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Joyce Assaad

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Riesz transforms associated to Schrödinger operators with negative potentials

Joyce ASSAAD*

Abstract

The goal of this paper is to study the Riesz transforms $\nabla A^{-1/2}$ where A is the Schrödinger operator $-\Delta - V$, $V \geq 0$, under different conditions on the potential V . We prove that if V is strongly subcritical, $\nabla A^{-1/2}$ is bounded on $L^p(\mathbb{R}^N)$, $N \geq 3$, for all $p \in (p'_0; 2]$ where p'_0 is the dual exponent of p_0 where $2 < \frac{2N}{N-2} < p_0 < \infty$; and we give a counterexample to the boundedness on $L^p(\mathbb{R}^N)$ for $p \in (1; p'_0) \cup (p_{0*}; \infty)$ where $p_{0*} := \frac{p_0 N}{N+p_0}$ is the reverse Sobolev exponent of p_0 . If the potential is strongly subcritical in the Kato subclass K_N^∞ , then $\nabla A^{-1/2}$ is bounded on $L^p(\mathbb{R}^N)$ for all $p \in (1; 2]$, moreover if it is in $L^{N/2}(\mathbb{R}^N)$ then $\nabla A^{-1/2}$ is bounded on $L^p(\mathbb{R}^N)$ for all $p \in (1; N)$. We prove also boundedness of $V^{1/2}A^{-1/2}$ with the same conditions on the same spaces. Finally we study these operators on manifolds. We prove that our results hold on a class of Riemannian manifolds.

keywords: Riesz transforms, Schrödinger operators, off-diagonal estimates, singular operators, Riemannian manifolds.

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1 Introduction and definitions

Let A be a Schrödinger operator $-\Delta + V$ where $-\Delta$ is the nonnegative Laplace operator and the potential $V : \mathbb{R}^N \rightarrow \mathbb{R}$ such that $V = V^+ - V^-$ (where V^+ and V^- are the positive and negative parts of V , respectively). The operator is defined via the sesquilinear form method. We define

$$\mathfrak{a}(u, v) = \int_{\mathbb{R}^N} \nabla u(x) \nabla v(x) dx + \int_{\mathbb{R}^N} V^+(x) u(x) v(x) dx - \int_{\mathbb{R}^N} V^-(x) u(x) v(x) dx$$

*Université Bordeaux 1, Institut de Mathématiques (IMB). CNRS UMR 5251. Equipe d'Analyse et Géométrie. 351 Cours de la Libération 33405 Talence, France. Tel: (33) 05 40 00 21 71, Fax: (33) 05 40 00 26 26. e-mail adress: joyce.assaad@math.u-bordeaux1.fr

$$D(\mathbf{a}) = \left\{ u \in W^{1,2}(\mathbb{R}^N), \int_{\mathbb{R}^N} V^+(x)u^2(x)dx < \infty \right\}.$$

Here we assume $V^+ \in L^1_{loc}(\mathbb{R}^N)$ and V^- satisfies (for all $u \in D(\mathbf{a})$):

$$\int_{\mathbb{R}^N} V^-(x)u^2(x)dx \leq \alpha \left[\int_{\mathbb{R}^N} |\nabla u|^2(x)dx + \int_{\mathbb{R}^N} V^+(x)u^2(x)dx \right] + \beta \int_{\mathbb{R}^N} u^2(x)dx \quad (1)$$

where $\alpha \in (0, 1)$ and $\beta \in \mathbb{R}$. By the well-known KLMN theorem (see for example [21] Chapter VI), the form \mathbf{a} is closed (and bounded from below). Its associated operator is A . If in addition $\beta \leq 0$, then A is nonnegative.

We can define the Riesz transforms associated to A by

$$\nabla A^{-1/2} := \frac{1}{\Gamma(\frac{1}{2})} \int_0^\infty \sqrt{t} \nabla e^{-tA} \frac{dt}{t}.$$

The boundedness of Riesz transforms on $L^p(\mathbb{R}^N)$ implies that the domain of $A^{1/2}$ is included in the Sobolev space $W^{1,p}(\mathbb{R}^N)$. Thus the solution of the corresponding evolution equation will be in the Sobolev space $W^{1,p}(\mathbb{R}^N)$ for initial data in $L^p(\mathbb{R}^N)$.

It is our aim to study the boundedness on $L^p(\mathbb{R}^N)$ of the Riesz transforms $\nabla A^{-1/2}$. We are also interested in the boundedness of the operator $V^{1/2}A^{-1/2}$. If $\nabla A^{-1/2}$ and $V^{1/2}A^{-1/2}$ are bounded on $L^p(\mathbb{R}^N)$, we obtain for some positive constant C

$$\|\nabla u\|_p + \|V^{1/2}u\|_p \leq C\|(-\Delta + V)^{1/2}u\|_p.$$

By a duality argument, we obtain

$$\|(-\Delta + V)^{1/2}u\|_{p'} \leq C(\|\nabla u\|_{p'} + \|V^{1/2}u\|_{p'})$$

where p' is the dual exponent of p .

Riesz transforms associated to Schrödinger operators with nonnegative potentials were studied by Ouhabaz [24], Shen [27], and Auscher and Ben Ali [2]. Ouhabaz proved that Riesz transforms are bounded on $L^p(\mathbb{R}^N)$ for all $p \in (1, 2]$, for all potential V locally integrable. Shen and Auscher and Ben Ali proved that if the potential V is in the reverse Hölder class B_q , then the Riesz transforms are bounded on $L^p(\mathbb{R}^N)$ for all $p \in (1, p_1)$ where $2 < p_1 \leq \infty$ depends on q . The result of Auscher and Ben Ali generalize that of Shen because Shen has restrictions on the dimension N and on the class B_q . Recently, Badr and Ben Ali [5] extend the result of Auscher and Ben Ali

[2] to Riemannian manifolds of homogeneous type with polynomial volume growth where Poincaré inequalities hold and Riesz transforms associated to the Laplace-Beltrami operator are bounded. They also prove that a smaller range is possible if the volume growth is not polynomial.

With negative potentials new difficulties appear. If we take $V \in L^\infty(\mathbb{R}^N)$, and apply the method in [24] to the operator $A + \|V\|_\infty$, we obtain boundedness of $\nabla(A + \|V\|_\infty)^{-1/2}$ on $L^p(\mathbb{R}^N)$ for all $p \in (1; 2]$. This is weaker than the boundedness of $\nabla A^{-1/2}$ on the same spaces. Guillarmou and Hassell [17] studied Riesz transforms $\nabla(A \circ P_+)^{-1/2}$ where A is the Schrödinger operator with negative potential and P_+ is the spectral projection on the positive spectrum. They prove that, on asymptotically conic manifolds M of dimension $N \geq 3$, if V is smooth and satisfies decay conditions, and the Schrödinger operator has no zero-modes nor zero-resonances, then Riesz transforms $\nabla(A \circ P_+)^{-1/2}$ are bounded on $L^p(M)$ for all $p \in (1, N)$. They also prove (see [18]) that when zero-modes are present, Riesz transforms $\nabla(A \circ P_+)^{-1/2}$ are bounded on $L^p(M)$ for all $p \in (\frac{N}{N-2}, \frac{N}{3})$, with bigger range possible if the zero modes have extra decay at infinity.

In this paper we consider only negative potentials. From now on, we denote by A the Schrödinger operator with negative potential,

$$A := -\Delta - V, \quad V \geq 0.$$

Our purpose is, first, to find optimal conditions on V allowing the boundedness of Riesz transforms $\nabla A^{-1/2}$ and that of $V^{1/2} A^{-1/2}$ on $L^p(\mathbb{R}^N)$ second, to find the best possible range of p 's.

Let us take the following definition

Definition 1.1. We say that the potential V is strongly subcritical if for some $\varepsilon > 0$, $A \geq \varepsilon V$. This means that for all $u \in W^{1,2}(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N} V u^2 \leq \frac{1}{1 + \varepsilon} \int_{\mathbb{R}^N} |\nabla u|^2.$$

For more information on strongly subcritical potentials see [15] and [33]. With this condition, V satisfies assumption (1) where $\beta = 0$ and $\alpha = \frac{1}{1+\varepsilon}$. Thus A is well defined, nonnegative and $-A$ generates an analytic contraction semigroup $(e^{-tA})_{t \geq 0}$ on $L^2(\mathbb{R}^N)$.

Since $-\Delta - V \geq \varepsilon V$ we have $(1 + \varepsilon)(-\Delta - V) \geq \varepsilon(-\Delta)$. Therefore

$$\|\nabla u\|_2^2 \leq \left(1 + \frac{1}{\varepsilon}\right) \|A^{1/2} u\|_2^2. \quad (2)$$

Thus, $\nabla A^{-1/2}$ is bounded on $L^2(\mathbb{R}^N)$. Conversely, it is clear that if $\nabla A^{-1/2}$ is bounded on $L^2(\mathbb{R}^N)$ then V is strongly subcritical.

We observe also that $-\Delta - V \geq \varepsilon V$ is equivalent to

$$\|V^{1/2}u\|_2^2 \leq \frac{1}{\varepsilon} \|A^{1/2}u\|_2^2. \quad (3)$$

Thus, $V^{1/2}A^{-1/2}$ is bounded on $L^2(\mathbb{R}^N)$ if and only if V is strongly subcritical.

