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## Riesz transforms associated to Schrödinger operators with negative potentials

Joyce ASSAAD

**Abstract** The goal of this paper is to study the Riesz transforms  $\nabla A^{-1/2}$  where  $A$  is the Schrödinger operator  $-\Delta - V$ ,  $V \geq 0$ , under different conditions on the potential  $V$ . We prove that if  $V$  is strongly subcritical,  $\nabla A^{-1/2}$  is bounded on  $L^p(\mathbb{R}^N)$ ,  $N \geq 3$ , for all  $p \in (p'_0; 2]$  where  $p'_0$  is the dual exponent of  $p_0$  where  $2 < \frac{2N}{N-2} < p_0 < \infty$ ; and we give a counterexample to the boundedness on  $L^p(\mathbb{R}^N)$  for  $p \in (1; p'_0) \cup (p_{0*}; \infty)$  where  $p_{0*} := \frac{p_0 N}{N+p_0}$  is the reverse Sobolev exponent of  $p_0$ . In addition, if the potential is in  $L^{N/2}(\mathbb{R}^N)$ , then  $\nabla A^{-1/2}$  is bounded on  $L^p(\mathbb{R}^N)$  for all  $p \in (p'_0; p_{0*})$ . If the potential is strongly subcritical in the Kato subclass  $K_N^\infty$ , then  $\nabla A^{-1/2}$  is bounded on  $L^p(\mathbb{R}^N)$  for all  $p \in (1; 2]$ , moreover if it is in  $L^{N/2}(\mathbb{R}^N)$  then  $\nabla A^{-1/2}$  is bounded on  $L^p(\mathbb{R}^N)$  for all  $p \in (1; N)$ . We prove also boundedness of  $V^{1/2}A^{-1/2}$  with the same conditions on the same spaces.

**Keywords** Riesz transforms · Schrödinger operators · off-diagonal estimates · singular operators.

**Mathematics Subject Classification (2010):** 42B20 · 35J10.

### 1 Introduction and definitions

Let  $A$  be a Schrödinger operator  $-\Delta + V$  where  $-\Delta$  is the nonnegative Laplace operator and the potential  $V : \mathbb{R}^N \rightarrow \mathbb{R}$  such that  $V = V^+ - V^-$  (where  $V^+$  and  $V^-$  are the positive and negative parts of  $V$ , respectively). The operator is defined via the sesquilinear form method. We define

$$\mathfrak{a}(u, v) = \int_{\mathbb{R}^N} \nabla u(x) \nabla v(x) dx + \int_{\mathbb{R}^N} V^+(x) u(x) v(x) dx - \int_{\mathbb{R}^N} V^-(x) u(x) v(x) dx$$

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$$D(\mathbf{a}) = \left\{ u \in W^{1,2}(\mathbb{R}^N), \int_{\mathbb{R}^N} V^+(x)u^2(x)dx < \infty \right\}.$$

Here we assume  $V^+ \in L^1_{loc}(\mathbb{R}^N)$  and  $V^-$  satisfies (for all  $u \in D(\mathbf{a})$ ):

$$\int_{\mathbb{R}^N} V^-(x)u^2(x)dx \leq \alpha \left[ \int_{\mathbb{R}^N} |\nabla u|^2(x)dx + \int_{\mathbb{R}^N} V^+(x)u^2(x)dx \right] + \beta \int_{\mathbb{R}^N} u^2(x)dx \quad (1)$$

where  $\alpha \in (0,1)$  and  $\beta \in \mathbb{R}$ . By the well-known KLMN theorem (see for example [14] Chapter VI), the form  $\mathbf{a}$  is closed (and bounded from below). Its associated operator is  $A$ . If in addition  $\beta \leq 0$ , then  $A$  is nonnegative.

We can define the Riesz transforms associated to  $A$  by

$$\nabla A^{-1/2} := \frac{1}{\Gamma(\frac{1}{2})} \int_0^\infty \sqrt{t} \nabla e^{-tA} \frac{dt}{t}.$$

The boundedness of Riesz transforms on  $L^p(\mathbb{R}^N)$  implies that the domain of  $A^{1/2}$  is included in the Sobolev space  $W^{1,p}(\mathbb{R}^N)$ . Thus the solution of the corresponding evolution equation will be in the Sobolev space  $W^{1,p}(\mathbb{R}^N)$  for initial data in  $L^p(\mathbb{R}^N)$ .

It is our aim to study the boundedness on  $L^p(\mathbb{R}^N)$  of the Riesz transforms  $\nabla A^{-1/2}$ . We are also interested in the boundedness of the operator  $V^{1/2}A^{-1/2}$ . If  $\nabla A^{-1/2}$  and  $V^{1/2}A^{-1/2}$  are bounded on  $L^p(\mathbb{R}^N)$ , we obtain for some positive constant  $C$

$$\|\nabla u\|_p + \|V^{1/2}u\|_p \leq C\|(-\Delta + V)^{1/2}u\|_p.$$

By a duality argument, we obtain

$$\|(-\Delta + V)^{1/2}u\|_{p'} \leq C(\|\nabla u\|_{p'} + \|V^{1/2}u\|_{p'})$$

where  $p'$  is the dual exponent of  $p$ .

Riesz transforms associated to Schrödinger operators with nonnegative potentials were studied by Ouhabaz [16], Shen [17], and Auscher and Ben Ali [2]. Ouhabaz proved that Riesz transforms are bounded on  $L^p(\mathbb{R}^N)$  for all  $p \in (1; 2]$ , for all potential  $V$  locally integrable. Shen and Auscher and Ben Ali proved that if the potential  $V$  is in the reverse Hölder class  $B_q$ , then the Riesz transforms are bounded on  $L^p(\mathbb{R}^N)$  for all  $p \in (1, p_1)$  where  $2 < p_1 \leq \infty$  depends on  $q$ . The result of Auscher and Ben Ali generalize that of Shen because Shen has restrictions on the dimension  $N$  and on the class  $B_q$ .

With negative potentials new difficulties appear. If we take  $V \in L^\infty(\mathbb{R}^N)$ , and apply the method in [16] to the operator  $A + \|V\|_\infty$ , we obtain boundedness of  $\nabla(A + \|V\|_\infty)^{-1/2}$  on  $L^p(\mathbb{R}^N)$  for all  $p \in (1; 2]$ . This is weaker than the boundedness of  $\nabla A^{-1/2}$  on the same spaces. Guillarmou and Hassell [12] studied Riesz transforms  $\nabla(A \circ P_+)^{-1/2}$  where  $A$  is the Schrödinger operator with negative potential and  $P_+$  is the spectral projection on the positive spectrum.

They prove that, on asymptotically conic manifolds  $M$  of dimension  $N \geq 3$ , if  $V$  is smooth and satisfies decay conditions, and the Schrödinger operator has no zero-modes nor zero-resonances, then Riesz transforms  $\nabla(A \circ P_+)^{-1/2}$  are bounded on  $L^p(M)$  for all  $p \in (1, N)$ . They also prove (see [13]) that when zero-modes are present, Riesz transforms  $\nabla(A \circ P_+)^{-1/2}$  are bounded on  $L^p(M)$  for all  $p \in (\frac{N}{N-2}, \frac{N}{3})$ , with bigger range possible if the zero modes have extra decay at infinity.

In this paper we consider only negative potentials. From now on, we denote by  $A$  the Schrödinger operator with negative potential,

$$A := -\Delta - V, \quad V \geq 0.$$

Our purpose is to find optimal conditions on  $V$  allowing the boundedness of Riesz transforms  $\nabla A^{-1/2}$  and that of  $V^{1/2} A^{-1/2}$  on  $L^p(\mathbb{R}^N)$ .

Let us take the following definition

**Definition 11** We say that the potential  $V$  is strongly subcritical if for some  $\varepsilon > 0$ ,  $A \geq \varepsilon V$ . This means that for all  $u \in W^{1,2}(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N} V u^2 \leq \frac{1}{1 + \varepsilon} \int_{\mathbb{R}^N} |\nabla u|^2.$$

For more information on strongly subcritical potentials see [10] and [21].

With this condition,  $V$  satisfies assumption (1) where  $\beta = 0$  and  $\alpha = \frac{1}{1+\varepsilon}$ . Thus  $A$  is well defined, nonnegative and  $-A$  generates an analytic contraction semigroup  $(e^{-tA})_{t \geq 0}$  on  $L^2(\mathbb{R}^N)$ .

Since  $-\Delta - V \geq \varepsilon V$  we have  $(1 + \varepsilon)(-\Delta - V) \geq \varepsilon(-\Delta)$ . Therefore

$$\|\nabla u\|_2^2 \leq (1 + \frac{1}{\varepsilon}) \|A^{1/2} u\|_2^2. \quad (2)$$

Thus,  $\nabla A^{-1/2}$  is bounded on  $L^2(\mathbb{R}^N)$ . Conversely, it is clear that if  $\nabla A^{-1/2}$  is bounded on  $L^2(\mathbb{R}^N)$  then  $V$  is strongly subcritical.

We observe also that  $-\Delta - V \geq \varepsilon V$  is equivalent to

$$\|V^{1/2} u\|_2^2 \leq \frac{1}{\varepsilon} \|A^{1/2} u\|_2^2. \quad (3)$$

Thus,  $V^{1/2} A^{-1/2}$  is bounded on  $L^2(\mathbb{R}^N)$  if and only if  $V$  is strongly subcritical.

So we can conclude that

$$\|\nabla u\|_2 + \|V^{1/2} u\|_2 \leq C \|(-\Delta - V)^{1/2} u\|_2$$

if and only if  $V$  is strongly subcritical. Then by duality argument we have

$$\|\nabla u\|_2 + \|V^{1/2} u\|_2 \approx \|(-\Delta - V)^{1/2} u\|_2$$

if and only if  $V$  is strongly subcritical.

To study Riesz transforms on  $L^p(\mathbb{R}^N)$  for  $1 \leq p \leq \infty$  with  $p \neq 2$  we use the results on the uniform boundedness of the semigroup on  $L^p(\mathbb{R}^N)$ . Taking central potentials which are equivalent to  $c/|x|^2$  as  $|x|$  tends to infinity where  $0 < c < (\frac{N-2}{2})^2$ ,  $N \geq 3$ , Davies and Simon [10] proved that for all  $t > 0$  and all  $p \in (p'_0; p_0)$ ,

$$\|e^{-tA}\|_{p-p} \leq C$$

where  $2 < \frac{2N}{N-2} < p_0 < \infty$  and  $p'_0$  its dual exponent. Next Liskevich, Sobol, and Vogt [15] proved the uniform boundedness on  $L^p(\mathbb{R}^N)$  for all  $p \in (p'_0; p_0)$  where  $2 < \frac{2N}{N-2} < p_0 = \frac{2N}{(N-2)(1-\sqrt{1-\frac{1}{1+\varepsilon}})}$ , for general strongly subcritical potentials. They also proved that the range  $(p'_0, p_0)$  is optimal and the semigroup does not even act on  $L^p(\mathbb{R}^N)$  for  $p \notin (p'_0, p_0)$ . Under additional condition on  $V$ , Takeda [20] used stochastic methods to prove a Gaussian estimate of the associated heat kernel. Thus the semigroup acts boundedly on  $L^p(\mathbb{R}^N)$  for all  $p \in [1, \infty]$ .

In this paper we prove that when  $V$  is strongly subcritical and  $N \geq 3$ , Riesz transforms are bounded on  $L^p(\mathbb{R}^N)$  for all  $p \in (p'_0; 2]$ . We also give a counterexample to the boundedness of Riesz transforms on  $L^p(\mathbb{R}^N)$  when  $p \in (1; p'_0) \cup (p_{0*}; \infty)$  where  $2 < p_{0*} := \frac{p_0 N}{N+p_0} < p_0 < \infty$ . This example is a counterexample to the boundedness of  $V^{1/2}A^{-1/2}$  on  $L^p(\mathbb{R}^N)$  when  $p \in (p_{0*}; \infty)$ . If, in addition,  $V \in L^{N/2}(\mathbb{R}^N)$  then Riesz transforms are bounded on  $L^p(\mathbb{R}^N)$  for all  $p \in (p'_0, p_{0*})$ . If  $V$  is strongly subcritical in the Kato subclass  $K_N^\infty$ ,  $N \geq 3$  (see Section 5), then  $\nabla A^{-1/2}$  is bounded on  $L^p(\mathbb{R}^N)$  for all  $p \in (1, 2]$ . If, in addition,  $V \in L^{N/2}(\mathbb{R}^N)$  then it is bounded on  $L^p(\mathbb{R}^N)$  for all  $p \in (1, N)$ . With the same conditions, we prove similar results for the operator  $V^{1/2}A^{-1/2}$ . Hence if  $V$  is strongly subcritical and  $V \in L^{N/2}(\mathbb{R}^N)$ ,  $N \geq 3$ , then

$$\|\nabla u\|_p + \|V^{1/2}u\|_p \approx \|(-\Delta - V)^{1/2}u\|_p \quad (4)$$

for all  $p \in ((p_{0*})'; p_{0*})$ . If in addition  $V \in K_N^\infty$ , then (4) holds for all  $p \in (N'; N)$ .

