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# Moment estimates for solutions of linear stochastic differential equations driven by analytic fractional Brownian motion

J er mie Unterberger

As a general rule, differential equations driven by a multi-dimensional irregular path  $\Gamma$  are solved by constructing a rough path over  $\Gamma$ . The domain of definition – and also estimates – of the solutions depend on upper bounds for the rough path; these general, deterministic estimates are too crude to apply e.g. to the solutions of stochastic differential equations with linear coefficients driven by a Gaussian process with H older regularity  $\alpha < 1/2$ .

We prove here (by showing convergence of Chen’s series) that linear stochastic differential equations driven by analytic fractional Brownian motion [7, 8] with arbitrary Hurst index  $\alpha \in (0, 1)$  may be solved on the closed upper half-plane, and that the solutions have finite variance.

**Keywords:** stochastic differential equations, fractional Brownian motion, analytic fractional Brownian motion, rough paths, H older continuity, Chen series

**Mathematics Subject Classification (2000):** 60G15, 60H05, 60H10

## 0 Introduction

Assume  $\Gamma_t = (\Gamma_t(1), \dots, \Gamma_t(d))$  is a smooth  $d$ -dimensional path, and  $V_1, \dots, V_d : \mathbb{R}^r \rightarrow \mathbb{R}^r$  be smooth vector fields. Then (by the classical Cauchy-Lipschitz theorem for instance) the differential equation driven by  $\Gamma$

$$dy(t) = \sum_{i=1}^d V_i(y(t)) d\Gamma_i(t) \quad (0.1)$$

admits a unique solution with initial condition  $y(0) = y_0$ . The usual way to prove this is by showing (by a functional fixed-point theorem) that iterated integrals

$$y_n(t) \rightarrow y_{n+1}(t) := y_0 + \int_0^t \sum_i V_i(y_n(s)) d\Gamma_i(s) \quad (0.2)$$

converge when  $n \rightarrow \infty$ .

Expanding this expression to all orders yields formally for an arbitrary analytic function  $f$

$$f(y_t) = f(y_s) + \sum_{n=1}^{\infty} \sum_{1 \leq i_1, \dots, i_n \leq d} [V_{i_1} \dots V_{i_n} f](y_s) \mathbf{\Gamma}_{ts}^n(i_1, \dots, i_n), \quad (0.3)$$

where

$$\mathbf{\Gamma}_{ts}^n(i_1, \dots, i_n) := \int_s^t d\Gamma_{t_1}(i_1) \int_s^{t_1} d\Gamma_{t_2}(i_2) \dots \int_s^{t_{n-1}} d\Gamma_{t_n}(i_n), \quad (0.4)$$

provided, of course, the series converges. By specializing to the identity function  $f = \text{Id} : \mathbb{R}^r \rightarrow \mathbb{R}^r$ ,  $x \rightarrow x$ , one gets a series expansion for the solution  $(y_t)$ .

This formula, somewhat generalized, has been used in a variety of contexts:

1. Let

$$\mathcal{E}_V^{N,t,s}(y_s) = y_s + \sum_{n=1}^N \sum_{1 \leq i_1, \dots, i_n \leq d} [V_{i_1} \dots V_{i_n} \text{Id}](y_s) \mathbf{\Gamma}_{ts}^n(i_1, \dots, i_n) \quad (0.5)$$

be the  $N$ -th order truncation of (0.3). It may be interpreted as *one* iteration of the numerical Euler scheme of order  $N$ , which is defined by

$$y_{t_k}^{\text{Euler};D} := \mathcal{E}_V^{N,t_k,t_{k-1}} \circ \dots \circ \mathcal{E}_V^{N,t_1,t_0}(y_0) \quad (0.6)$$

for an arbitrary partition  $D = \{0 = t_0 < \dots < t_n = T\}$  of the interval  $[0, T]$ . When  $\Gamma$  is only  $\alpha$ -Hölder with  $\frac{1}{N+1} < \alpha \leq \frac{1}{N}$ , the iterated integrals  $\mathbf{\Gamma}^n(i_1, \dots, i_n)$ ,  $n = 2, \dots, N$  do not make sense a priori and must be substituted with a *geometric rough path* over  $\Gamma$ . A *geometric rough path* over  $\Gamma$  is a family

$$\left( (\mathbf{\Gamma}_{ts}^1(i_1))_{1 \leq i_1 \leq d}, (\mathbf{\Gamma}_{ts}^2(i_1, i_2))_{1 \leq i_1, i_2 \leq d}, \dots, (\mathbf{\Gamma}_{ts}^N(i_1, \dots, i_N))_{1 \leq i_1, \dots, i_N \leq d} \right) \quad (0.7)$$

of functions of two variables such that:

