The karaṇī: How to use integers to make accurate calculations on square roots

François Patte

THE *KARAṆĪ*: HOW TO USE INTEGERS TO MAKE ACCURATE CALCULATIONS ON SQUARE ROOTS

This paper intends to present a mathematical construction made by Indian mathematicians in order to use integers in the calculations involving square roots: these mathematical objects are called *karaṇī*.

To begin with, it may be useful to introduce the authors and the texts on which this paper is based. According to the Indian tradition, knowledge develops and is transmitted through a fundamental text and the commentaries made about it. Here, the main text is the *Bījagaṇita* of Bhāskarācārya, and two commentaries are used: the *Sūryaprakāśa* by Sūryadāsa and the *Bījapallava* by Kṛṣṇadaivajña.

There are two Indian mathematicians by the name of Bhāskara: one who lived in the 7th century and is often referred to as Bhāskara I and one who lived in the 12th century and is known as Bhāskarācārya (Bhāskara-ācārya: Bhāskara the master) or Bhāskara II; in this paper, his name will be shortened to Bhāskara because there is no ambiguity here: we shall never use the works of the first Bhāskara. In the same way, Kṛṣṇadaivajña, is abbreviated as Kṛṣṇa.

1. The author: Bhāskara

Bhāskarācārya was born in 1036 śaka, which is 1114 A.D., as he puts it himself at the end of the third part of his main work, the *Siddhāntaśiromaṇi*: *"I was born in the year 1036 of kings* śaka*, during my thirty-sixth year I composed the* Siddhāntaśiromaṇi*"*.[1]

Then, he gives the name of a town and its location, the name of the *gotra* he belongs to and the name of his father: *"There was at Vijjaḍaviḍa, a town located in the Sahya mountains... a twice-born from the Śāṇḍilya lineage... the virtuous Maheśvara..., born from the latter, the clever poet Bhāskara..."*[2]

What remains a mystery up to now is the exact name and location of the town: Vijjaḍaviḍa. The Sahya mountains are located in the northern part of the Maharashtra state and hold the well-known sites of Ajanta and Ellora; in these mountains, an inscription was discovered about 1850 in the basement of a temple at Pātnādevī, a small place near the modern Chalisgaon; this inscription gives Bhāskara's genealogy from the 10th to the 13th century, approximately, and was engraved

[1] rasaguṇapūrṇamahīsamaśakanṛpasamaye 'bhavan mamotpattiḥ । rasaguṇavarṣeṇa mayā siddhāntaśiromaṇiḥ racitaḥ ॥

[2] āsīt sahyakulācalāśritapure... vijjaḍaviḍe śāṇḍilyagotro dvijaḥ... maheśvarakṛtī... tajjas... suddhīḥ kavir bhāskaraḥ ।

to celebrate the foundation by his grandson of a school dedicated to the studies of Bhāskara's works in this very place. The name of Vijjaḍaviḍa is not quoted in this inscription which explains that king Jaitrapāla, from the Yādava dynasty, made Bhāskara's son, Lakṣmīdāsa, his astrologer and took him from *"this town"* to his capital; at that time, the capital of the Yādava was Devagiri which has been identified as Daulatabad near Aurangabad and is not that far from Pātnādevī.

1.1. **Bhāskara's works.** The main work of Bhāskara is the *Siddhāntaśiromaṇi*; it is composed of four parts: the first two are mathematical ones, the last two are astronomical ones. The two mathematical parts are respectively entitled: *Līlāvatī* and *Bījagaṇita*.

Let us briefly explain the title *Siddhāntaśiromaṇi*: "The Diadem on the *Siddhānta*". *Siddhānta* means "settled opinion". In India this name was given to the fundamental astronomical works; there were five major *Siddhānta* but there remains only one of them: the *Sūryasiddhānta*, "The *Siddhānta* of the Sun", thus called because it is assumed to have been revealed by the god Sun. The work of Bhāskara is based on it.

The *Līlāvatī* is a treatise of elementary calculus; numeration, operations, rules of proportions, calculation of areas, volumes and so on; it belongs to the class of *pāṭī-gaṇita* or *vyakta-gaṇita. pāṭī* means method, *gaṇita* means calculus and *vyakta* means manifested; this is a *"method of calculus"* or a *"calculus on manifested numbers"*.

We can find the name of the title at the end of the first stanza of the *Līlāvatī* which describes the method *(pāṭī)*: *"(...) I proclaim with soft and correct words, using short syllables, a method of good computation, which causes great satisfaction, which is clear and* ***possesses the grace of the play****."*[3]

The Mogul Abū al Fayḍ Fayḍī reports a legend about this title, which is also a name given to a girl in India, in his translation of this work into Persian, in 1587. He says that Līlāvatī is the name of one of Bhāskara's daughter to whom this book was dedicated. We did not find anything about this in Sanskrit commentaries; commentators merely explain the formation of the word *līlā-vat*: "that which possesses the play, that which is like a game".

