On a new characterisation of Besov spaces with negative exponents

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Dedicated to Vladimir Maz'ya with high esteem

1 Introduction

Let B denote the unit N-ball and $\Sigma = \partial B$. If μ is a distribution on Σ we denote by $\mathbb{P}(\mu)$ its Poisson potential in B, that is

$$\mathbb{P}(\mu)(x) =< \mu, P(x,.) >_{\Sigma}, \quad \forall x \in B, \tag{1.1}$$

where $< , >_{\Sigma}$ denotes the pairing between distributions on Σ and functions in $C^{\infty}(\Sigma)$. In the particular case where μ is a measure, this can be written as follows

$$\mathbb{P}(\mu)(x) = \int_{\Sigma} P(x,y)d\mu(y), \quad \forall x \in B. \tag{1.2}$$

In [4] it is proved that for $q > 1$ the Besov space $W^{-2/q,q}(\Sigma)$ is characterized by an integrability condition on $\mathbb{P}(\mu)$ with respect to a wheight function involving the distance to the boundary, and more precisely that there exists a positive constant $C = C(N,q)$ such that for any distribution μ on Σ there holds

$$C^{-1}\|\mu\|_{W^{-2/q,q}(\Sigma)} \leq \left(\int_B |\mathbb{P}(\mu)|^q(1-|x|)dx\right)^{1/q} \leq C\|\mu\|_{W^{-2/q,q}(\Sigma)}. \tag{1.3}$$

The aim of this article is to prove that for all $1 < q < \infty$ any negative Besov spaces $B^{-s,q}(\Sigma)$ can be described by an integrability condition on the Poisson potential of its elements. More precisely, we prove

Theorem 1.1 *Let $s > 0$, $q > 1$ and μ be a distribution on Σ. Then*

$$\mu \in B^{-s,q}(\Sigma) \Longleftrightarrow \mathbb{P}(\mu) \in L^q(B;(1-|x|)^{sq-1}dx).$$

Moreover there exists a constant $C > 0$ such that for any $\mu \in B^{-s,q}(\Sigma)$,

$$C^{-1}\|\mu\|_{B^{-s,q}(\Sigma)} \leq \left(\int_B |\mathbb{P}(\mu)|^q(1-|x|)^{sq-1}dx\right)^{1/q} \leq C\|\mu\|_{B^{-s,q}(\Sigma)}. \tag{1.4}$$

The key idea for proving such a result is to use a lifting operator which reduces the estimate question to an estimate between Besov spaces with positive exponents. In one direction the main technique relies on interpolation theory between domain of powers of analytic semigroups. In the other direction we use a new representation formula for harmonic functions in a ball.

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2 The left-hand side inequality (1.4)

We recall that for $1 \leq p < \infty$, $r \notin \mathbb{N}$, $r = k + \eta$ with $k \in \mathbb{N}$ and $0 < \eta < 1$,

$$B^{r,p}(\mathbb{R}^d) = \left\{ \varphi \in W^{k,p}(\mathbb{R}^d) : \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|D^\alpha\varphi(x) - D^\alpha\varphi(y)|^p}{|x-y|^{d+\eta p}} dxdy < \infty, \forall \alpha \in \mathbb{N}^d, |\alpha| = k, \right\}$$

with norm

$$\|\varphi\|^p_{B^{r,p}} = \|\varphi\|^p_{W^{k,p}} + \sum_{|\alpha|=k} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|D^\alpha\varphi(x+y) - D^\alpha\varphi(x)|^p}{|y|^{d+\eta p}} dxdy.$$

When $r \in \mathbb{N}$,

$$B^{r,p}(\mathbb{R}^d) = \Bigg\{ \varphi \in W^{r-1,p}(\mathbb{R}^d) : \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|D^\alpha\varphi(x+2y) + D^\alpha\varphi(x) - 2D^\alpha\varphi(x+y)|^p}{|y|^{p+d}} dxdy < \infty, \forall \alpha \in \mathbb{N}^d, |\alpha| = r-1, \Bigg\},$$

with norm

$$\|\varphi\|^p_{B^{r,p}} = \|\varphi\|^q_{W^{k,p}} + \sum_{|\alpha|=r-1} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|D^\alpha\varphi(x+2y) + D^\alpha\varphi(x) - 2D^\alpha\varphi(x+y)|^p}{|y|^{p+d}} dxdy.$$

