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On the topology of fillings of contact manifolds and applications*

ALEXANDRU OANCEA ¹ CLAUDE VITERBO ²

¹ IRMA

UMR 7501, Université de Strasbourg
7 rue René Descartes, 67084 Strasbourg
oancea@math.u-strasbg.fr

² Centre de Mathématiques Laurent Schwartz
UMR 7640 du CNRS

École Polytechnique - 91128 Palaiseau, France
viterbo@math.polytechnique.fr

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Abstract

The aim of this paper is to address the following question: given a contact manifold (Σ, ξ) , what can be said about the aspherical symplectic manifolds (W, ω) bounded by (Σ, ξ) ? We first extend a theorem of Eliashberg, Floer and McDuff to prove that under suitable assumptions the map from $H_*(\Sigma)$ to $H_*(W)$ induced by inclusion is surjective. We then apply this method in the case of contact manifolds having a contact embedding in \mathbb{R}^{2n} or in a subcritical Stein manifold. We prove in many cases that the homology of the fillings is uniquely determined. Finally we use more recent methods of symplectic topology to prove that, if a contact hypersurface has a Stein subcritical filling, then all its weakly subcritical fillings have the same homology.

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A number of applications are given, from obstructions to the existence of Lagrangian or contact embeddings, to the exotic nature of some contact structures. We refer to the table in section 7 for a summary of our results.

1 Introduction

In this paper all symplectic manifolds will be assumed to be connected, of dimension $2n$ and aspherical, that is $[\omega]\pi_2(M) = 0$. All contact manifolds are connected and have dimension $2n - 1$. The form σ_0 denotes the standard symplectic form on \mathbb{R}^{2n} or $\mathbb{C}P^n$, and α_0 the standard contact form on S^{2n-1} .

In a celebrated paper, Eliashberg, Floer and McDuff ([McDuff]) proved that if (W, ω) is a symplectic manifold with contact boundary (S^{2n-1}, α_0) , then W is diffeomorphic to the unit ball B^{2n} . In the case of dimension 4, Gromov had earlier proved in ([Gromov]) that W is actually symplectomorphic to (B^4, σ_0) , but this relies heavily on positivity of intersection for holomorphic curves that is special to dimension 4.

One can ask more generally, given a fillable contact manifold (Σ, ξ) and a symplectic filling (W, ω) , what can be said about the topology or the homology of W . Is it uniquely determined by the contact structure (Σ, ξ) ? Is it determined by the topology of Σ ? Do we have lower bounds? Upper bounds? It turns out that all these possibilities actually occur.

For example, if (Σ, ξ) has an exact contact embedding into $(\mathbb{R}^{2n}, \sigma_0)$ - many such examples can be found in [Laudenbach] -, it readily follows from the Eliashberg-Floer-McDuff theorem and some elementary algebraic topology that all fillings have the same homology. This gives easy examples of contact manifolds with no exact embedding in $(\mathbb{R}^{2n}, \sigma_0)$. As far as the authors know, there are only few previously known examples of fillable manifolds not embeddable in \mathbb{R}^{2n} , with the exception of recent results in [Cieliebak-Frauenfelder-Oancea] and [Albers-McLean], which however assume the exactness of the embeddings, an assumption we usually can dispense with. More general results relating the homology follow from the same methods, and a generalization of the Eliashberg-Floer-McDuff theorem to the Stein subcritical¹ manifold, that is a manifold admitting an exhausting plurisubharmonic function with no critical points of index $n = \frac{1}{2} \dim(W)$.

Our last result uses more sophisticated tools. One of them will be symplectic homology of W , and the positive part, defined in [Viterbo]. It turns out that this positive part, under mild assumptions on the Conley-Zehnder index of closed characteristics, is independent of the filling. In [Cieliebak-Frauenfelder-Oancea], this is proved as a consequence of arguments in [Bourgeois-Oancea-1]. One can then

¹see the definition 2.3

prove that if (Σ, ξ) bounds a subcritical Stein manifold (W, ω) , any other weakly subcritical filling will have the same homology.

Of course many questions remain open. As far as we can see, nothing can be said about the symplectic topology of fillings outside the subcriticality/non-subcriticality alternative. Are there examples of compact manifolds L such that ST^*L has fillings with homology different from $H_*(L)$? Is there an embedding of the Brieskorn sphere of a singularity of Milnor number μ in the Milnor fibre of a singularity of Milnor number $\mu' < \mu$?

2 The Eliashberg-Floer-McDuff theorem revisited

Conventions. In this section we denote by (W, ω) an aspherical symplectic manifold of dimension $2n$, and by (M, ξ) a contact manifold of dimension $2n - 1$. We assume that ξ is co-orientable, and fix a co-orientation. The contact structure ξ is then defined by a contact form α and since $\alpha \wedge (d\alpha)^{n-1}$ does not vanish, it defines an orientation of M .

All homology and cohomology groups are taken with field coefficients.

Definition 2.1. A *contact embedding* of (Σ, ξ) in (W, ω) is a codimension 1 embedding such that there exists a positive contact form α which extends to a neighbourhood of Σ as a primitive of ω . We shall say that the embedding is an *exact contact embedding* if α extends to the whole of W as a primitive of ω .

Definition 2.2. A *symplectic filling* of (Σ, ξ) is a symplectic manifold (W, ω) without closed components, such that $\partial W = \Sigma$ and there exists a positive contact form α which extends to a neighbourhood of Σ as a primitive of ω . We shall say that the symplectic filling is an *exact symplectic filling* if α extends to the whole of W as a primitive of ω .

Definition 2.3. A symplectic filling (W, ω) of (Σ, ξ) is a *Stein symplectic filling* if W has an almost complex structure J , and a non-positive plurisubharmonic function ψ such that $\Sigma = \psi^{-1}(0)$ and $-J^*d\psi$ is a contact form defining ξ . Note that ψ can always be chosen to be a Morse function. Its critical points must then have index at most n , hence W has the homotopy type of a CW complex of dimension at most n . If we can find the function ψ with no critical points of index n , then W is said to be *Stein subcritical*.

Remark 2.4. A contact embedding of (Σ, ξ) in (W, ω) which is *separating* – i.e. $W \setminus \Sigma$ consists of two connected components – yields a filling of (Σ, α) by the connected component of $W \setminus \Sigma$ for which the boundary orientation of Σ coincides with the

orientation induced by α . This filling we shall call *the interior of Σ* . If W is non-compact, this is the bounded component of $W \setminus \Sigma$. Note that Σ is always separating if $H_{2n-1}(W; \mathbb{Z})$ is torsion.

Our goal in this section is to prove the following theorem

Theorem 2.5. *Assume (Σ, ξ) admits a contact embedding in a subcritical Stein manifold (M, ω_0) . If (W, ω) is **any** symplectic filling of Σ and $(M \setminus Z) \sqcup_{\Sigma} W$ is aspherical, then the map $H_j(\Sigma) \rightarrow H_j(W)$ is onto. This holds in particular if $[\omega]\pi_2(W) = 0$ and one of the following conditions is satisfied*

- (a). $H_2(W, \Sigma) = 0$.
- (b). (Σ, ξ) is of restricted contact type in (M, ω_0) and in (W, ω) . Moreover, we require that the same contact form α is a primitive of both ω and ω_0 (this last condition holds if $H^1(\Sigma, \mathbb{R}) = 0$).
- (c). The maps $\pi_1(\Sigma) \rightarrow \pi_1(W)$ and $\pi_1(\Sigma) \rightarrow \pi_1(M \setminus Z)$ are injective.

Remark 2.6. Note that (c) holds if Σ is simply connected and (a) holds if W is Stein and $n \geq 3$.

Remark 2.7. When Σ is a sphere, we get that W has vanishing homology. This is the original Eliashberg-Floer-McDuff theorem (see [McDuff]), since an application of the h -cobordism theorem (plus the fact due to Eliashberg that $\pi_1(W)$ vanishes) implies that W is diffeomorphic to the ball B^{2n} . Indeed, since $H_j(S^{2n-1}) = 0$, for $1 \leq j \leq 2n - 2$, the same holds for $H_j(W)$, and since Σ is a boundary in W , the map $H_{2n-1}(S^{2n-1}) \rightarrow H_{2n-1}(W)$ vanishes. When $n = 2$ Gromov (see [Gromov]) proved that W is symplectomorphic to the ball B^4 , but this relies heavily on purely 4-dimensional arguments (positivity of intersection of holomorphic curves).