Bījagaṇita: this is the generic name for algebra. *Bīja* means seed, so *Bījagaṇita* is calculus on seeds, the seeds which potentially contain calculus on manifested numbers; another name for this is *avyakta-gaṇita*: calculus on non-manifested numbers. We can find in the commentaries

[3](...) pāṭīṃ sadgaṇitasya vacmi caturaprītipradām prasphuṭām । saṃkṣiptākṣarakomalāmalapadair lālitya**līlāvatīm** ॥

some parallel with the *vyakta* and *avyakta* worlds, the manifested and non-manifested worlds of the *Sāṃkhya* philosophy, the non-manifested world containing the manifested world before the creation.

The *Bījagaṇita* expounds calculus on negative and positive numbers, calculus on unknown quantities — these are the *avyakta*, the non-manifested numbers which contain the possibility of making "real computations" if you replace them by numbers (manifested-numbers); another name for unknown quantities is *varṇa*: color, letter — and it explains resolution methods of equations: linear, Diophantine, algebraic... The fourth chapter, following the chapter on unknown quantities, is devoted to the *karaṇī*.

The two astronomical parts of the *Siddhāntaśiromaṇi* are entitled: *Grahagaṇitādhyāya*, "Lesson on the Computation of Planets", and *Golādhyāya*, "Lesson on Spheres". This part contains some trigonometry: calculation of additions of sines etc.

The *Siddhāntaśiromaṇi* was written in verse, as is usually done for this kind of treatises, and it is often difficult to understand without the help of commentaries. Here, we shall use two commentaries on the *Bījagaṇita*: one is the *Sūryaprakāśa* by Sūryadāsa, composed around 1530; the other is the *Bījapallava* by Kṛṣṇadaivajña, composed around 1604.

The other Bhāskara's known works are: a commentary on his *Siddhāntaśiromaṇi*, the *Mitākṣara* "Having Measured Syllables" or *Vāsanābhāṣya*; these are the names given for small commentaries to explain works very briefly. The *Karaṇakutūhala*: "The Wonder of Astronomical Calculations" is a practical treatise on astronomy. It is dated 1183 by Bhāskara himself; this is the last date that is known about him and, for this reason, some people think that he died about 1185. We do not know exactly when and where. Some think that he was in charge of the astronomical observatory in Ujjain and that he died there, but there are no texts and no inscriptions to support these facts. The astronomical instruments that we can see nowadays in Ujjain were erected in the 18th century by the maharajah of Jaipur, Jai Singh II. The last work is a commentary on a mathematical work by Lalla, an astronomer from the 8th century: the *Śiṣyadhīvṛddhidatantra*: "A Treatise to Increase the Understanding of Students".

2. The commentators: Sūryadāsa and Kṛṣṇadaivajña

2.1. **Sūryadāsa.** We do not know much about Sūryadāsa. As is usually the case, he belonged to a family of astronomers and, very likely, lived in the western part of the Godāvarī valley during the first part of the 16th century. He is known for two commentaries: one on the *Līlāvatī*, entitled *Gaṇitāmṛtakūpikā*, "The Well of Nectar which is The Calculus", and one on the *Bījagaṇita*: the *Sūryaprakāśa*, "The Brightness of the Sun"; this title is a play on words: his own name, Sūryadāsa, but also the name of Bhāskara, which also means the Sun, as does the word *sūrya*. The name of his father is also known, Jñānarāja. He wrote an astronomical treatise too: the *Siddhāntasundaraprakṛti* or "The Charming Foundation of the Siddhānta", and a mathematical work, in imitation of Bhāskara's *Bījagaṇita*: the *Siddhāntasundarabīja*.

2.2. **Kṛṣṇadaivajña.** Kṛṣṇadaivajña was born in a family of astronomers who had settled in Varanasi at the end of the 16th century. He was a protégé of the Mogul emperor Jahāngir (1605-1627). His commentary on the *Bījagaṇita*, the *Bījapallava* or "The Sprout of the Seed", is dated Saturday, the fourth *tithi* of the dark fortnight of the *Caitra* month, in the year 1523 of the *śaka* era, namely: Tuesday, March 12th, 1602. He also wrote some examples *(udāharaṇa)* using the horoscopes he made about members of the Mogul dynasty and he may have composed a commentary on the *Līlāvatī*. Unlike the others, Bhāskara and Sūryadāsa, he seems to have had as a *guru*, not his own father, but a nephew of Gaṇeśa, another great Indian mathematician.