The relation of the Besov spaces with integer order of differentiation and the classical Sobolev spaces is the following [3], [2]

$$\begin{array}{ll} B^{r,p}(\mathbb{R}^d) \subset W^{r,p}(\mathbb{R}^d) & \text{if } 1 \leq p \leq 2, \\ W^{r,2}(\mathbb{R}^d) = B^{r,2}(\mathbb{R}^d), & \\ W^{r,p}(\mathbb{R}^d) \subset B^{r,p}(\mathbb{R}^d) & \text{if } p \geq 2. \end{array} \tag{2.1}$$

Since for $r\mathbb{N}_*$ and $1 \leq p < \infty$, the space $B^{-r,p}(\mathbb{R}^d)$ is the space of derivatives of $L^p(\mathbb{R}^d)$-functions, up to the total order k, for noninteger r, $r = k + \eta$ with $k \in \mathbb{N}$ and $0 < \eta < 1$ $B^{-r,p}(\mathbb{R}^d)$ can be defined by using the real interpolation method [3] by

$$\left[W^{-k,p}(\mathbb{R}^d), W^{-k-1,p}(\mathbb{R}^d)\right]_{\eta,p} = B^{-r,p}(\mathbb{R}^d).$$

The spaces $B^{-r,p}(\mathbb{R}^d)$, or $1 < p < \infty$ and $r > 0$ can also be defined by duality with $B^{-r,p'}(\mathbb{R}^d)$. The Sobolev and Besov spaces $W^{k,p}(\Sigma)$ and $B^{r,p}(\Sigma)$ are defined by using local charts from the same spaces in $\mathbb{R}^{N-1}$.

Now we present the proof of the left-hand side inequality in the case $N \geq 3$. However, with minor modifications, the proof applies also to the case $N = 2$ (see the remark). Let $(r, \sigma) \in [0, \infty) \times S^{N-1}$ (with $S^{N-1} \approx \Sigma$) be spherical coordinates in B and put $t = -\ln r$. Suppose that $\mu \in B^{-s,q}(S^{N-1})$, let $u = \mathbb{P}(\mu)$ and denote by $\tilde{u}$ the function u expressed in terms of the coordinates (t, σ). Then

$$u_{rr} + \frac{N-1}{r} u_r + \frac{1}{r^2} \Delta_\sigma u = 0, \quad \text{in } (0,1) \times S^{N-1}, \tag{2.2}$$

and

$$\tilde{u}_{tt} - (N-2)\tilde{u}_t + \Delta_\sigma \tilde{u} = 0, \quad \text{in } (0,\infty) \times S^{N-1}. \tag{2.3}$$

Then the right inequality in (1.4) obtains the form

$$\int_0^\infty \int_{S^{N-1}} |\tilde{u}|^q \, (1 - e^{-t})^{sq-1} e^{-Nt} d\sigma \, dt \leq C \, \|\mu\|^q_{B^{-s,q}(S^{N-1})}. \tag{2.4}$$

Clearly it is sufficient to establish this inequality in the case that $\mu \in \mathfrak{M}(S^{N-1})$ (or even $\mu \in C^\infty(S^{N-1})$), which is assumed in the sequel. We define $k \in \mathbb{N}^*$ by

$$2(k-1) \leq s < 2k, \tag{2.5}$$

with the restriction $s > 0$ if $k = 1$. We denote by $\mathbb{B}$ the elliptic operator of order 2k