Our proof of theorem 2.5 closely follows the original proof in [McDuff], except for the final homological argument.

We shall start by working in the following special situation. We will then prove that this is enough to deal with the general case.

Let (P, ω_P) be a symplectic manifold, and H a codimension two symplectic submanifold with pseudo-convex complement (i.e. the first Chern class of its normal bundle is a positive multiple of ω) such that $P \setminus H$ is aspherical. Consider the symplectic manifold $(P \times S^2, \omega_P \oplus \sigma)$. Let (Σ, α) be a separating contact manifold contained in $(P \setminus H) \times D_-^2$, where D_-^2 (resp. D_+^2) is the southern (resp. northern) hemisphere. Let Z be the interior of Σ , $Y = (P \times S^2 \setminus Z)$, and

$$V = Y \sqcup_{\Sigma} W = (P \times S^2 \setminus Z) \sqcup_{\Sigma} W$$

where (W, ω) is a filling of (Σ, α) . Then V has a symplectic form ω_V obtained by gluing $\omega_P \oplus \sigma$ on Y and ω on W . Let p_0 be a point in H , and \mathcal{M} be the

space of maps $u : \mathbb{C}P^1 \longrightarrow V$ homologous to $\{p_0\} \times S^2$ (notice that for $p_0 \in H$, $\{p_0\} \times S^2 \subset Y$) such that $u(z) \in P \times \{z\}$ for $z \in \{-1, 1, \infty\}$. This makes sense since $P \times D_+^2 \subset Y \subset V$. Note that \mathcal{M} coincides with the set of holomorphic maps $u : \mathbb{C}P^1 \longrightarrow V$ homologous to $\{p_0\} \times S^2$ divided by the conformal group $PSL(2, \mathbb{C})$.

Proposition 2.8. *Under the assumptions of the previous paragraph, the map $H_j(\Sigma) \longrightarrow H_j(W)$ induced by inclusion is surjective.*

Proof. Assuming for the moment the compactness of \mathcal{M} , our first task is to prove that the map

$$\begin{aligned} ev : \mathcal{M} \times S^2 &\longrightarrow V \\ (u, z) &\longrightarrow u(z) \end{aligned}$$

has degree one. Indeed, the Riemann-Roch formula implies that the two spaces have the same dimension. It is then enough to count algebraically the number of preimages over a point in V such that no curve through this point goes through W (or through Z , which is equivalent by connectedness). Choosing a point $p_0 \in H$ and counting the number of curves through $p_0 \times \infty$, we will get the same result as for $P \times S^2$. Indeed $p_0 \times \infty$ is contained in Y , as well as the curve $p_0 \times S^2$, and we get that provided H is a complex submanifold, which we can assume without loss of generality, any holomorphic curve either intersects positively with $H \times S^2$, or is contained in $H \times S^2$. The first case is impossible for homological reasons, since $[p_0 \times S^2] \cap [H \times S^2] = 0$, the second case implies that our count is the same as the number of rational curves through $p_0 \times \infty$ in $P \times S^2$, and then such a curve is unique.

Let us now deal with the compactness issue. For this we have to understand the possible bubbling off in V of sequences of holomorphic spheres in the homology class of $\{p_0\} \times S^2$. First notice that such bubbling off will produce at least two pieces: one that has intersection $+1$ with the divisor $P \times \{\infty\}$, and a second one (that could have several components), that is contained in $V \setminus (H \times S^2 \cup P \times \{\infty\})$. Indeed, we may assume both $H \times S^2$ and $P \times \{\infty\}$ are complex submanifolds, and positivity of intersection implies that only one component will intersect $P \times \{\infty\}$. As for $H \times S^2$, the homological intersection of the curve with $H \times S^2$ is zero, hence the curves in the sequence are either completely contained in $H \times S^2$ or in its complement. In the first case there can be no bubbling off, since the class $\{p_0\} \times S^2$ is J -simple in $H \times S^2$. In the second case, no bubble can intersect $H \times S^2$ by pseudoconvexity of $P \setminus H$. So the question is that of bubbling off in $M \sqcup_{\Sigma} W$. Bubbles contained completely in M or W are already ruled out by asphericity.

Under assumption (b) we have $\omega_0 = d\alpha = \omega$, so that $M \sqcup_{\Sigma} W$ is exact and there can be no bubble.

Under assumption (c), consider a bubble C , and let C_1, C_2 be the parts of the bubble separated by Σ . At least one of the components of C_1 or C_2 is a disc, with boundary in Σ . By assumption, we can cap it by a disc in Σ to get a sphere in M or in W , which by asphericity has zero area. We can thus inductively remove each component of C_1, C_2 and finally prove that C has zero area, a contradiction.

Under assumption (a), we again consider the two pieces of the rational curve separated by Σ . We denote by C_1 the piece in W and by C_2 its complement. Then $\int_{C_1 \cup C_2} \omega = \int_{C_1} \omega + \int_{C_2} \omega$. But $C_1 \in H_2(W, \Sigma)$ so by our assumption there is a cycle Γ in Σ such that $\partial C_1 = \partial \Gamma$. Because $C_1 \cup \Gamma$ is a cycle in $H_2(W)$, and the map $H_2(\Sigma) \rightarrow H_2(W)$ is onto (using again $H_2(W, \Sigma) = 0$), we obtain that $C_1 \cup \Gamma$ is homologous in W to a cycle C_3 contained in Σ . Thus $\int_{C_1} \omega - \int_{\Gamma} \omega = \int_{C_3} \omega$ which vanishes because ω is exact near Σ . Finally $C_2 \cup \Gamma$ is a cycle in M with the same area as $C_1 \cup C_2$, but since (M, ω_0) is exact (Stein) this area is zero, a contradiction.

We have thus proved that $ev : \mathcal{M} \times S^2 \rightarrow V$ has degree one. This implies that the map

$$(ev)_* : H_j(\mathcal{M} \times S^2) \rightarrow H_j(V)$$

is surjective. Denote by i_U , for any subset U of V , the inclusion map of U in V . Consider a class C_1 in $H_j(W)$ and $C = (i_W)_*(C_1) \in H_j(V)$. It is enough to prove that C is also in the image of $(i_Y)_*$. Indeed, the Mayer-Vietoris exact sequence writes

$$\rightarrow H_j(\Sigma) \xrightarrow{(i_\Sigma^W)_* \oplus (i_\Sigma^Y)_*} H_j(W) \oplus H_j(Y) \xrightarrow{(i_W)_* - (i_Y)_*} H_j(V) \rightarrow H_{j-1}(\Sigma) \rightarrow$$

where i_Σ^W (resp. i_Σ^Y) denotes the inclusion of Σ in W (resp. Y). If there exists C_2 in $H_j(Y)$ such that $C = (i_Y)_*(C_2)$, then (C_1, C_2) is in the kernel of $(i_W)_* - (i_Y)_*$, hence in the image of $(i_\Sigma^W)_* \oplus (i_\Sigma^Y)_*$. Thus $C_1 \in \text{Im}(i_\Sigma^W)_*$, and $H_j(\Sigma) \rightarrow H_j(W)$ is surjective.

We now prove that C lies in the image of $(i_Y)_*$. By surjectivity of the map $(ev)_* : H_j(\mathcal{M} \times S^2) \rightarrow H_j(V)$, the class C is homologous to the image of some Γ_C in

$$H_j(\mathcal{M} \times S^2) \simeq H_j(\mathcal{M}) \otimes H_0(S^2) \oplus H_{j-2}(\mathcal{M}) \otimes H_2(S^2).$$

If $\Gamma_C = A \otimes \{pt\} + B \otimes [S^2]$, where $A \in H_j(\mathcal{M})$ and $B \in H_{j-2}(\mathcal{M})$, we claim that B must vanish. Arguing by contradiction, let B' be Poincaré dual to B in $H_*(\mathcal{M})$, then

$$(B \otimes [S^2]) \cdot (B' \otimes \{pt\}) = (B \cdot B') \otimes \{pt\} = \{pt\} \otimes \{pt\}.$$

Thus $\Gamma_C \cdot (B' \otimes \{pt\}) = \{pt\} \otimes \{pt\}$. This implies that

$$\{pt\} = (ev)_*(\Gamma_C \cdot (B' \otimes \{pt\})) = C \cdot (ev)_*(B' \otimes \{pt\}) = C \cdot ev^\infty(B'),$$

where $ev^z(u) = u(z)$. Since $ev^\infty(B') \subset P \times \{\infty\}$, we get $C \cdot ev^\infty(B') = 0$, a contradiction.

