An index theorem for manifolds with boundary

Paulo Carrillo Rouse, Bertrand Monthubert

An index theorem for manifolds with boundary

by Paulo Carrillo Rouse and Bertrand Monthubert

Abstract

In [2] II.5, Connes gives a proof of the Atiyah-Singer index theorem for closed manifolds by using deformation groupoids and appropiate actions of these on $\mathbb{R}^N$. Following these ideas, we prove an index theorem for manifolds with boundary.

Résumé

Dans [2] II.5, Connes donne une preuve du théorème de l'indice d'Atiyah-Singer pour des variétés fermées en utilisant des groupoïdes de déformation et des actions appropriées de ceux-ci dans $\mathbb{R}^N$. Nous suivons ces idées pour montrer un théorème d'indice pour des variétés à bord.

Version française abrégée

Dans [2], II.5, Alain Connes donna une preuve du théorème d'Atiyah-Singer pour une variété fermée entièrement fondée sur l'utilisation de groupoïdes, grâce à une action du groupoïde tangent de la variété sur $\mathbb{R}^N$. L'idée centrale est de remplacer des groupoïdes qui ne sont pas (Morita) équivalents à des espaces, par des groupoïdes obtenus par produit croisé et qui possèdent cette propriété, ce qui permet ensuite de donner une formule.

Si X est une variété à bord ∂X, nous construisons le groupoïde $\mathcal{T}_bX := ({}^{ad}G_{\partial X} \times \mathbb{R}) \bigcup_\partial TX$ en recollant ${}^{ad}G_{\partial X} \times \mathbb{R}$ avec TX le long de leur bord commun $T\partial X \times \mathbb{R}$ (ici ${}^{ad}G_{\partial X} = T\partial X \cup \partial X \times \partial X \times (0,1)$ est le groupoïde adiabatique). Nous le recollons alors avec le groupoïde tangent de l'intérieur de X, ${}^TG_{\overset{\circ}{X}} = T\overset{\circ}{X} \cup \overset{\circ}{X} \times \overset{\circ}{X} \times (0,1]$: ${}^TG_X := \mathcal{T}_bX \bigcup_0 {}^TG_{\overset{\circ}{X}}$.

Il existe un homomorphisme ${}^TG_X \overset{h}{\longrightarrow} \mathbb{R}^N$ induit par un plongement de X dans $\mathbb{R}^{N-1} \times \mathbb{R}_+$, tel que ∂X se plonge dans $\mathbb{R}^{N-1} \times \mathbb{R}_+ \times \{0\}$ et $\overset{\circ}{X}$ se plonge dans $\mathbb{R}^{N-1} \times \mathbb{R}_+^*$. Le produit croisé de ${}^TG_X$ par h (noté ${}^T(G_X)_h$) est un groupoïde propre dont les groupes d'isotropie sont triviaux, il est donc Morita-équivalent à son espace d'orbites.

Soit $V(\overset{\circ}{X})$ le fibré normal de $\overset{\circ}{X}$ dans $\mathbb{R}^N$, et $V(\partial X)$ le fibré normal de ∂X dans $\mathbb{R}^{N-1}$; soit enfin $V(X) = V(\overset{\circ}{X}) \bigcup V(\partial X)$. En notant $\mathscr{D}_\partial = V(\partial X) \times \{0\} \bigsqcup \mathbb{R}^{N-1} \times (0,1)$ et $\mathscr{D}_\circ = V(\overset{\circ}{X}) \times \{0\} \bigsqcup \mathbb{R}^N \times (0,1]$ les déformations au cône normal, on construit les espaces $\mathscr{B}_\partial := V(X) \bigcup_\partial \mathscr{D}_\partial$ et $\mathscr{B} := \mathscr{B}_\partial \bigcup_\circ \mathscr{D}_\circ$.