3. The karaṇī

3.1. **The "concept" of *karaṇī*.** The word *karaṇī* means producer. It is in the feminine for, in *Śulbasūtra* texts, "The Aphorisms on the Rope", it was originally in relation to the word *rajju*, or rope, which is feminine too. This *rajju karaṇī* has been used to produce geometric squares, that is a right angle; there are some aphorisms in the *Śulbasūtra* explaining how to place some marks on a rope, at the distance of 3, 4 and 5, in order to produce a right-angled triangle and hence a square. So, the word *karaṇī* was used to designate the side of a square and because of that, it can mean the square root of a number. From the 5th century before the common era to the 17th century, the word *karaṇī* seems to have had many meanings related to squares and square roots and consequently it is not easy to translate. Moreover, in this paper, the mathematical concept expressed by this word has no equivalent in modern mathematics. For this reason we will not translate it. One common translation into English is surd. The word surd is ambiguous, for it means either irrational number or square root; there is no idea of irrational numbers in *karaṇī*, no idea of a number which cannot be expressed by a ratio, by a fraction of integers. As for the meaning "square root", this is not a correct translation for *karaṇī*, as *karaṇī* are not merely numbers, but numbers with a set of operations.

Bhāskara does not give any definition of *karaṇī*; the first rule, *"sūtra"*, as this kind of stanza is called in Sanskrit, deals straight with the rules of addition and multiplication, but we can find some information in the commentaries. Here is what Sūryadāsa says about this first *sūtra*[4]:

"Now, [the author] who examines the nature of these karaṇī*, under the pretext of explaining the rule of multiplication, says:* "vargeṇa"*.*

One will multiply a square ***vargeṇa*** *by a square number; likewise, one will divide a square by a square only; on the contrary, one will not multiply or divide a square by a number. It is pointed out by this, that what is of the nature of* karaṇī *by name is of the nature of a number accepted with the quality of square; this is said by Nārayaṇa:*

> *"The name of* karaṇī *will be for that [number] the square root of which must be taken.*[5]*"*

We shall examine the first part of this text later, and use now only the underlined part. What is important here is that if you consider the number two as a *karaṇī*, you must accept that this number is a square; this is reinforced by the above quotation from Nārāyaṇa who is a mathematician from the 14th century. A *karaṇī* is a number for which you have to compute the square root, so you must think that it is a square, you must think of a mental squaring operation.

Here is now what Kṛṣṇa says as an introduction to his commentary on this chapter:

"Now are commented the six operations on the karaṇī*. In this matter, one should understand that the six operations on* karaṇī*, are six operations done through the squares of two square root quantities. Because the origin of these six operations is preceded by a state of square, therefore, [there will be] also the use of the expression "state of* karaṇī*", in relation with these six operations, for a quantity producing a square root; this usage will not be possible if the calculation proceeds with the state of square root as the first step. The technical word "six operations on* karaṇī*" must be understood because of the necessity of such calculations about square root numbers; in this [case], that quantity for which a square root without a remainder is not possible, when its square root is required, is a* karaṇī*, but this is not merely a quantity which does not*

[4]We will see the *sūtra* itself later on. The Sanskrit transliteration of all quoted texts is given at the end of the article, see page 15.

[5]Quoted by Sūryadāsa from Nārayaṇa's *Bījagaṇitāvataṃsa.*

produce a square root; if it were so, there would always be the use of the expression "state of karaṇī*" for two, three, five, six, and so on.*

— Let it be so!

No! If the operations were done on that basis, for instance, eight added to two would be eighteen, etc."

This commentary requires some explanations. Note that the *"six operations"* are the operations which are studied in the *Bījagaṇita*, whatever the objects (negative numbers, unknown quantities, *karaṇī*), namely: addition, subtraction, multiplication, division, squaring and square root. First, we recognise the same idea as in the commentary by Sūryadāsa: a *karaṇī* is a number which is the square of its square root. Of course, this is a truism. For a modern mathematician, every number is the square of its square root; but at the time of Bhāskara, and his commentators, it was not so obvious that a number which was not a square could have a square root; we can see that in this text: Kṛṣṇa speaks of a *quantity producing a square root*, the Sanskrit compound *mūla-da* (which gives a root) is used by Indian mathematicians to designate a perfect square. Here, we are told that a square can be called a *karaṇī* but — and it is the second fact to be noticed — *"in relation with these six operations"*, that is to say that we cannot separate, for a number, the state of being a *karaṇī* from the class of operations defined for them.