As a result, we obtain $C = ev^z(A)$ (for any $z \in S^2$), with $A \in H_j(\mathcal{M})$. Choosing $z = \infty$ we get that $C \in (i_{P \times \{\infty\}})_*(H_j(P)) \subset (i_Y)_*(H_j(Y))$. This concludes the proof. \square

Remark 2.9. The above proof still works provided we have compactness of the set of holomorphic curves in $(P \setminus H) \times S^2$ homologous to $\{pt\} \times S^2$. For example this will hold if $[\omega]\pi_2(P \setminus H) = a\mathbb{Z}$ and $a > 4\pi = \int_{S^2} \sigma$, since the class $\{pt\} \times S^2$ is then J -simple (see [McDuff] p.659, 3.1).

Remark 2.10. If the image of the boundary map $H_{j+1}(Y, \Sigma) \rightarrow H_j(\Sigma)$ coincides with the image of the boundary map $H_{j+1}(W, \Sigma) \rightarrow H_j(\Sigma)$, then $\dim H_j(W) \leq \dim H_j(P)$. Indeed, it follows from the commutative diagram below that the map $H_j(W) \rightarrow H_j(V)$ is injective. Since its image is contained in $\text{Im}(i_{P \times \{\infty\}})_*$, the conclusion follows.

$$\begin{array}{ccccc}
 H_{j+1}(V, W) & \longrightarrow & H_j(W) & \longrightarrow & H_j(V) \\
 \simeq \uparrow \text{excision} & & \uparrow & & \\
 H_{j+1}(Y, \Sigma) & \xrightarrow{\partial} & H_j(\Sigma) & \longrightarrow & H_j(Y) \\
 & & \uparrow \partial & & \\
 & & H_{j+1}(W, \Sigma) & &
 \end{array}$$

Proof of theorem 2.5. We use a result of Cieliebak (see [Cieliebak1]) stating that a Stein subcritical manifold is symplectomorphic to $N \times \mathbb{C}$ where N is Stein, and a result of Lisca and Matič ([Lisca-Matič], section 3, theorem 3.2), stating that any Stein manifold embeds symplectically in a smooth projective manifold, P , with ample canonical bundle. Moreover N is contained in the complement of a hyperplane section, H . We can thus assume that we have embeddings $\Sigma \subset N \times D^2 \subset P \times S^2$ such that N and P carry symplectic forms ω_N, ω_P for which the second embedding is symplectic, and of course the image of Σ is contained in the complement of $P \times \{\infty\} \cup H \times S^2$. We may now apply proposition 2.8 and this concludes our proof. \square

Remark 2.11. Of course the condition that $H_j(\Sigma) \rightarrow H_j(W)$ is onto is equivalent to the claim that $H^j(W) \rightarrow H^j(\Sigma)$ is injective, or that $H_j(W) \rightarrow H_j(W, \Sigma)$ vanishes, etc.

The case when Σ is a sphere leads to the following variant of the Eliashberg-Floer-McDuff theorem ([McDuff]): the assumptions that we impose are weaker, but so is the conclusion.

Corollary 2.12. *Let (Σ, ξ) be a simply connected contact manifold admitting an embedding in a Stein subcritical manifold, and assume that Σ is a homology sphere (resp. rational homology sphere). Then if it admits a filling, W is a homology ball (resp. rational homology ball).*

Proof. Indeed apply theorem 2.5 to the case where $H_j(\Sigma) = 0$. We conclude that $H_j(W) = 0$. \square

Examples are given by Brieskorn spheres (see corollary 6.2). Note that if (Σ, α) is the standard contact sphere, it has an obvious embedding in \mathbb{R}^{2n} . In this situation, using an argument by Eliashberg, it is proved in [McDuff] that W is simply connected. Thus we get, using Smale's h -cobordism theorem ([Smale]) that W is diffeomorphic to the ball. This is the original Eliashberg-Floer-McDuff theorem.

Remark 2.13. Here is a more precise statement. Remember that in the proof of our theorem 2.5, we showed that the image of $(i_W)_*$ is contained in the image of $(i_P)_* := (i_{P \times \{\infty\}})_*$ in $H_*(V)$. Now the following commutative diagram

$$\begin{array}{ccccc} H_j(W) & \xrightarrow{(i_W)_*} & H_j(V) & \xleftarrow{(i_P)_*} & H_j(P) \\ \uparrow & & \uparrow & & \parallel \\ H_j(\Sigma) & \xrightarrow{(i_\Sigma)_*} & H_j(Y) & \xleftarrow{(i_P)_*} & H_j(P) \end{array}$$

shows that

$$\dim((i_P)_*H_j(P)/(i_W)_*H_j(W)) \geq \dim((i_P)_*(H_j(P)/(i_\Sigma)_*H_j(\Sigma))).$$

3 The case of $(\mathbb{R}^{2n}, \sigma_0)$

In this section we denote $b_p(X)$ the Betti numbers of a manifold X with coefficients in a given field. Thus $b_p(X)$ is the rank of the p -th homology/cohomology group. We denote $B(X)$ the total Betti number of X , that is $B(X) = \sum_{j=0}^{\dim X} b_j(X)$.

It is convenient in this section to slightly change the point of view and use the following definitions. Let (Σ, ξ) be a contact manifold and α a contact form for ξ .

Definition 3.1. A *contact embedding* of (Σ, α) in (W, ω) is a codimension 1 embedding such that α extends to a neighbourhood of Σ as a primitive of ω . We shall say that the embedding is an *exact contact embedding* if α extends to the whole of W as a primitive of ω .

We call (Σ, α) an (*exact*) *contact hypersurface* of (W, ω) .

Definition 3.2. A *symplectic filling* of (Σ, α) is a symplectic manifold (W, ω) without closed components, such that $\partial W = \Sigma$ and such that α extends to a neighbourhood of Σ as a primitive of ω . We shall say that the symplectic filling is an *exact symplectic filling* if α extends to the whole of W as a primitive of ω .

Theorem 3.3. *Let (Σ, α) be a contact hypersurface in $(\mathbb{R}^{2n}, \sigma_0)$, and let W_0 be the bounded component of $\mathbb{R}^{2n} \setminus \Sigma$. Let W be a symplectic filling of (Σ, α) such that $(\mathbb{R}^{2n} \setminus W_0) \sqcup W$ is aspherical. This holds in particular if W is aspherical and one of the following properties is satisfied*

(a). $H_2(W, \Sigma) = 0$.

(b). (Σ, α) is of restricted contact type in (W, ω) and in $(\mathbb{R}^{2n}, \sigma_0)$.

(c). The maps $\pi_1(\Sigma) \longrightarrow \pi_1(W)$ and $\pi_1(\Sigma) \longrightarrow \pi_1(\mathbb{R}^{2n} \setminus W_0)$ are injective.

Then

(1) any two aspherical symplectic fillings of (Σ, α) which satisfy either of the conditions (a)–(b) have the same cohomology.

(2) given an aspherical symplectic filling W which satisfies one of the conditions (a)–(b), the inclusion of Σ in W induces an injection in cohomology

$$H^p(W) \longrightarrow H^p(\Sigma).$$

Moreover, we have

$$b_p(\Sigma) = b_p(W) + b_{2n-p-1}(W).$$

Remark 3.4. Note that again, (a) holds if W is Stein and $n \geq 3$, while (c) holds if Σ is simply connected.