So we can conclude that

$$\|\nabla u\|_2 + \|V^{1/2}u\|_2 \leq C \|(-\Delta - V)^{1/2}u\|_2$$

if and only if V is strongly subcritical. Then by duality argument we have

$$\|\nabla u\|_2 + \|V^{1/2}u\|_2 \approx \|(-\Delta - V)^{1/2}u\|_2$$

if and only if V is strongly subcritical.

To study Riesz transforms on $L^p(\mathbb{R}^N)$ for $1 \leq p \leq \infty$ with $p \neq 2$ we use the results on the uniform boundedness of the semigroup on $L^p(\mathbb{R}^N)$. Taking central potentials which are equivalent to $c/|x|^2$ as $|x|$ tends to infinity where $0 < c < (\frac{N-2}{2})^2$, $N \geq 3$, Davies and Simon [15] proved that for all $t > 0$ and all $p \in (p'_0; p_0)$,

$$\|e^{-tA}\|_{p \rightarrow p} \leq C$$

where $2 < \frac{2N}{N-2} < p_0 < \infty$ and p'_0 its dual exponent. Next Liskevich, Sobol, and Vogt [23] proved the uniform boundedness on $L^p(\mathbb{R}^N)$ for all $p \in (p'_0; p_0)$ where $2 < \frac{2N}{N-2} < p_0 = \frac{2N}{(N-2)(1-\sqrt{1-\frac{1}{1+\varepsilon}})}$, for general strongly subcritical potentials. They also proved that the range (p'_0, p_0) is optimal and the semigroup does not even act on $L^p(\mathbb{R}^N)$ for $p \notin (p'_0, p_0)$. Under additional condition on V , Takeda [31] used stochastic methods to prove a Gaussian estimate of the associated heat kernel. Thus the semigroup acts boundedly on $L^p(\mathbb{R}^N)$ for all $p \in [1, \infty]$.

In this paper we prove that when V is strongly subcritical and $N \geq 3$, Riesz transforms are bounded on $L^p(\mathbb{R}^N)$ for all $p \in (p'_0; 2]$. We also give a counterexample to the boundedness of Riesz transforms on $L^p(\mathbb{R}^N)$ when $p \in (1; p'_0) \cup (p_{0*}; \infty)$ where $2 < p_{0*} := \frac{p_0 N}{N+p_0} < p_0 < \infty$. If V is strongly subcritical in the Kato subclass K_N^∞ , $N \geq 3$ (see Section 4), then $\nabla A^{-1/2}$ is bounded on $L^p(\mathbb{R}^N)$ for all $p \in (1, 2]$. If, in addition, $V \in L^{N/2}(\mathbb{R}^N)$ then it

is bounded on $L^p(\mathbb{R}^N)$ for all $p \in (1, N)$. With the same conditions, we prove similar results for the operator $V^{1/2}A^{-1/2}$. Hence if V is strongly subcritical and $V \in K_N^\infty \cap L^{N/2}(\mathbb{R}^N)$, $N \geq 3$, then

$$\|\nabla u\|_p + \|V^{1/2}u\|_p \approx \|(-\Delta - V)^{1/2}u\|_p \quad (4)$$

for all $p \in (N'; N)$.

In the last section, we extend our results to Riemannian manifolds. We denote by $-\Delta$ the Laplace-Beltrami operator on a complete non-compact Riemannian manifold M of dimension $N \geq 3$. We prove that when V is strongly subcritical on M , $\nabla(-\Delta - V)^{-1/2}$ and $V^{1/2}(-\Delta - V)^{-1/2}$ are bounded on $L^p(M)$ for all $p \in (p'_0; 2]$ if M is of homogeneous type and the Sobolev inequality holds on M . If in addition Poincaré inequalities hold on M and V belongs to the Kato class K_∞ then $\nabla(-\Delta - V)^{-1/2}$ and $V^{1/2}(-\Delta - V)^{-1/2}$ are bounded on $L^p(M)$ for all $p \in (1; 2]$. When V is in addition in $L^{N/2}(M)$ and the Riesz transforms associated to the Laplace-Beltrami operator are bounded on $L^r(M)$ for some $r \in (2; N]$, then $\nabla(-\Delta - V)^{-1/2}$ and $V^{1/2}(-\Delta - V)^{-1/2}$ are bounded on $L^p(M)$ for all $p \in (1; r)$.

For the proof of the boundedness of Riesz transforms we use off-diagonal estimates (for properties and more details see [4]). These estimates are a generalization of the Gaussian estimates used by Coulhon and Duong in [13] to study the Riesz transforms associated to the Laplace-Beltrami operator on Riemannian manifolds, and by Duong, Ouhabaz and Yan in [16] to study the magnetic Schrödinger operator on \mathbb{R}^N . We also use the approach of Blunck and Kunstmann in [8] and [9] to weak type (p, p) -estimates. In [1], Auscher used these tools to divergence-form operators with complex coefficients. For $p \in (2; N)$ we use a complex interpolation method (following an idea in Auscher and Ben Ali [2]).

In contrast to [17] and [18], we do not assume decay nor smoothness conditions on V .

In the following sections, we denote by L^p the Lebesgue space $L^p(\mathbb{R}^N)$ with the Lebesgue measure dx , $\|\cdot\|_p$ its usual norm, (\cdot, \cdot) the inner product of L^2 , $\|\cdot\|_{p \rightarrow q}$ the norm of operators acting from L^p to L^q . We denote by p' the dual exponent to p , $p' := \frac{p}{p-1}$. We denote by C, c the positive constants even if their values change at each occurrence. Through this paper, $\nabla A^{-1/2}$ denotes one of the partial derivative $\frac{\partial}{\partial x_k} A^{-1/2}$ for any fixed $k \in \{1, \dots, N\}$.

2 Off-diagonal estimates

In this section, we show that $(e^{-tA})_{t>0}$, $(\sqrt{t}\nabla e^{-tA})_{t>0}$ and $(\sqrt{t}V^{1/2}e^{-tA})_{t>0}$ satisfy $L^p - L^2$ off-diagonal estimates provided that V is strongly subcritical.

Definition 2.1. Let $(T_t)_{t>0}$ be a family of uniformly bounded operators on L^2 . We say that $(T_t)_{t>0}$ satisfies $L^p - L^q$ off-diagonal estimates for $p, q \in [1; \infty]$ with $p \leq q$ if there exist positive constants C and c such that for all closed sets E and F of \mathbb{R}^N and all $h \in L^p(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ with support in E , we have for all $t > 0$:

$$\|T_t h\|_{L^q(F)} \leq C t^{-\gamma_{pq}} e^{-\frac{cd(E,F)^2}{t}} \|h\|_p,$$

where d is the Euclidean distance and $\gamma_{pq} := \frac{N}{2}(\frac{1}{p} - \frac{1}{q})$.

Proposition 2.1. Let $A = -\Delta - V$ where $V \geq 0$ and V is strongly subcritical. Then $(e^{-tA})_{t>0}$, $(\sqrt{t}\nabla e^{-tA})_{t>0}$, and $(\sqrt{t}V^{1/2}e^{-tA})_{t>0}$ satisfy $L^2 - L^2$ off-diagonal estimates, and we have for all $t > 0$ and all $f \in L^2$ supported in E :

- (i) $\|e^{-tA} f\|_{L^2(F)} \leq e^{-d^2(E,F)/4t} \|f\|_2,$
- (ii) $\|\sqrt{t}\nabla e^{-tA} f\|_{L^2(F)} \leq C e^{-d^2(E,F)/16t} \|f\|_2,$
- (iii) $\|\sqrt{t}V^{1/2}e^{-tA} f\|_{L^2(F)} \leq C e^{-d^2(E,F)/8t} \|f\|_2.$

Proof: The ideas are classical and rely on the well known Davies perturbation technique. Let $A_\rho := e^{\rho\phi} A e^{-\rho\phi}$ where $\rho > 0$ and ϕ is a Lipschitz function with $|\nabla\phi| \leq 1$ a.e.. Here A_ρ is the associated operator to the sesquilinear form \mathbf{a}_ρ defined by

$$\mathbf{a}_\rho(u, v) := \mathbf{a}(e^{-\rho\phi} u, e^{\rho\phi} v)$$

for all $u, v \in D(\mathbf{a})$.

By the strong subcriticality property of V we have for all $u \in W^{1,2}$

$$\begin{aligned} ((A_\rho + \rho^2)u, u) &= - \int \rho^2 |\nabla\phi|^2 u^2 + \int |\nabla u|^2 - \int V u^2 + \rho^2 \|u\|_2^2 \\ &\geq \varepsilon \|V^{1/2} u\|_2^2. \end{aligned} \tag{5}$$

Using (2), we obtain

$$\begin{aligned} ((A_\rho + \rho^2)u, u) &= - \int \rho^2 |\nabla\phi|^2 u^2 + \int |\nabla u|^2 - \int V u^2 + \rho^2 \|u\|_2^2 \\ &\geq \frac{\varepsilon}{\varepsilon + 1} \|\nabla u\|_2^2. \end{aligned} \tag{6}$$

In particular $(A_\rho + \rho^2)$ is a maximal accretive operator on L^2 , and this implies

$$\|e^{-tA_\rho}u\|_2 \leq e^{t\rho^2}\|u\|_2. \quad (7)$$

Now we want to estimate

$$\|(A_\rho + 2\rho^2)e^{-t(A_\rho + 2\rho^2)}\|_{2-2}.$$

First, let us prove that $A_\rho + 2\rho^2$ is a sectorial operator.