This becomes clearer with the end of the text: Kṛṣṇa explains that a *"quantity for which a square root without a remainder is not possible, when its square root is required, is a* karaṇī ***but*** *this is not merely a quantity which does not produce a square root".* Indeed! For if we keep in mind that every number which is a square could be called a *karaṇī*, every integer should be a *karaṇī*. The explanation comes right after, taking the form of a small dialogue, as is often the case in Sanskrit commentaries: an opponent argues *"Let it be so!"*: and what if we take this definition for granted? Kṛṣṇa answers: *"If the operations were be done on that basis, for instance, eight added to two would be eighteen."* So we cannot call a number a *karaṇī* without thinking of how to make operations with it.

3.2. **Rules of computation.** We can now analyse the text of Bhāskara and see why *"eight added to two would be eighteen"*; here are the two first rules of this chapter:

> *Let us fix the* mahatī *as the sum of the two* karaṇī *and the* laghu *as twice the square root of their product, the*

> *sum and the difference of these two are as for the integers. One will multiply and divide a square by a square. –1– But the square root of the larger [number] divided by the smaller [number], plus one or minus one, multiplied by itself and by the smaller [number], will also be respectively, the sum and the difference of these two* [karaṇī]. *One will leave it apart if there is no square root. –2–*[6]

These *sūtra* give two ways of calculation for the sum and the difference of two *karaṇī*: in the first one, two new objects are defined: the *mahatī*, the greater, which is the sum of the two *karaṇī*, that is to say the sum of the two integers which measure the two *karaṇī*; the second object is the *laghu*, the smaller, which is twice the square root of the product of the same integers. The names chosen for these technical words are obvious, for the sum of two integers is always greater than twice the square root of their product. Once the *mahatī* and the *laghu* have been calculated, the two integers produced are added and the result is the *karaṇī* sum of the two *karaṇī*, that is to say: the square of the sum of the square roots of the two integers measuring the *karaṇī*.

If the difference of the *mahatī* and the *laghu* is computed, the result is the difference of the two *karaṇī*. There is a problem with this definition: the difference between two *karaṇī* is the same whether you compute the difference between the greater one and the smaller one or the smaller minus the greater.

Before going deeper in the explanations of these rules, let us see the example given by Bhāskara, and its solution by Sūryadāsa.

> *Say the sum and the difference of the two* karaṇī *measured by two and eight and by three and twenty-seven and after a long while, if you know the six operations on* karaṇī, *say, dear, [the sum and the difference] of these two measured by three and seven.*[7]

Now the commentary by Sūryadāsa:

In this case, putting down **ka** *2* **ka** *8, by:* "Let us fix the *mahatī* as the sum of the two *karaṇī*", *the mahatī is 10.*

Now, the product of these two karaṇī *is 16, its square root 4, when multiplied by two, the* laghu *is produced: 8.*

[6]yogaṃ karaṇyor mahatīṃ prakalpya ghātasya mūlaṃ dviguṇaṃ laghuṃ ca । yogāntare rūpavad etayoḥ sto vargeṇa vargaṃ guṇayed bhajec ca ॥ laghvyā hṛtāyās tu padaṃ mahatyāḥ saikaṃ nirekaṃ svahataṃ laghughnam । yogāntare staḥ kramaśas tayor vā pṛthaksthitiḥ syād yadi nāsti mūlam ॥

[7]dvikāṣṭamityos tribhasaṃkhyayoś ca yogāntare brūhi pṛthak karaṇyoḥ । trisaptamityoś ca ciraṃ vicintya cet ṣaḍvidhaṃ vetsi sakhe karaṇyāḥ ॥

"The sum and the difference of these two are as for the integers"*: 18 and 2; these two* karaṇī*:* **ka** *18 and* **ka** *2 are the sum and the difference.*

The solution of this exercise put forward by Bhāskara is clear. We can observe how the mathematicians proceeded to differentiate a simple number and a number which must be considered as a *karaṇī*: they wrote the first syllable of the word *karaṇī* before the number: ***ka*** 2 and this means that this "2" is the square of the square root of 2, so, in this addition we have to find out the square of the square root which is the sum of the square root of 2 and the square root of 8; with our modern notations, we have:

$$(\sqrt{2}+\sqrt{8})^2 = 2+8+2\sqrt{2\times 8} = (\sqrt{18})^2$$

If we could mix our modern formalism with the ancient Indian one, we would write that the set of *karaṇī* is a set of numbers with this particular addition:

$$\boldsymbol{ka}\ a \boxplus \boldsymbol{ka}\ b = \boldsymbol{ka}\ (a+b+2\sqrt{ab})$$

It is obvious that this addition (and subtraction) is only defined if the product of the two integers measuring the *karaṇī* is a square. What happens if this is not the case is expressed by the last sentence of Bhāskara's rule: *"One will leave it apart if there is no square root."* This situation is illustrated by the third example given: the sum of the *karaṇī* measured by three and seven is impossible because 21 is not a square, so the solution of this exercise is merely: ***ka*** 3 ***ka*** 7.