$$\mathbb{B} = \left(\frac{(N-2)^2}{4} - \Delta_\sigma \right)^k$$

and call f the unique solution of

$$\mu = \mathbb{B} f \quad \text{in } S^{N-1}.$$

Then $f \in W^{2k-s,q}(S^{N-1})$ since $\mathbb{B}$ is an isomorphism between the spaces $B^{2k-s,q}(S^{N-1})$ and $B^{-s,q}(S^{N-1})$. Put $v = \mathbb{P}(f)$ in B, then v satisfies the same equation as u in $(0,1) \times S^{N-1}$. Let $\tilde{v}$ denote this function in terms of the coordinates (t, σ). Then

$$\begin{cases} \tilde{L}\tilde{v} := \tilde{v}_{tt} - (N-2)\tilde{v}_t + \Delta_\sigma \tilde{v} = 0 & \text{in } \mathbb{R}_+ \times S^{N-1}, \\ \tilde{v}|_{t=0} = f, & \text{in } S^{N-1}. \end{cases} \tag{2.6}$$

Since the operator $\mathbb{B}$ commutes with Δ_σ and $\partial/\partial t$, and this problem has a unique solution which is bounded near $t = \infty$, it follows that

$$\mathbb{P}(\mathbb{B} f) = \mathbb{B}\tilde{v}. \tag{2.7}$$

Hence,

$$\tilde{u} = \mathbb{P}(\mu) = \mathbb{P}(\mathbb{B} f) = \mathbb{B}\tilde{v}. \tag{2.8}$$

If $v^* := e^{-t(N-2)/2}\tilde{v}$, then

$$\begin{cases} v^*_{tt} - \frac{(N-2)^2}{4} v^* + \Delta_\sigma v^* = 0, & \text{in } \mathbb{R}_+ \times S^{N-1}, \\ v^*(0,\cdot) = f, & \text{in } S^{N-1}. \end{cases} \tag{2.9}$$

Note that

$$v^* = e^{tA}(f) \quad \text{where} \quad A = -\left(\frac{(N-2)^2}{4} I - \Delta_\sigma\right)^{1/2} \Longleftrightarrow A^{2k} = \mathbb{B},$$

where e^{tA} is the semigroup generated by A in $L^q(S^{N-1})$. By the Lions-Peetre real interpolation method [3],

$$\left[W^{2k,q}(S^{N-1}), L^q(S^{N-1})\right]_{1-s/2k,q} = B^{2k-s,q}(S^{N-1}).$$

Since $D(A^2) = W^{2,q}(S^{N-1})$, $D(A^{2k}) = W^{2k,q}(S^{N-1})$. The semi-group generated by A is analytic as any semi-group generated by the square root of a closed operator, therefore by [8] p 96,

$$\begin{aligned} \|f\|^q_{W^{2k-s,q}} &\sim \|f\|^q_{L^q(S^{N-1})} + \int_0^\infty \left(t^{(2kqs/2kq)} \left\|A^{2k} v^*\right\|_{L^q(S^{N-1})}\right)^q \frac{dt}{t} \\ &\sim \|f\|^q_{L^q(S^{N-1})} + \int_0^1 \left(t^s \left\|A^{2k} v^*\right\|_{L^q(S^{N-1})}\right)^q \frac{dt}{t} \\ &= \|f\|^q_{L^q(S^{N-1})} + \int_0^1 \left(t^s e^{-t(N-2)/2} \left\|\mathbb{B}\tilde{v}\right\|_{L^q(S^{N-1})}\right)^q \frac{dt}{t} \end{aligned} \tag{2.10}$$

where the symbol $\sim$ denotes equivalence of norms. Therefore, by (2.10),

$$\begin{aligned} \|f\|^q_{W^{2k-s,q}(S^{n-1})} &\geq C\|f\|^q_{L^q(S^{N-1})} + C\int_0^1 \left(t^s e^{-t(N-2)/2} \|\tilde{u}\|_{L^q(S^{N-1})}\right)^q \frac{dt}{t} \\ &\geq C\|f\|^q_{L^q(S^{N-1})} + C\int_0^1 \|\tilde{u}\|^q_{L^q(S^{N-1})} e^{-Nt} t^{sq-1} dt. \end{aligned} \tag{2.11}$$