- (i)  $\mathbf{\Gamma}_{ts}^1 = \Gamma_t^1 - \Gamma_s^1$ ;
- (ii) (*Hölder continuity*) each component of  $\mathbf{\Gamma}^k$ ,  $k = 1, \dots, N$  is  $k\alpha$ -Hölder continuous, that is to say,  $\sup_{s \in \mathbb{R}} \left( \sup_{t \in \mathbb{R}} \frac{|\mathbf{\Gamma}_{ts}^k(i_1, \dots, i_k)|}{|t-s|^{k\alpha}} \right) < \infty$ .
- (iii) (*multiplicativity*) letting  $\delta \mathbf{\Gamma}_{tus}^k := \mathbf{\Gamma}_{ts}^k - \mathbf{\Gamma}_{tu}^k - \mathbf{\Gamma}_{us}^k$ , one requires

$$\delta \mathbf{\Gamma}_{tus}^k(i_1, \dots, i_k) = \sum_{k_1 + k_2 = k} \mathbf{\Gamma}_{tu}^{k_1}(i_1, \dots, i_{k_1}) \mathbf{\Gamma}_{us}^{k_2}(i_{k_1+1}, \dots, i_k). \quad (0.8)$$

- (iii) (*geometricity*)

$$\mathbf{\Gamma}_{ts}^{n_1}(i_1, \dots, i_{n_1}) \mathbf{\Gamma}_{ts}^{n_2}(j_1, \dots, j_{n_2}) = \sum_{\mathbf{k} \in \text{Sh}(\mathbf{i}, \mathbf{j})} \mathbf{\Gamma}^{n_1+n_2}(k_1, \dots, k_{n_1+n_2}) \quad (0.9)$$

where  $\text{Sh}(\mathbf{i}, \mathbf{j})$  is the subset of permutations of  $i_1, \dots, i_{n_1}, j_1, \dots, j_{n_2}$  which do not change the orderings of  $(i_1, \dots, i_{n_1})$  and  $(j_1, \dots, j_{n_2})$ .

Properties (i)-(iv) are true when  $\Gamma$  is regular; the multiplicative property measures in some sense the defect of additivity of iterated integrals, which is easy to measure when one represents these as geometric quantities (areas, volumes, etc.) Under these conditions, it is possible to integrate a 1-form along the path  $\Gamma$  (or, more precisely, along the rough path  $\mathbf{\Gamma}$ ); we refer the reader either to [2] or to [3].

It is also possible to solve differential equations driven by  $\Gamma$  like (0.1), either by using eq. (0.2) and a fixed-point theorem in a class of  $\Gamma$ -controlled processes [3], or by using the above Euler scheme [2]. Assuming either (i) the vector fields  $V$  and their derivatives up to order  $N$  are bounded or (ii) they are linear <sup>1</sup>, then the solution is

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<sup>1</sup>Similar results are also expected when the vector fields  $V$  are uniformly Lipschitz on  $\mathbb{R}$  [4]. See also [5] for investigations on the same subject using a fixed point method.

globally defined, and the solution at time  $T$  is bounded (i) by a polynomial in  $|||\mathbf{\Gamma}|||$  or (ii) by something like  $\exp C|||\mathbf{\Gamma}|||^N$ , where  $|||\mathbf{\Gamma}||| = \max_{n=1,\dots,N} \sup_{1 \leq i_1, \dots, i_n \leq d} \sup_{0 \leq s, t \leq T} \frac{|\mathbf{\Gamma}_{ts}^n(i_1, \dots, i_n)|}{|t-s|^{n\alpha}}$  is the maximum of the Hölder norms. It seems there is no way (using either approach) to improve these bounds in the general deterministic setting. Unfortunately, they do not give a control of the solution as a stochastic process in the linear case (ii) when  $\Gamma$  is a Gaussian process (such as fBm or analytic fBm, see below) with Hölder regularity  $\alpha < \frac{1}{2}$ , since  $\mathbb{E} \exp C(\Gamma_t - \Gamma_s)^2 = \infty$  for  $C$  large enough, and in any case  $\mathbb{E} \exp C|\Gamma_t - \Gamma_s|^N = \infty$  if  $N \geq 3$ .