For the proof of the boundedness of Riesz transforms we use the off-diagonal estimates (for properties and more details see [4]). These estimates are a generalization of the Gaussian estimates used by Coulhon and Duong in [9] to study the Riesz transforms associated to the Laplace-Beltrami operator on Riemannian manifolds, and by Duong, Ouhabaz and Yan in [11] to study the magnetic Schrödinger operator on  $\mathbb{R}^N$ . We also use the approach of Blunck and Kunstmann in [6] to weak type  $(p, p)$ -estimates. In [1], Auscher used these tools to divergence-form operators with complex coefficients. We study also the Berry-Esseen estimate. Coulhon and Dungey [8] used these estimates to study the Riesz transforms associated to the divergence-form operators with real coefficients. In contrast to [12] and [13], we do not assume decay nor

smoothness conditions on  $V$ .

In the following sections, we denote by  $L^p$  the Lebesgue space  $L^p(\mathbb{R}^N)$  with the Lebesgue measure  $dx$ ,  $\|\cdot\|_p$  its usual norm,  $(\cdot, \cdot)$  the inner product of  $L^2$ ,  $\|\cdot\|_{p \rightarrow q}$  the norm of operators acting from  $L^p$  to  $L^q$ . We denote by  $C, c$  the positive constants even if their values change at each occurrence. Through this paper,  $\nabla A^{-1/2}$  denotes one of the partial derivative  $\frac{\partial}{\partial x_k} A^{-1/2}$  for any fixed  $k \in \{1, \dots, N\}$ .

## 2 Off-diagonal estimates

In this section, we show that  $(e^{-tA})_{t>0}$ ,  $(\sqrt{t}\nabla e^{-tA})_{t>0}$  and  $(\sqrt{t}V^{1/2}e^{-tA})_{t>0}$  satisfy  $L^p - L^2$  off-diagonal estimates provided that  $V$  is strongly subcritical.

**Definition 21** Let  $(T_t)_{t>0}$  be a family of uniformly bounded operators on  $L^2$ . We say that  $(T_t)_{t>0}$  satisfies  $L^p - L^q$  off-diagonal estimates for  $p, q \in [1; \infty]$  with  $p \leq q$  if there exist positive constants  $C$  and  $c$  such that for all closed sets  $E$  and  $F$  of  $\mathbb{R}^N$  and all  $h \in L^p(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$  with support in  $E$ , we have for all  $t > 0$ :

$$\|T_t h\|_{L^q(F)} \leq C t^{-\gamma_{pq}} e^{-\frac{cd(E,F)^2}{t}} \|h\|_p,$$

where  $d$  is the Euclidean distance and  $\gamma_{pq} := \frac{N}{2} \left( \frac{1}{p} - \frac{1}{q} \right)$ .

**Proposition 21** Let  $A = -\Delta - V$  where  $V \geq 0$  and  $V$  is strongly subcritical. Then  $(e^{-tA})_{t>0}$ ,  $(\sqrt{t}\nabla e^{-tA})_{t>0}$ , and  $(\sqrt{t}V^{1/2}e^{-tA})_{t>0}$  satisfy  $L^2 - L^2$  off-diagonal estimates, and we have for all  $t > 0$  and all  $f \in L^2$  supported in  $E$ :

- (i)  $\|e^{-tA} f\|_{L^2(F)} \leq e^{-d^2(E,F)/4t} \|f\|_2,$
- (ii)  $\|\sqrt{t}\nabla e^{-tA} f\|_{L^2(F)} \leq C e^{-d^2(E,F)/16t} \|f\|_2,$
- (iii)  $\|\sqrt{t}V^{1/2}e^{-tA} f\|_{L^2(F)} \leq C e^{-d^2(E,F)/8t} \|f\|_2.$

**Proof** The ideas are classical and rely on the well known Davies perturbation technique. Let  $A_\rho := e^{\rho\phi} A e^{-\rho\phi}$  where  $\rho > 0$  and  $\phi$  is a Lipschitz function with  $|\nabla\phi| \leq 1$  a.e.. Here  $A_\rho$  is the associated operator to the sesquilinear form  $\mathfrak{a}_\rho$  defined by

$$\mathfrak{a}_\rho(u, v) := \mathfrak{a}(e^{-\rho\phi} u, e^{\rho\phi} v)$$

for all  $u, v \in D(\mathfrak{a})$ .

By the strong subcriticality property of  $V$  we have for all  $u \in W^{1,2}$

$$\begin{aligned} ((A_\rho + \rho^2)u, u) &= - \int \rho^2 |\nabla\phi|^2 u^2 + \int |\nabla u|^2 - \int V u^2 + \rho^2 \|u\|_2^2 \\ &\geq \varepsilon \|V^{1/2} u\|_2^2. \end{aligned} \tag{5}$$

Using (2), we obtain

$$\begin{aligned} ((A_\rho + \rho^2)u, u) &= - \int \rho^2 |\nabla \phi|^2 u^2 + \int |\nabla u|^2 - \int V u^2 + \rho^2 \|u\|_2^2 \\ &\geq \frac{\varepsilon}{\varepsilon + 1} \|\nabla u\|_2^2. \end{aligned} \quad (6)$$

In particular  $(A_\rho + \rho^2)$  is a maximal accretive operator on  $L^2$ , and this implies

$$\|e^{-tA_\rho} u\|_2 \leq e^{t\rho^2} \|u\|_2. \quad (7)$$

Now we want to estimate

$$\|(A_\rho + 2\rho^2)e^{-t(A_\rho + 2\rho^2)}\|_{2-2}.$$

First, let us prove that  $A_\rho + 2\rho^2$  is a sectorial operator.

For  $u$  complex-valued,

$$\mathfrak{a}_\rho(u, u) := \mathfrak{a}(u, u) + \rho \int u \nabla \phi \overline{\nabla u} - \rho \int \bar{u} \nabla \phi \nabla u - \rho^2 \int |\nabla \phi|^2 |u|^2.$$

Then

$$\begin{aligned} \mathfrak{a}_\rho(u, u) + 2\rho^2 \|u\|_2^2 &\geq \mathfrak{a}(u, u) + \rho \int u \nabla \phi \overline{\nabla u} - \rho \int \bar{u} \nabla \phi \nabla u + \rho^2 \|u\|_2^2 \\ &= \mathfrak{a}(u, u) + 2i\rho \Im \int u \nabla \phi \overline{\nabla u} + \rho^2 \|u\|_2^2. \end{aligned}$$

This implies that

$$\Re(\mathfrak{a}_\rho(u, u) + 2\rho^2 \|u\|_2^2) \geq \mathfrak{a}(u, u), \quad (8)$$

and

$$\Re(\mathfrak{a}_\rho(u, u) + 2\rho^2 \|u\|_2^2) \geq \rho^2 \|u\|_2^2. \quad (9)$$

On the other hand,

$$\begin{aligned} \mathfrak{a}_\rho(u, u) &= \mathfrak{a}(u, u) + \rho \int u \nabla \phi \overline{\nabla u} - \rho \int \bar{u} \nabla \phi \nabla u - \rho^2 \int |\nabla \phi|^2 |u|^2 \\ &= \mathfrak{a}(u, u) + 2i\rho \Im \int u \nabla \phi \overline{\nabla u} - \rho^2 \int |\nabla \phi|^2 |u|^2. \end{aligned}$$

So

$$\begin{aligned} |\Im(\mathfrak{a}_\rho(u, u) + 2\rho^2 \|u\|_2^2)| &= 2|\rho| \int |u| |\nabla \phi| |\overline{\nabla u}| \\ &\leq 2|\rho| \|u\|_2 \|\nabla u\|_2. \end{aligned}$$

Using (2) we obtain that

$$\begin{aligned} |\Im(\mathfrak{a}_\rho(u, u) + 2\rho^2 \|u\|_2^2)| &\leq 2|\rho| \|u\|_2 c_\varepsilon^{\frac{1}{2}}(u, u) \\ &\leq c_\varepsilon^2 \mathfrak{a}(u, u) + \rho^2 \|u\|_2^2, \end{aligned}$$

where  $c_\varepsilon = (1 + \frac{1}{\varepsilon})^{\frac{1}{2}}$ . Now using estimates (8) and (9), we deduce that there exists a constant  $C > 0$  depending only on  $\varepsilon$  such that

$$|\Im(\mathbf{a}_\rho(u, u) + 2\rho^2 \|u\|_2^2)| \leq C \Re(\mathbf{a}_\rho(u, u) + 2\rho^2 \|u\|_2^2).$$

We conclude that (see [14] or [16])

$$\|e^{-z(A_\rho + 2\rho^2)}\|_{2-2} \leq 1$$

for all  $z$  in the open sector of angle  $\arctan(1/C)$ . Hence by the Cauchy formula

$$\|(A_\rho + 2\rho^2)e^{-t(A_\rho + 2\rho^2)}\|_{2-2} \leq \frac{C}{t}. \quad (10)$$

The constant  $C$  is independent of  $\rho$ .

By estimate (5) and (6) we have

$$((A_\rho + 2\rho^2)u, u) \geq ((A_\rho + \rho^2)u, u) \geq \varepsilon \|V^{1/2}u\|_2^2,$$

and

$$((A_\rho + 2\rho^2)u, u) \geq ((A_\rho + \rho^2)u, u) \geq \frac{\varepsilon}{\varepsilon + 1} \|\nabla u\|_2^2.$$

Setting  $u = e^{-t(A_\rho + 2\rho^2)}f$  and using (10) and (7) we obtain

$$\|\sqrt{t}\nabla e^{-tA_\rho}f\|_2 \leq Ce^{2t\rho^2}\|f\|_2. \quad (11)$$

and

$$\|\sqrt{t}V^{1/2}e^{-tA_\rho}f\|_2 \leq Ce^{2t\rho^2}\|f\|_2. \quad (12)$$

Let  $E$  and  $F$  be two closed subsets of  $\mathbb{R}^N$ ,  $f \in L^2(\mathbb{R}^N)$  supported in  $E$ , and let  $\phi(x) := d(x, E)$  where  $d$  is the Euclidean distance. Since  $e^{\rho\phi}f = f$ , we have the following relation

$$e^{-tA}f = e^{-\rho\phi}e^{-tA_\rho}f.$$

Then

$$\nabla e^{-tA}f = -\rho\nabla\phi e^{-\rho\phi}e^{-tA_\rho}f + e^{-\rho\phi}\nabla e^{-tA_\rho}f,$$

and

$$V^{1/2}e^{-tA}f = e^{-\rho\phi}V^{1/2}e^{-tA_\rho}f.$$

Now taking the norm on  $L^2(F)$ , we obtain from (7), (11) and (12)

$$\|e^{-tA}f\|_{L^2(F)} \leq e^{-\rho d(E, F)}e^{\rho^2 t}\|f\|_2, \quad (13)$$

$$\|\nabla e^{-tA}f\|_{L^2(F)} \leq \rho e^{-\rho d(E, F)}e^{\rho^2 t}\|f\|_2 + \frac{C}{\sqrt{t}}e^{-\rho d(E, F)}e^{2t\rho^2}\|f\|_2, \quad (14)$$



and

$$\|V^{1/2}e^{-tA}f\|_{L^2(F)} \leq \frac{C}{\sqrt{t}}e^{-\rho d(E,F)}e^{2\rho^2 t}\|f\|_2. \quad (15)$$

We set  $\rho = d(E, F)/2t$  in (13) and  $\rho = d(E, F)/4t$  in (15), then we get the  $L^2 - L^2$  off-diagonal estimates (i) and (iii).