Furthermore,

$$\begin{aligned} \int_0^\infty \|\tilde{u}\|^q_{L^q(S^{N-1})} (1-e^t)^{sq-1} e^{-Nt} dt &\leq C\int_0^1 \|\tilde{u}\|^q_{L^q(S^{N-1})} (1-e^{-t})^{sq-1} e^{-Nt} dt \\ &\leq C\int_0^1 \|\tilde{u}\|^q_{L^q(S^{N-1})} e^{-Nt} t^{sq-1} dt. \end{aligned} \tag{2.12}$$

This is a consequence of the inequality

$$\int_{\partial B_r} |u|^q dS \leq (r/\rho)^{N-1} \int_{\partial B_\rho} |u|^q dS,$$

which holds for $0 < r < \rho$, for every harmonic function u in B. By a straightforward computation, this inequality implies that

$$\int_{|x|<1} |u|^q (1-r)\, dx \leq c(\gamma) \int_{\gamma<|x|<1} |u|^q (1-r)\, dx,$$

for every $\gamma \in (0,1)$.

In view of the definition of f,

$$\|\mu\|^q_{B^{-s,q}(S^{n-1})} \sim \|f\|^q_{W^{2k-s,q}(S^{n-1})}\,. \tag{2.13}$$

Therefore, the right hand side inequality in (2.4) follows from (2.11), (2.12) and (2.13).

3 The right-hand side inequality (1.4)

Suppose that μ is a distribution on S^{N-1} such that $\mathbb{P}(\mu) \in L^q(B;(1-|x|^{sq-1})$. Then we claim that $\mu \in B^{-s,q}(S^{N-1})$ and

$$C^{-1}\|\mu\|_{B^{-s,q}(\Sigma)} \le \left(\int_B |\mathbb{P}(\mu)|^q (1-|x|)^{sq-1} dx\right)^{1/q}. \tag{3.1}$$

Because of estimate (2.10) it is sufficient to prove that

$$\|f\|_{L^q(S^{N-1})} \le C\,\|u\|_{L^q(B,(1-r)^{sq-1}\,dx)}\,. \tag{3.2}$$

With $u = \mathbb{B}v$ this relation becomes

$$\begin{aligned}\|f\|_{L^q(S^{N-1})} &\le C\,\|\mathbb{B}v\|_{L^q(B;(1-r)^{sq-1}\,dx)}\\ &\le C\left(\int_0^1 \|v\|^q_{W^{2k,q}(S^{N-1})}(1-r)^{sq-1}\,r^{N-1}dr\right)^{1/q}.\end{aligned} \tag{3.3}$$

In order to simplify the exposition, we shall first present the case where $0 < s < 2$.

3.1 The case $0 < s < 2$

We take $k = 1$. Since the imbedding of $B^{2-s,q}(S^{N-1})$ into $L^q(S^{N-1})$ is compact, for any $\varepsilon > 0$ there is $C_\varepsilon > 0$ such that

$$\|\varphi\|_{L^q(S^{N-1})} \le \varepsilon\|\varphi\|_{B^{2-s,q}(S^{N-1})} + C_\varepsilon\|\varphi\|_{L^1(S^{N-1})}, \quad \forall\,\varphi \in B^{2-s,q}(S^{N-1}).$$

Therefore the following norm for $B^{2-s,q}(S^{N-1})$ is equivalent to the one given in (2.4)

$$\|f\|^q_{B^{2-s,q}} = \|f\|^q_{L^1(S^{N-1})} + \int_0^1 \left(t^s\left\|A^2v^*\right\|_{L^q(S^{N-1})}\right)^q \frac{dt}{t}, \tag{3.4}$$

and estimate (3.3) will be a consequence of

$$\|f\|^q_{L^1(S^{N-1})} \le \int_0^1 \left(t^s\left\|A^2v^*\right\|_{L^q(S^{N-1})}\right)^q \frac{dt}{t}. \tag{3.5}$$