For u complex-valued,

$$\mathbf{a}_\rho(u, u) := \mathbf{a}(u, u) + \rho \int u \nabla \phi \overline{\nabla u} - \rho \int \bar{u} \nabla \phi \nabla u - \rho^2 \int |\nabla \phi|^2 |u|^2.$$

Then

$$\begin{aligned} \mathbf{a}_\rho(u, u) + 2\rho^2 \|u\|_2^2 &\geq \mathbf{a}(u, u) + \rho \int u \nabla \phi \overline{\nabla u} - \rho \int \bar{u} \nabla \phi \nabla u + \rho^2 \|u\|_2^2 \\ &= \mathbf{a}(u, u) + 2i\rho \Im \int u \nabla \phi \overline{\nabla u} + \rho^2 \|u\|_2^2. \end{aligned}$$

This implies that

$$\Re(\mathbf{a}_\rho(u, u) + 2\rho^2 \|u\|_2^2) \geq \mathbf{a}(u, u), \quad (8)$$

and

$$\Re(\mathbf{a}_\rho(u, u) + 2\rho^2 \|u\|_2^2) \geq \rho^2 \|u\|_2^2. \quad (9)$$

On the other hand,

$$\begin{aligned} \mathbf{a}_\rho(u, u) &= \mathbf{a}(u, u) + \rho \int u \nabla \phi \overline{\nabla u} - \rho \int \bar{u} \nabla \phi \nabla u - \rho^2 \int |\nabla \phi|^2 |u|^2 \\ &= \mathbf{a}(u, u) + 2i\rho \Im \int u \nabla \phi \overline{\nabla u} - \rho^2 \int |\nabla \phi|^2 |u|^2. \end{aligned}$$

So

$$\begin{aligned} |\Im(\mathbf{a}_\rho(u, u) + 2\rho^2 \|u\|_2^2)| &= 2|\rho| \int |u| |\nabla \phi| |\overline{\nabla u}| \\ &\leq 2|\rho| \|u\|_2 \|\nabla u\|_2. \end{aligned}$$

Using (2) we obtain that

$$\begin{aligned} |\Im(\mathbf{a}_\rho(u, u) + 2\rho^2 \|u\|_2^2)| &\leq 2|\rho| \|u\|_2 c_\varepsilon \mathbf{a}^{\frac{1}{2}}(u, u) \\ &\leq c_\varepsilon^2 \mathbf{a}(u, u) + \rho^2 \|u\|_2^2, \end{aligned}$$

where $c_\varepsilon = (1 + \frac{1}{\varepsilon})^{\frac{1}{2}}$. Now using estimates (8) and (9), we deduce that there exists a constant $C > 0$ depending only on ε such that

$$|\Im(\mathbf{a}_\rho(u, u) + 2\rho^2\|u\|_2^2)| \leq C\Re(\mathbf{a}_\rho(u, u) + 2\rho^2\|u\|_2^2).$$

We conclude that (see [21] or [24])

$$\|e^{-z(A_\rho + 2\rho^2)}\|_{2-2} \leq 1$$

for all z in the open sector of angle $\arctan(1/C)$. Hence by the Cauchy formula

$$\|(A_\rho + 2\rho^2)e^{-t(A_\rho + 2\rho^2)}\|_{2-2} \leq \frac{C}{t}. \quad (10)$$

The constant C is independent of ρ .

By estimate (5) and (6) we have

$$((A_\rho + 2\rho^2)u, u) \geq ((A_\rho + \rho^2)u, u) \geq \varepsilon\|V^{1/2}u\|_2^2,$$

and

$$((A_\rho + 2\rho^2)u, u) \geq ((A_\rho + \rho^2)u, u) \geq \frac{\varepsilon}{\varepsilon + 1}\|\nabla u\|_2^2.$$

Setting $u = e^{-t(A_\rho + 2\rho^2)}f$ and using (10) and (7) we obtain

$$\|\sqrt{t}\nabla e^{-tA_\rho}f\|_2 \leq Ce^{2t\rho^2}\|f\|_2. \quad (11)$$

and

$$\|\sqrt{t}V^{1/2}e^{-tA_\rho}f\|_2 \leq Ce^{2t\rho^2}\|f\|_2. \quad (12)$$

Let E and F be two closed subsets of \mathbb{R}^N , $f \in L^2(\mathbb{R}^N)$ supported in E , and let $\phi(x) := d(x, E)$ where d is the Euclidean distance. Since $e^{\rho\phi}f = f$, we have the following relation

$$e^{-tA}f = e^{-\rho\phi}e^{-tA_\rho}f.$$

Then

$$\nabla e^{-tA}f = -\rho\nabla\phi e^{-\rho\phi}e^{-tA_\rho}f + e^{-\rho\phi}\nabla e^{-tA_\rho}f,$$

and

$$V^{1/2}e^{-tA}f = e^{-\rho\phi}V^{1/2}e^{-tA_\rho}f.$$

Now taking the norm on $L^2(F)$, we obtain from (7), (11) and (12)

$$\|e^{-tA}f\|_{L^2(F)} \leq e^{-\rho d(E,F)} e^{\rho^2 t} \|f\|_2, \quad (13)$$

$$\|\nabla e^{-tA}f\|_{L^2(F)} \leq \rho e^{-\rho d(E,F)} e^{\rho^2 t} \|f\|_2 + \frac{C}{\sqrt{t}} e^{-\rho d(E,F)} e^{2t\rho^2} \|f\|_2, \quad (14)$$

and

$$\|V^{1/2}e^{-tA}f\|_{L^2(F)} \leq \frac{C}{\sqrt{t}} e^{-\rho d(E,F)} e^{2\rho^2 t} \|f\|_2. \quad (15)$$

We set $\rho = d(E, F)/2t$ in (13) and $\rho = d(E, F)/4t$ in (15), then we get the $L^2 - L^2$ off-diagonal estimates (i) and (iii).

We set $\rho = d(E, F)/4t$ in (14), we get

$$\|\nabla e^{-tA}f\|_{L^2(F)} \leq \frac{C}{\sqrt{t}} \left(1 + \frac{d(E, F)}{4\sqrt{t}}\right) e^{-d^2(E,F)/8t} \|f\|_2.$$

This gives estimate (ii). \square \square

Now, we study the $L^p - L^2$ boundedness of the semigroup, of its gradient, and of $(V^{1/2}e^{-tA})_{t>0}$.

Proposition 2.2. Suppose that $A \geq \varepsilon V$, then $(e^{-tA})_{t>0}$, $(\sqrt{t}\nabla e^{-tA})_{t>0}$ and $(\sqrt{t}V^{1/2}e^{-tA})_{t>0}$ are $L^p - L^2$ bounded for all $p \in (p'_0; 2]$. Here p'_0 is the dual exponent of p_0 where $p_0 = \frac{2N}{(N-2)(1-\sqrt{1-\frac{1}{1+\varepsilon}})}$, and the dimension $N \geq 3$. More precisely we have for all $t > 0$:

- i) $\|e^{-tA}f\|_2 \leq Ct^{-\gamma_p} \|f\|_p,$
- ii) $\|\sqrt{t}\nabla e^{-tA}f\|_2 \leq Ct^{-\gamma_p} \|f\|_p,$
- iii) $\|\sqrt{t}V^{1/2}e^{-tA}f\|_2 \leq Ct^{-\gamma_p} \|f\|_p,$

where $\gamma_p = \frac{N}{2} \left(\frac{1}{p} - \frac{1}{2}\right)$.

Proof. i) We apply the Gagliardo-Nirenberg inequality

$$\|u\|_2^2 \leq C_{a,b} \|\nabla u\|_2^{2a} \|u\|_p^{2b},$$

where $a + b = 1$ and $(1 + 2\gamma_p)a = 2\gamma_p$, to $u = e^{-tA}f$ for all $f \in L^2 \cap L^p$, all $t > 0$, and all $p \in (p'_0; 2]$. We obtain

$$\|e^{-tA}f\|_2^2 \leq C_{a,b} \|\nabla e^{-tA}f\|_2^{2a} \|e^{-tA}f\|_p^{2b}.$$

At present we use the boundedness of the semigroup on L^p for all $p \in (p'_0; 2]$ proved in [23], and the fact that $\|\nabla u\|_2^2 \leq (1 + 1/\varepsilon)(Au, u)$ from the strong subcriticality condition, then we obtain that

$$\|e^{-tA}f\|_2^{2/a} \leq -C\psi'(t)\|f\|_p^{2b/a}$$

where $\psi(t) = \|e^{-tA}f\|_2^2$. This implies

$$\|f\|_p^{-2b/a} \leq C(\psi(t)^{\frac{a-1}{a}})'$$

Since $\frac{2b}{a} = \frac{1}{\gamma_p}$ and $\frac{a-1}{a} = -\frac{1}{2\gamma_p}$, integration between 0 and t yields

$$t\|f\|_p^{-1/\gamma_p} \leq C\|e^{-tA}f\|_2^{-1/\gamma_p},$$

which gives i).