We set  $\rho = d(E, F)/4t$  in (14), we get

$$\|\nabla e^{-tA}f\|_{L^2(F)} \leq \frac{C}{\sqrt{t}}\left(1 + \frac{d(E, F)}{4\sqrt{t}}\right)e^{-d^2(E, F)/8t}\|f\|_2.$$

This gives estimate (ii).  $\square$

Now, we study the  $L^p - L^2$  boundedness of the semigroup, of its gradient, and of  $(V^{1/2}e^{-tA})_{t>0}$ .

**Proposition 22** Suppose that  $A \geq \varepsilon V$ , then  $(e^{-tA})_{t>0}$ ,  $(\sqrt{t}\nabla e^{-tA})_{t>0}$  and  $(\sqrt{t}V^{1/2}e^{-tA})_{t>0}$  are  $L^p - L^2$  bounded for all  $p \in (p'_0; 2]$ . Here  $p'_0$  is the dual exponent of  $p_0$  where  $p_0 = \frac{2N}{(N-2)(1-\sqrt{1-\frac{1}{1+\varepsilon}})}$ , and the dimension  $N \geq 3$ . More precisely we have for all  $t > 0$ :

- i)  $\|e^{-tA}f\|_2 \leq Ct^{-\gamma_p}\|f\|_p$ ,
- ii)  $\|\sqrt{t}\nabla e^{-tA}f\|_2 \leq Ct^{-\gamma_p}\|f\|_p$ ,
- iii)  $\|\sqrt{t}V^{1/2}e^{-tA}f\|_2 \leq Ct^{-\gamma_p}\|f\|_p$ ,

where  $\gamma_p = \frac{N}{2}\left(\frac{1}{p} - \frac{1}{2}\right)$ .

**Proof** i) We apply the Gagliardo-Nirenberg inequality

$$\|u\|_2^2 \leq C_{a,b}\|\nabla u\|_2^{2a}\|u\|_p^{2b},$$

where  $a + b = 1$  and  $(1 + 2\gamma_p)a = 2\gamma_p$ , to  $u = e^{-tA}f$  for all  $f \in L^2 \cap L^p$ , all  $t > 0$ , and all  $p \in (p'_0; 2]$ . We obtain

$$\|e^{-tA}f\|_2^2 \leq C_{a,b}\|\nabla e^{-tA}f\|_2^{2a}\|e^{-tA}f\|_p^{2b}.$$

At present we use the boundedness of the semigroup on  $L^p$  for all  $p \in (p'_0; 2]$  proved in [15], and the fact that  $\|\nabla u\|_2^2 \leq (1 + 1/\varepsilon)(Au, u)$  from the strong subcriticality condition, then we obtain that

$$\|e^{-tA}f\|_2^{2/a} \leq -C\psi'(t)\|f\|_p^{2b/a}$$

where  $\psi(t) = \|e^{-tA}f\|_2^2$ . This implies

$$\|f\|_p^{-2b/a} \leq C(\psi(t))^{\frac{a-1}{a}}.$$

Since  $\frac{2b}{a} = \frac{1}{\gamma_p}$  and  $\frac{a-1}{a} = -\frac{1}{2\gamma_p}$ , integration between 0 and  $t$  yields

$$t\|f\|_p^{-1/\gamma_p} \leq C\|e^{-tA}f\|_2^{-1/\gamma_p},$$

which gives i).

We obtain ii) by using the following decomposition:

$$\sqrt{t}\nabla e^{-tA} = \sqrt{t}\nabla A^{-1/2}A^{1/2}e^{-tA/2}e^{-tA/2},$$

the boundedness of  $\nabla A^{-1/2}$  and of  $(\sqrt{t}A^{1/2}e^{-tA})_{t>0}$  on  $L^2$ , and the fact that  $(e^{-tA})_{t>0}$  is  $L^p - L^2$  bounded for all  $p \in (p'_0; 2]$  proved in i).

We prove iii) by using the following decomposition:

$$\sqrt{t}V^{1/2}e^{-tA} = \sqrt{t}V^{1/2}A^{-1/2}A^{1/2}e^{-tA/2}e^{-tA/2},$$

the boundedness of  $V^{1/2}A^{-1/2}$

and of  $(\sqrt{t}A^{1/2}e^{-tA})_{t>0}$  on  $L^2$ , and the fact that  $(e^{-tA})_{t>0}$  is  $L^p - L^2$  bounded for all  $p \in (p'_0; 2]$  proved in i).  $\square$

We invest the previous results to obtain :

**Theorem 21** Assume that  $A \geq \varepsilon V$  then  $(e^{-tA})_{t>0}$ ,  $(\sqrt{t}\nabla e^{-tA})_{t>0}$  and  $(\sqrt{t}V^{1/2}e^{-tA})_{t>0}$  satisfy  $L^p - L^2$  off-diagonal estimates for all  $p \in (p'_0; 2]$ . Here  $p'_0$  is the dual exponent of  $p_0$  where  $p_0 = \frac{2N}{(N-2)(1-\sqrt{1-\frac{1}{1+\varepsilon}})}$ , and the dimension  $N \geq 3$ . Then we have for all  $t > 0$ , all  $p \in (p'_0; 2]$ , and all  $f \in L^2 \cap L^p$  with  $\text{supp} f \subseteq E$

i)

$$\|e^{-tA}f\|_{L^2(F)} \leq Ct^{-\gamma_p} e^{-\frac{cd^2(E,F)}{t}} \|f\|_p, \quad (16)$$

ii)

$$\|\sqrt{t}\nabla e^{-tA}f\|_{L^2(F)} \leq Ct^{-\gamma_p} e^{-\frac{cd^2(E,F)}{t}} \|f\|_p, \quad (17)$$

iii)

$$\|\sqrt{t}V^{1/2}e^{-tA}f\|_{L^2(F)} \leq Ct^{-\gamma_p} e^{-\frac{cd^2(E,F)}{t}} \|f\|_p, \quad (18)$$

where  $\gamma_p = \frac{N}{2}(\frac{1}{p} - \frac{1}{2})$  and  $C, c$  are positive constants.

**Remark:** By duality, we deduce from (16) a  $L^2 - L^p$  off-diagonal estimate of the norm of the semigroup for all  $p \in [2; p_0)$ , but we cannot deduce from (17) and (18) the same estimate of the norm of  $\sqrt{t}\nabla e^{-tA}f$  and of  $\sqrt{t}V^{1/2}e^{-tA}f$  because they are not selfadjoint. This affect the boundedness of Riesz transforms and of  $V^{1/2}A^{-1/2}$  on  $L^p$  for  $p > 2$ .

**Proof** i) In the previous proposition we have proved that

$$\|e^{-tA}f\|_2 \leq Ct^{-\gamma_p} \|f\|_p$$

for all  $p \in (p'_0; 2]$ . This implies that for all  $t > 0$

$$\|\chi_F e^{-tA} \chi_E f\|_2 \leq Ct^{-\gamma_p} \|f\|_p$$

where  $\chi_M$  is the characteristic function of  $M$ . The  $L^2 - L^2$  off-diagonal estimate proved in the Proposition 21 implies that

$$\|\chi_F e^{-tA} \chi_E f\|_2 \leq e^{-d^2(E,F)/4t} \|f\|_2.$$

Hence we can apply the Riesz-Thorin interpolation theorem and we obtain the off-diagonal estimate (16).

Assertions ii) and iii) are proved in a similar way. We use  $L^2 - L^2$  off-diagonal estimates of Proposition 21 and assertions ii) and iii) of Proposition 22.  $\square$

### 3 Boundedness of $\nabla A^{-1/2}$ and $V^{1/2} A^{-1/2}$ on $L^p$ for $p < 2$

This section is devoted to the study of the boundedness of  $V^{1/2} A^{-1/2}$  and Riesz transforms associated to Schrödinger operators with negative and strongly subcritical potentials. We prove that  $\nabla A^{-1/2}$  and  $V^{1/2} A^{-1/2}$  are bounded on  $L^p(\mathbb{R}^N)$ ,  $N \geq 3$ , for all  $p \in (p'_0; 2]$ , where  $p'_0$  is the exponent mentioned in Theorem 21.

**Theorem 31** Assume that  $A \geq \varepsilon V$ , then  $\nabla A^{-1/2}$  is bounded on  $L^p(\mathbb{R}^N)$  for  $N \geq 3$ , for all  $p \in (p'_0; 2]$  where  $p'_0 = \left( \frac{2N}{(N-2)(1-\sqrt{1-\frac{1}{1+\varepsilon}})} \right)'$ .

*Remark:* In the proof of this theorem, we do not use the expression of  $A$  explicitly, we use the boundedness of Riesz transforms associated to  $A$  on  $L^2$ , the  $L^p - L^2$  off-diagonal estimates of  $(e^{-tA})_{t>0}$  and of  $(\sqrt{t}\nabla e^{-tA})_{t>0}$ . Hence  $\nabla H^{-1/2}$  is bounded on  $L^p$  for all  $p \in (r; 2]$ , where  $r \in [1; 2)$ , if  $\nabla H^{-1/2}$  is bounded on  $L^2$  where  $H$  is an operator such that  $(e^{-tH})_{t>0}$  and  $(\sqrt{t}\nabla e^{-tH})_{t>0}$  satisfy  $L^r - L^2$  (or  $L^p - L^2$  for all  $p \in (r; 2]$ ) off-diagonal estimates.

To prove Theorem 31, we prove that  $\nabla A^{-1/2}$  is of weak type  $(p, p)$  for all  $p \in (p'_0; 2)$  by using the following theorem of Blunck and Kunstmann [6]. Then by the boundedness of  $\nabla A^{-1/2}$  on  $L^2$ , and the Marcinkiewicz interpolation theorem, we obtain boundedness on  $L^p$  for all  $p \in (p'_0; 2]$ .

**Theorem 32** Let  $p \in [1; 2)$ . Suppose that  $T$  is sublinear operator of strong type  $(2, 2)$ , and let  $(A_r)_{r>0}$  be a family of linear operators acting on  $L^2$ .

Assume that for  $j \geq 2$

$$\left( \frac{1}{|2^{j+1}B|} \int_{C_j(B)} |T(I - A_{r(B)})f|^2 \right)^{1/2} \leq g(j) \left( \frac{1}{|B|} \int_B |f|^p \right)^{1/p}, \quad (19)$$

and for  $j \geq 1$

$$\left( \frac{1}{|2^{j+1}B|} \int_{C_j(B)} |A_{r(B)}f|^2 \right)^{1/2} \leq g(j) \left( \frac{1}{|B|} \int_B |f|^p \right)^{1/p}, \quad (20)$$

for all ball  $B$  with radius  $r(B)$  and all  $f$  supported in  $B$ . If  $\Sigma := \sum g(j)2^{Nj} < \infty$ , then  $T$  is of weak type  $(p, p)$ , with a bound depending only on the strong type  $(2, 2)$  bound of  $T$ ,  $p$ , and  $\Sigma$ .

Here  $C_1 = 4B$  and  $C_j(B) = 2^{j+1}B \setminus 2^jB$  for  $j \geq 2$ , where  $\lambda B$  is the ball of radius  $\lambda r(B)$  with the same center as  $B$ , and  $|\lambda B|$  its Lebesgue measure.

Proof (of Theorem 31) Let  $T = \nabla A^{-1/2}$ . We prove assumptions (19) and (20) with  $A_r = I - (I - e^{-r^2 A})^m$  for some  $m > N/4 - \gamma_p$ , using arguments similar to Auscher [1] Theorem 4.2.