Proposition 0.1. *Le groupoïde $({}^TG_X)_h$ est Morita équivalent à l'espace $\mathscr{B}$.*

Soit

$$ind_f^X = (e_1)_* \circ (e_0)_*^{-1} : K^0(\mathcal{T}_bX) \longrightarrow K^0(\overset{\circ}{X} \times \overset{\circ}{X}) \approx \mathbb{Z}.$$

Définition 0.1 (Indice topologique pour une variété à bord). *Soit X une variété à bord. L'indice topologique de X est le morphisme*

$$ind_t^X : K^0(\mathcal{T}_bX) \longrightarrow \mathbb{Z}$$

défini comme la composition des trois morphismes suivants

(1) *L'isomorphisme de Connes-Thom* CT_0 *suivi de l'équivalence de Morita* $\mathscr{M}_0$ *:*
$$K^0(\mathcal{T}_bX) \overset{CT_0}{\longrightarrow} K^0((\mathcal{T}_bX)_{h_0}) \overset{\mathscr{M}_0}{\longrightarrow} K^0(\mathscr{B}_\partial),$$
où $(\mathcal{T}_bX)_{h_0}$ *est le produit croisé de* $\mathcal{T}_bX$ *par* h_0 *(l'homomorphisme* h *en* $t=0$*).*

(2) *Le morphisme indice de l'espace de déformation* $\mathscr{B}$ *:* $K^0(\mathscr{B}_\partial) \underset{\approx}{\overset{(e_0)_*}{\longleftarrow}} K^0(\mathscr{B}) \overset{(e_1)_*}{\longrightarrow} K^0(\mathbb{R}^N)$

(3) *Le morphisme de périodicité de Bott :* $K^0(\mathbb{R}^N) \overset{Bott}{\longrightarrow} \mathbb{Z}$.

Theorem 0.2. *Pour toute variété à bord, on a l'égalité*
$$ind_f^X = ind_t^X.$$

1. Actions of $\mathbb{R}^N$

All the groupoids considered here will be continuous family groupoids [5, 10]. Hence we can consider their convolution and C^*-algebras without any problem. If G is such a groupoid, we will denote by $K^0(G)$ the K-theory group of its C^*-algebra (unless explicetely written otherwise). This is consistent with the usual notation when G is a space (a groupoid made only of units). In the sequel, given a smooth manifold N, we will denote by ${}^{ad}G_N : TN \times \{0\} \bigsqcup N \times N \times \mathbb{R}^* \rightrightarrows N \times \mathbb{R}$, the deformation to normal cone of N in $N \times N$(for complete details about this deformation functor see [1]). At each time, we will need to restrict it to some interval, e.g. $[0,1]$ gives the tangent groupoid, and $[0,1)$ gives the adiabatic groupoid.

Let $G \rightrightarrows M$ be a groupoid and $h : G \to \mathbb{R}^N$ a (smooth or continuous) homomorphism of groupoids, ($\mathbb{R}^N$ as an additive group). Connes defined the semi-direct product groupoid $G_h = G \times \mathbb{R}^N \rightrightarrows M \times \mathbb{R}^N$ ([2], II.5) with structure maps $t(\gamma, X) = (t(\gamma), X)$, $s(\gamma, X) = (s(\gamma), X + h(\gamma))$ and product $(\gamma, X) \circ (\eta, X + h(\gamma)) = (\gamma \circ \eta, X)$ for composable arrows.

At the level of C^*-algebras, $C^*(G_h)$ can be seen as the crossed product algebra $C^*(G) \rtimes \mathbb{R}^N$ where $\mathbb{R}^N$ acts on $C^*(G)$ by automorphisms by the formula: $\alpha_X(f)(\gamma) = e^{i\cdot(X\cdot h(\gamma))} f(\gamma)$, $\forall f \in C_c(G)$, (see [2], propostion II.5.7 for details). In particular, in the case N is even, we have a Connes-Thom isomorphism in K-theory $K^0(G) \overset{\approx}{\to} K^0(G_h)$ ([2], II.C).

Using this groupoid, Connes gives a conceptual, simple proof of the Atiyah-Singer Index theorem for closed smooth manifolds. Let M be a smooth manifold, $G_M = M \times M$ its groupoid, and consider the tangent groupoid ${}^TG_M$. It is well known that the index morphism provided by this deformation groupoid is precisely the analytic index of Atiyah-Singer, [2, 8]. In other words, the analytic index of M is the morphism

$$K^0(TM) \xrightarrow{(e_0)_*^{-1}} K^0({}^TG_M) \xrightarrow{(e_1)_*} K^0(M \times M) = K^0(\mathscr{K}(L^2(M))) \approx \mathbb{Z}, \tag{1}$$