Proof. Of course the first statement in (2) follows from the previous section, but we shall give an alternative proof based on the Eliashberg-Floer-McDuff theorem. The main point is that any of the conditions (a)–(b) guarantees that $W \sqcup (\mathbb{R}^{2n} \setminus W_0)$ is symplectically aspherical, hence diffeomorphic to \mathbb{R}^{2n} . Equivalently, since Σ is contained in some large ball, B , then $W \sqcup (B - W_0)$ is diffeomorphic to B .

Thus, the cohomology Mayer-Vietoris exact sequence may be written

$$\longrightarrow H^p(B) \longrightarrow H^p(W) \oplus H^p(B - W_0) \longrightarrow H^p(\Sigma) \longrightarrow H^{p+1}(B) \longrightarrow$$

Since $H^p(B) = 0$ for $p > 0$, we see that the map

$$H^p(W) \oplus H^p(B - W_0) \longrightarrow H^p(\Sigma)$$

is an isomorphism for $p \geq 1$. Since it is induced by the inclusion maps, the first claim follows.

Moreover, for $p > 0$ we have

$$b_p(\Sigma) = b_p(W) + b_p(B - W_0)$$

Since according to Alexander duality (see [Greenberg-Harper], theorem 27.5, p.233) for $2n - 1 > p > 0$ we have

$$b_p(B - W_0) = b_{2n-p-1}(W_0)$$

this last equality implies that

$$b_p(\Sigma) = b_p(W) + b_{2n-p-1}(W_0)$$

But of course this also holds when we replace W by W_0 , so that

$$b_p(\Sigma) - b_p(W_0) = b_{2n-p-1}(W_0)$$

hence

$$b_p(\Sigma) = b_p(W) + (b_p(\Sigma) - b_p(W_0))$$

This implies $b_p(W) = b_p(W_0)$ for $0 < p < 2n - 1$.

For $p = 2n - 1$, if $B(\varepsilon)$ is a very small ball inside W_0 , the inclusions

$$B \setminus B(\varepsilon) \supset B - W_0 \supset S^{2n-1}$$

imply that $b_{2n-1}(B - W_0) \geq 1$, and the exact sequence

$$0 \longrightarrow H^{2n-1}(W) \oplus H^{2n-1}(B - W_0) \longrightarrow H^{2n-1}(\Sigma) \longrightarrow 0$$

implies that $b_{2n-1}(B - W_0) = 1$ and $b_{2n-1}(W_0) = 0$. Finally it is easy to check that the equality still holds for $p = 0$, since $b_0(\Sigma) = b_0(W) = 1$ while $b_{2n-1}(W) = b_{2n-1}(\Sigma) - b_0(W) = 0$ (using the equality for $p = 2n - 1$). \square

Corollary 3.5. *Assume (Σ, ξ) has a Stein symplectic filling (W, ω) and has a contact embedding in $(\mathbb{R}^{2n}, \sigma_0)$. Then*

$$\begin{cases} b_p(\Sigma) = b_p(W) \text{ for } 0 \leq p \leq n - 2 \\ b_{n-1}(\Sigma) = b_n(\Sigma) = b_n(W) + b_{n-1}(W) \end{cases}$$

Thus the homology of W is completely determined by the homology of Σ except, maybe, for degree $n-1$. It is completely determined by the homology of Σ if $b_n(\Sigma) = 0$ or W is Stein subcritical.

Proof. We see from theorem 3.3 that $b_p(W)$ is determined by $b_p(\Sigma)$, except maybe in dimension $n, n-1$. But if $b_n(\Sigma) = 0$ this implies $b_n(W) = b_{n-1}(W) = 0$ and if W is subcritical, $b_n(W) = 0$ hence $b_{n-1}(\Sigma) = b_{n-1}(W)$. \square

Remark 3.6. Mei-Lin Yau proved (see [M.-L.Yau]) that if W is Stein subcritical, and the first Chern class of the complex vector bundle defined by ξ , $c_1(\xi)$, vanishes, then

$$HC_*(\Sigma, \alpha) \simeq H_*(W, \Sigma) \otimes H_*(\mathbb{C}P^\infty)$$

so that it is a general fact that the homology of a subcritical filling is determined by the contact structure (Σ, ξ) . It is however not clear whether in general it is already determined by the knowledge of the topology of Σ (i.e. independently from ξ or the topology of a filling).

As a first consequence of corollary 3.5 and Mei-Lin Yau's result we have:

Corollary 3.7. *Assume (Σ, ξ) satisfies $c_1(\xi) = 0$, has a Stein subcritical filling (W, ω) and has a contact embedding in $(\mathbb{R}^{2n}, \sigma_0)$. Then the rank of $HC_*(\Sigma, \alpha)$ is determined by $H_*(\Sigma)$. Indeed, we have*

$$\text{rank}(HC_k(\Sigma, \alpha)) = \sum_{\substack{2n-2-k \leq p \leq n-1 \\ p \equiv k \pmod{2}}} b_p(\Sigma)$$

Proof. Note that assumption (b) from theorem 3.3 is automatically satisfied: we are in the Stein case. The result is a straightforward application of the corollary, Mei-Lin Yau's theorem and the duality $H^{2n-k}(W) \simeq H_k(W, \partial W)$.

Thus

$$HC_k(\Sigma, \xi) = \bigoplus_{m \geq 0} H_{k-2m+2}(W, \Sigma) = \bigoplus_{m \geq 0} H^{2n-2-k+2m}(W)$$

and $b_{2n-2-k+2m}(W) = b_{2n-2-k+2m}(\Sigma)$ for $0 \leq 2n-2-k+2m \leq n-1$. Setting $p = 2n-2-k+2m$ yields the above formula. \square

As another application of our theorem, we can prove

Proposition 3.8. *Let Σ be the boundary of the disc bundle W associated to a complex line bundle over a symplectic manifold (N^{2n-2}, ω) with $[\omega]\pi_2(N) = 0$ and Chern class equal to minus the symplectic form, the contact structure being given by the kernel of the connection form α . Then (Σ, ξ) has no contact embedding in $(\mathbb{R}^{2n}, \sigma_0)$ with interior Z , such that $(\mathbb{R}^{2n} \setminus Z) \sqcup W$ is aspherical. The same holds for $n \geq 3$ and for any contact manifold obtained by contact surgery (as in [Eliashberg, Weinstein]) of index k for any $k \in [3, n]$.*

Proof. For details about the fact that Σ has a contact form appearing as the boundary of W , we refer to [Oancea], section 3.3.

The Gysin exact sequence can be written as

$$\longrightarrow H^{p-2}(N) \xrightarrow{\omega \cup} H^p(N) \longrightarrow H^p(\Sigma) \longrightarrow H^{p-1}(N) \longrightarrow$$

In degree 2, we get

$$H^2(\Sigma) = H^2(N)/\langle [\omega] \rangle \oplus \ker([\omega] \cup : H^1(N) \longrightarrow H^3(N))$$

hence

$$b_2(\Sigma) < b_2(N) + b_1(N) = b_2(N) + b_{2n-2-1}(N)$$

and this contradicts theorem 3.3.

Let us now see what happens when we make a contact surgery. We shall denote our hypersurface by Σ^- , W^- will denote its filling, and Σ^+ the result of the surgery on Σ^- along a $k-1$ -dimensional isotropic sphere. Let us denote by $A_k \simeq D^k \times D^{n-k}$ the attached handle, and denote $\partial^- A_k = S^{k-1} \times D^{2n-k}$, $\partial^+ A_k = D^k \times S^{2n-k-1}$, so that the new filling of Σ^+ is $W^+ = W^- \cup_{\partial^- A_k} A_k$. We first need to prove that W^+ is aspherical. But the homotopy exact sequence of the pair (W^+, W^-) is given by

$$\longrightarrow \pi_3(W^+, W^-) \longrightarrow \pi_2(W^-) \longrightarrow \pi_2(W^+) \longrightarrow \pi_2(W^+, W^-) \longrightarrow$$

and $\pi_2(W^+, W^-) \simeq \pi_2(A_k, \partial^- A_k) \simeq \pi_2(D^k, \partial D^k) = 0$ for $k \geq 3$. Thus the inclusion of W^- in W^+ induces a surjective map on π_2 , hence if $[\omega] \pi_2(W^-) = 0$, we also have $[\omega] \pi_2(W^+) = 0$.