Let us prove (20). For  $f$  supported in a ball  $B$  (with radius  $r$ ),

$$\begin{aligned} \frac{1}{|2^{j+1}B|^{1/2}} \|A_r f\|_{L^2(C_j(B))} &= \frac{1}{|2^{j+1}B|^{1/2}} \left\| \sum_{k=1}^m \binom{m}{k} (-1)^{k+1} e^{-kr^2 A} f \right\|_{L^2(C_j(B))} \\ &\leq \frac{1}{|2^{j+1}B|^{1/2}} \sum_{k=1}^m \binom{m}{k} C(kr^2)^{-\gamma_p} e^{-\frac{cd^2(B, C_j(B))}{kr^2}} \|f\|_p. \end{aligned}$$

for all  $p \in (p'_0; 2)$  and all  $f \in L^2 \cap L^p$  supported in  $B$ . Here we use the  $L^p - L^2$  off-diagonal estimates (16) for  $p \in (p'_0; 2]$ . Since  $\gamma_p = \frac{N}{2}(\frac{1}{p} - \frac{1}{2})$  we obtain

$$\begin{aligned} \left( \frac{1}{|2^{j+1}B|} \int_{C_j(B)} |A_r f|^2 \right)^{1/2} &\leq \frac{Cr^{-2\gamma_p}}{|2^{j+1}B|^{1/2}} e^{-\frac{cd^2(B, C_j(B))}{mr^2}} \|f\|_p \\ &\leq C2^{-jN/2} e^{-\frac{cd^2(B, C_j(B))}{r^2}} \left( \frac{1}{|B|} \int_B |f|^p \right)^{1/p}. \end{aligned}$$

This yields, for  $j = 1$ ,

$$\left( \frac{1}{|4B|} \int_{4B} |A_r f|^2 \right)^{1/2} \leq C2^{-N/2} \left( \frac{1}{|B|} \int_B |f|^p \right)^{1/p},$$

and for  $j \geq 2$

$$\left( \frac{1}{|2^{j+1}B|} \int_{C_j(B)} |A_r f|^2 \right)^{1/2} \leq C2^{-jN/2} e^{-c4^j} \left( \frac{1}{|B|} \int_B |f|^p \right)^{1/p}.$$

Thus assumption (20) of Theorem 32 holds with  $\sum_{j \geq 1} g(j)2^{jN} < \infty$ .

It remains to check the assumption (19):

We know that

$$\nabla A^{-1/2} f = C \int_0^\infty \nabla e^{-tA} f \frac{dt}{\sqrt{t}}$$

then, using the Newton binomial, we get

$$\begin{aligned} \nabla A^{-1/2} (I - e^{-r^2 A})^m f &= C \int_0^\infty \nabla e^{-tA} (I - e^{-r^2 A})^m f \frac{dt}{\sqrt{t}} \\ &= C \int_0^\infty g_{r^2}(t) \nabla e^{-tA} f dt \end{aligned}$$

where

$$g_{r^2}(t) = \sum_{k=0}^m \binom{m}{k} (-1)^k \frac{\chi_{(t-kr^2>0)}}{\sqrt{t-kr^2}}.$$

Hence, using the  $L^p-L^2$  off-diagonal estimate (17), we obtain for all  $p \in (p'_0; 2)$ , all  $j \geq 2$ , and all  $f \in L^2 \cap L^p$  supported in  $B$

$$\|\nabla A^{-1/2}(I - e^{-r^2 A})^m f\|_{L^2(C_j(B))} \leq C \int_0^\infty |g_{r^2}(t)| t^{-\gamma_p-1/2} e^{-c4^j r^2/t} dt \|f\|_p.$$

We observe that (see [1] p. 27)

$$|g_{r^2}(t)| \leq \frac{C}{\sqrt{t-kr^2}} \quad \text{if } kr^2 < t \leq (k+1)r^2 \leq (m+1)r^2$$

and

$$|g_{r^2}(t)| \leq Cr^{2m} t^{-m-1/2} \quad \text{if } t > (m+1)r^2.$$

This yields

$$\begin{aligned} \|\nabla A^{-\frac{1}{2}}(I - e^{-r^2 A})^m f\|_{L^2(C_j(B))} &\leq C \sum_{k=0}^m \int_{kr^2}^{(k+1)r^2} \frac{t^{-\gamma_p-1/2}}{\sqrt{t-kr^2}} e^{-\frac{c4^j r^2}{t}} dt \|f\|_p \\ &\quad + C \int_{(m+1)r^2}^\infty r^{2m} t^{-\gamma_p-1-m} e^{-\frac{c4^j r^2}{t}} dt \|f\|_p \\ &\leq I_1 + I_2. \end{aligned} \quad (21)$$

We have

$$I_2 := C \int_{(m+1)r^2}^\infty r^{2m} t^{-\gamma_p-1-m} e^{-\frac{c4^j r^2}{t}} dt \|f\|_p \leq Cr^{-2\gamma_p} 2^{-2j(m+\gamma_p)} \|f\|_p,$$

by the Laplace transform formula, and

$$\begin{aligned} I_1 &:= C \|f\|_p \sum_{k=0}^m \int_{kr^2}^{(k+1)r^2} \frac{t^{-\gamma_p-1/2}}{\sqrt{t-kr^2}} e^{-\frac{c4^j r^2}{t}} dt \\ &= C \|f\|_p \left( \sum_{k=1}^m \int_{kr^2}^{(k+1)r^2} \frac{t^{-\gamma_p-1/2}}{\sqrt{t-kr^2}} e^{-\frac{c4^j r^2}{t}} dt + \int_0^{r^2} t^{-\gamma_p-1} e^{-\frac{c4^j r^2}{t}} dt \right) \\ &= J_1 + J_2. \end{aligned}$$

In the preceding equation

$$\begin{aligned} J_1 &:= C \|f\|_p \sum_{k=1}^m \int_{kr^2}^{(k+1)r^2} \frac{t^{-\gamma_p-1/2}}{\sqrt{t-kr^2}} e^{-\frac{c4^j r^2}{t}} dt \\ &\leq C \|f\|_p e^{-\frac{c4^j}{m+1}} \sum_{k=1}^m (kr^2)^{-\gamma_p-1/2} \int_{kr^2}^{(k+1)r^2} (t-kr^2)^{-1/2} dt \\ &\leq Cr^{-2\gamma_p} 2^{-2j(m+\gamma_p)} \|f\|_p, \end{aligned}$$

and

$$\begin{aligned}
J_2 &:= C \int_0^{r^2} t^{-\gamma_p-1} e^{-\frac{c_4^j r^2}{t}} dt \|f\|_p \\
&\leq C \|f\|_p e^{-\frac{c_4^j}{2(m+1)}} \int_0^{r^2} t^{-\gamma_p-1} e^{-\frac{c_4^j r^2}{2t}} dt \\
&\leq C \|f\|_p 2^{-2jm} \int_0^{r^2} t^{-1-\gamma_p} C(2^{-2j} r^{-2} t)^{\gamma_p} e^{-\frac{c_4^j r^2}{4t}} dt \\
&\leq C \|f\|_p 2^{-2j(m+\gamma_p)} r^{-2\gamma_p} \int_0^{r^2} t^{-1} e^{-\frac{c_4^j r^2}{4t}} dt \\
&\leq C r^{-2\gamma_p} 2^{-2j(m+\gamma_p)} \|f\|_p.
\end{aligned}$$

Here, for the last inequality, we use the fact that  $j \geq 2$  to obtain the convergence of the integral without dependence on  $r$  nor on  $j$ .

We can therefore employ these estimates in (21) to conclude that

$$\|\nabla A^{-1/2}(I - e^{-r^2 A})^m f\|_{L^2(C_j(B))} \leq C r^{-2\gamma_p} 2^{-2j(m+\gamma_p)} \|f\|_p,$$

which implies

$$\left( \frac{1}{|2^{j+1}B|} \int_{C_j(B)} |\nabla A^{-\frac{1}{2}}(I - e^{-r^2 A})^m f|^2 \right)^{\frac{1}{2}} \leq C 2^{-2j(m+\gamma_p+\frac{N}{4})} \left( \frac{1}{|B|} \int_B |f|^p \right)^{\frac{1}{p}}$$

where  $\sum g(j)2^{jN} < \infty$  because we set  $m > N/4 - \gamma_p$ .  $\square$

**Proposition 31** Assume that  $A \geq \varepsilon V$ , then  $V^{1/2}A^{-1/2}$  is bounded on  $L^p(\mathbb{R}^N)$  for  $N \geq 3$ , for all  $p \in (p'_0; 2]$  where  $p'_0$  is the dual exponent of  $p_0$  with  $p_0 = \frac{2N}{(N-2)(1-\sqrt{1-\frac{1}{1+\varepsilon}})}$ .

**Proof** We have seen in (3) that the operator  $V^{1/2}A^{-1/2}$  is bounded on  $L^2$ . To prove its boundedness on  $L^p$  for all  $p \in (p'_0; 2]$  we prove that it is of weak type  $(p, p)$  for all  $p \in (p'_0; 2)$  by checking assumptions (19) and (20) of Theorem 32, where  $T = V^{1/2}A^{-1/2}$ . Then, using the Marcinkiewicz interpolation theorem, we deduce boundedness on  $L^p$  for all  $p \in (p'_0; 2]$ .

We check assumptions of Theorem 32 similarly as we did in the proof of Theorem 31, using the  $L^p - L^2$  off-diagonal estimate (18) instead of (17).  $\square$

Let us now move on, setting  $V = c|x|^{-2}$  where  $0 < c < (\frac{N-2}{2})^2$ , which is strongly subcritical thanks to the Hardy inequality, we prove that the associated Riesz transforms are not bounded on  $L^p$  for  $p \in (1; p'_0)$  neither for  $p \in (p_{0*}; \infty)$ . We also prove that  $V^{1/2}A^{-1/2}$  is not bounded on  $L^p$  for  $p \in (p_{0*}; \infty)$ . Here  $p_{0*} = \frac{p_0 N}{N+p_0}$  is the reverse Sobolev exponent of  $p_0$ .

**Proposition 32** Set  $V$  strongly subcritical and  $N \geq 3$ . Assume that  $\nabla A^{-1/2}$  is bounded on  $L^p$  for some  $p \in (1; p'_0)$ . Then there exists an exponent  $q_1 \in [p; p'_0)$  such that  $(e^{-tA})_{t>0}$  is bounded on  $L^r$  for all  $r \in (q_1; 2)$ .

Consider now  $V = c|x|^{-2}$  where  $0 < c < (\frac{N-2}{2})^2$ . It is proved in [15] that the semigroup does not act on  $L^p$  for  $p \notin (p'_0; p_0)$ . Therefore we obtain from this proposition that the Riesz transform  $\nabla A^{-1/2}$  is not bounded on  $L^p$  for  $p \in (1; p'_0)$ .

Proof Assume that  $\nabla A^{-1/2}$  is bounded on  $L^p$  for some  $p \in (1; p'_0)$ . By the boundedness on  $L^2$  and the Riesz-Thorin interpolation theorem, we get the boundedness of  $\nabla A^{-1/2}$  on  $L^q$  for all  $q \in [p; 2]$ . Now we apply the Sobolev inequality

$$\|f\|_{q^*} \leq C\|\nabla f\|_q \quad (22)$$

where  $q^* = \frac{Nq}{N-q}$  if  $q < N$  to  $f := A^{-1/2}u$ , so we get

$$\|A^{-1/2}u\|_{q^*} \leq C\|\nabla A^{-1/2}u\|_q \leq C\|u\|_q$$

for all  $q \in [p; 2]$ . In particular,  $\|A^{-1/2}\|_{q_1 - q_1^*} \leq C$  where  $p \leq q_1 < p'_0$  such that  $q_1^* > p'_0$ .

Decomposing the semigroup as follows

$$e^{-tA} = A^{1/2}e^{-tA/2}e^{-tA/2}A^{-1/2} \quad (23)$$

where  $A^{-1/2}$  is  $L^{q_1} - L^{q_1^*}$  bounded,  $e^{-tA/2}$  has  $L^{q_1^*} - L^2$  norm bounded by  $Ct^{-\gamma_{q_1^*}}$  (Proposition 22) and  $A^{1/2}e^{-tA/2}$  is  $L^2 - L^2$  bounded by  $Ct^{-1/2}$  because of the analyticity of the semigroup on  $L^2$ . Therefore, we obtain

$$\|e^{-tA}\|_{q_1 - 2} \leq Ct^{-\gamma_{q_1^*} - 1/2} = Ct^{-\gamma_{q_1}}.$$

We now interpolate this norm with the  $L^2 - L^2$  off-diagonal estimate of the norm of  $e^{-tA}$ , as we did in the proof of Theorem 21, so we get a  $L^r - L^2$  off-diagonal estimate for all  $r \in (q_1; 2)$ . Then Lemma 3.3 of [1] yields that  $(e^{-tA})_{t>0}$  is bounded on  $L^r$  for all  $r \in (q_1; 2)$  for some  $q_1 \in [p; p'_0)$ .  $\square$

Proposition 33 Set  $V$  strongly subcritical and  $N \geq 3$ . Assume that  $\nabla A^{-1/2}$  is bounded on  $L^p$  for some  $p \in (p_{0*}; \infty)$ . Then there exists an exponent  $q_2 > p_{0*}$  such that the semigroup  $(e^{-tA})_{t>0}$  is bounded on  $L^s$  for all  $s \in (2; q_2^*)$ . Here  $q_2^* > p_0$ .