2. Assume  $\Gamma$  is a stochastic process, and let  $P_t(f) = \mathbb{E}f(y_t)$ . When (0.1) is a diffusion driven by usual Brownian motion,  $P_t$  is the associated semi-group operator. Assume now  $\Gamma$  is more general, for instance fBm or analytic fBm. Even though the process is not Markov, the operator  $P_t$  is interesting in itself. The small-time expansion of  $P_t$  (corresponding to an arbitrary truncation of the above series) has been studied [1] when  $\Gamma$  is fBm with Hurst index  $\alpha > 1/3$ . When  $\alpha > 1/2$ , it has been proved [6] that the series converges for functions and vector fields  $V$  satisfying somewhat drastic conditions.

In any case, it seems difficult to get moment estimates for the solutions of stochastic differential equations driven by stochastic processes  $\Gamma$  with Hölder regularity  $\alpha < \frac{1}{2}$ . One of the reasons [6] is the difficulty of getting estimates for the iterated integrals  $\mathbf{\Gamma}$ ; another reason lies in the essence of the rough path method which relies on pathwise estimates; a third reason is, of course, that the Chen series diverges even in the simplest cases (one-dimensional usual Brownian motion for instance) as soon as the vector fields are unbounded and non-linear, e.g. quadratic.

In this article, we prove convergence of the series (0.3) when the vector fields  $V_i$  are linear and  $\Gamma$  is *analytic fBm* (afBm for short). This process – first defined in [8] –, depending on an index  $\alpha \in (0, 1)$ , is a complex-valued process, a.s.  $\kappa$ -Hölder for every  $\kappa < \alpha$ , which has an analytic continuation to the upper half-plane  $\Pi^+ := \{z = x + iy \mid x \in \mathbb{R}, y > 0\}$ . Its real part ( $2\text{Re } \Gamma_t, t \in \mathbb{R}$ ) has the same law as fBm with Hurst index  $\alpha$ . Trajectories of  $\Gamma$  on the closed upper half-plane  $\bar{\Pi}^+ = \Pi^+ \cup \mathbb{R}$  have the same regularity as those of fBm, namely, they are  $\kappa$ -Hölder for every  $\kappa < \alpha$ . As shown in [7], the regularized rough path – constructed by moving inside the upper half-plane through an imaginary translation  $t \rightarrow t + i\varepsilon$  – converges in the

limit  $\varepsilon \rightarrow 0$  to a geometric rough path over  $\Gamma$  for *any*  $\alpha \in (0, 1)$ , which makes it possible to produce strong, local pathwise solutions of stochastic differential equations driven by  $\Gamma$  with analytic coefficients.

We do not enquire about the convergence of the series (0.3) in the general case (as mentioned before, it diverges e.g. when  $V$  is quadratic), but only in the *linear* case. One obtains:

**Main theorem.**

*Let  $V_1, \dots, V_d$  be linear vector fields on  $\mathbb{C}^r$ . Then the series (0.3) converges in  $L^2(\Omega)$  on the closed upper half-plane  $\bar{\Pi}^+ = \Pi^+ \cup \mathbb{R}$ . Furthermore, the solution  $(y_t)_{t \in \bar{\Pi}^+}$ , defined as the limit of the series, has finite variance. More precisely, there exists a constant  $C$  such that*

$$\mathbb{E}|y_t - y_s|^2 \leq C|t - s|^{2\alpha}, \quad s, t \in \bar{\Pi}^+. \quad (0.10)$$

**Notation.** Constants (possibly depending on  $\alpha$ ) are generally denoted by  $C, C', C_1, c_\alpha$  and so on.

## 1 Definition of afBm and first estimates

We briefly recall to begin with the definition of the analytic fractional Brownian motion  $\Gamma$ , which is a complex-valued process defined on the closed upper half-plane  $\bar{\Pi}^+$  [7]. Its introduction was initially motivated by the possibility to construct quite easily iterated integrals of  $\Gamma$  by a contour deformation. Alternatively, its Fourier transform is supported on  $\mathbb{R}_+$ , which makes the regularization procedure in [9, 10] void.