We obtain ii) by using the following decomposition:

$$\sqrt{t}\nabla e^{-tA} = \sqrt{t}\nabla A^{-1/2}A^{1/2}e^{-tA/2}e^{-tA/2},$$

the boundedness of $\nabla A^{-1/2}$ and of $(\sqrt{t}A^{1/2}e^{-tA})_{t>0}$ on L^2 , and the fact that $(e^{-tA})_{t>0}$ is $L^p - L^2$ bounded for all $p \in (p'_0; 2]$ proved in i).

We prove iii) by using the following decomposition:

$$\sqrt{t}V^{1/2}e^{-tA} = \sqrt{t}V^{1/2}A^{-1/2}A^{1/2}e^{-tA/2}e^{-tA/2},$$

the boundedness of $V^{1/2}A^{-1/2}$ and of $(\sqrt{t}A^{1/2}e^{-tA})_{t>0}$ on L^2 , and the fact that $(e^{-tA})_{t>0}$ is $L^p - L^2$ bounded for all $p \in (p'_0; 2]$ proved in i). $\square \quad \square$

We invest the previous results to obtain :

Theorem 2.1. Assume that $A \geq \varepsilon V$ then $(e^{-tA})_{t>0}$, $(\sqrt{t}\nabla e^{-tA})_{t>0}$ and $(\sqrt{t}V^{1/2}e^{-tA})_{t>0}$ satisfy $L^p - L^2$ off-diagonal estimates for all $p \in (p'_0; 2]$. Here p'_0 is the dual exponent of p_0 where $p_0 = \frac{2N}{(N-2)(1-\sqrt{1-\frac{1}{1+\varepsilon}})}$, and the dimension $N \geq 3$. Then we have for all $t > 0$, all $p \in (p'_0; 2]$, all closed sets E and F of \mathbb{R}^N and all $f \in L^2 \cap L^p$ with $\text{supp } f \subseteq E$

i)

$$\|e^{-tA}f\|_{L^2(F)} \leq Ct^{-\gamma_p}e^{-\frac{cd^2(E,F)}{t}}\|f\|_p, \quad (16)$$

ii)

$$\|\sqrt{t}\nabla e^{-tA}f\|_{L^2(F)} \leq Ct^{-\gamma_p}e^{-\frac{cd^2(E,F)}{t}}\|f\|_p, \quad (17)$$

iii)

$$\|\sqrt{t}V^{1/2}e^{-tA}f\|_{L^2(F)} \leq Ct^{-\gamma_p}e^{-\frac{cd^2(E,F)}{t}}\|f\|_p, \quad (18)$$

where $\gamma_p = \frac{N}{2}(\frac{1}{p} - \frac{1}{2})$ and C, c are positive constants.

Remark: By duality, we deduce from (16) a $L^2 - L^p$ off-diagonal estimate of the norm of the semigroup for all $p \in [2; p_0)$, but we cannot deduce from (17) and (18) the same estimate of the norm of $\sqrt{t}\nabla e^{-tA}f$ and of $\sqrt{t}V^{1/2}e^{-tA}f$ because they are not selfadjoint. This affects the boundedness of Riesz transforms and of $V^{1/2}A^{-1/2}$ on L^p for $p > 2$.

Proof. i) In the previous proposition we have proved that

$$\|e^{-tA}f\|_2 \leq Ct^{-\gamma_p}\|f\|_p$$

for all $p \in (p'_0; 2]$. This implies that for all $t > 0$

$$\|\chi_F e^{-tA} \chi_E f\|_2 \leq Ct^{-\gamma_p}\|f\|_p$$

where χ_M is the characteristic function of M . The $L^2 - L^2$ off-diagonal estimate proved in the Proposition 2.1 implies that

$$\|\chi_F e^{-tA} \chi_E f\|_2 \leq e^{-d^2(E,F)/4t}\|f\|_2.$$

Hence we can apply the Riesz-Thorin interpolation theorem and we obtain the off-diagonal estimate (16).

Assertions ii) and iii) are proved in a similar way. We use $L^2 - L^2$ off-diagonal estimates of Proposition 2.1 and assertions ii) and iii) of Proposition 2.2. \square \square

3 Boundedness of $\nabla A^{-1/2}$ and $V^{1/2}A^{-1/2}$ on L^p for $p \in (p'_0; 2]$

This section is devoted to the study of the boundedness of $V^{1/2}A^{-1/2}$ and Riesz transforms associated to Schrödinger operators with negative and strongly subcritical potentials. We prove that $\nabla A^{-1/2}$ and $V^{1/2}A^{-1/2}$ are bounded on $L^p(\mathbb{R}^N)$, $N \geq 3$, for all $p \in (p'_0; 2]$, where p'_0 is the exponent mentioned in Theorem 2.1.

Theorem 3.1. Assume that $A \geq \varepsilon V$, then $\nabla A^{-1/2}$ is bounded on $L^p(\mathbb{R}^N)$ for $N \geq 3$, for all $p \in (p'_0; 2]$ where $p'_0 = \left(\frac{2N}{(N-2)(1-\sqrt{1-\frac{1}{1+\varepsilon}})} \right)'$.

To prove Theorem 3.1, we prove that $\nabla A^{-1/2}$ is of weak type (p, p) for all $p \in (p'_0; 2)$ by using the following theorem of Blunck and Kunstmann [8]. Then by the boundedness of $\nabla A^{-1/2}$ on L^2 , and the Marcinkiewicz interpolation theorem, we obtain boundedness on L^p for all $p \in (p'_0; 2]$. This result can also be deduced from Theorem 2.1 together with Theorem 1.1 of [9].

Theorem 3.2. Let $p \in [1; 2)$. Suppose that T is sublinear operator of strong type $(2, 2)$, and let $(A_r)_{r>0}$ be a family of linear operators acting on L^2 .

Assume that for $j \geq 2$

$$\left(\frac{1}{|2^{j+1}B|} \int_{C_j(B)} |T(I - A_{r(B)})f|^2 \right)^{1/2} \leq g(j) \left(\frac{1}{|B|} \int_B |f|^p \right)^{1/p}, \quad (19)$$

and for $j \geq 1$

$$\left(\frac{1}{|2^{j+1}B|} \int_{C_j(B)} |A_{r(B)}f|^2 \right)^{1/2} \leq g(j) \left(\frac{1}{|B|} \int_B |f|^p \right)^{1/p}, \quad (20)$$

for all ball B with radius $r(B)$ and all f supported in B . If $\Sigma := \sum g(j)2^{Nj} < \infty$, then T is of weak type (p, p) , with a bound depending only on the strong type $(2, 2)$ bound of T , p , and Σ .

Here $C_1 = 4B$ and $C_j(B) = 2^{j+1}B \setminus 2^jB$ for $j \geq 2$, where λB is the ball of radius $\lambda r(B)$ with the same center as B , and $|\lambda B|$ its Lebesgue measure.

of Theorem 3.1. Let $T = \nabla A^{-1/2}$. We prove assumptions (19) and (20) with $A_r = I - (I - e^{-r^2 A})^m$ for some $m > N/4 - \gamma_p$, using arguments similar to Auscher [1] Theorem 4.2.

Let us prove (20). For f supported in a ball B (with radius r),

$$\begin{aligned} \frac{1}{|2^{j+1}B|^{1/2}} \|A_r f\|_{L^2(C_j(B))} &= \frac{1}{|2^{j+1}B|^{1/2}} \left\| \sum_{k=1}^m \binom{m}{k} (-1)^{k+1} e^{-kr^2 A} f \right\|_{L^2(C_j(B))} \\ &\leq \frac{1}{|2^{j+1}B|^{1/2}} \sum_{k=1}^m \binom{m}{k} C(kr^2)^{-\gamma_p} e^{\frac{-cd^2(B, C_j(B))}{kr^2}} \|f\|_p. \end{aligned}$$

for all $p \in (p'_0; 2)$ and all $f \in L^2 \cap L^p$ supported in B . Here we use the $L^p - L^2$ off-diagonal estimates (16) for $p \in (p'_0; 2]$. Since $\gamma_p = \frac{N}{2}(\frac{1}{p} - \frac{1}{2})$ we obtain

$$\begin{aligned} \left(\frac{1}{|2^{j+1}B|} \int_{C_j(B)} |A_r f|^2 \right)^{1/2} &\leq \frac{C r^{-2\gamma_p}}{|2^{j+1}B|^{1/2}} e^{\frac{-cd^2(B, C_j(B))}{mr^2}} \|f\|_p \\ &\leq C 2^{-jN/2} e^{\frac{-cd^2(B, C_j(B))}{r^2}} \left(\frac{1}{|B|} \int_B |f|^p \right)^{1/p}. \end{aligned}$$

This yields, for $j = 1$,

$$\left(\frac{1}{|4B|} \int_{4B} |A_r f|^2 \right)^{1/2} \leq C 2^{-N/2} \left(\frac{1}{|B|} \int_B |f|^p \right)^{1/p},$$

and for $j \geq 2$

$$\left(\frac{1}{|2^{j+1}B|} \int_{C_j(B)} |A_r f|^2 \right)^{1/2} \leq C 2^{-jN/2} e^{-c4^j} \left(\frac{1}{|B|} \int_B |f|^p \right)^{1/p}.$$

Thus assumption (20) of Theorem 3.2 holds with $\sum_{j \geq 1} g(j) 2^{jN} < \infty$.