where e_t are the obvious evaluation algebra morphisms at t. As discussed by Connes, if the groupoids appearing in this interpretation of the index were equivalent to spaces then we would immediately have a geometric interpretation of the index. Now, $M \times M$ is equivalent to a point (hence to a space), but the other fundamental groupoid playing a role is not, that is, TM is a groupoid whose fibers are the groups T_xM, which are not equivalent (as groupoids) to a space. The idea of Connes is to use an appropriate action of the tangent groupoid in some $\mathbb{R}^N$ in order to translate the index (via a Thom isomorphism) in an index associated to a deformation groupoid which will be equivalent to some space.

2. Groupoids and Manifolds with boundary

Let X be a manifold with boundary ∂X. We denote, as usual, $\overset{\circ}{X}$ the interior which is a smooth manifold. Let X_∂ be the smooth manifold obtained by glueing X with $\partial X \times [0,1)$ along their common boundary, $\partial X \sim \partial X \times \{0\}$. Set $TX := TX_\partial|_X$, and consider the smooth manifold $\mathcal{T}_b X := ({}^{ad}G_{\partial X} \times \mathbb{R}) \bigcup_\partial TX$ obtained by glueing ${}^{ad}G_{\partial X} \times \mathbb{R}$ and TX along their common boundary $T\partial X \times \mathbb{R}$ (${}^{ad}G_{\partial X} = T\partial X \cup \partial X \times \partial X \times (0,1)$ is the adiabatic groupoid). Now, we have a continuous family groupoid over X_∂: $\mathcal{T}_b X \rightrightarrows X_\partial$. As a groupoid it is the union of the groupoids ${}^{T}G_{\partial X} \times \mathbb{R} \rightrightarrows \partial X \times [0,1)$ and $TX \rightrightarrows X$. For the topology, it is very easy to see that all the groupoid structures are compatible with the glueings we considered.

We are going to consider a deformation groupoid ${}^{T}G_X$ ([9]). This will be a natural generalisation of the Connes tangent groupoid of a smooth manifold, to the case with boundary. The space of arrows ${}^{T}G_X := \mathcal{T}_b X \bigcup_\circ {}^{T}G_{\overset{\circ}{X}}$ is obtained by glueing at $T\overset{\circ}{X}$ ($T\overset{\circ}{X} \times \{0\} \subset {}^{T}G_{\overset{\circ}{X}}$ is closed). The space of units $X_{g_0} := X_\partial \bigcup_\circ \overset{\circ}{X} \times [0,1]$ is obtained by glueing $\overset{\circ}{X} \sim \overset{\circ}{X} \times \{0\}$ ($\overset{\circ}{X} \times \{0\} \subset \overset{\circ}{X} \times [0,1]$ is closed). Using the groupoid structures of $\mathcal{T}_b X \rightrightarrows X_\partial$ and of ${}^{T}G_{\overset{\circ}{X}} \rightrightarrows \overset{\circ}{X} \times [0,1]$, we have a continuous family groupoid ${}^{T}G_X \rightrightarrows X_{g_0}$. Again, all the groupoid structures are compatible with the considered glueings.

To define a homomorphism ${}^{T}G_X \xrightarrow{h} \mathbb{R}^N$ we will need as in the nonboundary case an appropiate embedding. It is possible to find an embedding $i : X \hookrightarrow \mathbb{R}^{N-1} \times \mathbb{R}_+$ such that its restrictions to the interior and to the boundary are (smooth embeddings) of the following form $i_\circ : \overset{\circ}{X} \hookrightarrow \mathbb{R}^{N-1} \times \mathbb{R}_+^*$ and $i_\partial : \partial X \hookrightarrow \mathbb{R}^{N-1} \times \{0\}$. We define the homomorphism $h : {}^{T}G_X \to \mathbb{R}^N$ as follows.