This creates an extension of the meaning of *karaṇī*. In Sanskrit texts there are some statements like: *"Let the* karaṇī *be measured by 2, 3 and 5"* as if a *karaṇī* was also a composition of several *karaṇī* the sum of which is not possible but which can be used for other calculations.

The second way for adding, or subtracting, the *karaṇī* given by Bhāskara (see page 6) is very simply derived from the first one and there is no need to comment upon it:

$$(\sqrt{8}\pm\sqrt{2})^2 = 2\Big(\sqrt{\frac{8}{2}}\pm 1\Big)^2 = \left\{\begin{array}{c}18\\ 2\end{array}\right.$$

Let us focus on the sentence of the rule: *"One will multiply and divide a square by a square."* We have seen how one of the commentators, Sūryadāsa, had explained it (page 5) in order to put forward the nature of square of the *karaṇī*.

Here is now what Kṛṣṇa says about it: "vargeṇa vargaṃ guṇayed bhajec ca"*: here is what is said: when you want to multiply* karaṇī*, if there is the state of multiplicand or the state of multiplier for some* *integer — or if you want to divide* karaṇī, *if there is the state of dividend or the state of divisor for some integer — then, having squared the integer, the multiplication and the division can be performed, because a* karaṇī *has a nature of square."*

Another idea springs from this commentary: how to "embed" the integers in the *karaṇī* set; if we want to make operations mixing integers and *karaṇī*, we have to square the integers in order to give them a state of *karaṇī* and this will change the general rules given for the operations. For instance, the rule for squaring the integers given in the *Līlāvatī* uses the identity $(a+b)^2 = a^2+b^2+2\,ab$. So Kṛṣṇa explains afterwards: *"for the square also, the fulfillment is in like manner because it (the square) is a kind of multiplication according to its nature of product of two equal [numbers]. Or, according to the method stated for the manifested numbers:* "The square of the last [digit] must be placed and [the other digit must be] multiplied by the last one increased two times..."[8]*, there will be a fulfillment [of this method] for the squaring of the* karaṇī *also but, as has been said:* "One will multiply and divide a square by a square"*, when it is said:* "multiplied by the last increased two times"*, we must understand: "multiplied by the last increased four times."*

Let us now make a digression. Notice the extreme degree of conciseness of Sanskrit works such as the *Bījagaṇita*: in one single rule, Bhāskara describes four of the six operations: addition, subtraction, multiplication and squaring. Although he describes the addition and the subtraction, by giving two methods, because these operations differ from the same operations applied to numbers or unknown quantities, a simple sentence is enough for him to say how to handle the multiplication and consequently, the squaring.

This method of exposition is usual in the fundamental works of Sanskrit literature and the works of Bhāskara are of this kind; they were used for centuries as a basis for mathematical teachings: students learned them by heart, then their masters made commentaries which became original lectures. In order to be easily remembered, they were composed in verse, *"using short syllables"*, as seen earlier (page 2), and any unnecessary rule was avoided. So, in this chapter on the *karaṇī*, there will not be any proper rule for multiplication: once it is understood — with the help of the commentator — that for any "simple" *karaṇī*[9] the multiplication is merely given by:

[8]sthāpyo 'ntyavargo dviguṇāntyanighnāḥ । svasvopariṣṭāc ca tathāpare 'ṅkās tyaktvāntyam utsārya punaś ca rāśim — *Līlāvatī*

[9]Let us call "simple" a *karaṇī* measured by a single integer

$$(\sqrt{2})^2\,(\sqrt{8})^2 = (\sqrt{2\times 8})^2$$

the general rule for *karaṇī* measured by more than one integer has already been given in a preceding chapter, the chapter on unknown quantities:

> *One must think here, in like manner, of the rule of multiplication by parts stated for manifested numbers, in the case of the square of non-manifested numbers and in the case of the multiplication of* karaṇī.[10]

And even here, the rule refers to another rule given in the *Līlāvatī* which describes the property of distributiveness of the multiplication with respect to the addition — the "multiplication by parts":

> *(...) Or the multiplicand, equal in number to the number of parts of the multiplier, being placed under each of them, is multiplied by these parts and added up.*[11]

In fact, it is clear from the text of the *Līlāvatī* and its commentaries that the "parts" in question are either the result of splitting an integer in two (or more) parts in order to perform a mental calculation or the digits of the number with their decimal place value; so, this last rule can be used to perform calculations on polynomial-like quantities: a number considered in such a way being merely a polynomial in the powers of ten.