Integrating (2.9) and using the fact that

$$\lim_{t\to\infty}\|v^*\|_{L^\infty(S^{N-1})} = \lim_{t\to\infty}\|v^*_t\|_{L^\infty(S^{N-1})} = 0, \tag{3.6}$$

yields to

$$v_t^*(t,\sigma) = -\int_t^\infty A^2 v^*(s,\sigma)ds, \quad \forall (t,\sigma) \in (0,\infty)\times S^{N-1},$$

and

$$\begin{aligned} v^*(t,\sigma) &= \int_t^\infty \int_s^\infty A^2 v^*(\tau,\sigma)d\tau ds, \\ &= \int_t^\infty A^2 v^*(\tau,\sigma)(\tau - t)d\tau, \quad \forall (t,\sigma) \in (0,\infty)\times S^{N-1}. \end{aligned} \tag{3.7}$$

Letting $t \to 0$ and integrating over S^{N-1}, one obtains

$$\begin{aligned} \int_{S^{N-1}} |f|\, d\sigma &\le \int_0^\infty \int_{S^{N-1}} \left|A^2 v^*\right| \tau d\sigma d\tau \\ &\le C(N,s,q,\delta) \left(\int_0^\infty \int_{S^{N-1}} \left|A^2 v^*\right|^q e^{\delta\tau} \tau^{sq-1} d\sigma d\tau \right)^{1/q} \end{aligned} \tag{3.8}$$

for any $\delta > 0$ (δ will be taken smaller that $(N-2)q/2$) is the sequel), where

$$C(N,s,q,\delta) = \left(\left|S^{N-1}\right| \int_0^\infty \tau^{(q+1-sq)/(q-1)} e^{-\delta\tau/(q-1)} d\tau \right)^{1/q'}.$$

Notice that the integral is convergent since $(q+1-sq)/(q-1) > -1 \iff s < 2$. Going back to $\tilde{v}$

$$\int_0^\infty \int_{S^{N-1}} \left|A^2 v^*\right|^q e^{\delta\tau} \tau^{sq-1} d\sigma d\tau = \int_0^\infty \int_{S^{N-1}} \left|A^2 \tilde{v}\right|^q e^{(\delta-(N-2)q/2)\tau} \tau^{sq-1} d\sigma d\tau.$$

Since u is harmonic

$$\int_{S^{N-1}} |\tilde{u}(\tau_1,.)|^q d\sigma \le \int_{S^{N-1}} |\tilde{u}(\tau_2,.)|^q d\sigma, \quad \forall 0 < \tau_2 \le \tau_1,$$

or equivalently,

$$\int_{S^{N-1}} \left|A^2 \tilde{v}(\tau_1,.)\right|^q d\sigma \le \int_{S^{N-1}} \left|A^2 \tilde{v}(\tau_2,.)\right|^q d\sigma, \quad \forall 0 < \tau_2 \le \tau_1. \tag{3.9}$$

Applying (3.9) between τ and $1/\tau$ for $\tau \ge 1$ yields to

$$\int_1^\infty \int_{S^{N-1}} \left|A^2 \tilde{v}\right|^q e^{(\delta-(N-2)q/2)\tau} \tau^{sq-1} d\sigma d\tau \le \int_0^1 \int_{S^{N-1}} \left|A^2 \tilde{v}\right|^q e^{(\delta-(N-2)q/2)\tau^{-1}} \tau^{-sq-1} d\sigma d\tau \tag{3.10}$$

Moreover there exists $C = C(N,q,\delta) > 0$ such that

$$e^{(\delta-(N-2)q/2)t^{-1}} t^{-sq-1} \le C e^{(\delta-(N-2)q/2)t} t^{sq-1}, \quad \forall 0 < t \le 1.$$

Plugging this inequality into (3.9) and using (3.8), one derives

$$\int_{S^{N-1}} |f|\, d\sigma \le C \left(\int_0^1 \int_{S^{N-1}} \left|A^2 v^*\right|^q e^{\delta\tau} \tau^{sq-1} d\sigma d\tau \right)^{1/q} \tag{3.11}$$

for some positive constant C, from which (3.5) follows.