(2)

$$h : \begin{cases} h(x,X,0) = d_x i_\circ(X) \text{ and } h(x,y,\epsilon) = \frac{i_\circ(x) - i_\circ(y)}{\epsilon} \text{ on } {}^{T}G_{\overset{\circ}{X}} \\ h(x,\xi,0,\lambda) = (d_x i_\partial(\xi), \lambda) \text{ and } h(x,y,\epsilon,\lambda) = (\frac{i_\partial(x) - i_\partial(y)}{\epsilon}, \lambda) \text{ on } {}^{T}G_{\partial X} \times \mathbb{R} \\ h(x,X) = d_x i_\circ(X) \text{ on } T\overset{\circ}{X} \end{cases}$$

Since all these morphisms are compatible with the glueings, one has:

Proposition 2.1. *With the formulas defined above, $h : {}^{T}G_X \to \mathbb{R}^N$ defines a homomorphism of continuous family groupoids.*

The action of ${}^{T}G_X$ on $\mathbb{R}^N$ defined by h is free because i is an immersion. It is not necessarily proper (in the case of Connes [2] II.5 it is since M was supposed closed), however we can prove the following:

Proposition 2.2. *The groupoid $({}^{T}G_X)_h$ is a proper groupoid with trivial isotropy groups.*

Notice that the groupoid G_h is not the action groupoid (if not, the properness of the action would be equivalent to the properness of the groupoid). It is very important that the units of the groupoid G_h be the units of G times $\mathbb{R}^N$.

As an immediate consequence of the propositions above, the groupoid $({}^TG_X)_h$ is Morita equivalent to its space of orbits. Let us specify this space.

Let $V(\overset{\circ}{X})$ be the total space of the normal bundle of $\overset{\circ}{X}$ in $\mathbb{R}^N$. Similarly, let $V(\partial X)$ be the total space of the normal bundle of ∂X in $\mathbb{R}^{N-1}$. Observe that they have the same fiber vector dimension. In fact, their union $V(X) = V(\overset{\circ}{X}) \bigcup V(\partial X)$, is a vector bundle over X, the normal bundle of X in $\mathbb{R}^N$.

Take $\mathscr{D}_\partial = V(\partial X) \times \{0\} \bigsqcup \mathbb{R}^{N-1} \times (0,1)$ the deformation to the normal cone associated to the embedding $\partial X \overset{i_\partial}{\hookrightarrow} \mathbb{R}^{N-1}$. We consider the space $\mathscr{B}_\partial := V(X) \bigcup_\partial \mathscr{D}_\partial$ glued over their common boundary $V(\partial X) \sim V(\partial X) \times \{0\}$. On the other hand, take $\mathscr{D}_\circ = V(\overset{\circ}{X}) \times \{0\} \bigsqcup \mathbb{R}^N \times (0,1]$ the deformation to the normal cone associated to the embedding $\overset{\circ}{X} \overset{i_\circ}{\hookrightarrow} \mathbb{R}^N$. We consider the space $\mathscr{B} := \mathscr{B}_\partial \bigcup_\circ \mathscr{D}_\circ$ glued over $V(\overset{\circ}{X})$ by the identity map.

Proposition 2.3. *The space of orbits of the groupoid* $({}^TG_X)_h$ *is* $\mathscr{B}$.

We can give the explicit homeomorphism. The orbit space of $({}^TG_X)_h$ is a quotient of $X_{g_0} \times \mathbb{R}^N$. To define a map $\Psi : X_{g_0} \times \mathbb{R}^N \to \mathscr{B}$ it is enough to define it for each component of X_{g_0}. Let

(3)
$$\Psi : \begin{cases} \partial X \times (0,1) \times \mathbb{R}^{N-1} \times \mathbb{R} \to \mathbb{R}^{N-1} \times (0,1) \\ \Psi(a,t,\xi,\lambda) := (\frac{i_\partial(a)}{t} + \xi, t) \end{cases} \quad \begin{cases} \partial X \times \{0\} \times \mathbb{R}^{N-1} \times \mathbb{R} \to V(\partial X) \\ \Psi(a,0,\xi,\lambda) := \overline{(i_\partial(a), \xi)} \end{cases}$$

$$\begin{cases} \overset{\circ}{X} \times (0,1] \times \mathbb{R}^N \to \mathbb{R}^N \times (0,1] \\ \Psi(x,t,X) := (\frac{i_\circ(x)}{t} + X, t) \end{cases} \quad \begin{cases} \overset{\circ}{X} \times \{0\} \times \mathbb{R}^N \to V(\overset{\circ}{X}) \\ \Psi(x,0,X) := \overline{(i_\circ(x), X)} \end{cases}$$