Let us now first consider the case $k \geq 4$. We claim that we have $b_2(\Sigma^+) = b_2(\Sigma^-)$ and $b_2(W^+) = b_2(W^-)$. Indeed the homology exact sequence for the pair (W^+, W^-) writes

$$\longrightarrow H_3(W^+, W^-) \longrightarrow H_2(W^-) \longrightarrow H_2(W^+) \longrightarrow H_2(W^+, W^-) \longrightarrow$$

but $H_j(W^+, W^-) \simeq H_j(A_k, \partial^- A_k) \simeq H_j(D^k, \partial D^k) = 0$ for $j = 2, 3$ and $k \geq 4$, so $b_2(W^+) = b_2(W^-)$.

Similarly the Mayer-Vietoris exact sequence for $\Sigma^\pm = \Sigma^- \setminus (\partial^- A_k) \cup \partial^\pm A_k$ reads

$$(3.1) \quad \begin{aligned} H_2(S^{k-1} \times S^{2n-k-1}) &\longrightarrow H_2(\Sigma \setminus \partial^- A_k) \oplus H_2(\partial^\pm A_k) \longrightarrow \\ H_2(\Sigma^\pm) &\longrightarrow H_1(S^{k-1} \times S^{2n-k-1}) = 0 \end{aligned}$$

When $k \geq 4$, the groups $H_2(S^{k-1} \times S^{2n-k-1})$ and $H_2(\partial^\pm A_k)$ vanish, so that we have isomorphisms

$$H_2(\Sigma \setminus \partial^- A_k) \simeq H_2(\Sigma^\pm)$$

and therefore $b_2(\Sigma^+) = b_2(\Sigma^-)$.

Let us now deal with the case $k = 3, n \geq 4$.

In case “−”, the first map in the exact sequence 3.1 is injective (since its projection on the second summand is induced by the inclusion $S^2 \times S^{2n-4} \rightarrow S^2 \times D^{2n-3}$) and since $2n - 4 > 2$, $b_2(\Sigma^-) = b_2(\Sigma^- \setminus \partial^- A_3)$. In the “+” case, $H_2(\partial^+ A_3) = 0$ and $b_2(\Sigma^+) \leq b_2(\Sigma \setminus \partial^- A_3) = b_2(\Sigma^-)$.

We write the homology exact sequences of the pairs (W^+, W^-) and $(\Sigma^+, \Sigma^+ \cap W^-)$

$$\begin{array}{ccccccc}
H_3(W^+, W^-) & \xrightarrow{\partial_W} & H_2(W^-) & \longrightarrow & H_2(W^+) & \longrightarrow & H_2(W^+, W^-) = 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
H_3(\Sigma^+, \Sigma^+ \cap W^-) & \xrightarrow{\partial_\Sigma} & H_2(\Sigma^+ \cap W^-) & \longrightarrow & H_2(\Sigma^+) & \longrightarrow & H_2(\Sigma^+, \Sigma^+ \cap W^-) = 0 \\
& & \parallel & & & & \\
& & H_2(\Sigma^- \setminus \partial^- A_3) \simeq H_2(\Sigma^-) & & & &
\end{array}$$

The left hand side vertical map is an isomorphism since

$$\begin{aligned}
H_3(\Sigma^+, \Sigma^+ \cap W^-) &\simeq H_3(D^3 \times S^{2n-4}, S^2 \times S^{2n-4}) \xrightarrow{\simeq} \\
&H_3(D^3 \times D^{2n-3}, S^2 \times D^{2n-3}) \simeq H_3(W^+, W^-)
\end{aligned}$$

Therefore either the ∂_W is injective, and thus so is ∂_Σ and consequently $b_2(W^+) = b_2(W^-) - 1$ and $b_2(\Sigma^+) = b_2(\Sigma^-) - 1$, or it is zero, and then $b_2(W^+) = b_2(W^-)$ and $b_2(\Sigma^+) \leq b_2(\Sigma^-)$.

For $n = k = 3$, we leave it to the reader to check that

$$H_2(\Sigma^-) \simeq H_2(\Sigma \setminus \partial^- A_3) / \text{Im}(H_2(\{pt\} \times S^2))$$

$$H_2(\Sigma^+) \simeq H_2(\Sigma \setminus \partial^- A_3) / \text{Im}(H_2(S^2 \times S^2))$$

so that $b_2(\Sigma^+)$ equals either $b_2(\Sigma^-)$ or $b_2(\Sigma^-) - 1$. Again using the same argument as above, whenever $b_2(W^+) = b_2(W^-) - 1$ we have $b_2(\Sigma^+) = b_2(\Sigma^-) - 1$. This concludes our proof. \square

Remark 3.9. According to [Laudenbach], if (Σ, ξ) has a contact embedding in \mathbb{R}^{2n} , the same holds for any manifold obtained by contact surgery over an isotropic sphere of dimension $\leq n - 1$. In contrast we display here an obstruction to embedding Σ in \mathbb{R}^{2n} that survives such a surgery.

Examples 3.10. The asphericity condition is really necessary: $(\mathbb{C}P^n, \sigma_0)$ does not satisfy the asphericity condition, and indeed (S^{2n-1}, α_0) has an embedding into $(\mathbb{R}^{2n}, \sigma_0)$.

This generalizes to higher dimensional bundles as follows

Proposition 3.11. *Let Σ be the boundary of a negative rank r vector bundle ($r \geq 2$) W over a symplectic manifold (N^{2n-2r}, ω) with $[\omega]\pi_2(N) = 0$, and its canonical contact structure ξ . Then (Σ, ξ) has no contact embedding in $(\mathbb{R}^{2n}, \sigma_0)$. The same holds for $n \geq 2r + 1$ and for any contact manifold obtained by contact surgery of index k for any $k \in [2r + 1, n]$.*

Proof. For details on the contact structure on Σ , we refer to [Oancea], section 3.4. Let as usual Z denote the interior of Σ in \mathbb{R}^{2n} . The asphericity assumption on $(\mathbb{R}^{2n} \setminus Z) \sqcup W$ is automatically satisfied here, since a 2-sphere in $(\mathbb{R}^{2n} \setminus Z) \sqcup W$ generically avoids N , hence deforms into a 2-sphere contained in $(\mathbb{R}^{2n} \setminus Z)$, where ω vanishes on spheres.

Again the Gysin exact sequence reads

$$\longrightarrow H^{p-2r}(N) \xrightarrow{e \cup} H^p(N) \longrightarrow H^p(\Sigma) \longrightarrow H^{p-2r+1}(N) \xrightarrow{e \cup}$$

where e is the Euler class of W . Hence

$$H^{2r}(\Sigma) = H^{2r}(N) / \langle e \rangle \oplus \ker(e \cup : H^1(N) \longrightarrow H^{2r+1}(N))$$

and negativity implies that e is nonzero, so

$$b_{2r}(\Sigma) < b_{2r}(N) + b_1(N) = b_{2r}(W) + b_{2n-2r-1}(W).$$

This contradicts Theorem 3.3.

The case of surgery is treated as in the previous proposition, details are left to the reader. \square

Proposition 3.12. *Let L be a compact manifold admitting a Lagrangian embedding into \mathbb{R}^{2n} and $n \geq 3$. Then any symplectic filling of ST^*L has the same homology as DT^*L (and hence the homology of L).*

Proof. Indeed, the hypothesis implies that ST^*L has a contact (non exact !) embedding into \mathbb{R}^{2n} , so that we can apply theorem 3.3. The condition $H_2(DT^*L, ST^*L)$ is clearly satisfied using Thom's isomorphism. \square

Let now ST^*L be the unit cotangent bundle of L . Then the spectral sequence of this sphere bundle yields the following dichotomy:

- either the Euler class vanishes, and then

$$b_p(ST^*L) = b_p(L) + b_{p-(n-1)}(L)$$

- or the Euler class is non zero and then

$$\begin{cases} b_p(ST^*L) = b_p(L) + b_{p-(n-1)}(L) \text{ for } p \neq n-1, n \\ b_n(ST^*L) = b_{n-1}(ST^*L) = b_{n-1}(L) = b_1(L) \end{cases}$$

The formula

$$b_p(\Sigma) = b_p(W) + b_{2n-p-1}(W)$$

becomes

- (a). in the first case

$$b_p(L) + b_{p-(n-1)}(L) = b_p(L) + b_{2n-p-1}(L)$$

hence

$$b_{p-(n-1)}(L) = b_{2n-p-1}(L)$$

that is the Poincaré duality formula

- (b). in the second case

$$b_1(L) = b_{n-1}(L) = b_n(L) + b_{2n-n-1}(L) = b_n(L) + b_{n-1}(L)$$

This implies $b_n(L) = 0$, which is impossible (at least for orientable L).