3.2 the general case

We assume that $k \geq 1$. Since the imbedding of $B^{2k-s,q}(S^{N-1})$ into $L^q(S^{N-1})$ is compact, for any $\varepsilon > 0$ there is $C_\varepsilon > 0$ such that

$$\|\varphi\|_{L^q(S^{N-1})} \leq \varepsilon \|\varphi\|_{B^{2k-s,q}(S^{N-1})} + C_\varepsilon \|\varphi\|_{L^1(S^{N-1})}, \quad \forall\, \varphi \in B^{2k-s,q}(S^{N-1}).$$

Thus the following norm for $B^{2k-s,q}(S^{N-1})$ is equivalent to the one given in (2.4)

$$\|f\|^q_{B^{2k-s,q}} = \|f\|^q_{L^1(S^{N-1})} + \int_0^1 \left(t^s \left\| A^{2k} v^* \right\|_{L^q(S^{N-1})} \right)^q \frac{dt}{t}, \tag{3.12}$$

and estimate (3.3) will follow from

$$\|f\|^q_{L^1(S^{N-1})} \leq \int_0^1 \left(t^s \left\| A^{2k} v^* \right\|_{L^q(S^{N-1})} \right)^q \frac{dt}{t}. \tag{3.13}$$

From (3.7),

$$v^*(t,\sigma) = \int_t^\infty A^2 v^*(\tau,\sigma)(\tau - t) d\tau, \quad \forall (t,\sigma) \in (0,\infty) \times S^{N-1}. \tag{3.14}$$

Since the operator A^2 is closed,

$$A^2 v^*(t,\sigma) = \int_t^\infty A^4 v^*(\tau,\sigma)(\tau - t) d\tau,$$

and

$$\begin{aligned} v^*(t,\sigma) &= \int_t^\infty (t_1 - t) \int_{t_1}^\infty A^4 v^*(t_2,\sigma)(t_2 - t_1) dt_2 dt_1, \\ &= \int_t^\infty \int_{t_1}^\infty (t_1 - t)(t_2 - t_1) A^4 v^*(t_2,\sigma) dt_2 dt_1, \quad \forall (t,\sigma) \in (0,\infty) \times S^{N-1}. \end{aligned} \tag{3.15}$$

Iterating this process one gets, for every $(t,\sigma) \in (0,\infty) \times S^{N-1}$,

$$v^*(t,\sigma) = \int_t^\infty \int_{t_1}^\infty \dots \int_{t_{k-1}}^\infty \prod_{j=1}^k (t_j - t_{j-1}) A^{2k} v^*(t_k,\sigma) dt_k dt_{k-1} \dots dt_1. \tag{3.16}$$

where we have set $t = t_0$ in the product symbol. The following representation formula is valid for any $k \in \mathbb{N}_*$.

Lemma 3.1 *For any $(t,\sigma) \in (0,\infty) \times S^{N-1}$,*

$$v^*(t,\sigma) = \int_t^\infty \frac{(s-t)^{2k-1}}{(2k-1)!} A^{2k} v^*(s,\sigma) ds. \tag{3.17}$$

Proof. We proceed by induction. By Fubini's theorem

$$\int_t^\infty \int_{t_1}^\infty (t_1-t)(t_2-t_1)A^4v^*(t_2,\sigma)dt_2dt_1 = \int_t^\infty A^4v^*(t_2,\sigma)\int_t^{t_2}(t_1-t)(t_2-t_1)dt_1dt_2$$
$$= \int_t^\infty \frac{(t_2-t)^3}{6}A^4v^*(t_2,\sigma)dt_2.$$

Suppose now that for $t>0$, $\ell<k$ and any smooth function φ defined on $(,\infty)$,

$$\int_t^\infty \int_{t_1}^\infty \dots \int_{t_{\ell-1}}^\infty \prod_{j=1}^{\ell}(t_j-t_{j-1})\varphi(t_\ell)dt_\ell dt_{\ell-1}\dots dt_1 = \int_t^\infty \frac{(t_\ell-t)^{2\ell-1}}{(2\ell-1)!}\varphi(t_\ell)dt_\ell. \tag{3.18}$$