**Proposition 1.1** *Let  $\{\xi_k^1, \xi_k^2; k \geq 0\}$  be two families of independent standard Gaussian random variables, defined on a complete probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , and for  $k \geq 0$ , set  $\xi_k = \xi_k^1 + i\xi_k^2$ . Consider the process  $\Gamma'$  defined for  $z \in \Pi^+$  by  $\Gamma'_z = \sum_{k \geq 0} f_k(z)\xi_k$ . Then:*

(1)  $\Gamma'$  is a well-defined analytic process on  $\Pi^+$ , with Hermitian covariance kernel

$$\mathbb{E}\Gamma'_z\Gamma'_w = 0, \quad \mathbb{E}\Gamma'_z\bar{\Gamma}'_w = \frac{\alpha(1-2\alpha)}{2\cos\pi\alpha}(-i(z-\bar{w}))^{2\alpha-2}. \quad (1.1)$$

(2) Let  $\gamma : (0, 1) \rightarrow \Pi^+$  be any continuous path with endpoints  $\gamma(0) = 0$  and  $\gamma(1) = z$ , and set  $\Gamma_z = \int_\gamma \Gamma'_u du$ . Then  $\Gamma$  is an analytic process on  $\Pi^+$ . Furthermore, as  $z$  runs along any path in  $\Pi^+$  going to  $t \in \mathbb{R}$ , the random variables  $\Gamma_z$  converge almost surely to a random variable called again  $\Gamma_t$ .

(3) The family  $\{\Gamma_t; t \in \mathbb{R}\}$  defines a Gaussian centered complex-valued process, whose covariance function is given by:

$$\mathbb{E}[\Gamma_s \Gamma_t] = 0, \quad \mathbb{E}[\Gamma_s \bar{\Gamma}_t] = \frac{e^{-i\pi\alpha \operatorname{sgn}(s)} |s|^{2\alpha} + e^{i\pi\alpha \operatorname{sgn}(t)} |t|^{2\alpha} - e^{i\pi\alpha \operatorname{sgn}(t-s)} |s-t|^{2\alpha}}{4 \cos(\pi\alpha)}.$$

The paths of this process are almost surely  $\kappa$ -Hölder for any  $\kappa < \alpha$ .

(4) Both real and imaginary parts of  $\{\Gamma_t; t \in \mathbb{R}\}$  are (non independent) fractional Brownian motions indexed by  $\mathbb{R}$ , with covariance given by

$$\mathbb{E}[\operatorname{Re} \Gamma_s \operatorname{Im} \Gamma_t] = -\frac{\tan \pi\alpha}{8} [-\operatorname{sgn}(s)|s|^{2\alpha} + \operatorname{sgn}(t)|t|^{2\alpha} - \operatorname{sgn}(t-s)|t-s|^{2\alpha}]. \quad (1.2)$$

**Definition 1.2** Let  $Y_t := \operatorname{Re} \Gamma_{it}$ ,  $t \in \mathbb{R}_+$ . More generally,  $Y_t = (Y_t(1), \dots, Y_t(d))$  is a vector-valued process with  $d$  independent, identically distributed components.

The above results imply that  $Y_t$  is real-analytic on  $\mathbb{R}_+^*$ .

**Lemma 1.3** The infinitesimal covariance function of  $Y_t$  is:

$$\mathbb{E}Y'_s Y'_t = \frac{\alpha(1-2\alpha)}{4 \cos \pi\alpha} (s+t)^{2\alpha-2}. \quad (1.3)$$

**Proof.** Let  $X_t := \operatorname{Im} \Gamma_{it}$ . Since  $\mathbb{E}\Gamma_s \Gamma_t = 0$ ,  $(Y_s, s \geq 0)$  and  $(X_s, s \geq 0)$  have same law, with covariance kernel  $\mathbb{E}Y_s Y_t = \mathbb{E}X_s X_t = \frac{1}{2} \operatorname{Re} \Gamma_{is} \bar{\Gamma}_{it}$ . Hence

$$\mathbb{E}[Y'_s Y'_t] = \frac{1}{2} \operatorname{Re} \mathbb{E}\Gamma'_{is} \bar{\Gamma}'_{it} = \frac{\alpha(1-2\alpha)}{4 \cos \pi\alpha} (s+t)^{2\alpha-2}. \quad (1.4)$$