Consider now  $V = c|x|^{-2}$  where  $0 < c < (\frac{N-2}{2})^2$ . It is proved in [15] that the semigroup does not act on  $L^p$  for  $p \notin (p'_0; p_0)$ . Therefore we obtain from this proposition that the Riesz transforms  $\nabla A^{-1/2}$  are not bounded on  $L^p$  for  $p \in (p_{0*}; \infty)$ .

Proof Assume that  $\nabla A^{-1/2}$  is bounded on  $L^p$  for some  $p \in (p_{0*}; \infty)$ . Then by interpolation we obtain the boundedness of  $\nabla A^{-1/2}$  on  $L^q$  for all  $q \in [2; p]$ . In particular,

$$\|\nabla A^{-1/2}\|_{q_2 - q_2} \leq C$$

where  $p_{0*} < q_2 < p_0$ ,  $q_2 \leq p$ ,  $q_2 < N$ . Using the Sobolev inequality (22), we obtain that  $A^{-1/2}$  is  $L^{q_2} - L^{q_2^*}$  bounded where  $q_2^* > p_0$ .

Now we decompose the semigroup as follows

$$e^{-tA} = A^{-1/2} e^{-tA/2} A^{1/2} e^{-tA/2}. \quad (24)$$

Thus we remark that it is  $L^2 - L^{q_2^*}$  bounded where  $q_2^* > p_0$ .

Then, using similar arguments as in the previous proof, we conclude that  $(e^{-tA})_{t>0}$  is bounded on  $L^s$  for all  $s \in (2; q_2^*)$  for some  $q_2 > p_{0*}$ .  $\square$

**Proposition 34** Set  $V$  strongly subcritical and  $N \geq 3$ . Assume that  $V^{1/2}A^{-1/2}$  is bounded on  $L^p$  for some  $p \in [2; p_0)$ . Then  $\nabla A^{-1/2}$  is bounded on  $L^r$  for all  $r \in [2; p)$ .

We prove this result after the proof of Theorem 43.

Consider now  $p > p_{0*}$  and  $V = c|x|^{-2}$  where  $0 < c < (\frac{N-2}{2})^2$ . It is proved in [15] that the semigroup does not act on  $L^p$  for  $p \notin (p'_0; p_0)$ . Therefore we obtain from this proposition and Proposition 33 that  $V^{1/2}A^{-1/2}$  is not bounded on  $L^p$  when  $p \in (p_{0*}; p_0)$ . Now if  $V^{1/2}A^{-1/2}$  is bounded on  $L^q$  for some  $q \geq p_0$ , then, by interpolation with its boundedness on  $L^2$ ,  $V^{1/2}A^{-1/2}$  will be bounded on  $L^p$  for all  $p \in (2; q)$ , in particular for  $p \in (p_{0*}; p_0)$ . Therefore we conclude that  $V^{1/2}A^{-1/2}$  is not bounded on  $L^p$  when  $p \in (p_{0*}; \infty)$ .

#### 4 The case $p \in (2; p_{0*})$

In this section we prove that if the strongly subcritical potential is, in addition, in  $L^{N/2}$ ,  $N \geq 3$ , then  $\nabla A^{-1/2}$  and  $V^{1/2}A^{-1/2}$  are bounded on  $L^p$  for all  $p \in (p'_0; p_{0*})$ . In the previous section we have proved the boundedness on  $L^p$  for all  $p \in (p'_0; 2]$  when the potential is strongly subcritical, so it remains to study the case of  $L^p$  when  $p \in (2; p_{0*})$ .

First we prove the following result:

**Theorem 41** If  $(\sqrt{t}\nabla e^{-tA})_{t>0}$  is bounded on  $L^{p_1}$  for some  $p_1 \in [2; p_0)$ , then  $\nabla A^{-1/2}$  is bounded on  $L^p$  for all  $p \in (2; p_1)$  provided that  $V$  is strongly subcritical, and  $N \geq 3$ .

Let us prove that  $\nabla A^{-1/2}$  satisfies the two estimates of the following theorem proved in [3] (Theorem 2.1).

**Theorem 42** Let  $q \in (2; \infty]$ . Suppose that  $T$  is a sublinear operator bounded on  $L^2$ , and let  $(A_r)_{r>0}$  be a family of linear operators on  $L^2$ . Assume that

$$\left( \frac{1}{|B|} \int_B |T(I - A_{r(B)})f|^2 \right)^{1/2} \leq C(M(|f|^2))^{1/2}(x) \quad (25)$$

$$\left( \frac{1}{|B|} \int_B |TA_{r(B)}f|^q \right)^{1/q} \leq C(M(|Tf|^2))^{1/2}(x) \quad (26)$$



for all  $f \in L^2$ , all ball  $B$  and all  $x \in B$  where  $r(B)$  is the radius of  $B$ . Here  $M$  is the Hardy-Littlewood maximal operator

$$Mf(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f|.$$

If  $2 < p < q$  and  $Tf \in L^p$  when  $f \in L^p$ , then  $T$  is of strong type  $(p, p)$ . More precisely, for all  $f \in L^2 \cap L^p$ ,  $\|Tf\|_p \leq c\|f\|_p$  where  $c$  depends only on the dimension  $N, p, q$  and  $C$ .

Proof (of Theorem 41) Set  $T = \nabla A^{-1/2}$ ,  $A_r = I - (I - e^{-r^2 A})^m$  where  $m > N/4$ , and take  $f \in L^2$ . We use similar arguments as Auscher [1] Theorem 4.6. Decompose  $f = \sum_j f \chi_{C_j(B)}$  where  $C_1 = 4B$  and  $C_j(B) = 2^{j+1}B \setminus 2^j B$  for  $j \geq 2$ , where  $B$  is the ball of radius  $r$ .

Let us verify (25): By the Minkowski inequality, we have

$$\|\nabla A^{-1/2}(I - e^{-r^2 A})^m f\|_{L^2(B)} \leq \sum_{j \geq 1} \|\nabla A^{-1/2}(I - e^{-r^2 A})^m f \chi_{C_j(B)}\|_{L^2(B)} \quad (27)$$

We study  $\|\nabla A^{-1/2}(I - e^{-r^2 A})^m f \chi_{C_j(B)}\|_{L^2(B)}$  for  $j \geq 1$ . We have

$$\begin{aligned} \nabla A^{-1/2}(I - e^{-r^2 A})^m f \chi_{C_j(B)} &= C \int_0^\infty \nabla e^{-tA}(I - e^{-r^2 A})^m f \chi_{C_j(B)} \frac{dt}{\sqrt{t}} \\ &= C \int_0^\infty g_{r^2}(t) \nabla e^{-tA} f \chi_{C_j(B)} dt \end{aligned}$$

where

$$g_{r^2}(t) = \sum_{k=0}^m \binom{m}{k} (-1)^k \frac{\chi_{(t - kr^2 > 0)}}{\sqrt{t - kr^2}}.$$

Hence, by the  $L^2 - L^2$  off-diagonal estimate of the gradient of the semigroup proved in Proposition 21, we obtain for  $j \geq 2$

$$\|\nabla A^{-1/2}(I - e^{-r^2 A})^m f \chi_{C_j(B)}\|_{L^2(B)} \leq C \int_0^\infty |g_{r^2}(t)| t^{-1/2} e^{-c4^j r^2/t} dt \|f \chi_{C_j(B)}\|_2.$$

As we have mentioned in the previous section

$$|g_{r^2}(t)| \leq \frac{C}{\sqrt{t - kr^2}} \quad \text{if } kr^2 < t \leq (k+1)r^2 \leq (m+1)r^2,$$

and

$$|g_{r^2}(t)| \leq Cr^{2m} t^{-m-1/2} \quad \text{if } t > (m+1)r^2.$$

This yields

$$\begin{aligned}
& \|\nabla A^{-\frac{1}{2}}(I - e^{-r^2 A})^m f \chi_{C_j(B)}\|_{L^2(B)} \\
& \leq C \sum_{k=0}^m \int_{kr^2}^{(k+1)r^2} \frac{t^{-1/2}}{\sqrt{t - kr^2}} e^{-\frac{c4^j r^2}{t}} dt \|f \chi_{C_j(B)}\|_2 \\
& + C r^{2m} \int_{(m+1)r^2}^{\infty} t^{-1-m} e^{-\frac{c4^j r^2}{t}} dt \|f \chi_{C_j(B)}\|_2 \\
& \leq I_1 + I_2. \tag{28}
\end{aligned}$$

$$I_2 := C r^{2m} \int_{(m+1)r^2}^{\infty} t^{-1-m} e^{-\frac{c4^j r^2}{t}} dt \|f \chi_{C_j(B)}\|_2 \leq C 4^{-jm} \|f \chi_{C_j(B)}\|_2$$

by the Laplace transform formula, and

$$\|f \chi_{C_j(B)}\|_2 \leq \|f\|_{L^2(2^{j+1}B)} \leq |2^{j+1}B|^{1/2} (M(|f|^2))^{1/2}(x) \tag{29}$$

for all  $x \in B$  and  $j \geq 1$ . So

$$I_2 \leq C 4^{-jm} |2^{j+1}B|^{1/2} (M(|f|^2))^{1/2}(x).$$

At present, we can turn to  $I_1$ :

$$\begin{aligned}
I_1 & := C \sum_{k=0}^m \int_{kr^2}^{(k+1)r^2} \frac{t^{-\frac{1}{2}}}{\sqrt{t - kr^2}} e^{-\frac{c4^j r^2}{t}} dt \|f \chi_{C_j(B)}\|_2 \\
& = C \left( \sum_{k=1}^m \int_{kr^2}^{(k+1)r^2} \frac{t^{-\frac{1}{2}}}{\sqrt{t - kr^2}} e^{-\frac{c4^j r^2}{t}} dt + \int_0^{r^2} t^{-1} e^{-\frac{c4^j r^2}{t}} dt \right) \|f \chi_{C_j(B)}\|_2 \\
& = J_1 + J_2
\end{aligned}$$

where

$$\begin{aligned}
J_1 & := C \sum_{k=1}^m \int_{kr^2}^{(k+1)r^2} \frac{t^{-\frac{1}{2}}}{\sqrt{t - kr^2}} e^{-\frac{c4^j r^2}{t}} dt \|f \chi_{C_j(B)}\|_2 \\
& \leq C \|f \chi_{C_j(B)}\|_2 e^{-\frac{c4^j}{m+1}} \sum_{k=1}^m (kr^2)^{-1/2} \int_{kr^2}^{(k+1)r^2} (t - kr^2)^{-1/2} dt \\
& \leq C 4^{-jm} \|f \chi_{C_j(B)}\|_2 \\
& \leq C 4^{-jm} |2^{j+1}B|^{1/2} (M(|f|^2))^{1/2}(x)
\end{aligned}$$

where we use estimate (29). For the second term

$$\begin{aligned}
J_2 &:= \int_0^{r^2} t^{-1} e^{-\frac{c_4^j r^2}{t}} dt \|f \chi_{C_j(B)}\|_2 \\
&\leq C \|f \chi_{C_j(B)}\|_2 e^{-\frac{c_4^j}{2(m+1)}} \int_0^{r^2} t^{-1} e^{-\frac{c_4^j r^2}{2t}} dt \\
&\leq C \|f \chi_{C_j(B)}\|_2 4^{-jm} \int_0^{r^2} t^{-1} e^{-\frac{c_4^j r^2}{2t}} dt \\
&\leq C 4^{-jm} \|f \chi_{C_j(B)}\|_2 \\
&\leq C 4^{-jm} |2^{j+1} B|^{1/2} (M(|f|^2))^{1/2}(x).
\end{aligned}$$

Here we use the fact that  $j \geq 2$  to obtain the convergence of the integral without dependence on  $r$  nor on  $j$ , and for the last inequality we use (29). Inserting these estimates in (28) we find

$$\|\nabla A^{-1/2} (I - e^{-r^2 A})^m f \chi_{C_j(B)}\|_{L^2(B)} \leq C 4^{-jm} |2^{j+1} B|^{1/2} (M(|f|^2))^{1/2}(x)$$

for all  $j \geq 2$ . Now for  $j = 1$  we show that

$$\begin{aligned}
\|\nabla A^{-1/2} (I - e^{-r^2 A})^m f \chi_{C_1(B)}\|_{L^2(B)} &\leq \sum_{k=0}^m \binom{m}{k} \|\nabla A^{-1/2} e^{-kr^2 A} f \chi_{4B}\|_{L^2(B)} \\
&\leq C \sum_{k=0}^m \binom{m}{k} \|e^{-kr^2 A} f \chi_{4B}\|_{L^2(\mathbb{R}^N)} \\
&\leq C \sum_{k=0}^m \binom{m}{k} \|f \chi_{4B}\|_{L^2(\mathbb{R}^N)} \\
&\leq C |4B|^{1/2} (M(|f|^2))^{1/2}(x),
\end{aligned}$$

where we use the boundedness of Riesz transforms on  $L^2$  from estimate (2), and that of the semigroup on  $L^2$ .