Let us see now the example given by Bhāskara for the multiplication of the *karaṇī*:

> *Set the multiplier as the* karaṇī *counted by two, three and eight and the multiplicand as counted by [the karaṇī] three with the integer five; say the product quickly. Or the multiplier is the two* karaṇī *measured by three and twelve less the integer five.*[12]

and the solution given by Sūryadāsa:

Here, the multiplier is: **ka** *2* **ka** *3* **ka** *8.*

In like manner, the multiplicand is counted by three with five units; in this multiplicand, there are: **ka** *3* **rū** *5.*

One notices an integer: after taking its square, the state of karaṇī *must be brought about because it has been said:* "One will multiply and divide a square by a square"*. By so doing,* **ka** *3* **ka** *25 are produced.*

[10] avyaktavargakaraṇīguṇanāsu cintyo vyaktoktakhaṇḍaguṇanāvidhir evam atra ।

[11] guṇyas tv adho 'dho guṇakhaṇḍatulyas taiḥ khaṇḍakaiḥ saṃguṇito yuto vā ।

[12] dvitryaṣṭasaṃkhyā guṇakaḥ karaṇyo guṇyas trisaṃkhyā ca sapañcarūpā । vadhaṃ pracakṣvāśu vipañcarūpe guṇo 'thavā tryarkamite karaṇyau ॥

Now, according to the method of the rule: "One must think here, in like manner, of the rule of multiplication by parts stated for manifested numbers, in the case of the square of non-manifested numbers and in the case of the multiplication of *karaṇī*", *after multiplication,* **ka** *54* **ka** *450* **ka** *9* **ka** *75 are produced.*

The rule *"One will leave it apart if there is no square root"* is used in this example and we discover a new formalism to note integers: ***rū***, first syllable of the Sanskrit word *rūpa*, the meaning of which is "unity", "integer". The commentator squares this integer to transform it into a *karaṇī* before he performs the multiplication.

According to Bhāskara's rule for the multiplication "by parts" (see page 10), we could represent the way to do this multiplication with the following table:

ka 2	***ka*** 3	***ka*** 8
ka 3 ***ka*** 25	***ka*** 3 *ka* 25	***ka*** 3 ***ka*** 25
ka 6 ***ka*** 50	***ka*** 9 ***ka*** 75	***ka*** 24 ***ka*** 200
† ‡		† ‡

The multiplier is split in three parts, as said in Bhāskara's rule, and placed in the first row; then the multiplicand is put under each part of the multiplier and the multiplication is performed in each cell of the table, the results being written in the third row.

We have put an identical symbol under the *karaṇī* for which the addition is possible.

†: $6 + 24 + 2\sqrt{6 \times 24} = 54$
‡: $50 + 200 + 2\sqrt{50 \times 200} = 450$

Once the addition is done, the result is the one given by Sūryadāsa.

The second example raises a problem because there is no mathematical notations for addition or subtraction; the Indians have developed formalisms in some branches of knowledge like grammar and mathematics but there are no signs to denote the operations. In this chapter on the *karaṇī*, putting two *karaṇī* side by side indicates that it is the sum of these two *karaṇī* which is considered. This is the meaning of: *"One will leave it apart if there is no square root"*. For the subtraction, the notation is almost the same because it has been explained at the beginning of the *Bījagaṇita* that a subtracted positive number becomes a negative number, therefore, subtracting a number is only adding its opposite. There is a sign to denote negative numbers: a dot is placed over them; applying this notation to the *karaṇī* leads to:

ka 8 ***ka*** $\dot{2}$

meaning that the *karaṇī* 2 is subtracted from the *karaṇī* 8.

As long as only *karaṇī* are considered, no problems occur: the subtraction rule applies as it is formulated: the difference is the *karaṇī* measured by the number $8 + 2 - 2\sqrt{2 \times 8} = 2$. But in his example, Bhāskara says: *"(...) Or the multiplier is the two* karaṇī *measured by three and twelve less the integer five"* and according to the Indian notation system, we have to write:

$$\boldsymbol{ka}\ 3\ \boldsymbol{ka}\ 12\ \boldsymbol{r\bar{u}}\ \dot{5}$$

Because *karaṇī* and integers are mixed in this multiplier, we have to square the integer in order to transform it into a *karaṇī* and when doing this, we will loose the "negative sign" showing that the last component must be subtracted.