□

Note that  $\mathbb{E}Y'_s Y'_t > 0$ . From this simple remark follows (see proof of a similar statement in [6] concerning usual fractional Brownian motion with Hurst index  $\alpha > 1/2$ ):

**Lemma 1.4** Let  $\mathbf{Y}_{ts}^n(i_1, \dots, i_n)$ ,  $n \geq 2$  be the iterated integrals of  $Y$ . Then there exists a constant  $C > 0$  such that

$$\operatorname{Var} \mathbf{Y}_{ts}^n(i_1, \dots, i_n) \leq C \frac{(C|t-s|)^{2n\alpha}}{n!}. \quad (1.5)$$

**Proof.** Let  $\Pi$  be the set of all pairings  $\pi$  of the set  $\{1, \dots, 2n\}$  such that  $((k_1, k_2) \in \pi) \Rightarrow (i_{k'_1} = i_{k'_2})$ , where  $k'_1 = k_1$  if  $k_1 \leq n$ ,  $k_1 - n$  otherwise, and similarly for  $k'_2$ . By Wick's formula,

$$\begin{aligned} & \text{Var} \mathbf{Y}_{ts}^n(i_1, \dots, i_n) \\ &= \sum_{\pi \in \Pi} \left( \int_s^t dx_1 \dots \int_s^{x_{n-1}} dx_n \right) \left( \int_s^t dx_{n+1} \dots \int_s^{x_{2n-1}} dx_{2n} \right) \\ & \quad \prod_{(k_1, k_2) \in \pi} \mathbb{E}[Y'_{x_{k_1}} Y'_{x_{k_2}}]. \end{aligned} \quad (1.6)$$

Since the process  $Y'$  is positively correlated, and  $\Pi$  is largest when all indices  $i_1, \dots, i_n$  are equal, one gets  $\text{Var} \mathbf{Y}_{ts}^n(i_1, \dots, i_n) \leq \text{Var} \mathbf{Y}_{ts}^n(1, \dots, 1)$ . On the other hand,  $\mathbf{Y}_{ts}^n(1, \dots, 1) = \frac{1}{n!} (Y_t - Y_s)^n$ , hence

$$\text{Var} \mathbf{Y}_{ts}^n(1, \dots, 1) = \frac{[\text{Var}(Y_t - Y_s)]^n}{(n!)^2} \cdot \frac{(2n)!}{2^n \cdot n!} \leq \frac{[2\text{Var}(Y_t - Y_s)]^n}{n!}. \quad (1.7)$$

Now (assuming for instance  $0 < s < t$ )

$$\text{Var}(Y_t - Y_s) = c_\alpha \int_s^t \int_s^t (u+v)^{2\alpha-2} dudv \leq c_\alpha s^{2\alpha-2} (t-s)^2 \leq c_\alpha (t-s)^{2\alpha} \quad (1.8)$$

if  $\frac{t}{2} \leq s \leq t$ , and

$$\text{Var}(Y_t - Y_s) = \frac{c_\alpha}{2\alpha(2\alpha-1)} [(2t)^{2\alpha} + (2s)^{2\alpha} - 2(t+s)^{2\alpha}] \leq Ct^{2\alpha} \leq C'(t-s)^{2\alpha} \quad (1.9)$$

if  $s < t/2$ . Hence the result.  $\square$

## 2 Estimates for iterated integrals of $\Gamma$

The main tool for the study of  $\Gamma$  is the use of contour deformation. Iterated integrals of  $\Gamma$  are particular cases of analytic iterated integrals, see [8] or [7]. In particular, the following holds:

**Lemma 2.1** *Let  $\gamma : (0, 1) \rightarrow \Pi^+$  be the piecewise linear contour with affine parametrization defined by :*

- (i)  $\gamma([0, 1/3]) = [s, s + i|\text{Re}(t-s)|]$ ;
- (ii)  $\gamma([1/3, 2/3]) = [s + i|\text{Re}(t-s)|, t + i|\text{Re}(t-s)|]$ ;



(iii)  $\gamma([2/3, 1]) = [t + i|\operatorname{Re}(t - s)|, t]$ .