Using these estimates in the inequality (27) we obtain

$$\begin{aligned}
\|\nabla A^{-\frac{1}{2}} (I - e^{-r^2 A})^m f\|_{L^2(B)} &\leq C \left( |4B|^{\frac{1}{2}} + \sum_{j \geq 2} 4^{-jm} |2^{j+1} B|^{\frac{1}{2}} \right) (M(|f|^2))^{\frac{1}{2}}(x) \\
&\leq C |B|^{\frac{1}{2}} \left( 2^{N/2} + \sum_{j \geq 2} 2^{-j(2m-N/2)} \right) (M(|f|^2))^{\frac{1}{2}}(x) \\
&\leq C |B|^{1/2} (M(|f|^2))^{1/2}(x)
\end{aligned}$$

because we set  $m > N/4$ . Finally, this implies that

$$\left( \frac{1}{|B|} \int_B |\nabla A^{-1/2} (I - e^{-r^2(B)A})^m f|^2 \right)^{1/2} \leq C (M(|f|^2))^{1/2}(x)$$

for all  $f \in L^2$  and all  $x$  in  $B$ , which is assumption (25).

Let us now prove (26):

$$\nabla A^{-1/2}(I - (I - e^{-r^2 A})^m)f = \sum_{k=1}^m \binom{m}{k} (-1)^{k+1} \nabla e^{-kr^2 A} A^{-1/2} f.$$

So to check (26) it suffices to prove that for  $q \in (2; p_1)$

$$\sum_{k=1}^m \binom{m}{k} \left( \frac{1}{|B|} \int_B |\nabla e^{-kr^2(B)A} g|^q \right)^{1/q} \leq C(M(|\nabla g|^2))^{1/2}(x). \quad (30)$$

This applied to  $g = A^{-1/2}f$ , gives us (26). Decompose  $g = \sum_j g\chi_{C_j(B)}$  where  $C_1 = 4B$  and  $C_j(B) = 2^{j+1}B \setminus 2^j B$  for  $j \geq 2$ . Here  $B$  is the ball of radius  $r$ . By the Minkowski inequality, we have

$$\left( \frac{1}{|B|} \int_B |\nabla e^{-kr^2 A} g|^q \right)^{1/q} \leq \sum_{j \geq 1} \left( \frac{1}{|B|} \int_B |\nabla e^{-kr^2 A} g\chi_{C_j(B)}|^q \right)^{1/q}. \quad (31)$$

By duality of the  $L^p - L^2$  boundedness of the semigroup proved in Proposition 22 we have

$$\|e^{-tA}\|_{2-p} \leq Ct^{-\gamma_p} \quad (32)$$

where  $\gamma_p := \frac{N}{2}(\frac{1}{2} - \frac{1}{p})$  for all  $p \in [2; p_0)$ . We have assumed that for all  $t > 0$ ,  $\|\sqrt{t}\nabla e^{-tA}\|_{p_1-p_1} \leq C$  for some  $p_1 \in [2; p_0)$ . Thus we obtain for all  $t > 0$

$$\|\sqrt{t}\nabla e^{-tA}\|_{2-p_1} \leq Ct^{-\gamma_{p_1}}$$

where  $\gamma_{p_1} = \frac{N}{2}(\frac{1}{2} - \frac{1}{p_1})$ . After interpolation of this boundedness with the  $L^2 - L^2$  off-diagonal estimate (Proposition 21 (ii)), as we did in the proof of Theorem 21, we obtain the following  $L^2 - L^q$  off-diagonal estimate

$$\|\sqrt{t}\nabla e^{-tA} f\|_{L^q(F)} \leq Ct^{-\gamma_q} e^{-c\frac{d^2(E;F)}{t}} \|f\|_2,$$

for all  $q \in (2; p_1)$ , all  $t > 0$ , and for all  $f \in L^2$  supported in  $E$ , where  $\gamma_q = \frac{N}{2}(\frac{1}{2} - \frac{1}{q})$ . Hence

$$\left( \frac{1}{|B|} \int_B |\nabla e^{-kr^2 A} g\chi_{C_j(B)}|^q \right)^{\frac{1}{q}} \leq \frac{C}{|B|^{\frac{1}{q}}} (kr^2)^{-\gamma_q - \frac{1}{2}} e^{-c\frac{d^2(B;C_j(B))}{kr^2}} \|g\chi_{C_j(B)}\|_2. \quad (33)$$

Now, applying the Hardy inequality, we find the following upper bound for  $\|g\chi_{C_j(B)}\|_2$ :

$$\begin{aligned} \left( \frac{1}{16r^2} \int_{\mathbb{R}^N} |g(x)\chi_{4B}|^2 dx \right)^{1/2} &\leq \left( \int_{\mathbb{R}^N} \frac{|g(x)\chi_{4B}|^2}{|x|^2} dx \right)^{1/2} \\ &\leq \left( \frac{4}{(N-2)^2} \int_{\mathbb{R}^N} |\nabla g(x)\chi_{4B}|^2 dx \right)^{1/2} \end{aligned}$$

for  $j = 1$ , and

$$\begin{aligned} \left( \frac{1}{4^{j+1}r^2} \int_{\mathbb{R}^N} |g(x)\chi_{C_j(B)}|^2 dx \right)^{1/2} &\leq \left( \int_{\mathbb{R}^N} \frac{|g(x)\chi_{C_j(B)}|^2}{|x|^2} dx \right)^{1/2} \\ &\leq \left( \frac{4}{(N-2)^2} \int_{\mathbb{R}^N} |\nabla g(x)\chi_{C_j(B)}|^2 dx \right)^{1/2} \end{aligned}$$

for  $j \geq 2$ . Employing these estimates of  $\|g\chi_{C_j(B)}\|_2$  in (33) we conclude that, for  $j = 1$ ,

$$\begin{aligned} \left( \frac{1}{|B|} \int_B |\nabla e^{-r^2 k A} g \chi_{4B}|^q \right)^{\frac{1}{q}} &\leq \frac{C}{|B|^{\frac{1}{q}}} k^{-(\gamma_q + \frac{1}{2})} r^{-2\gamma_q} \|\nabla g \chi_{4B}\|_2 \\ &\leq \frac{C}{|4B|^{\frac{1}{2}}} k^{-(\gamma_q + \frac{1}{2})} r^{-2\gamma_q} |B|^{\frac{1}{2} - \frac{1}{q}} \|\nabla g\|_{L^2(4B)} \\ &\leq C k^{-(\gamma_q + \frac{1}{2})} (M(|\nabla g|^2))^{\frac{1}{2}}(x), \end{aligned}$$

and for  $j \geq 2$ ,

$$\begin{aligned} \left( \frac{1}{|B|} \int_B |\nabla e^{-r^2 k A} g \chi_{C_j(B)}|^q \right)^{\frac{1}{q}} &\leq \frac{C 2^{j(N/2+1)}}{|2^{j+1}B|^{\frac{1}{2}}} k^{-(\gamma_q + \frac{1}{2})} e^{-\frac{c_4 j}{k}} \|\nabla g\|_{L^2(C_j(B))} \\ &\leq C k^{-(\gamma_q + \frac{1}{2})} 2^{j(\frac{N}{2}+1)} e^{-\frac{c_4 j}{m}} (M(|\nabla g|^2))^{\frac{1}{2}}(x). \end{aligned}$$

At this stage, we insert these two estimates in the sum on  $j$ , in (31), then in the sum on  $k$  in (30), to obtain the upper bound desired in (30). This gives us estimate (26) of Theorem 42.

Hence by Theorem 42, we obtain that  $\nabla A^{-1/2}$  is bounded on  $L^p$  for all  $p \in (2; p_1)$ , if  $(\sqrt{t}\nabla e^{-tA})_{t>0}$  is bounded on  $L^{p_1}$  for some  $p_1 \in [2; p_0)$ .  $\square$

Next we will prove the following result. It can be seen also as a consequence of [18] Theorem 5. This is useful to prove the boundedness of  $\nabla A^{-1/2}$  on  $L^p$  for all  $p \in (2; p_{0*})$ .

**Proposition 41** If  $V$  is strongly subcritical, then  $\|V^{1/2}(-\Delta)^{-1/2}\|_{p-p} \leq C$  for all  $p \in (1; 2]$ .

**Proof** We set  $A_0 = -\Delta$ , and we assume that  $V$  is strongly subcritical. This implies that

$$\|V^{1/2}u\|_2^2 \leq \frac{1}{1+\varepsilon} \|\nabla u\|_2^2 \leq C \|A_0^{1/2}u\|_2^2,$$

which is

$$\|V^{1/2}A_0^{-1/2}u\|_2 \leq C \|u\|_2. \quad (34)$$

Now we prove that  $V^{1/2}A_0^{-1/2}$  is of weak type  $(1, 1)$  to deduce by the Marcinkiewicz interpolation theorem that it is bounded on  $L^p$  on all  $p \in (1; 2]$ .

First, we search for a  $L^1 - L^2$  off-diagonal estimate of  $(\sqrt{t}V^{1/2}e^{-tA_0})_{t>0}$ . We write for all  $f \in L^1$

$$\begin{aligned}\chi_F V^{\frac{1}{2}} e^{-tA_0} \chi_E f(x) &= \chi_F(x) V^{\frac{1}{2}}(x) \frac{1}{(4\pi t)^{\frac{N}{2}}} \int e^{-\frac{|x-y|^2}{4t}} \chi_E(y) f(y) dy \\ &\leq C e^{-\frac{d^2(E,F)}{8t}} V^{\frac{1}{2}}(x) \frac{1}{(4\pi t)^{\frac{N}{2}}} \int e^{-\frac{|x-y|^2}{8t}} \chi_E(y) |f(y)| dy \\ &\leq C e^{-\frac{d^2(E,F)}{8t}} V^{\frac{1}{2}}(x) e^{-2tA_0} \chi_E |f|(x).\end{aligned}$$

Hence

$$\begin{aligned}\|\chi_F V^{\frac{1}{2}} e^{-tA_0} \chi_E f\|_2 &\leq C e^{-\frac{d^2(E,F)}{8t}} \|V^{\frac{1}{2}} e^{-2tA_0} \chi_E |f|\|_2 \\ &\leq C e^{-\frac{d^2(E,F)}{8t}} \|V^{\frac{1}{2}} A_0^{-\frac{1}{2}} A_0^{\frac{1}{2}} e^{-tA_0} e^{-tA_0} \chi_E |f|\|_2.\end{aligned}$$

Now using (34) and the analyticity of the semigroup we get

$$\|\chi_F V^{\frac{1}{2}} e^{-tA_0} \chi_E f\|_2 \leq \frac{C}{t^{\frac{1}{2}}} e^{-\frac{d^2(E,F)}{8t}} \|e^{-tA_0} \chi_E |f|\|_2.$$

Thus we obtain

$$\|\chi_F \sqrt{t} V^{\frac{1}{2}} e^{-tA_0} \chi_E f\|_2 \leq C t^{-\gamma_1} e^{-\frac{d^2(E,F)}{8t}} \|f\|_1 \quad (35)$$

for all  $t > 0$ , where  $\gamma_1 = \frac{N}{2}(\frac{1}{1} - \frac{1}{2}) = \frac{N}{4}$ . Now we use same arguments as in the previous section. We check estimates (19) and (20) of Theorem 32, by setting  $T = V^{1/2} A_0^{-1/2}$  and  $A_r = I - (I - e^{-r^2 A_0})^m$ .