where $\overline{\xi}$ denotes the class in $V_a(\partial X) := \mathbb{R}^{N-1}/T_{i_\partial(a)}\partial X$ (resp. $\overline{X}$ denotes the class in $V_x(\overset{\circ}{X}) := \mathbb{R}^N/T_{i_\circ(x)}\overset{\circ}{X}$). This gives a continuous map $\Psi : X_{g_0} \times \mathbb{R}^N \to \mathscr{B}$ that passes to the quotient into a homeomorphism $\overline{\Psi} : (X_{g_0} \times \mathbb{R}^N)/\sim \to \mathscr{B}$, where $(X_{g_0} \times \mathbb{R}^N)/\sim$ is the orbit space of the groupoid $({}^TG_X)_h$.

3. The index theorem for manifolds with boundary

Deformation groupoids induce index morphisms. The groupoid ${}^TG_X$ is naturally parametrized by the closed interval $[0,1]$. Its algebra comes equipped with evaluations to the algebra of $\mathcal{T}_bM$ (at t=0) and to the algebra of $\overset{\circ}{X} \times \overset{\circ}{X}$ (for $t \neq 0$). We have a short exact sequence of C^*-algebras

$$0 \longrightarrow C^*(\overset{\circ}{X} \times \overset{\circ}{X} \times (0,1]) \longrightarrow C^*({}^TG_X) \xrightarrow{e_0} C^*(\mathcal{T}_bM) \longrightarrow 0 \tag{4}$$

where the algebra $C^*(\overset{\circ}{X} \times \overset{\circ}{X} \times (0,1])$ is contractible. Hence applying the K-theory functor to this sequence we obtain an index morphism

$$ind_f^X = (e_1)_* \circ (e_0)_*^{-1} : K^0(\mathcal{T}_bX) \longrightarrow K^0(\overset{\circ}{X} \times \overset{\circ}{X}) \approx \mathbb{Z}.$$

The morphism $h : {}^TG_X \to \mathbb{R}^N$ is by definition also parametrized by $[0,1]$, *i.e.*, we have morphisms $h_0 : \mathcal{T}_bM \to \mathbb{R}^N$ and $h_t : \overset{\circ}{X} \times \overset{\circ}{X} \to \mathbb{R}^N$, for $t \neq 0$. We can consider the associated groupoids, which satisfy the same properties as in proposition 2.2

(in fact, for proving such proposition it is better to do it for each t, and to check all the compatibilities).

Définition 3.1. *[Topological index morphism for a manifold with boundary] Let X be a manifold with boundary. The topological index morphism of X is the morphism*

$$ind_t^X : K^0(\mathcal{T}_b X) \longrightarrow \mathbb{Z}$$

defined (using an embedding as above) as the composition of the following three morphisms

(1) *The Connes-Thom isomorphism CT_0 followed by the Morita equivalence $\mathscr{M}_0$:*

$$K^0(\mathcal{T}_b X) \xrightarrow{CT_0} K^0((\mathcal{T}_b X)_{h_0}) \xrightarrow{\mathscr{M}_0} K^0(\mathscr{B}_\partial)$$

(2) *The index morphism of the deformation space $\mathscr{B}$:* $K^0(\mathscr{B}_\partial) \underset{\approx}{\xleftarrow{(e_0)_*}} K^0(\mathscr{B}) \xrightarrow{(e_1)_*} K^0(\mathbb{R}^N)$

(3) *The usual Bott periodicity morphism:* $K^0(\mathbb{R}^N) \xrightarrow{Bott} \mathbb{Z}$.