It remains to check the assumption (19):

We know that

$$\nabla A^{-1/2} f = C \int_0^\infty \nabla e^{-tA} f \frac{dt}{\sqrt{t}}$$

then, using the Newton binomial, we get

$$\begin{aligned} \nabla A^{-1/2} (I - e^{-r^2 A})^m f &= C \int_0^\infty \nabla e^{-tA} (I - e^{-r^2 A})^m f \frac{dt}{\sqrt{t}} \\ &= C \int_0^\infty g_{r^2}(t) \nabla e^{-tA} f dt \end{aligned}$$

where

$$g_{r^2}(t) = \sum_{k=0}^m \binom{m}{k} (-1)^k \frac{\chi_{(t-kr^2 > 0)}}{\sqrt{t-kr^2}}.$$

Hence, using the $L^p - L^2$ off-diagonal estimate (17), we obtain for all $p \in (p'_0; 2)$, all $j \geq 2$, and all $f \in L^2 \cap L^p$ supported in B

$$\|\nabla A^{-1/2} (I - e^{-r^2 A})^m f\|_{L^2(C_j(B))} \leq C \int_0^\infty |g_{r^2}(t)| t^{-\gamma_p - 1/2} e^{-c4^j r^2/t} dt \|f\|_p.$$

We observe that (see [1] p. 27)

$$|g_{r^2}(t)| \leq \frac{C}{\sqrt{t-kr^2}} \quad \text{if} \quad kr^2 < t \leq (k+1)r^2 \leq (m+1)r^2$$

and

$$|g_{r^2}(t)| \leq Cr^{2m}t^{-m-1/2} \quad \text{if } t > (m+1)r^2.$$

This yields

$$\begin{aligned} \|\nabla A^{-\frac{1}{2}}(I - e^{-r^2A})^m f\|_{L^2(C_j(B))} &\leq C \sum_{k=0}^m \int_{kr^2}^{(k+1)r^2} \frac{t^{-\gamma_p-1/2}}{\sqrt{t-kr^2}} e^{-\frac{c4^j r^2}{t}} dt \|f\|_p \\ &+ C \int_{(m+1)r^2}^{\infty} r^{2m} t^{-\gamma_p-1-m} e^{-\frac{c4^j r^2}{t}} dt \|f\|_p \\ &\leq I_1 + I_2. \end{aligned} \quad (21)$$

We have

$$I_2 := C \int_{(m+1)r^2}^{\infty} r^{2m} t^{-\gamma_p-1-m} e^{-\frac{c4^j r^2}{t}} dt \|f\|_p \leq Cr^{-2\gamma_p} 2^{-2j(m+\gamma_p)} \|f\|_p,$$

by the Laplace transform formula, and

$$\begin{aligned} I_1 &:= C \|f\|_p \sum_{k=0}^m \int_{kr^2}^{(k+1)r^2} \frac{t^{-\gamma_p-1/2}}{\sqrt{t-kr^2}} e^{-\frac{c4^j r^2}{t}} dt \\ &= C \|f\|_p \left(\sum_{k=1}^m \int_{kr^2}^{(k+1)r^2} \frac{t^{-\gamma_p-1/2}}{\sqrt{t-kr^2}} e^{-\frac{c4^j r^2}{t}} dt + \int_0^{r^2} t^{-\gamma_p-1} e^{-\frac{c4^j r^2}{t}} dt \right) \\ &= J_1 + J_2. \end{aligned}$$

In the preceding equation

$$\begin{aligned} J_1 &:= C \|f\|_p \sum_{k=1}^m \int_{kr^2}^{(k+1)r^2} \frac{t^{-\gamma_p-1/2}}{\sqrt{t-kr^2}} e^{-\frac{c4^j r^2}{t}} dt \\ &\leq C \|f\|_p e^{-\frac{c4^j}{m+1}} \sum_{k=1}^m (kr^2)^{-\gamma_p-1/2} \int_{kr^2}^{(k+1)r^2} (t-kr^2)^{-1/2} dt \\ &\leq Cr^{-2\gamma_p} 2^{-2j(m+\gamma_p)} \|f\|_p, \end{aligned}$$

and

$$\begin{aligned} J_2 &:= C \int_0^{r^2} t^{-\gamma_p-1} e^{-\frac{c4^j r^2}{t}} dt \|f\|_p \\ &\leq C \|f\|_p e^{-\frac{c4^j}{2(m+1)}} \int_0^{r^2} t^{-\gamma_p-1} e^{-\frac{c4^j r^2}{2t}} dt \\ &\leq C \|f\|_p 2^{-2jm} \int_0^{r^2} t^{-1-\gamma_p} C(2^{-2j} r^{-2} t)^{\gamma_p} e^{-\frac{c4^j r^2}{4t}} dt \\ &\leq C \|f\|_p 2^{-2j(m+\gamma_p)} r^{-2\gamma_p} \int_0^{r^2} t^{-1} e^{-\frac{c4^j r^2}{4t}} dt \\ &\leq Cr^{-2\gamma_p} 2^{-2j(m+\gamma_p)} \|f\|_p. \end{aligned}$$

Here, for the last inequality, we use the fact that $j \geq 2$ to obtain the convergence of the integral without dependence on r nor on j .

We can therefore employ these estimates in (21) to conclude that

$$\|\nabla A^{-1/2}(I - e^{-r^2 A})^m f\|_{L^2(C_j(B))} \leq Cr^{-2\gamma_p} 2^{-2j(m+\gamma_p)} \|f\|_p,$$

which implies

$$\left(\frac{1}{|2^{j+1}B|} \int_{C_j(B)} |\nabla A^{-1/2}(I - e^{-r^2 A})^m f|^2 \right)^{\frac{1}{2}} \leq C 2^{-2j(m+\gamma_p+\frac{N}{4})} \left(\frac{1}{|B|} \int_B |f|^p \right)^{\frac{1}{p}}$$

where $\sum g(j)2^{jN} < \infty$ because we set $m > N/4 - \gamma_p$. □ □

Proposition 3.1. Assume that $A \geq \varepsilon V$, then $V^{1/2}A^{-1/2}$ is bounded on $L^p(\mathbb{R}^N)$ for $N \geq 3$, for all $p \in (p'_0; 2]$ where p'_0 is the dual exponent of p_0 with $p_0 = \frac{2N}{(N-2)(1-\sqrt{1-\frac{1}{1+\varepsilon}})}$.

Proof. We have seen in (3) that the operator $V^{1/2}A^{-1/2}$ is bounded on L^2 . To prove its boundedness on L^p for all $p \in (p'_0; 2]$ we prove that it is of weak type (p, p) for all $p \in (p'_0; 2)$ by checking assumptions (19) and (20) of Theorem 3.2, where $T = V^{1/2}A^{-1/2}$. Then, using the Marcinkiewicz interpolation theorem, we deduce boundedness on L^p for all $p \in (p'_0; 2]$.

We check assumptions of Theorem 3.2 similarly as we did in the proof of Theorem 3.1, using the $L^p - L^2$ off-diagonal estimate (18) instead of (17). □ □

Let us now move on, setting $V = c|x|^{-2}$ where $0 < c < (\frac{N-2}{2})^2$, which is strongly subcritical thanks to the Hardy inequality, we prove that the associated Riesz transforms are not bounded on L^p for $p \in (1; p'_0)$ neither for $p \in (p_{0*}; \infty)$. Here $p_{0*} = \frac{p_0 N}{N+p_0}$ is the reverse Sobolev exponent of p_0 .

Proposition 3.2. Set V strongly subcritical and $N \geq 3$. Assume that $\nabla A^{-1/2}$ is bounded on L^p for some $p \in (1; p'_0)$. Then there exists an exponent $q_1 \in [p; p'_0)$ such that $(e^{-tA})_{t>0}$ is bounded on L^r for all $r \in (q_1; 2)$.

Consider now $V = c|x|^{-2}$ where $0 < c < (\frac{N-2}{2})^2$. It is proved in [23] that the semigroup does not act on L^p for $p \notin (p'_0; p_0)$. Therefore we obtain from this proposition that the Riesz transform $\nabla A^{-1/2}$ is not bounded on L^p for $p \in (1; p'_0)$.

Proof. Assume that $\nabla A^{-1/2}$ is bounded on L^p for some $p \in (1; p'_0)$. By the boundedness on L^2 and the Riesz-Thorin interpolation theorem, we get the boundedness of $\nabla A^{-1/2}$ on L^q for all $q \in [p; 2]$. Now we apply the Sobolev inequality

$$\|f\|_{q^*} \leq C \|\nabla f\|_q \quad (22)$$

where $q^* = \frac{Nq}{N-q}$ if $q < N$ to $f := A^{-1/2}u$, so we get

$$\|A^{-1/2}u\|_{q^*} \leq C \|\nabla A^{-1/2}u\|_q \leq C \|u\|_q$$

for all $q \in [p; 2]$. In particular, $\|A^{-1/2}\|_{q_1 - q_1^*} \leq C$ where $p \leq q_1 < p'_0$ such that $q_1^* > p'_0$.