If  $z = \gamma(x) \in \gamma([0, 1])$ , we let  $\gamma_z$  be the same path stopped at  $z$ , i.e.  $\gamma_z = \gamma([0, x])$ , with the same parametrization. Then (letting  $c_\alpha = \frac{\alpha(1-2\alpha)}{2\cos\pi\alpha}$ )

$$\begin{aligned} \operatorname{Var}\mathbf{\Gamma}_{ts}^n(i_1, \dots, i_n) = & c_\alpha^n \sum_{\sigma \in \Sigma_I} \int_{\gamma} dz_1 \int_{\bar{\gamma}} d\bar{w}_1 (-i(z_1 - \bar{w}_{\sigma(1)}))^{2\alpha-2} \cdot \int_{\gamma_{z_1}} dz_2 \int_{\bar{\gamma}_{\bar{w}_1}} d\bar{w}_2 (-i(z_2 - \bar{w}_{\sigma(2)}))^{2\alpha-2} \dots \\ & \int_{\gamma_{z_{n-1}}} dz_n \int_{\bar{\gamma}_{\bar{w}_{n-1}}} d\bar{w}_n (-i(z_n - \bar{w}_{\sigma(n)}))^{2\alpha-2} \end{aligned} \quad (2.1)$$

where  $\Sigma_I$  is the subset of permutations of  $\{1, \dots, n\}$  such that  $(i_j = i_k) \Rightarrow (\sigma(j) = \sigma(k))$ .

**Proof.** Note first that, similarly to eq. (1.6),

$$\begin{aligned} \operatorname{Var}\mathbf{\Gamma}_{ts}^n(i_1, \dots, i_n) = & \sum_{\sigma \in \Sigma_I} \left( \int_1^t dz_1 \dots \int_s^{z_{n-1}} dz_n \right) \left( \int_{\bar{s}}^{\bar{t}} d\bar{w}_1 \dots \int_{\bar{s}}^{\bar{w}_{n-1}} d\bar{w}_n \right) \\ & \prod_{j=1}^n \mathbb{E} \left[ \Gamma'_{z_j} \bar{\Gamma}'_{\bar{w}_{\sigma(j)}} \right] \end{aligned} \quad (2.2)$$

(the difference with respect to eq. (1.6) comes from the fact that contractions only operate between  $\Gamma$ 's and  $\bar{\Gamma}$ 's, since  $\mathbb{E}[\Gamma_{z_j} \Gamma_{z_k}] = \mathbb{E}[\bar{\Gamma}_{\bar{w}_j} \bar{\Gamma}_{\bar{w}_k}] = 0$  by Proposition 1.1). Now the result comes from a deformation of contour, see [8].  $\square$

**Lemma 2.2** *There exists a constant  $C'$  such that, for every  $s, t \in \bar{\Pi}^+ = \Pi^+ \cup \mathbb{R}$ ,*

$$\operatorname{Var}\mathbf{\Gamma}_{ts}^n(i_1, \dots, i_n) \leq \frac{(C'|t - s|)^{2n\alpha}}{n!}. \quad (2.3)$$

**Proof.** We assume (without loss of generality) that  $\operatorname{Im} s \leq \operatorname{Im} t$ . If  $|\operatorname{Im}(t - s)| \geq c\operatorname{Re}|t - s|$  for some positive constant  $c$  (or equivalently  $|\operatorname{Re}(t - s)| \leq c'|t - s|$  for some  $0 \leq c' < 1$ ) then it is preferable to integrate along the straight line  $[s, t] = \{z \in \mathbb{C} \mid z = (1 - u)s + ut, 0 \leq u \leq 1\}$  instead of  $\gamma$ , and use the parametrization  $y = \operatorname{Im} z$ . If  $z_1, z_2 \in [s, t]$ ,  $y_1 = \operatorname{Im} z_1, y_2 = \operatorname{Im} z_2$ , then  $|(-i(z_1 - \bar{z}_2))^{2\alpha-2}| \leq C(y_1 + y_2)^{2\alpha-2}$ , hence  $\operatorname{Var}\mathbf{\Gamma}_{ts}^n(i_1, \dots, i_n) \leq C'^n \operatorname{Var}\mathbf{Y}_{y_2, y_1}^n(i_1, \dots, i_n)$ , which yields the result by Lemma 1.4. So we shall assume that  $|\operatorname{Re}(t - s)| > c|t - s|$  for some constant  $c > 0$ .