Then

$$\int_t^\infty \int_{t_1}^\infty \dots \int_{t_\ell}^\infty \prod_{j=1}^{\ell+1}(t_j-t_{j-1})\varphi(t_{\ell+1})dt_{\ell+1}dt_\ell\dots dt_1$$
$$= \int_t^\infty \int_{t_1}^\infty \dots \int_{t_{\ell-1}}^\infty \prod_{j=1}^{\ell}(t_j-t_{j-1})\Phi(t_\ell)dt_\ell dt_{\ell-1}\dots dt_1,$$
$$= \int_t^\infty \frac{(t_\ell-t)^{2\ell-1}}{(2\ell-1)!}\Phi(t_\ell)dt_\ell,$$

with

$$\Phi(t_\ell) = \int_{t_\ell}^\infty (t_{\ell+1}-t_\ell)\varphi(t_{\ell+1})dt_{\ell+1}.$$

But

$$\int_t^\infty \frac{(t_\ell-t)^{2\ell-1}}{(2\ell-1)!}\int_{t_\ell}^\infty (t_{\ell+1}-t_\ell)\varphi(t_{\ell+1})dt_{\ell+1}dt_\ell$$
$$= \int_t^\infty \varphi(t_{\ell+1})\int_t^{t_{ell+1}} \frac{(t_\ell-t)^{2\ell-1}}{(2\ell-1)!}(t_{\ell+1}-t_\ell)dt_\ell dt_{\ell+1}$$
$$= \int_t^\infty \varphi(t_{\ell+1})\int_0^{t_{ell+1}-t} \frac{\tau^{2\ell-1}}{(2\ell-1)!}(t_{\ell+1}-t-\tau)d\tau dt_{\ell+1}$$
$$= \int_t^\infty \varphi(t_{\ell+1})\frac{(t_{2\ell+1}-\tau)^{2\ell+1}}{(2\ell+1)!}dt_{\ell+1}$$

as $\frac{1}{(2\ell-1)!}(\frac{1}{2\ell}-\frac{1}{2\ell+1}) = \frac{1}{(2\ell+1)!}$. Taking $\varphi(t_{\ell+1}) = A^{2\ell}v^*(t_{\ell+1},\sigma)$ implies (3.17).

End of the proof. From (3.16) and Lemma 3.1 with $t=0$, we get

$$\int_{S^{N-1}} |f|\,d\sigma \le \int_0^\infty \int_{S^{N-1}} \left|A^{2k}v^*\right| \frac{\tau^{2k-1}}{(2k-1)!}d\sigma d\tau$$
$$\le C(N,s,k,q,\delta)\left(\int_0^\infty \int_{S^{N-1}} \left|A^{2k}v^*\right|^q e^{\delta\tau}\tau^{sq-1}d\sigma d\tau\right)^{1/q} \tag{3.19}$$

for any $\delta > 0$ (δ will be taken smaller that $(N-2)q/2$) is the sequel), where

$$C(N,s,k,q,\delta) = \left(\left|S^{N-1}\right| \int_0^\infty \tau^{(2k-s-1/q')q'} e^{-\delta\tau/(q-1)} d\tau \right)^{1/q'}.$$

Notice that the integral is convergent since $(2k-s-1/q')q' > -1 \Longleftrightarrow s < 2k$. As in the case $s < 2$ we return to $\tilde{v}$ and $\tilde{u} = A^{2k}\tilde{u}$, use the harmonicity of u in order to derive

$$\begin{aligned} \int_1^\infty \int_{S^{N-1}} \left|A^{2k}\tilde{v}\right|^q e^{(\delta-(N-2)q/2)\tau} \tau^{sq-1} d\sigma d\tau \\ \leq \int_0^1 \int_{S^{N-1}} \left|A^{2k}\tilde{v}\right|^q e^{(\delta-(N-2)q/2)\tau^{-1}} \tau^{-sq-1} d\sigma d\tau \end{aligned} \tag{3.20}$$