Remark 1. *The topological index defined above is a natural generalisation of the topological index theorem defined by Atiyah-Singer. Indeed, in the boundaryless case, they coincide. The index of the deformation space $\mathscr{B}$ is quite easy to understand because we are dealing now with spaces (as groupoids the product is trivial), then the group $K^0(\mathscr{B})$ is the K-theory of the algebra of continuous functions vanishing at infinity $C_0(\mathscr{B})$ and the evaluation maps are completely explicit. In particular, if we identify $\mathscr{B}_\partial$ with an open subset of $\mathbb{R}^N$ (in the natural way), then the morphism (ii) above correspond to the canonical extension of functions of $C_0(\mathscr{B}_\partial)$ to $C_0(\mathbb{R}^N)$.*

The following diagram, in which the morphisms CT and $\mathscr{M}$ are the Connes-Thom and Morita isomorphisms respectively, is trivially commutative:

$$(5)\qquad \begin{array}{ccccc}
K^0(\mathcal{T}_b X) & \underset{\approx}{\xleftarrow{e_0}} & K^0({}^T G_X) & \xrightarrow{e_1} & K^0(\overset{\circ}{X} \times \overset{\circ}{X}) \\
{\scriptstyle CT}\downarrow{\scriptstyle \approx} & & {\scriptstyle CT}\downarrow{\scriptstyle \approx} & & {\scriptstyle CT}\downarrow{\scriptstyle \approx} \\
K^0((\mathcal{T}_b X)_{h_0}) & \underset{\approx}{\xleftarrow{e_0}} & K^0(({}^T G_X)_h) & \xrightarrow{e_1} & K^0((\overset{\circ}{X} \times \overset{\circ}{X}))_{h_1}) \\
{\scriptstyle \mathscr{M}}\downarrow{\scriptstyle \approx} & & {\scriptstyle \mathscr{M}}\downarrow{\scriptstyle \approx} & & {\scriptstyle \mathscr{M}}\downarrow{\scriptstyle \approx} \\
K^0(\mathscr{B}_\partial) & \underset{\approx}{\xleftarrow{e_0}} & K^0(\mathscr{B}) & \xrightarrow{e_1} & K^0(\mathbb{R}^N),
\end{array}$$

The left vertical line gives the first part of the topological index map. The bottom line is the morphism induced by the deformation space $\mathscr{B}$. And the right vertical line is precisely the inverse of the Bott isomorphism $\mathbb{Z} = K^0(\{pt\}) \approx K^0(\overset{\circ}{X} \times \overset{\circ}{X}) \to K^0(\mathbb{R}^N)$. Since the top line gives ind_f^X, we obtain the following result:

Theorem 3.1. *For any manifold with boundary X, we have the equality of morphisms*

$$ind_f^X = ind_t^X.$$

4. Perspectives

As discussed in [3, 4, 5], the index map ind_f^X computes the Fredholm index of a fully elliptic operator in the b-calculus of Melrose. We shall use the result proven here to give a formula in relation to that of Atiyah-Patodi-Singer ([6]).

References

[1] Carrillo-Rouse, P. A Schwartz type algebra for the tangent groupoid. In K-theory and Noncommutative Geometry *EMS series of congress and reports.* (2008), 181–200.

[2] Connes, A. Non commutative geometry. *Academic Press,Inc,San Diego, CA* (1994).

[3] Debord, C., Lescure, J.M. K-duality for pseudomanifolds with isolated singularities. *J. of Functional Analysis.* **219** (2005), 109–133.

[4] Debord, C., Lescure, J.M., Nistor, V. Groupoids and an index theorem for conical pseudo-manifolds. *Preprint* arxiv:math.OA/0609438.

[5] Lauter, R., Monthubert, B., Nistor, V. Pseudodifferential Analysis on Continuous Family Groupoids. *Documenta Mathematica.* **5** (2000), 625–656.

[6] Melrose, R. The Atiyah-Patodi-Singer index theorem. (English summary) *Research Notes in Mathematics*, 4. A K Peters, Ltd., Wellesley, MA, (1993). xiv+377 pp.

[7] Monthubert, B. Groupoids and pseudodifferential calculus on manifolds with corners. *J. of Functional Analysis* **199**, no. I (2003), 243–286.

[8] Monthubert, B., Pierrot, F. Indice analytique et groupoïdes de Lie. *C.R. Acad.Sci.Paris* **325** Série I (1997), 193–198.

[9] Monthubert, B. Contribution of noncommutative geometry to index theory on singular manifolds. *Geometry and topology of manifolds*, 221–237, Banach Center Publ., 76, Polish Acad. Sci., Warsaw, 2007.

[10] Paterson, A. Groupoids, inverse semigroups, and their operator algebras. *Progress in Mathematics*, 170. Birkhäuser Boston, Inc., Boston, MA, (1999). xvi+274 pp.