Decomposing the semigroup as follows

$$e^{-tA} = A^{1/2} e^{-tA/2} e^{-tA/2} A^{-1/2} \quad (23)$$

where $A^{-1/2}$ is $L^{q_1} - L^{q_1^*}$ bounded, $e^{-tA/2}$ has $L^{q_1^*} - L^2$ norm bounded by $Ct^{-\gamma_{q_1^*}}$ (Proposition 2.2) and $A^{1/2} e^{-tA/2}$ is $L^2 - L^2$ bounded by $Ct^{-1/2}$ because of the analyticity of the semigroup on L^2 . Therefore, we obtain

$$\|e^{-tA}\|_{q_1 - 2} \leq Ct^{-\gamma_{q_1^*} - 1/2} = Ct^{-\gamma_{q_1}}.$$

We now interpolate this norm with the $L^2 - L^2$ off-diagonal estimate of the norm of e^{-tA} , as we did in the proof of Theorem 2.1, so we get a $L^r - L^2$ off-diagonal estimate for all $r \in (q_1; 2)$. Then Lemma 3.3 of [1] yields that $(e^{-tA})_{t>0}$ is bounded on L^r for all $r \in (q_1; 2)$ for $q_1 \in [p; p'_0)$ such that $q_1^* > p'_0$. \square

Proposition 3.3. Set V strongly subcritical and $N \geq 3$. Assume that $\nabla A^{-1/2}$ is bounded on L^p for some $p \in (p_{0*}; \infty)$. Then there exists an exponent $q_2 > p_{0*}$ such that the semigroup $(e^{-tA})_{t>0}$ is bounded on L^s for all $s \in (2; q_2^*)$. Here $q_2^* > p_0$.

Consider now $V = c|x|^{-2}$ where $0 < c < (\frac{N-2}{2})^2$. It is proved in [23] that the semigroup does not act on L^p for $p \notin (p'_0; p_0)$. Therefore we obtain from this proposition that the Riesz transforms $\nabla A^{-1/2}$ are not bounded on L^p for $p \in (p_{0*}; \infty)$.

Proof. Assume that $\nabla A^{-1/2}$ is bounded on L^p for some $p \in (p_{0*}; \infty)$. Then by interpolation we obtain the boundedness of $\nabla A^{-1/2}$ on L^q for all $q \in [2; p]$. In particular,

$$\|\nabla A^{-1/2}\|_{q_2 - q_2} \leq C$$

where $p_{0*} < q_2 < p_0$, $q_2 \leq p$, $q_2 < N$. Using the Sobolev inequality (22), we obtain that $A^{-1/2}$ is $L^{q_2} - L^{q_2^*}$ bounded where $q_2^* > p_0$.

Now we decompose the semigroup as follows

$$e^{-tA} = A^{-1/2} e^{-tA/2} A^{1/2} e^{-tA/2}. \quad (24)$$

Thus we remark that it is $L^2 - L^{q_2^*}$ bounded where $q_2^* > p_0$.

Then, using similar arguments as in the previous proof, we conclude that $(e^{-tA})_{t>0}$ is bounded on L^s for all $s \in (2; q_2^*)$ for $p_{0*} < q_2 < \inf(p_0, p, N)$. \square

4 Boundedness of $\nabla A^{-1/2}$ and $V^{1/2} A^{-1/2}$ on L^p for all $p \in (1; N)$

In this section we assume that V is strongly subcritical in the Kato subclass K_N^∞ , $N \geq 3$. Following Zhao [33], we define

$$K_N^\infty := \left\{ V \in K_N^{loc}; \lim_{B \uparrow \infty} \left[\sup_{x \in \mathbb{R}^N} \int_{|y| \geq B} \frac{|V(y)|}{|y-x|^{N-2}} dy \right] = 0 \right\},$$

where K_N^{loc} is the class of potentials that are locally in the Kato class K_N . For necessary background of the Kato class see [29] and references therein.

We use results proved by stochastic methods to deduce a $L^1 - L^\infty$ off-diagonal estimate of the norm of the semigroup which leads to the boundedness of $\nabla A^{-1/2}$ and $V^{1/2} A^{-1/2}$ on L^p for all $p \in (1; N)$.

Theorem 4.1. Let A be the Schrödinger operator $-\Delta - V$, $V \geq 0$. Assume that V is strongly subcritical in the class K_N^∞ , ($N \geq 3$), then $\nabla A^{-1/2}$ and $V^{1/2} A^{-1/2}$ are of weak type $(1, 1)$, they are bounded on L^p for all $p \in (1; 2]$. If in addition $V \in L^{N/2}$, then $\nabla A^{-1/2}$ and $V^{1/2} A^{-1/2}$ are bounded on L^p for all $p \in (1; N)$.

Proof. We assume that V is strongly subcritical in the class K_N^∞ . Therefore V satisfies assumptions of Theorem 2 of [31] (The classes K_∞ and S_∞ mentioned in [31] are equivalent to the class K_N^∞ (see Chen [11] Theorem 2.1 and Section 3.1)). Thus the heat kernel associated to $(e^{-tA})_{t>0}$ satisfies a Gaussian estimate. Therefore $(e^{-tA})_{t>0}$, $(\sqrt{t} \nabla e^{-tA})_{t>0}$, and $(\sqrt{t} V^{1/2} e^{-tA})_{t>0}$ satisfy $L^1 - L^2$ off-diagonal estimates. Arguing now as in the proof of Theorem 3.1 (or using Theorem 5 of [28]) we conclude that $\nabla A^{-1/2}$ and $V^{1/2} A^{-1/2}$ are of weak type $(1, 1)$ and they are bounded on L^p for all $p \in (1; 2]$.

To prove the boundedness of $\nabla A^{-1/2}$ on L^p for higher p we use the Stein complex interpolation theorem (see [30] Section V.4).

Let us first mention that $D := R(A) \cap L^1 \cap L^\infty$ is dense in L^p for all $p \in (1; \infty)$ provided that V is strongly subcritical in K_N^∞ , $N \geq 3$. We prove the density as in [2], where in our case we have the following estimate

$$|f_k - f| \leq k(c(-\Delta) + k)^{-1}f \quad (25)$$

where $f_k := A(A + k)^{-1}f$ and c is a positive constant. This estimate holds from the Gaussian estimate of the heat kernel associated to the semigroup $(e^{-tA})_{t>0}$.

Set $F(z) := \langle (-\Delta)^z A^{-z} f, g \rangle$ where $f \in D$, $g \in C_0^\infty(\mathbb{R}^N)$ and $z \in S := \{x + iy \text{ such that } x \in [0; 1] \text{ and } y \in \mathbb{R}^N\}$. $F(z)$ is admissible. Indeed, the function $z \mapsto F(z)$ is continuous in S and analytic in its interior. In addition

$$|F(z)| = |\langle A^{-z} f, (-\Delta)^{\bar{z}} g \rangle| \leq \|A^{-z} f\|_2 \|(-\Delta)^{\bar{z}} g\|_2. \quad (26)$$

For $\Re z \in (0; 1)$, $D(-\Delta) \subset D((-\Delta)^{\bar{z}})$, so

$$\|(-\Delta)^{\bar{z}} g\|_2 \leq C \|g\|_{W^{2,2}} \quad (27)$$

for all $z \in S$.

When V is strongly subcritical, A is non-negative self-adjoint operator on L^2 , hence $\|A^{iy}\|_{2-2} \leq 1$ for all $y \in \mathbb{R}$. Therefore for all $z = x + iy \in S$ and $f = Au \in R(A)$ we have

$$\begin{aligned} \|A^{-z} f\|_2 &\leq \|A^{-iy}\|_{2-2} \|A^{1-x} u\|_2 \\ &\leq C(\|u\|_2 + \|Au\|_2). \end{aligned} \quad (28)$$

Here we use $D(A) \subset D(A^{1-x})$ because $(1-x) \in (0; 1)$.

Now we employ (27) and (28) in (26) to deduce the admissibility of $F(z)$ in S . Thus we can apply the Stein complex interpolation theorem to $F(z)$.

Since V is strongly subcritical and belongs to the class K_N^∞ , $N \geq 3$, we obtain a Gaussian estimate of the heat kernel of A . Thus A has a H^∞ -bounded calculus on L^p for all $p \in (1; \infty)$ (see e.g. [8] Theorem 2.2). Hence

$$|F(iy)| \leq \|A^{-iy} f\|_{p_0} \|(-\Delta)^{-iy} g\|_{p'_0} \leq C_{\gamma, p_0} e^{2\gamma|y|} \|f\|_{p_0} \|g\|_{p'_0}$$

for all $\gamma > 0$, all $p_0 \in (1; \infty)$.