Then we deduce that  $V^{1/2} A_0^{-1/2}$  is of weak type  $(1, 1)$ .

By interpolation with (34), we obtain

$$\|V^{1/2} A_0^{-1/2}\|_{p-p} \leq C$$

for all  $p \in (1; 2]$ .  $\square$

Next we prove the boundedness of Riesz transforms when  $V \in L^{N/2}$ .

**Theorem 43** Let  $A = -\Delta - V$  where  $0 \leq V \in L^{N/2}$ ,  $N \geq 3$ . Assume that  $A \geq \varepsilon V$ , then  $\nabla A^{-1/2}$  is bounded on  $L^p$  for all  $p \in (p'_0; p_{0*})$  where  $p_0 := \frac{2N}{(N-2)(1-\sqrt{1-\frac{1}{1+\varepsilon}})}$ ,  $p'_0$  its dual exponent, and  $2 < p_{0*} = \frac{p_0 N}{N+p_0}$  its reverse Sobolev exponent.

**Proof** In the previous section, we have proved the boundedness of  $\nabla A^{-1/2}$  on  $L^p$  for all  $p \in (p'_0; 2]$  when  $V$  is strongly subcritical and when the dimension  $N \geq 3$ . Let us now prove the boundedness on  $L^p$  for all  $p \in (2; p_{0*})$ .

Take  $A_0$  and  $A$  two operators such that

$$\mathbf{a}_0(u, v) := \langle A_0 u, v \rangle \quad \text{and} \quad \mathbf{a}(u, v) := \langle A u, v \rangle$$

where  $\langle, \rangle$  is the pairing  $\langle, \rangle_{W^{-1,2}, W^{1,2}}$ .  $D(\Lambda_0) := D(\mathfrak{a}_0) = W^{1,2}$  and  $D(\Lambda) := D(\mathfrak{a}) = W^{1,2}$ . Here  $\mathfrak{a}_0$  and  $\mathfrak{a}$  are the sesquilinear forms associated to  $A_0 := -\Delta$  and  $A := -\Delta - V$ , respectively. The operator  $\Lambda_0$  restricted to  $D(\Lambda_0)$  is  $A_0$ , and  $\Lambda$  restricted to  $D(\Lambda)$  is  $A$ . It is known that  $(I + tA_0)^{-1}$  restricted to  $L^2$  coincides with  $(I + tA_0)^{-1}$ , and  $(I + tA)^{-1}$  restricted to  $L^2$  coincides with  $(I + tA)^{-1}$ . (see [16] Chapter I).

Set  $G(t)$  the following difference:

$$G(t) = \nabla(I + tA_0)^{-1} - \nabla(I + tA)^{-1} \quad \text{for } t > 0.$$

The proof is divided into three steps:

Step 1: We prove that for all  $t > 0$ , all  $f \in L^2 \cap L^p$ , all  $\phi \in C_0^\infty$ , and all  $p \in (2; p_{0*})$

$$| \langle G(t)f, \phi \rangle | \leq Ct^{-1/2} \|f\|_p \|\phi\|_{p'}.$$

Step 2: We prove that for all  $t > 0$  and all  $p \in (2; p_{0*})$

$$\|\nabla(I + tA)^{-1}\|_{p-p} \leq Ct^{-1/2}.$$

Step 3: We prove that for all  $t > 0$  and all  $p \in (2; p_{0*})$

$$\|\sqrt{t}\nabla e^{-tA}\|_{p-p} \leq C.$$

Then, we apply Theorem 41 and conclude the result of this one.

Step 1:

$$\begin{aligned} G(t) &= \nabla(I + tA_0)^{-1} - \nabla(I + tA)^{-1} \\ &= \nabla\left((I + tA_0)^{-1} - (I + tA)^{-1}\right) \\ &= t\nabla(I + tA_0)^{-1}(A - A_0)(I + tA)^{-1} \\ &= -t\nabla(I + tA_0)^{-1}V(I + tA)^{-1}. \end{aligned}$$

Since  $D(\Lambda_0) = D(\Lambda)$ ,  $G(t)$  is well defined. For  $f \in L^2 \cap L^p$ ,  $p > 2$ , and  $\phi \in C_0^\infty$  we have

$$| \langle G(t)f, \phi \rangle | \leq t \|\nabla(I + tA_0)^{-1}V(I + tA)^{-1}f\|_p \|\phi\|_{p'}.$$

Thus

$$\nabla(I + tA_0)^{-1}V^{1/2} = \nabla(I + tA_0)^{-1/2}(I + tA_0)^{-1/2}V^{1/2}.$$

Let us study each of the following operators:  $\nabla(I + tA_0)^{-1/2}$ ,  $(I + tA_0)^{-1/2}V^{1/2}$ , and  $V^{1/2}(I + tA)^{-1}$ .

We begin with the first one: we know that Riesz transforms associated to  $A_0$  are bounded on  $L^p$  for all  $p \in (1; \infty)$ . We use the boundedness of the operator  $(tA_0)^{1/2}(I + tA_0)^{-1/2}$  on all  $L^p$ , proved by the study of the functional calculus theory, to estimate for all  $p \in (1; \infty)$ , and all  $t > 0$

$$\begin{aligned} \|\nabla(I + tA_0)^{-1/2}\|_{p-p} &\leq \|\nabla A_0^{-1/2}\|_{p-p} \cdot \|A_0^{1/2}(I + tA_0)^{-1/2}\|_{p-p} \\ &\leq Ct^{-1/2}. \end{aligned} \tag{36}$$

Let us now estimate the norm of the second operator,  $(I + tA_0)^{-1/2}V^{1/2}$ . We remark that this operator is the dual of  $V^{1/2}(I + tA_0)^{-1/2}$ , so we study the dual. We consider the following decomposition

$$\|V^{1/2}(I + tA_0)^{-1/2}\|_{p-p} \leq \|V^{1/2}A_0^{-1/2}\|_{p-p} \|A_0^{1/2}(I + tA_0)^{-1/2}\|_{p-p}.$$

By the previous proposition, we have  $\|V^{1/2}A_0^{-1/2}\|_{p-p} \leq C$  for all  $p \in (1; 2]$ . Hence, once again, using the boundedness of  $(tA_0)^{1/2}(I + tA_0)^{-1/2}$  on all  $L^p$ , we conclude that for all  $t > 0$  and all  $p \in (1; 2]$

$$\|V^{1/2}(I + tA_0)^{-1/2}\|_{p-p} \leq Ct^{-1/2}.$$

By duality this yields, for all  $t > 0$  and all  $p \in [2; \infty)$ ,

$$\|(I + tA_0)^{-1/2}V^{1/2}\|_{p-p} \leq Ct^{-1/2} \quad (37)$$

Now it remains to study the third operator,  $V^{1/2}(I + tA)^{-1}$ . With the assumption  $V \in L^{N/2}$ , we obtain for all  $p \in (p'_0; p_{0*})$  and all  $t > 0$

$$\|V^{1/2}(I + tA)^{-1}\|_{p-p} \leq Ct^{-1/2}.$$

In fact, by Hölder's inequality, we have

$$\|V^{1/2}(I + tA)^{-1}f\|_p \leq \|V^{1/2}\|_N \|(I + tA)^{-1}f\|_q$$

where  $p'_0 < p < q < p_0$  such that  $\frac{1}{p} - \frac{1}{q} = \frac{1}{N}$ . Interpolation of the uniform boundedness of the semigroup generated by  $A$  with its  $L^p - L^2$  boundedness (Proposition 22) yields

$$\|e^{-tA}\|_{p-q} \leq Ct^{-\gamma_{pq}} \quad (38)$$

for all  $p, q \in (p'_0; p_0)$  such that  $p \leq q$ , where  $\gamma_{pq} = \frac{N}{2}(\frac{1}{p} - \frac{1}{q}) = \frac{1}{2}$ . Hence

$$\|(I + tA)^{-1}f\|_q \leq \int_0^\infty e^{-s} \|e^{-stA}f\|_q ds \leq Ct^{-1/2} \|f\|_p \int_0^\infty e^{-s} s^{-1/2} ds$$

where  $p \in (p'_0; p_{0*})$  (to keep  $q$  in  $(p'_0; p_0)$ ). This yields, for all  $t > 0$ ,

$$\|(I + tA)^{-1}f\|_q \leq Ct^{-\frac{1}{2}} \|f\|_p.$$

Hence

$$\|V^{1/2}(I + tA)^{-1}f\|_p \leq C \|V\|_{\frac{N}{2}} t^{-\frac{1}{2}} \|f\|_p,$$

for all  $p \in (p'_0; p_{0*})$ , which is

$$\|V^{1/2}(I + tA)^{-1}\|_{p-p} \leq Ct^{-1/2}. \quad (39)$$

Going back to  $G(t)$ : employing estimates (36), (37) and (39), we deduce that for all  $t > 0$ , all  $f \in L^2 \cap L^p$ , all  $\phi \in C_0^\infty$  and all  $p \in (2; p_{0*})$ ,

$$|\langle G(t)f, \phi \rangle| \leq Ct^{-1/2} \|f\|_p \|\phi\|_{p'}. \quad (40)$$



Step 2: Now for all  $f \in L^2 \cap L^p$

$$\begin{aligned} G(t)f &:= \nabla(I + tA_0)^{-1}f - \nabla(I + tA)^{-1}f \\ &= \nabla(I + tA_0)^{-1}f - \nabla(I + tA)^{-1}f. \end{aligned}$$

Then for all  $\phi \in C_0^\infty$  we get

$$|\langle \nabla(I + tA)^{-1}f, \phi \rangle| = |\langle G(t)f - \nabla(I + tA_0)^{-1}f, \phi \rangle|. \quad (41)$$

We use the boundedness of Riesz transforms associated to  $A_0$ , and the analyticity of the semigroup associated to  $A_0$ , on  $L^p$  for all  $p \in (1; \infty)$ , to come to

$$\begin{aligned} \|\nabla(I + tA_0)^{-1}\|_{p-p} &\leq \int_0^\infty e^{-s} \|\nabla A_0^{-1/2} A_0^{1/2} e^{-stA_0}\|_{p-p} ds \\ &\leq Ct^{-1/2} \int_0^\infty e^{-s} s^{-1/2} ds \\ &\leq Ct^{-1/2} \end{aligned}$$

for all  $t > 0$  and all  $p \in (1; \infty)$ . Then, from (41) and (40) we obtain, for all  $t > 0$ , all  $f \in L^2 \cap L^p$ , all  $\phi \in C_0^\infty$  and all  $p \in (2; p_{0*})$ ,

$$|\langle \nabla(I + tA)^{-1}f, \phi \rangle| \leq Ct^{-1/2} \|f\|_p \|\phi\|_{p'}.$$

Thus we obtain

$$\|\nabla(I + tA)^{-1}\|_{p-p} \leq Ct^{-1/2} \quad (42)$$

for all  $t > 0$  and all  $p \in (2; p_{0*})$ .