To solve this problem, Bhāskara introduced a restriction to the general rule which says that the square of a negative quantity is positive:

> *The square of negative integers will also be negative if it is calculated for the reason of a state of* karaṇī. *Likewise, the square root of a* karaṇī *the nature of which is negative will be negative for the reason of creation of a state of integer.*[13]

With this rule, the multiplier becomes: ***ka*** 3 ***ka*** 12 ***ka*** $\dot{2}5$, which can be simplified as ***ka*** 27 ***ka*** $\dot{2}5$, by the addition of the *karaṇī* 3 and 12 $(3 + 12 + 2\sqrt{3 \times 12} = 27)$. Now, the multiplication can be performed in the same way as in the first example; let us summarize this with a table:

ka $\dot{2}5$	***ka*** 27
ka 3 ***ka*** 25	***ka*** 3 ***ka*** 25
ka $\dot{7}5$ ***ka*** $6\dot{2}5$	***ka*** 81 ***ka*** 675

Depending on the commentator, the result can be simplified in more than one way: noticing that 81 and 625 are squares, we can give them back their state of integers and subtract one from the other, because the square root of the "negative square", 625, remains negative according to the last given rule; we get ***rū*** $\dot{1}6$. The two remaining *karaṇī*, ***ka*** $\dot{7}5$ and ***ka*** 675 can be subtracted, for $675 \times 75 = 50625$ is a square and we get ***ka*** 300.

It may seem strange for a modern mathematician to state one general rule, such as: *"the square of a positive or a negative number is a positive*

[13]kṣayo bhavec ca kṣayarūpavargaś cet sādhyate 'sau karaṇītvahetoḥ | ṛṇātmikāyāś ca tathā karaṇyā mūlaṃ kṣayo rūpavidhānahetoḥ ||

number", then to restrict its range of application by another rule which may even contradict the general one. Nevertheless this is found very often in mathematical Sanskrit texts because the paradigm of logic in Sanskrit scientific knowledge is grammar rather than mathematics. We can see an example of this here and this process is constant in Pāṇini's grammar which is the fundamental text of this discipline.

There are two more operations to complete the six operations described in the *Bījagaṇita*: division and square root.

Division is easy to perform — and Bhāskara does not give any rule for this, only examples — because the algorithm given for the multiplication is the same as the one given for the unknown quantities and thus, as is shown by the two preceding tables, the division is very similar to today's Euclidean division of polynomials: it is sufficient to read these tables in the reverse order, making the third row the dividend and the first one the divisor to find out that the middle row is the quotient of the division. All the examples given for the multiplication are used in this way to explain the division in both chapters: the one about unknown quantities and the present one on the *karaṇī*.

The square root is rather difficult and we shall not discuss it in this paper. Let us just say that its algorithm is based on the identity:

$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2\,ab + 2\,ac + 2\,bc$$

which the mathematicians inverted in order to find out the quantities a, b, c from the left member of the identity. We have just written down three quantities, but there are some examples given by Bhāskara with more than three *karaṇī*.

3.3. **The use of the *karaṇī*.** What was the purpose of Indian mathematicians when they constructed these *karaṇī*? They knew perfectly well how to calculate the approximate values of a square root $\sqrt{A}$, using the first two or three values of the sequence:

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{A}{a_n}\right)$$

Let us consider what Kṛṣṇa says about it (this text is the paragraph that follows the text quoted on page 5):

"But it may be argued that these are only words! Why, then, trouble yourself to study these operations on karaṇī *for, in common practice, there is no use of* karaṇī *but only of the approximate values of their square roots and, with the use of six operations on numbers, these six*

operations [on karaṇī*] are meaningless. Moreover, even if the calculation with* karaṇī *is done, in common use, [calculation] with approximate square roots from the beginning is better than this and is preferable to it.*

— This is not correct. If a rough square root is taken from the beginning, there will be a big roughness in its multiplication and so on; but if the calculation of karaṇī*, which is minute, is performed, later, when the approximate square root is taken, there will be some difference but not very much; for this great distinction, the six operations with* karaṇī *must necessarily be undertaken."*

As already seen this justification for the construction of *karaṇī* takes the form of a dialogue; an opponent develops the idea that this construction is useless and that only approximate calculations of square roots is enough for everyday transactions. He is told in return that if many operations are done with approximate values of square roots the final error is much bigger than if the calculations were done through the sophisticated construction of *karaṇī* and if the approximate value of the result is taken at the very end of the calculation.

The next paragraph justifies the location of this chapter in the complete book of Bhāskara's mathematical works; in Sanskrit commentaries, it is mandatory to give the reasons why a particular subject is studied and to justify its place in the succession of the topics developed by an author.