Let us now estimate $\|VA^{-1}\|_{p_1-p_1}$. By Hölder's inequality

$$\|VA^{-1}u\|_{p_1} \leq \|V\|_{N/2} \|A^{-1}u\|_q \quad (29)$$

where $p_1 < N$ and $\frac{1}{p_1} = \frac{1}{q} + \frac{2}{N}$. As mentioned above we have a Gaussian upper bound for the heat kernel. In particular

$$\|e^{-tA}\|_{1-\infty} \leq Ct^{-N/2}$$

for all $t > 0$. Therefore A^{-1} extends to a bounded operator from L^s to L^q such that $s < \frac{N}{2}$ and $\frac{1}{s} = \frac{1}{q} + \frac{2}{N}$, and we have

$$\|A^{-1}u\|_q \leq C\|u\|_s.$$

(see Coulhon [12]). Thus $s = p_1$, $D(A) \subseteq D(V)$ and (29) implies

$$\|VA^{-1}\|_{p_1-p_1} \leq C$$

where C depends on $\|V\|_{N/2}$. Hence we can estimate

$$\begin{aligned} \|(-\Delta)A^{-1}u\|_{p_1} &= \|(-\Delta - V + V)A^{-1}u\|_{p_1} \\ &\leq \|u\|_{p_1} + \|VA^{-1}u\|_{p_1} \\ &\leq C\|u\|_{p_1} \end{aligned} \tag{30}$$

where C depends on $\|V\|_{N/2}$. We return to $F(z)$,

$$\begin{aligned} |F(1+iy)| &\leq \|(-\Delta)A^{-1}A^{-iy}f\|_{p_1} \|(-\Delta)^{-iy}g\|_{p'_1} \\ &\leq \|(-\Delta)A^{-1}\|_{p_1-p_1} \|A^{-iy}f\|_{p_1} \|(-\Delta)^{-iy}g\|_{p'_1} \\ &\leq C_{\gamma,p_1,\|V\|_{N/2}} e^{2\gamma|y|} \|f\|_{p_1} \|g\|_{p'_1} \end{aligned}$$

for all $p_1 \in (1; N/2)$ and all $\gamma > 0$.

From the Stein interpolation theorem it follows that for all $t \in [0; 1]$ there exists a constant M_t such that

$$|F(t)| \leq M_t \|f\|_{p_t} \|g\|_{p'_t}$$

where $\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1}$. Setting $t = \frac{1}{2}$ and using a density argument we conclude that $\nabla A^{-1/2}$ is bounded on L^p for all $p \in (1; N)$.

To prove boundedness of $V^{1/2}A^{-1/2}$ on L^p we use the following decomposition

$$V^{1/2}A^{-1/2} = V^{1/2}(-\Delta)^{-1/2}(-\Delta)^{1/2}A^{-1/2}.$$

Assuming $V \in L^{N/2}$ we have by Hölder's inequality

$$\|V^{1/2}u\|_p \leq \|V^{1/2}\|_N \|u\|_q$$

where $p < N$ and $\frac{1}{p} - \frac{1}{q} = \frac{1}{N}$. Then by Sobolev inequality and the boundedness of Riesz transforms associated to the Laplace operator we obtain

$$\|V^{1/2}u\|_p \leq C_{p,N,\|V\|_{N/2}} \|\nabla u\|_p \leq C_{p,N,\|V\|_{N/2}} \|(-\Delta)^{1/2}u\|_p \quad (31)$$

for all $p \in (1; N)$. Thus if $V \in L^{N/2}$ we have for all $p \in (1; N)$

$$\|V^{1/2}(-\Delta)^{-1/2}\|_{p-p} \leq C.$$

Using the boundedness of Riesz transforms associated to the Schrödinger operator A we have

$$\|(-\Delta)^{1/2}A^{-1/2}u\|_p \leq C\|u\|_p$$

for all $p \in (1; N)$.

Therefore $V^{1/2}A^{-1/2}$ is bounded on L^p for all $p \in (1; N)$ provided that V is strongly subcritical in the class $K_N^\infty \cap L^{N/2}$, $N \geq 3$. \square \square

Example: Set $N \geq 3$, and let us take potentials V in the Kato subclass $K_N \cap L^{N/2}$ such that $V \sim c|x|^{-\alpha}$ when x tends to infinity, where $\alpha > 2$. Suppose that $\|V\|_{\frac{N}{2}}$ is small enough. Let us prove that these potentials are strongly subcritical, so we should prove that

$$\|V^{1/2}u\|_2^2 \leq C\|\nabla u\|_2^2$$

where $C < 1$. This is (31) where $p = 2$, and $C < 1$ for $\|V\|_{\frac{N}{2}}$ is small enough. Hence these potentials are strongly subcritical. Z.Zhao [33] proved that they are in the subclass K_N^∞ . Hence they satisfy the assumptions of Theorem 4.1. Then $\nabla(-\Delta - V)^{-1/2}$ and $V^{1/2}(-\Delta - V)^{-1/2}$ are bounded on L^p for all $p \in (1; N)$.

Remarks: 1) The proof of the previous theorem shows that

$$\|Vu\|_{p_1} \leq C\|Au\|_{p_1}$$

and

$$\|\Delta u\|_{p_1} \leq C\|Au\|_{p_1}$$

for all $p_1 \in (1; N/2)$.

2) If we consider $H = -\Delta + V$ a Schrödinger operator with non-negative potential $V \in L^{N/2}$, we obtain by the previous arguments the L^{p_1} -boundedness of VH^{-1} and ΔH^{-1} for all $p_1 \in (1; N/2)$, and the L^p -boundedness of $V^{1/2}H^{-1/2}$ and $\nabla H^{-1/2}$ for all $p \in (1; N)$.

5 Schrödinger operators on Riemannian manifolds

Let M be a non-compact complete Riemannian manifold of dimension $N \geq 3$. Denote by $d\mu$ the Riemannian measure, ρ the geodesic distance on M and ∇ the Riemannian gradient. Denote by $|\cdot|$ the length in the tangent space, and by $\|\cdot\|_p$ the norm in $L^p(M, d\mu)$. Let $-\Delta$ be the positive self-adjoint Laplace-Beltrami operator on M . Take V a strongly subcritical positive potential on M , which means that there exists an $\varepsilon > 0$ such that

$$\int_M V u^2 d\mu \leq \frac{1}{1+\varepsilon} \int_M |\nabla u|^2 d\mu. \quad (32)$$

and set $A := -\Delta - V$ the associated Schrödinger operator on M . By the sesquilinear form method A is well defined, non-negative, and $-A$ generates a bounded analytic semigroup $(e^{-tA})_{t>0}$ on $L^2(M)$.

As in \mathbb{R}^N , we have the $L^2(M)$ -boundedness of $V^{1/2}A^{-1/2}$ and of the Riesz transforms $\nabla A^{-1/2}$ if and only if V is strongly subcritical.

We remark that methods used in [23] hold in manifolds. The semigroup $(e^{-tA})_{t>0}$ can be extrapolated to $L^p(M)$, and it is uniformly bounded for $p \in \left(\left(\frac{2}{1-\sqrt{1-\frac{1}{1+\varepsilon}}} \right)'; \left(\frac{2}{1-\sqrt{1-\frac{1}{1+\varepsilon}}} \right) \right)$. If in addition the Sobolev inequality

$$\|f\|_{L^{\frac{2N}{N-2}}(M)} \leq C \|\nabla f\|_{L^2(M)} \quad (33)$$

for all $f \in C_0^\infty(M)$ holds on M , then we obtain for all $t > 0$

$$\|e^{-tA}\|_{L^p(M)-L^{\frac{pN}{N-2}}(M)} \leq Ct^{-1/p}$$

for all $p \in \left(\left(\frac{2}{1-\sqrt{1-\frac{1}{1+\varepsilon}}} \right)'; \left(\frac{2}{1-\sqrt{1-\frac{1}{1+\varepsilon}}} \right) \right)$. Using the $L^2(M) - L^2(M)$ off-diagonal estimate we obtain as in [23] the fact that $(e^{-tA})_{t>0}$ is bounded on $L^p(M)$ for all $p \in (p'_0; p_0)$ where $p_0 := \frac{2N}{N-2} \frac{1}{1-\sqrt{1-\frac{1}{1+\varepsilon}}}$.

For classes of manifolds satisfying (33) see [26]. Note that (33) is equivalent to the following Gaussian upper bound of the heat kernel $p(t, x, y)$ of the Laplace-Beltrami operator (see [32] and [14])

$$p(t, x, y) \leq Ct^{-N/2} e^{-c\rho^2(x,y)/t} \quad \forall x, y \in M, t > 0. \quad (34)$$

We say that M is of homogeneous type if for all $x \in M$ and $r > 0$

$$\mu(B(x, 2r)) \leq C\mu(B(x, r)) \quad (35)$$

where $B(x, r) := \{y \in M \text{ such that } \rho(x, y) \leq r\}$.

