of the transverse cone $\mathfrak{t}(\mathfrak{p}, \mathfrak{f})$, as follows.

For every rational affine cone $\mathfrak{a} \subset V$, we construct in [2] an analytic function $\xi \mapsto \mu(\mathfrak{a})(\xi)$ on \mathbb{R}^d . This construction depends on the choice of a scalar product. Here we use the standard scalar product. These functions $\mu(\mathfrak{a})$ have nice properties which play a crucial role in our method. First, the assignment $\mathfrak{a} \mapsto \mu(\mathfrak{a})$ is a *valuation* on the set of affine cones with a given vertex. Second, it is *invariant under lattice translations*. Furthermore, $\mu(\mathfrak{a}) = 0$ if \mathfrak{a} contains a line.

We define

$$D(\mathfrak{p}, \mathfrak{f}) = D(\mu(\mathfrak{t}(\mathfrak{p}, \mathfrak{f})))$$

as the differential operator of infinite degree with constant coefficients, with symbol $\mu(\mathfrak{t}(\mathfrak{p}, \mathfrak{f}))(\xi)$. In other words, if $\xi = (\xi_1, \dots, \xi_d)$, we obtain $D(\mathfrak{p}, \mathfrak{f})$ by replacing ξ_i by $\frac{\partial}{\partial x_i}$ in the Taylor series of $\mu(\mathfrak{t}(\mathfrak{p}, \mathfrak{f}))(\xi)$.

For any positive integer t , we consider the dilated polytope $t\mathfrak{p}$ and the corresponding sum

$$S(t\mathfrak{p}, h) = \sum_{x \in t\mathfrak{p} \cap \Lambda} h(x).$$

From (6), it follows easily that the function $t \mapsto S(t\mathfrak{p}, h)$ is given by a quasi-polynomial: there exist periodic functions $t \mapsto E_i(\mathfrak{p}, h, t)$ on \mathbb{N} such that

$$(7) \quad S(t\mathfrak{p}, h) = \sum_{i=0}^{d+r} E_i(\mathfrak{p}, h, t) t^i$$

whenever t is a positive integer. Moreover the coefficients $E_i(\mathfrak{p}, h, t)$ are computed using the functions $\mu(\mathfrak{t}(t\mathfrak{p}, t\mathfrak{f}))$. Indeed, let s be the vertex of $\mathfrak{t}(\mathfrak{p}, \mathfrak{f})$ so that $\mathfrak{t}(\mathfrak{p}, \mathfrak{f}) = s + \mathfrak{t}_0$. Then the dilated transverse cone is $\mathfrak{t}(t\mathfrak{p}, t\mathfrak{f}) = ts + \mathfrak{t}_0$. As $\mathfrak{a} \mapsto \mu(\mathfrak{a})$ is invariant under lattice translations, we have

$$\mu(\mathfrak{t}((t+q)\mathfrak{p}, t\mathfrak{f})) = \mu(\mathfrak{t}(t\mathfrak{p}, t\mathfrak{f})),$$

if q is an integer such that qs is a lattice point for the projected lattice, or equivalently, such that $q \langle \mathfrak{f} \rangle$ contains a lattice point. Thus, the coefficients $E_i(\mathfrak{p}, h, t)$ depend only on $t \pmod q$, where q is the smallest integer such that $q\mathfrak{p}$ has integral vertices.

When \mathfrak{a} is a *unimodular* affine cone of dimension 1 or 2, the functions $\mu(\mathfrak{a})$ have an explicit form, in terms of the functions $B(X, u)$ introduced in (4).

Let \mathfrak{d} be a one dimensional affine cone of the form $(s + \mathbb{R}_+)V$ where V is a primitive vector and $s \in \mathbb{Q}$. We have

$$(8) \quad \mu(\mathfrak{d})(\xi) = B(\langle \xi, V \rangle, \text{ceil}(s) - s).$$

Let \mathfrak{a} be a two dimensional *unimodular* affine cone. Let V_1, V_2 be primitive generators of its edges, such that $\det(V_1, V_2) = 1$. For $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$, let $y_i = \langle \xi, V_i \rangle$, for $i = 1, 2$, be the coordinates of ξ relative to the dual basis (V_1^*, V_2^*) . We write the vertex of \mathfrak{a} as $s_1V_1 + s_2V_2$ with $s_i \in \mathbb{Q}$. Let $\epsilon_i = \text{ceil}(s_i) - s_i$, and let $C_i = \frac{\langle V_1, V_2 \rangle}{\langle V_i, V_i \rangle}$, for $i = 1, 2$. With these notations, we have

$$(9) \quad \mu(\mathfrak{a})(\xi) = \frac{e^{\epsilon_1 y_1 + \epsilon_2 y_2}}{(1 - e^{y_1})(1 - e^{y_2})} + \frac{1}{y_1} B(y_2 - C_1 y_1, \epsilon_2) + \frac{1}{y_2} B(y_1 - C_2 y_2, \epsilon_1) - \frac{1}{y_1 y_2}.$$

The function $\mu(\mathfrak{a})(\xi)$ is actually analytic, although this is not obvious on (9). In order to compute the contribution of a vertex s of \mathfrak{p} to the sum (6), we need to compute $\mu(\mathfrak{c})(\xi)$ when \mathfrak{c} is the two-dimensional supporting cone at s . The crucial tool here is Barvinok's decomposition into unimodular cones. The valuation property of $\mathfrak{a} \mapsto \mu(\mathfrak{a})$ makes it possible to reduce the computation to the unimodular case, and use (9). Notice that (9) returns a function of the relative coordinates (y_1, y_2) , which we must convert back to a function of the standard coordinates (ξ_1, ξ_2) , in order to add the contributions of the various unimodular cones in Barvinok's decomposition. Actually, since $\mu(\mathfrak{a}) = 0$ if the cone \mathfrak{a} contains a line, we use the variant of Barvinok's decomposition described in the first method.

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