"Although it is suitable that these [six operations on karaṇī*] should be undertaken before the six operations on* varṇa[14]*, because they are closer to the operations on manifested numbers, according to the maxim of the needle and the kettle*[15]*, it is however suitable to undertake them immediately after the operations on* varṇa *for the reason that a great effort is required by their examination and understanding."*

---

[14]Unknown quantities.

[15]This maxim is used to show that when two things — an easy one and a difficult one — must be done, the easier one should be first attended to just like when you have to prepare a needle and a kettle, you should first take in hand a needle as it is an easier work compared to the preparation of a kettle. See Apte's Practical Sanskrit-English Dictionary.

Appendix: Sanskrit texts

Sūryadāsa, page 5:

athaitasyāḥ karaṇyā guṇanavidhikathanavyājena svarūpaṃ nirūpayann āha *vargeṇe*ti—

vargeṇa vargāṅkena vargaṃ guṇayet tathā vargeṇaiva vargaṃ bhajen na paraṃ tu rūpeṇa vargaṃ guṇayed bhajed vety arthaḥ। anena karaṇītvaṃ nāma vargatvenābhimatāṅkatvaṃ sūcitaṃ bhavati tad uktaṃ nārāyaṇena

mūlaṃ grāhyaṃ rāśer yasya tu karaṇīnāma tasya syāt।

iti

Kṛṣṇa, page 5:

atha karaṇīṣaḍvidhaṃ vyākhyāyate। atredam avagantavyaṃ mūlarāśyor vargadvārā yat ṣaḍvidhaṃ tat karaṇīṣaḍvidham iti। asya ṣaḍvidhasya vargatvapuraskāreṇaiva pravṛter ata evāsmin ṣaḍvidhe mūladarāśāv api karaṇītvavyavahāraḥ karaṇītvapuraskāreṇa gaṇitapravṛttāv ayaṃ na syāt। karaṇīṣaḍvidham iti saṃjñā tu karaṇīrāśāv etasya gaṇitasyāvaśyakatvād dṛṣṭavyā। tatra yasya rāśer mūle 'pekṣite niragraṃ mūlaṃ na saṃbhavati sa karaṇī। na tv amūladarāśimātram। tathā sati dvitripañcaṣaḍādiṣu sarvadā karaṇītvavyavahāraḥ syāt।

— astu sa iti cet।

na। tathā sati tatprayuktaṃ kāryaṃ syāt। yathāṣṭau dvisaṃyutā aṣṭādaśaiva syur ity ādi॥

Kṛṣṇa page 8:

"vargeṇa vargaṃ guṇayed bhajec ca" iti। etad uktaṃ bhavati—karaṇīguṇane kartavye yadi rūpāṇāṃ guṇyatvaṃ guṇakatvaṃ vā syāt karaṇībhajane vā kartavye yadi rūpāṇāṃ bhājyatvaṃ bhajakatvaṃ vā syāt tadā rūpāṇāṃ vargaṃ kṛtvā guṇanabhajane kārye। karaṇyā vargarūpatvād iti।

Kṛṣṇa page 9:

vargasyāpi *samadvighāta*tayā guṇanaviśeṣatvād uktavat siddhiḥ। *"sthāpyo 'ntyavargo dviguṇāntyanighnā"* ityādinā vyaktoktaprakareṇa vā karaṇīvargasyāpi siddhiḥ syāt kiṃ tu *"vargeṇa vargaṃ guṇayed"* ityuktatvād *dviguṇāntyanighnā* ity atra caturguṇāntyanighnā iti draṣṭavyam।

Kṛṣṇa page 13:

nanv astu paribhāṣāmātram idaṃ tathāpi kim anena karaṇīṣaḍvidhanirūpaṇaśramena na hy asti loke karaṇībhir vyavahāraḥ kintu tadāsannamūlair eva tatṣaḍvidhaṃ ca rūpaṣaḍvidhenaiva gatārtham। kiṃ ca kṛte 'pi karaṇīgaṇite tatas tadāsannamūlenaiva vyavahāraḥ tadvaraṃ prāg eva tadādara iti cet॥

maivam | prāg eva sthūlamūlagrahaṇe tadguṇanādāv atisthūlatā syāt kṛte 'pi sūkṣme karaṇīgaṇite paścāt tadāsannamūlagrahaṇe kiṃ cid evāntaraṃ syān na mahad ity asti mahān viśeṣa iti karaṇīṣaḍvidham avaśyam arambhanīyam |

tad yady api vyaktaṣaḍvidhāntaraṅgatvād varṇaṣaḍvidhāt prāg evārabdhuṃ yuktaṃ tathāpy etasya nirūpaṇāvagamayoḥ prayāsagauravāt sūcikaṭāhanyāyena varṇaṣaḍvidhānantaram ārambho yukta eva |