Proposition 3.13. *Let L be an orientable manifold with non zero Euler class. Then ST^*L has no contact embedding in \mathbb{R}^{2n} . The same holds for any contact manifold obtained from such a ST^*L by surgery of index $3 \leq k \leq n-3$.*

Proof. The case of ST^*L has been already proved above. The surgery does not modify the conditions $H_2(W, \Sigma) = 0$ nor does it change $b_n(\Sigma)$ or $b_n(W), b_{n-1}(W)$. This concludes our proof. \square

Remark 3.14. The condition $e(L) = 0$ is exactly the condition needed to be able to find a Lagrangian immersion of L regularly homotopic to an embedding. We however suspect that there are no embeddings of ST^*L as a smooth hypersurface in \mathbb{R}^{2n} .

4 The Stein subcritical case

In this section we assume that (Σ_1, ξ_1) has a separating contact embedding in a Stein subcritical domain (W_2, ω_2) with boundary (Σ_2, ξ_2) , and we denote by V_1 the bounded component of $W_2 \setminus \Sigma_1$. We denote by (W_1, ω_1) an arbitrary aspherical symplectic filling of (Σ_1, ξ_1) such that $(W_2 \setminus V_1) \sqcup_{\Sigma_1} W_1$ is aspherical, which holds for example under one of the assumptions (b), (c), (a) of theorem 2.5.

Proposition 4.1. *Under the above assumptions, we have that*

$$b_j(W_1) \leq b_j(\Sigma_1) + \min(0, b_j(\Sigma_2) - b_j(W_2 \setminus V_1))$$

Proof. Note that given an exact sequence $A \xrightarrow{f} B \xrightarrow{g} C$ we have $\dim(B) = \dim(\ker(g)) + \dim(\text{Im}(g)) = \dim(\text{Im } f) + \dim(\text{Im}(g)) \leq \dim(A) + \dim(C)$. Using the Mayer-Vietoris exact sequence of $(W_2 \setminus V_1) \sqcup_{\Sigma_1} W_1$ and the inequality $\dim H_j(\Sigma_2) \geq \dim H_j((W_2 \setminus V_1) \sqcup_{\Sigma_1} W_1)$ proved in the previous section, we get that

$$b_j(W_2 \setminus V_1) + b_j(W_1) \leq b_j(\Sigma_2) + b_j(\Sigma_1).$$

Thus

$$b_j(W_1) \leq (b_j(\Sigma_2) - b_j(W_2 \setminus V_1)) + b_j(\Sigma_1).$$

Since according to theorem 2.5, $b_j(W_1) \leq b_j(\Sigma_1)$, our claim follows. \square

Note that $b_j(W_2 \setminus V_1) = b_{2n-j}(W_2, V_1 \cup \Sigma_2)$ by Poincaré duality and excision. Note also that the above result is stronger than Proposition 2.5 only when $b_j(\Sigma_2) - b_j(W_2 \setminus V_1) < 0$. This happens for example if Σ_2 is a homology sphere.

Proposition 4.2. *Let (M, ω) be a Stein subcritical manifold. Let Σ be the boundary of the disc bundle W associated to a complex line bundle over a symplectic manifold (N^{2n-2}, ω) with $[\omega]\pi_2(N) = 0$, Chern class equal to minus the symplectic form, the contact structure being given by the kernel of the connection form α . Assume moreover that $[\omega] \cup : H^1(N) \rightarrow H^3(N)$ is injective (this holds if N is projective complex manifold by the Hard Lefschetz theorem). Then (Σ, ξ) has no contact embedding in (M, ω) with interior Z , such that $(M \setminus Z) \sqcup W$ is aspherical. The same holds for $n \geq 3$ and for any contact manifold obtained by contact surgery (as in [Eliashberg, Weinstein]) of index k for any $k \in [3, n]$.*

Proof. The proof is the same as in the case of euclidean space, with the exception that our extra assumption implies using the Gysin exact sequence,

$$H^2(\Sigma) = H^2(N) / \langle [\omega] \rangle$$

(the term $\ker([\omega] \cup : H^1(N) \rightarrow H^3(N))$ vanishes). Hence

$$b_2(\Sigma) < b_2(N) = b_2(W)$$

and this contradicts theorem 2.5. The case of surgery is dealt with as in proposition 3.8. \square

There is an analogous theorem, with fewer assumptions, but which only states that Σ does not bound a subcritical Stein manifold (cf. Prop. 5.14).

Proposition 4.2 generalizes to higher dimensional bundles as in the euclidean case:

Proposition 4.3. *Let Σ be the boundary of a negative rank r vector bundle ($r \geq 2$) W over a symplectic manifold (N^{2n-2r}, ω) with $[\omega] \pi_2(N) = 0$, and its canonical contact structure ξ . Assume the Euler class of W induces an injective map $e \cup : H^1(N) \rightarrow H^{2r+1}(N)$. Then (Σ, ξ) has no contact embedding in a Stein subcritical manifold. The same holds for $n \geq 2r + 1$ and for any contact manifold obtained by contact surgery of index k for any $k \in [2r + 1, n]$.*

Proof. Again the Gysin exact sequence reads

$$\rightarrow H^{p-2r}(N) \xrightarrow{e \cup} H^p(N) \rightarrow H^p(\Sigma) \rightarrow H^{p-2r+1}(N) \xrightarrow{e \cup}$$

hence

$$H^{2r}(\Sigma) = H^{2r}(N) / \langle e \rangle$$

so

$$b_{2r}(\Sigma) < b_{2r}(N)$$

We check as in proposition 3.11 that the assumptions of Theorem 2.5 are satisfied, and thus get a contradiction. The case of surgery is dealt with as in the case of proposition 3.11. \square

The first part of the following result has been obtained by different methods in [Cieliebak-Frauenfelder-Oancea] (see also proposition 5.12).

Proposition 4.4. *Let L be an orientable closed manifold, with non-zero Euler class of dimension ≥ 3 . Then ST^*L has no contact embedding in a subcritical Stein manifold. As before this also holds for any manifold obtained from ST^*L by contact surgery of index $k \in [3, n - 1]$.*

Proof. Since $n \geq 3$, the group $H_2(DT^*L, ST^*L)$ is zero, so assumption (a) of theorem 2.5 is satisfied. The Gysin exact sequence of ST^*L shows that the map $H_n(ST^*L) \rightarrow H_n(L)$ vanishes. This contradicts Theorem 2.5. The case of manifolds obtained by surgery is dealt with as in proposition 3.8. \square

5 Obstructions from Symplectic homology

In this section we assume (Σ, ξ) is a contact manifold and denoting by c_1 the first Chern class, that $c_1(\xi) = 0$. All the symplectic fillings (W, ω) of (Σ, ξ) that we consider shall satisfy $c_1(TW) = 0$.

Definition 5.1. Let (W, ω) be symplectic with contact type boundary. We say that (W, ω) is *weakly subcritical* if $SH_*(W, \omega) = 0$.

Remark 5.2. This is of course a strengthening of *Algebraic Weinstein conjecture* defined in [Viterbo]. Note that a weakly subcritical also has the *Equivariant Algebraic Weinstein conjecture* property. This can be seen using the spectral sequence connecting the usual version of symplectic homology to the equivariant version [Viterbo], or using the Gysin long exact sequence [Bourgeois-Oancea-2].