We say that the L^2 -Poincaré inequalities hold on M if there exists a positive constant C such that

$$\int_{B(x,r)} |f(y) - f_r(x)|^2 d\mu(y) \leq Cr^2 \int_{B(x,r)} |\nabla f(y)|^2 d\mu(y) \quad (36)$$

for all $f \in C_0^\infty(M)$, $x \in M$, $r > 0$, where $f_r(x) := \frac{1}{\mu(B(x,r))} \int_{B(x,r)} f(y) d\mu(y)$. Saloff-Coste [25] proved that (35) and (36) hold if and only if the heat kernel $p(t, x, y)$ satisfies the following Li-Yau estimate

$$\frac{Ce^{-c\rho^2(x,y)/t}}{\mu(B(x, \sqrt{t}))} \leq p(t, x, y) \leq \frac{C_1 e^{-c_1\rho^2(x,y)/t}}{\mu(B(x, \sqrt{t}))}. \quad (37)$$

Arguing as in the Euclidean case we obtain the following theorem

Theorem 5.1. Let M be a non-compact complete Riemannian manifold of dimension $N \geq 3$. Assume (32) and (33). Then $(e^{-tA})_{t>0}$, $(\sqrt{t}\nabla e^{-tA})_{t>0}$ and $(\sqrt{t}V^{1/2}e^{-tA})_{t>0}$ satisfy $L^p(M) - L^2(M)$ off-diagonal estimates for all $p \in (p'_0; 2]$. Here p'_0 is the dual exponent of p_0 where $p_0 = \frac{2N}{(N-2)(1-\sqrt{1-\frac{1}{1+\varepsilon}})}$.

Then we have for all $t > 0$, all $p \in (p'_0; 2]$, all closed sets E and F of M , and all $f \in L^2(M) \cap L^p(M)$ with $\text{supp} f \subseteq E$

- i) $\|e^{-tA} f\|_{L^2(F)} \leq Ct^{-\gamma_p} e^{-\frac{c\rho^2(E,F)}{t}} \|f\|_p,$
- ii) $\|\sqrt{t}\nabla e^{-tA} f\|_{L^2(F)} \leq Ct^{-\gamma_p} e^{-\frac{c\rho^2(E,F)}{t}} \|f\|_p,$
- iii) $\|\sqrt{t}V^{1/2}e^{-tA} f\|_{L^2(F)} \leq Ct^{-\gamma_p} e^{-\frac{c\rho^2(E,F)}{t}} \|f\|_p,$

where $\gamma_p = \frac{N}{2}(\frac{1}{p} - \frac{1}{2})$ and C, c are positive constants.

We invest these off-diagonal estimates as in the proof of Theorem 3.1 to obtain the following result

Theorem 5.2. Let M be a non-compact complete Riemannian manifold of dimension $N \geq 3$. Assume (32), (33) and (35). Then $V^{1/2}A^{-1/2}$ and $\nabla A^{-1/2}$ are bounded on $L^p(M)$ for all $p \in (p'_0; 2]$ where $p'_0 = \left(\frac{2N}{(N-2)(1-\sqrt{1-\frac{1}{1+\varepsilon}})}\right)'$.

We say that the potential V is in the class $K_\infty(M)$, if for any $\varepsilon > 0$ there exists a compact set $K \subset M$ and $\delta > 0$ such that

$$\sup_{x \in M} \int_{K^c} G(x, y) |V(y)| d\mu(y) \leq \varepsilon$$

where $K^c := M \setminus K$, and for all measurable sets $B \subset K$ with $\mu(b) < \delta$,

$$\sup_{x \in M} \int_B G(x, y) |V(y)| d\mu(y) \leq \varepsilon.$$

Here $G(x, y) := \int_0^\infty p(t, x, y) dt$ is the Green function, and $p(t, x, y)$ is the heat kernel of the Laplace-Beltrami operator. This class is the generalization of K_N^∞ to manifolds (see [11] Section 2).

Since (35) and (36) imply the Li-Yau estimate (37), we can use Theorem 2 of [31] and obtain a Gaussian upper bound of the heat kernel of $-\Delta - V$. Thus arguing as in the Euclidean case, we obtain the following result

Theorem 5.3. Let M be a non-compact complete Riemannian manifold of dimension $N \geq 3$, and let A be the Schrödinger operator $-\Delta - V, 0 \leq V \in L^{N/2}(M) \cap K_\infty$. Assume that for all ball $B, \mu(B(x, r)) \geq Cr^N$. Assume (32), (35) and (36). Then $\Delta(-\Delta - V)^{-1}$ and $V(-\Delta - V)^{-1}$ are bounded on $L^p(M)$ for all $p \in (1; N/2)$.

Now using Theorem 2 of [31] and Theorem 5 of [28], then arguing as in the Euclidean case we obtain the following

Theorem 5.4. Let M be a non-compact complete Riemannian manifold of dimension $N \geq 3$, and let A be the Schrödinger operator $-\Delta - V, 0 \leq V \in K_\infty$. Assume (32), (35) and (36). Then $\nabla A^{-1/2}$ and $V^{1/2} A^{-1/2}$ are of weak type $(1, 1)$, thus they are bounded on $L^p(M)$ for all $p \in (1; 2]$.

If in addition we assume that for all ball $B, \mu(B(x, r)) \geq Cr^N$, and for some $r \in (2; N]$, the Riesz transforms $\nabla(-\Delta)^{-1/2}$ are bounded on $L^r(M)$ then $\nabla A^{-1/2}$ and $V^{1/2} A^{-1/2}$ are bounded on $L^p(M)$ for all $p \in (1; r)$ provided that $V \in L^{N/2}(M)$.

Remark Let M be a non-compact complete Riemannian manifold of dimension $N \geq 3$. Let $H = -\Delta + V$ be a Schrödinger operator with non-negative potential $V \in L^{N/2}(M)$. Assume that for some $r \in (2; N]$, the Riesz transforms $\nabla(-\Delta)^{-1/2}$ are bounded on $L^p(M)$ for all $p \in (2; r)$ or for $p = r$. Assume also (33). Then the heat kernel associated to H satisfies (34). Hence we obtain by the previous argument the L^p -boundedness of $V^{1/2} H^{-1/2}$ and $\nabla H^{-1/2}$ for all $p \in (1; r)$.

Note that (35) and (36) hold on manifolds with non-negative Ricci curvature (see [22]) as well as the boundedness on $L^p(M)$ for all $p \in (1, \infty)$ of Riesz transforms associated to the Laplace-Beltrami operator (see [6]). The

Sobolev inequality (33) is valid on manifolds with Ricci curvature bounded from below satisfying

$$\inf_{x \in M} \mu(B(x, 1)) > 0$$

(see [19] Theorem 3.14). Therefore manifolds with non-negative Ricci curvature satisfying $\inf_{x \in M} \mu(B(x, 1)) > 0$ are a class of manifolds where Theorem 5.4 holds.

We mention that Carron, Coulhon and Hassell [10] proved that the Riesz transforms $\nabla(-\Delta)^{-1/2}$ are bounded on $L^p(M)$ for all $p \in (2; N)$ on smooth complete Riemannian manifolds of dimension $N \geq 3$ which are the union of a compact part and a finite number of Euclidean ends. Ji, Kunstmann and Weber [20] proved that this boundedness holds for all $p \in (1; \infty)$, on the complete connected Riemannian manifolds whose Ricci curvature is bounded from below, if there is a constant $a > 0$ with $\sigma(-\Delta) \subset \{0\} \cup [a; \infty)$. They also give examples of manifolds that satisfy their conditions. Auscher, Coulhon, Duong and Hofmann [3] proved that on complete non-compact Riemannian manifolds satisfying assumption (37), the uniform boundedness of $(\sqrt{t}\nabla e^{-t(-\Delta)})_{t>0}$ on L^q for some $q \in (2; \infty]$ implies the boundedness on $L^p(M)$ of $\nabla(-\Delta)^{-1/2}$ for all $p \in (2; q)$. And we have equivalence if $(\sqrt{t}\nabla e^{-t(-\Delta)})_{t>0}$ is uniformly bounded on L^r for all $r \in (2; q)$.

Therefore we deduce the following propositions using our previous theorem and the criterion of [3]. We also use the fact that the semigroup $(e^{-t(-\Delta-V)})_{t>0}$ is bounded analytic on $L^p(M)$ for all $p \in (1; \infty)$. This is true on manifolds where assumptions (35) and (36) hold and when $V \in K_\infty$ satisfying (32) (see e.g. [7] Theorem 1.1).

Proposition 5.1. Let M be a non-compact complete Riemannian manifold of dimension $N \geq 3$. Assume that for all ball B $\mu(B(x, r)) \geq Cr^N$, assume (32), (35) and (36), and assume that $V \in K_\infty \cap L^{N/2}(M)$. If for some $r \in (2; N]$

$$\|\|\nabla e^{-t(-\Delta)}\|\|_{L^r(M)-L^r(M)} \leq C/\sqrt{t}$$

for all $t > 0$, then

$$\|\|\nabla e^{-t(-\Delta-V)}\|\|_{L^p(M)-L^p(M)} \leq C/\sqrt{t}$$

for all $t > 0$, all $p \in (1, r)$.

Proposition 5.2. Let M be a non-compact complete Riemannian manifold of dimension $N \geq 3$. Assume (33) and assume that $V \in L^{N/2}(M)$. If for some $r \in (2; N]$

$$\|\|\nabla e^{-t(-\Delta)}\|\|_{L^r(M)-L^r(M)} \leq C/\sqrt{t}$$

for all $t > 0$, then

$$\|\|\nabla e^{-t(-\Delta+V)}\|\|_{L^p(M)-L^p(M)} \leq C/\sqrt{t}$$

for all $t > 0$, all $p \in (1, r)$.

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