Step 3: We write

$$\|\nabla e^{-tA}\|_{p-p} \leq \|\nabla(I + tA)^{-1}\|_{p-p} \|(I + tA)e^{-tA}\|_{p-p}.$$

We use (42) and the analyticity of the semigroup associated to  $A = -\Delta - V$  on  $L^p$  for all  $p \in (p'_0; p_0)$  (see [5] Theorem 1.6 or [15]), to obtain that for all  $t > 0$  and all  $p \in (2; p_{0*})$

$$\|\sqrt{t}\nabla e^{-tA}\|_{p-p} \leq C.$$

Now we apply Theorem 41 and we conclude that the Riesz transforms associated to  $A$  are bounded on  $L^p$  for all  $p \in (2; p_{0*})$ .  $\square$

Proof (of Proposition 34) Let  $V$  strongly subcritical and  $N \geq 3$ . Assume that  $V^{1/2}A^{-1/2}$  is bounded on  $L^p$  for some  $p \in [2; p_0)$ . Then, in the previous proof, the third operator of the decomposition of  $G(t)$ ,  $V^{1/2}(I + tA)^{-1}$ , can be studied as follows:

$$\|V^{1/2}(I + tA)^{-1}\|_{p-p} \leq \|V^{1/2}A^{-1/2}\|_{p-p} \|A^{1/2}(I + tA)^{-1}\|_{p-p} \leq Ct^{-1/2}.$$

Here we use the fact that  $A$  has a bounded  $H^\infty$ -functional calculus on  $L^p$  for all  $p \in (p'_0; p_0)$  (see [6] Theorem 2.2) to get

$$\|A^{1/2}(I + tA)^{-1}\|_{p-p} \leq Ct^{-1/2}$$

for all  $t > 0$ . Employing this estimate in the previous proof, we obtain the result.  $\square$

**Proposition 42** Let  $A = -\Delta - V$  where  $0 \leq V \in L^{N/2}$ ,  $N \geq 3$ ,  $V$  is strongly subcritical. Then  $V^{1/2}A^{-1/2}$  is bounded on  $L^p$  for all  $p \in (p'_0; p_{0*})$  where  $p'_0$  is the dual exponent of  $p_0$ , and  $2 < p_{0*} = \frac{p_0 N}{N+p_0}$  is the reverse Sobolev exponent of  $p_0$ .

**Proof** We have proved in Proposition 31 that  $V^{1/2}A^{-1/2}$  is bounded on  $L^p$  for all  $p \in (p'_0; 2]$ , it remains to prove boundedness on  $L^p$  for  $p \in (2; p_{0*})$ . Let us first prove that, if  $V$  is in  $L^{N/2}$ ,  $(\sqrt{t}V^{1/2}e^{-tA})_{t>0}$  is bounded on  $L^p$  for all  $p \in (2; p_{0*})$ . By the Hölder inequality we can estimate for all  $p \in (2; p_{0*})$  and all  $t > 0$

$$\begin{aligned} \|\sqrt{t}V^{1/2}e^{-tA}f\|_p &\leq \sqrt{t}\|V^{1/2}\|_N \|e^{-tA}f\|_q \\ &\leq \sqrt{t}\|V\|_{\frac{N}{2}} Ct^{-\gamma_{pq}} \|f\|_p. \end{aligned}$$

This is for all  $p'_0 < p < q < p_0$  such that  $\frac{1}{p} - \frac{1}{q} = \frac{1}{N}$ . So  $\gamma_{pq} = \frac{N}{2}(\frac{1}{p} - \frac{1}{q}) = \frac{1}{2}$ . This yields that for all  $p \in (2; p_{0*})$

$$\|\sqrt{t}V^{1/2}e^{-tA}\|_{p-p} \leq C. \quad (43)$$

Now we take the following decomposition

$$\sqrt{t}V^{1/2}e^{-tA} = \sqrt{t}V^{1/2}e^{-tA/2}e^{-tA/2}$$

and we use estimates (32) and (43). Thus, we obtain, for all  $t > 0$  and all  $p \in (2; p_{0*})$ ,

$$\|\sqrt{t}V^{1/2}e^{-tA}\|_{2-p} \leq Ct^{-\gamma_p}.$$

At present, we interpolate this estimate with the  $L^2 - L^2$  off-diagonal estimate of the norm of  $\sqrt{t}V^{1/2}e^{-tA}$  proved in Proposition 21, as we did in the proof of Theorem 21. Then we get the following  $L^2 - L^p$  off-diagonal estimate

$$\|\sqrt{t}V^{1/2}e^{-tA}f\|_{L^p(F)} \leq Ct^{-\gamma_p} e^{-\frac{cd^2(E,F)}{t}} \|f\|_2, \quad (44)$$

for all  $t > 0$  and all  $p \in (2; p_{0*})$ .

At this step, we check assumptions (25) and (26) of Theorem 42 where  $T = V^{1/2}A^{-1/2}$  as we did in the proof of Theorem 41. We use the  $L^2 - L^2$  off-diagonal estimate proved in Proposition 21 (iii), and the estimate (44). Thus we deduce the boundedness of  $V^{1/2}A^{-1/2}$  on  $L^p$  for all  $p \in (2; p_{0*})$ .  $\square$

The following is an example of potentials satisfying conditions of Theorem 43 and Proposition 42. With these potentials,  $\nabla(-\Delta - V)^{-1/2}$  and  $V^{1/2}(-\Delta - V)^{-1/2}$  are bounded on  $L^p$  for all  $p \in (p'_0; p_{0*})$ .

Example: Set  $N \geq 3$ , and let us take  $V \in L^{N/2}$  such that  $V \sim c|x|^{-\alpha}$  when  $|x|$  tends to infinity, where  $c > 0$  and  $\alpha > 2$ . Assume that  $\|V\|_{\frac{N}{2}}$  is small enough and let us prove that these potentials are strongly subcritical, so we should prove that

$$\|V^{1/2}u\|_2^2 \leq C\|\nabla u\|_2^2$$

where  $C < 1$ . By the Hölder inequality, we have

$$\|V^{1/2}u\|_2^2 \leq \|V\|_{p'}\|u^2\|_p$$

where we set  $p' = \frac{N}{2}$  so  $p$  will be  $\frac{N}{N-2}$ . Now using the Sobolev inequality we obtain

$$\|V^{1/2}u\|_2^2 \leq \|V\|_{p'}\|u^2\|_p = \|V\|_{\frac{N}{2}}\|u\|_{2^*}^2 \leq C\|\nabla u\|_2^2,$$

where  $2^* := \frac{2N}{N-2}$  is the Sobolev exponent of 2, and  $C < 1$  for  $\|V\|_{\frac{N}{2}}$  is small enough.

## 5 Boundedness of $\nabla A^{-1/2}$ and $V^{1/2}A^{-1/2}$ on $L^p$ for all $p \in (1; N)$

In this section we assume that  $V$  is strongly subcritical in the Kato subclass  $K_N^\infty$ ,  $N \geq 3$ . Following Zhao [21], we define

$$K_N^\infty := \left\{ V \in K_N^{loc}; \lim_{B \uparrow \infty} \left[ \sup_{x \in \mathbb{R}^N} \int_{|y| \geq B} \frac{|V(y)|}{|y-x|^{N-2}} dy \right] = 0 \right\},$$

where  $K_N^{loc}$  is the class of potentials that are locally in the Kato class  $K_N$ . For necessary background of the Kato class see [19] and references therein.

We use results proved by stochastic methods to deduce a  $L^1 - L^\infty$  off-diagonal estimate of the norm of the semigroup which leads to the boundedness of  $\nabla A^{-1/2}$  and  $V^{1/2}A^{-1/2}$  on  $L^p$  for all  $p \in (1; N)$ .

**Theorem 51** Let  $A$  be the Schrödinger operator  $-\Delta - V$ ,  $V \geq 0$ . Assume that  $V$  is strongly subcritical in the class  $K_N^\infty$ , ( $N \geq 3$ ), then  $\nabla A^{-1/2}$  and  $V^{1/2}A^{-1/2}$  are bounded on  $L^p$  for all  $p \in (1; 2]$ . If in addition  $V \in L^{N/2}$ , then  $\nabla A^{-1/2}$  and  $V^{1/2}A^{-1/2}$  are bounded on  $L^p$  for all  $p \in (1; N)$ .

**Proof** We assume that  $V$  is strongly subcritical in the class  $K_N^\infty$ . Therefore  $V$  satisfies assumptions of Theorem 2 of [20] (The classes  $K_\infty$  and  $S_\infty$  mentioned in [20] are equivalent to the class  $K_N^\infty$  (see Chen [7] Theorem 2.1 and Section 3.1)). Thus the heat kernel associated to  $(e^{-tA})_{t>0}$  satisfies a Gaussian estimate. Now we apply Theorem 5 of [18] and we conclude that  $\nabla A^{-1/2}$

and  $V^{1/2}A^{-1/2}$  are of weak type  $(1, 1)$  and they are bounded on  $L^p$  for all  $p \in (1; 2]$ .

We have seen that when the potential is strongly subcritical in the class  $K_N^\infty$ , we obtain a Gaussian estimate of the heat kernel associated to the semigroup. This implies that the semigroup is bounded from  $L^1$  to  $L^\infty$  by  $Ct^{-N/2}$ . Thus, by interpolation,

$$\|e^{-tA}\|_{p-q} \leq Ct^{-\gamma_{pq}} \quad (45)$$

for all  $t > 0$ , all  $p, q \in [1; \infty]$ ,  $p \leq q$ , where  $\gamma_{pq} = \frac{N}{2}(\frac{1}{p} - \frac{1}{q})$ . Moreover the semigroup is bounded analytic on  $L^p$  for all  $p \in (1; \infty)$  (see [5] Theorem 1.6). Now (32) holds for all  $p \in (2; \infty)$ . Using the same proof as for Theorem 41 we deduce that

- if  $(\sqrt{t}\nabla e^{-tA})_{t>0}$  is bounded on  $L^{p_1}$  for some  $p_1 \in [2; \infty)$ , then  $\nabla A^{-1/2}$  is bounded on  $L^p$  for all  $p \in (2; p_1)$ .

We use (45) in the proof of Theorem 43: in the proof of the boundedness of the operator  $V^{1/2}(I + tA)^{-1}$  in *Step 1*:

$$\|V^{1/2}(I + tA)^{-1}f\|_p \leq \|V^{1/2}\|_{N \cdot} \|(I + tA)^{-1}f\|_q$$

where  $1 \leq p < q \leq \infty$  such that  $\frac{1}{p} - \frac{1}{q} = \frac{1}{N}$ . Thus, if  $p < N$  then  $q = p^*$ , if  $p = N$  then  $q = \infty$ ,  $p$  cannot be more than  $N$ .

$$\|(I + tA)^{-1}f\|_q \leq \int_0^\infty e^{-s} \|e^{-stA}f\|_q ds \leq Ct^{-1/2} \|f\|_p \int_0^\infty e^{-s} s^{-1/2} ds$$

where  $p \in [1; N]$  and  $q \in [1; \infty]$ . This yields for all  $t > 0$  and all  $p \in [1; N]$

$$\|V^{1/2}(I + tA)^{-1}\|_{p-p} \leq Ct^{-1/2}.$$

In the same proof, we use the fact that

$$\|tAe^{-tA}\|_{p-p} \leq C$$

for all  $p \in (1; \infty)$  in *Step 3*, because of the analyticity of the semigroup on  $L^p$  for all  $p \in (1; \infty)$ . Then we deduce

- if  $V \in L^{N/2} \cap K_N^\infty$  ( $N \geq 3$ ),  $V$  strongly subcritical, then  $\nabla A^{-1/2}$  is bounded on  $L^p$  for all  $p \in (1; N)$ .

We use the  $L^p - L^q$  bound of the semigroup for  $1 \leq p \leq q \leq \infty$  in the proof of Proposition 42 to deduce that

- If  $V \in L^{N/2} \cap K_N^\infty$  ( $N \geq 3$ ),  $V$  strongly subcritical, then  $V^{1/2}A^{-1/2}$  is bounded on  $L^p$  for all  $p \in (1; N)$ .  $\square$

Example: Set  $N \geq 3$ , and let us take potentials  $V$  in the Kato subclass  $K_N \cap L^{N/2}$  such that  $V \sim c|x|^{-\alpha}$  when  $x$  tends to infinity, where  $\alpha > 2$ . Suppose that  $\|V\|_{\frac{N}{2}}$  is small enough. As we have seen in the previous example, these potentials are strongly subcritical. Z.Zhao [21] proved that they are in the subclass  $K_N^\infty$ . Hence they satisfy the assumptions of Theorem 51. Then  $\nabla(-\Delta - V)^{-1/2}$  and  $V^{1/2}(-\Delta - V)^{-1/2}$  are bounded on  $L^p$  for all  $p \in (1; N)$ .

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