If we have an exact embedding (V_1, ω_1) into (W_1, ω_1) , there is an induced transfer map (see [Viterbo])

$$SH_*(W_1) \longrightarrow SH_*(V_1)$$

which according to Mark McLean (see [McLean]) is a unital ring homomorphism. This implies the following result:

Proposition 5.3 ([McLean]). *Let (V, ω) be an exact symplectic submanifold of (W, ω) . If (W, ω) is weakly subcritical then (V, ω) is also weakly subcritical.*

It is easy to find non-Stein weakly subcritical manifolds. For example we have

Proposition 5.4 ([Oancea]). *Let P be any exact symplectic manifold with contact type boundary. Then, for any exact weakly subcritical manifold W , we have that $P \times W$ is weakly subcritical. Also the total space of a symplectic fibration in the sense of [Oancea] with fiber W is weakly subcritical.*

Proposition 5.5 ([Cieliebak2]). *Let W' be obtained from W by attaching handles of index $\leq n - 1$. Then $SH_*(W) \simeq SH_*(W')$. In particular if W is weakly subcritical so is W' .*

The next statement is contained in [Cieliebak-Frauenfelder-Oancea], Corollary 1.15 and Remark 1.19.

Theorem 5.6. *Let (Σ, ξ) be a contact manifold for which there exists a contact form α with no closed characteristics of Conley-Zehnder index less than or equal to $3 - n$. Let $i : (\Sigma, \alpha) \hookrightarrow (W, \omega)$ be a separating exact embedding in a weakly subcritical manifold (W, ω) . Assume $i_* : \pi_1(\Sigma) \rightarrow \pi_1(W)$ is injective. Then the Betti numbers of the interior V of Σ in any coefficient field are determined by the contact structure ξ (and do not depend on the choice of the weakly subcritical W).*

Proof. By Proposition 5.3 we have $SH_*(V) = 0$. The relative exact sequence in Floer homology (see [Viterbo]) then implies that

$$SH_*^+(V) \simeq H_{*+n-1}(V, \partial V)$$

On the other hand, it was proved in [Bourgeois-Oancea-1] that we have a long exact sequence

$$(5.1) \quad \rightarrow SH_*^+(V) \rightarrow HC_*^{lin}(\partial V) \xrightarrow{\Delta} HC_{*-2}^{lin}(\partial V) \rightarrow SH_{*-1}^+(V) \rightarrow$$

If the Reeb vector field associated to the contact form α has no closed characteristic with Conley-Zehnder index $\leq 3 - n$, and if $i_* : \pi_1(\Sigma) \rightarrow \pi_1(V)$ is injective, then there is no rigid holomorphic plane in V bounding a closed characteristic. Thus $HC_*^{lin}(\partial V)$ and Δ only depend on the boundary (see [Bourgeois-Oancea-1]), and so does $SH_*^+(V)$. As a consequence, the Betti numbers $b_j(V, \partial V)$ only depend on ξ .

Now if we have another exact embedding of Σ in W' and W' is also weakly subcritical, the interior V' of Σ in W' must have the same cohomology as V . \square

Proposition 5.7. *Let (Σ, ξ) be the boundary of a Stein subcritical manifold (W, ω) . Let (M, ω) be a weakly subcritical manifold, such that (Σ, ξ) has an exact separating embedding into (M, ω) with interior Z . Then $H_*(Z) \simeq H_*(W)$.*

Proof. First of all, by proposition 5.3, we have $SH_*(Z) = 0$. On one hand the exact sequence ([Viterbo])

$$H_{*+n}(Z, \Sigma) \longrightarrow SH_*(Z) \longrightarrow SH_*^+(Z) \longrightarrow H_{*+n-1}(Z, \Sigma) \longrightarrow$$

shows that $H_*(Z, \Sigma) \simeq SH_{*+1-n}^+(Z)$. On the other hand, since Σ bounds a subcritical Stein manifold, there exists a contact form, α such that the Reeb orbits are all non-degenerate and of index $> 3 - n$ (cf. [M.-L.Yau]), so the stretch of the neck argument in [Bourgeois-Oancea-1] shows that the map Δ appearing in the exact sequence (5.1) depends only on the boundary. This implies that $SH_*^+(Z) \simeq SH_*^+(W)$. This last space is in turn isomorphic to $H_{*+n-1}(Z, \Sigma)$ by the same argument, and finally $H_*(Z, \Sigma) \simeq H_*(W, \Sigma)$, hence $H_*(Z) \simeq H_*(W)$. \square

Remark 5.8. The condition that W is subcritical is not really necessary. We only need W to be weakly subcritical provided there is a contact form defining ξ such that there is no Reeb orbit on (Σ, α) with index $\leq 3 - n$.

Remark 5.9. This can be compared to the following theorem of Mei-Lin Yau:

Corollary 5.10. ([M.-L.Yau]) *Let W be a subcritical Stein manifold with boundary ∂W such that $c_1(TW)|_{\pi_2(W)} = 0$. Then any subcritical Stein manifold with the same boundary ∂W and whose first Chern class vanishes on π_2 has the same homology as W .*

Proof. This follows from the main computation of [M.-L.Yau]

$$HC_*(\partial W, \alpha) \simeq H_*(W, \partial W) \otimes H_*(\mathbb{C}P^\infty),$$

which implies directly that the homology of a subcritical Stein filling is determined by the contact structure of the boundary. \square

Remark 5.11. Note that when $\Sigma = S^{2n-1}$, we may apply our proposition to $W = D^{2n}$. Thus we prove that any filling $H_*(Z) = 0$ in nonzero degree, so if Z is simply connected, and $n \geq 3$, it is diffeomorphic to a ball. This is a weak version of the Eliashberg-Floer-McDuff theorem mentioned in the previous section, but note that the above proof does not make use of it and also that it extends to many other contact manifolds.

Let us now use the above tools to find obstructions to embeddings. We first have:

Proposition 5.12 ([Cieliebak-Frauenfelder-Oancea]). *If $\Sigma = ST^*L$ where L is closed simply connected manifold, then (Σ, ξ) has no exact embedding in a weakly subcritical (M, ω) .*

Proof. Since the characteristic flow on ST^*L is the geodesic flow, it has all closed trajectories of index $\geq 0 > 3 - n$ if $n > 3$ (in cases $n = 2, 3$ L is a sphere and we can find a metric with geodesics of index $> 3 - n$). Assuming the existence of such an embedding, the map Δ in (5.1) has finite kernel and cokernel. As a result, (Σ, ξ) cannot be filled by a symplectic manifold with infinite dimensional $SH_*^+(W)$, provided this last space depends only on $\partial W = \Sigma$. In our case $SH_*^+(DT^*L) \simeq H_*(\Lambda L, L)$ is infinite-dimensional, so that ST^*L cannot be embedded in any subcritical Stein. \square

Remark 5.13. Let (M, ω) be obtained by attaching subcritical handles to DT^*L . Provided one can prove that the Reeb orbits on $(\partial M, \xi)$ still have index $> 3 - n$, our argument extends to show that $(\partial M, \xi)$ has no contact embedding in a weakly subcritical Stein manifold.

The case of a circle bundle over P can also be dealt with using contact and Floer homology. Indeed we have

Proposition 5.14. *Let (Σ, ξ) be the unit sphere bundle associated to a negative complex vector bundle E of rank r over a symplectic aspherical manifold (N^{2n-2r}, ω) with $c_1(TN) = 0$. Then for $n \geq 2r$, Σ does not bound a Stein subcritical manifold with vanishing first Chern class. The same holds for any contact manifold obtained by subcritical surgery on (Σ, ξ) of index $\neq 2r, 2r + 1$.*

Proof. Indeed, let W denote the manifold bounding Σ . Because if P is the unit disc bundle associated to Σ , we have $\partial P = \Sigma$ and since $SH_*(P) = 0$ ([Oancea]) we get an exact sequence:

$$\longrightarrow SH_*(P) \longrightarrow SH_*^+(\Sigma) \longrightarrow H_{*+n-1}(P, \Sigma) \longrightarrow$$

as a result

$$SH_*^+(\Sigma) \simeq H_{*+n-1}(P, \Sigma) \simeq H_{*+n-2r-1}(N)$$

while the same exact sequence with W yields

$$SH_*^+(\Sigma) \simeq H_{*+n-1}(W, \Sigma) \simeq H^{n-*+1}(W)$$

But this last space vanishes for $* \leq 1$ while $H_{*+n-2r-1}(N)$ is non-zero for $* = 2r - n + 1$. When $n \geq 2r$ we get a contradiction. Now since $k \neq 2r, 2r + 1$, $H_{2r}(P, \Sigma)$ does not change, so remains equal to $H_0(N) = \mathbb{Q}$. But we must have $H_{2r}(P, \Sigma) = SH_{2r-n+1}^+(\Sigma) = H^{2n-2r}(W) = 0$. A contradiction. \square

Remark 5.15. This partially answers a question of Biran in [Biran] who asked the same question in the Stein case (not subcritical). A different answer was given by Popescu-Pampu in [Popescu-Pampu]

6 Brieskorn spheres, McLean's examples

We consider an isolated singularity of holomorphic germ. For example, assume we are given V a complex submanifold in \mathbb{C}^n with an isolated singularity at the origin. We then consider the submanifold $\Sigma_\varepsilon = S_\varepsilon \cap V$ where $S_\varepsilon = \{z \in \mathbb{C}^n \mid |z|^2 = \varepsilon\}$. The maximal complex subspace of the tangent space defines a hyperplane distribution which happens to be a contact structure, and whose isotopy class is independent of ε . In case the singularity is smoothable, Σ_ε bounds a Stein manifold W .