as in (3.10) and finally

$$\begin{aligned} \int_{S^{N-1}} |f|\, d\sigma &\leq C \left(\int_0^1 \int_{S^{N-1}} |A|^{2k}\, {v^*}^q \tau^{sq-1} d\sigma d\tau \right)^{1/q}, \\ &\leq C' \left(\int_0^1 \int_{S^{N-1}} |\tilde{u}|^q \tau^{sq-1} d\sigma d\tau \right)^{1/q}, \end{aligned} \tag{3.21}$$

which ends the proof of Theorem 1.1.

Remark. If $N = 2$ the lifting operator is

$$\mathbb{B} = \left(1 - \frac{d^2}{d\sigma^2}\right)^k,$$

and the proof is similar. moreover, since $\mathbb{B}$ is an isomorphism between $B^{2k-s,1}(S^1)$ and $B^{-s,1}(S^1)$, the result of Theorem 1.1 holds also in the case $q = 1$.

4 A regularity result for the Green operator

Put $(1-|x|) = \delta(x)$. By duality between $L^q(B;\delta^{sq-1}dx)$ and $L^{q'}(B;\delta^{sq-1}dx)$, we write

$$\int_B \mathbb{P}(\mu)\psi\delta^{sq-1}dx = -\int_B \mathbb{P}(\mu)\Delta\zeta dx = -\int_\Sigma \frac{\partial\zeta}{\partial\nu} d\mu, \tag{4.1}$$

where ζ is the solution of

$$\begin{cases} -\Delta\zeta = \delta^{sq-1}\psi & \text{in } B, \\ \zeta = 0 & \text{on } \partial B. \end{cases} \tag{4.2}$$

In (4.1), the boundary term should be written $< \mu, \partial\zeta/\partial\nu >_\Sigma$ if μ is a distribution on Σ. Then the adjoint operator $\mathbb{P}^*$ is defined by

$$\mathbb{P}^*(\psi) = -\frac{\partial}{\partial\nu}\mathbb{G}(\delta^{sq-1}\psi), \tag{4.3}$$

where $\mathbb{G}(\delta^{sq-1}\psi)$ is the Green potential of $\delta^{sq-1}\psi$. Consequently, Theorem 1.1 implies that there exists a constant $C > 0$ such that

$$C^{-1}\left\|\psi\right\|_{L^{q'}(B;\delta^{sq-1}dx)}\leq\left\|\frac{\partial}{\partial\nu}\mathbb{G}(\delta^{sq-1}\psi)\right\|_{B^{s,q'}(\Sigma)}\leq C\left\|\psi\right\|_{L^{q'}(B;\delta^{sq-1}dx)}. \tag{4.4}$$

But

$$\psi\in L^{q'}(B;\delta^{sq-1}dx)\Longleftrightarrow\delta^{sq-1}\psi\in L^{q'}(B;\delta^{(sq-1)(1-q')}dx).$$

Putting $\varphi=\delta^{sq-1}\psi$ and replacing q' by p, implies the following result

Theorem 4.1 *Let $s>0$ and $1<p<\infty$. Then*

$$\varphi\in L^{p}(B;\delta^{p(1-s)-1})dx)\Longleftrightarrow\frac{\partial}{\partial\nu}\mathbb{G}(\varphi)\in B^{s,p}(\Sigma).$$

Moreover there exists a constant $C>0$ such that for any $\varphi\in L^{p}(B;\delta^{p(1-s)-1})dx)$

$$C^{-1}\|\varphi\|_{L^{p}(B;\delta^{p(1-s)-1})dx)}\leq\left\|\frac{\partial}{\partial\nu}\mathbb{G}(\varphi)\right\|_{B^{s,p}(\Sigma)}\leq C\|\varphi\|_{L^{p}(B;\delta^{p(1-s)-1})dx)}. \tag{4.5}$$

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