Example 6.1. If V is the hypersurface $f^{-1}(0)$ where f is polynomial, then the singularity is always smoothable. According to [Milnor], the manifold W has no topology in dimension other than n and $H^n(W) = \mathbb{Z}^\mu$. The number μ is called *the multiplicity* of the singularity.

The following is an immediate consequence of proposition 2.5 :

Corollary 6.2. *Let $n \geq 3$ and (Σ, ξ) be a Brieskorn sphere of a non trivial singularity (i.e. with Milnor number ≥ 1). Then Σ does not embed in a subcritical Stein manifold.*

Proof. Indeed, such a Brieskorn sphere is simply connected and it bounds a Stein manifold with $b_n(W) = \mu$. This is impossible since we should have $b_n(W) \leq b_n(\Sigma) = 0$ according to theorem 2.5. \square

Let us now study the manifolds of Mark McLean in [McLean]. These are Stein symplectic manifolds (M_k^{2n}, ω_k) diffeomorphic to \mathbb{R}^{2n} ($n \geq 4$), such that $(\partial M_k^{2n}, \xi_k^{2n})$ is a contact manifold diffeomorphic to S^{2n-1} . However he shows that $SH_n(M_k^n)$ contains N^k idempotent elements for some $N \geq 2$, therefore the manifolds M_k^{2n} are pairwise non symplectomorphic.

We now prove

Proposition 6.3. *The contact manifolds $(\partial M_k^{2n}, \xi_k^{2n})$ are never contactomorphic to the standard sphere.*

Proof. Let us denote for simplicity $W = M_k^{2n}$ and $(\Sigma, \xi) = (\partial M_k^{2n}, \xi_k^{2n})$. The exact sequence in symplectic homology reads

$$\longrightarrow H_{2n}(W, \Sigma) \longrightarrow SH_n(W) \longrightarrow SH_n^+(W) \longrightarrow 0$$

Assume (Σ, ξ) is the standard sphere. Then $SH_n^+(W)$ only depends on (Σ, ξ) so is the same as $SH_n^+(D^{2n}) = 0$. As a result we should have $\text{rank}(SH_n(W)) \leq 1$. But for $k \geq 2$, there are at least 3 idempotents, hence the rank is at least 2 and we get a contradiction. \square

If we knew that there is a contact form on $(\partial M_k^{2n}, \xi_k^{2n})$ with no closed characteristic of index less than $3 - n$, then we would get, by the above argument, that $(\partial M_k^{2n}, \xi_k^{2n})$ has no embedding in a weakly subcritical manifold.

7 Summary

A conceptual framework for the study of symplectic fillings is provided by the following definition of [Etnyre-Honda].

Definition 7.1 ([Etnyre-Honda]). Let (Σ_1, α_1) and (Σ_2, α_2) be two closed contact manifolds. We say that (Σ_1, α_1) is **dominated by** (Σ_2, α_2) if there exists an aspherical symplectic manifold (W, ω) such that (W, ω) has (Σ_1, α_1) as a concave boundary, (Σ_2, α_2) as a convex boundary and no other boundary component. We shall write

$$(\Sigma_1, \alpha_1) \prec (\Sigma_2, \alpha_2).$$

We shall say that (Σ_1, α_1) is **equivalent to** (Σ_2, α_2) if we both have $(\Sigma_1, \alpha_1) \prec (\Sigma_2, \alpha_2)$ and $(\Sigma_2, \alpha_2) \prec (\Sigma_1, \alpha_1)$, and this is denoted by

$$(\Sigma_1, \alpha_1) \simeq (\Sigma_2, \alpha_2).$$

Clearly, we have

$$(\Sigma_1, \alpha_1) \simeq (\Sigma_1, \alpha_1).$$

We would like to know if there are nonequivalent pairs of contact manifolds. Clearly, a contact manifold admits a filling if and only if it dominates the standard sphere. Which manifolds are dominated by the standard sphere? Our results give example of fillable manifolds which are not dominated by the standard sphere or, more generally, by the boundary of a subcritical Stein manifold. On the other hand, in dimension 4, any overtwisted contact manifold is dominated by any other contact manifold (see [Etnyre-Honda]). In particular, all overtwisted contact structures are equivalent!

The point of view of Definition 7.1 is also related to the work of [Chantraine] on the non-symmetry of Legendrian concordances.

We here try to summarize our results, but warn the reader that in the table below, the assumptions of the theorems are usually incomplete and the statements often not precise. One should refer to the relevant section of the paper for full details.

	Weakly subcritical case	Stein subcritical case	Case of \mathbb{R}^{2n}
<i>Hypothesis A</i>	(Σ, α) has a separating contact embedding in a weakly subcritical (M, ω) with bounded component Z .	(Σ, α) has a contact embedding in a Stein subcritical (M, ω) with bounded component Z .	(Σ, α) has a contact embedding in \mathbb{R}^{2n} with bounded component Z
<i>Assume</i>	$(\Sigma, \alpha) = \partial(W, \omega_1)$	$(\Sigma, \alpha) = \partial(W, \omega_1)$	$(\Sigma, \alpha) = \partial(W, \omega_1)$
<i>Conclusion 1</i>		The map $H_j(\Sigma) \rightarrow H_j(W)$ is onto (Thm. 2.5)	The homology of W is (almost) determined by the homology of Σ (Thm. 3.3)
<i>Hypothesis B</i>	W is Stein subcritical	W is weakly subcritical	W is Stein subcritical
<i>Conclusion</i>	(Σ, ξ) determines the homology of Z (Prop. 5.7) and the rank of $HC_*(\Sigma)$ is determined by $H_*(W)$ ([M.-L.Yau])	If the Conley-Zehnder indices of closed characteristics are $> 3 - n$, (Σ, ξ) determines the homology of Z (Prop. 5.7) and the rank of $HC_*(\Sigma)$ is determined by $H_*(W)$ ([M.-L.Yau])	The rank of $HC_*(\Sigma)$ is determined by $H_*(\Sigma)$ (Prop. 3.7)
<i>Examples: uniqueness of fillings</i>		Any filling of a simply connected homology sphere embeddable in a subcritical Stein is a homology ball.	If L has a Lagrange embedding in \mathbb{R}^{2n} , the fillings of ST^*L have the homology of L . (Prop. 3.12)
<i>Examples: obstructions to contact embeddings</i>	Obstructions to contact embedding ST^*L in a weakly subcritical Stein. (Prop. 5.12)	<ul style="list-style-type: none"> - Sphere bundles of negative complex bundles and some of their surgeries having no contact embedding in a Stein subcritical (Prop. 4.2, 4.3 and 5.14) - Obstructions to contact embedding ST^*L and the manifolds obtained from it by surgery in a subcritical Stein. (Prop. 4.4) - Brieskorn spheres do not embed in subcritical Stein (so their contact structure is exotic) (Cor 6.2) - Contact spheres obtained by [McLean] as boundaries of exotic symplectic \mathbb{R}^{2n} are exotic (Cor. 6.3) 	Sphere bundles of negative complex bundles and some of their surgeries having no contact embedding in \mathbb{R}^{2n} (Prop. 3.8 and 3.11)

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