Let us use as new variable the parametrization coordinate  $x$  along  $\gamma$ . Then formula (2.1) reads

$$\begin{aligned} \text{Var}\mathbf{\Gamma}_{ts}^n(i_1, \dots, i_n) &= c_\alpha^n \sum_{\sigma \in \Sigma_I} \int_0^1 dx_1 \int_0^1 dy_1 K'(x_1, y_{\sigma(1)}) \cdot \int_0^{x_1} dx_2 \int_0^{y_1} dy_2 K'(x_2, y_{\sigma(2)}) \dots \\ &\quad \int_0^{x_{n-1}} dx_n \int_0^{x_n} dy_n K'(x_n, y_{\sigma(n)}), \end{aligned} \quad (2.4)$$

where  $K'(x, y) = (3|\text{Re}(t-s)|)^2(3(x+y)|\text{Re}(t-s)| + 2\text{Im } s)^{2\alpha-2}$  if  $0 < x, y < 1/3$ ,  $(3|\text{Re}(t-s)|)^2(3((1-x) + (1-y))|\text{Re}(t-s)| + 2\text{Im } t)^{2\alpha-2}$  if  $2/3 < x, y < 1$ , and is bounded by a constant times  $|t-s|^{2\alpha}$  otherwise thanks to the condition  $|\text{Re}(t-s)| > c|t-s|$ . Note that  $(x+y)^{2\alpha-2} > 2^{2\alpha-2}$  if  $0 < x, y < 1$ . Hence (if  $0 < x, y < 1$ )  $|K'(x, y)| \leq (C_1|t-s|)^{2\alpha} [(x+y)^{2\alpha-2} + ((1-x) + (1-y))^{2\alpha-2}]$ , which is (up to a coefficient) the infinitesimal covariance of  $|t-s|^\alpha(Y_x + \tilde{Y}_{1-x}, 0 < x < 1)$  if  $\tilde{Y} \stackrel{(law)}{=} Y$  is independent of  $Y$ . A slight modification of the argument of Lemma 1.4 yields

$$\begin{aligned} \text{Var}\mathbf{\Gamma}_{ts}^n(i_1, \dots, i_n) &\leq (C_1|t-s|)^{2n\alpha} \frac{[\text{Var}(Y_1 - Y_0) + \text{Var}(\tilde{Y}_1 - \tilde{Y}_0)]^n}{(n!)^2} \cdot \frac{(2n)!}{2^n \cdot n!} \\ &\leq C_1^{2n\alpha} \cdot \frac{(2C|t-s|)^{2n\alpha}}{n!}. \end{aligned} \quad (2.5)$$

□

### 3 Proof of main theorem

We now prove the theorem stated in the introduction, which is really a simple corollary of Lemma 2.2.

Let  $C$  be the maximum of the matrix norms  $\|V_i\| = \sup_{\|x\|_\infty=1} \|V_i x\|_\infty$  for the supremum norm  $\|x\|_\infty = \max(|x_1|, \dots, |x_r|)$ . Rewrite eq. (0.5) as

$$\mathcal{E}_V^{N,t,s}(y_s) = y_s + \sum_{n=1}^N \sum_{1 \leq i_1, \dots, i_n \leq d} a_{i_1, \dots, i_n} \mathbf{\Gamma}_{ts}^n(i_1, \dots, i_n). \quad (3.1)$$

Then  $\|a_{i_1, \dots, i_n}\|_\infty \leq C^n$  and  $|\mathbb{E} \mathbf{\Gamma}_{ts}^n(i_1, \dots, i_n)|^2 \leq \frac{(C|t-s|)^{2n\alpha}}{n!}$ . Hence (by

the Cauchy-Schwarz inequality)

$$\begin{aligned} \mathbb{E} \left( \mathcal{E}_V^{N,t,s}(y_s) - y_s \right)^2 &\leq \sum_{m,n=1}^N \frac{(C''|t-s|)^{(m+n)\alpha}}{\sqrt{m!n!}} \\ &= \left( \sum_{m=1}^N \frac{(C''|t-s|)^{m\alpha}}{\sqrt{m!}} \right)^2 \leq C''' \cdot |t-s|^{2\alpha} \quad (3.2) \end{aligned}$$

independently of  $N$ . The series obviously converges and yields eq. (0.10) for  $p = 1$ .  $\square$

It should be easy to prove along the same lines that the series defining  $\mathbb{E}|y_t - y_s|^{2p}$  converges for every  $p \geq 1$ , and that there exists a constant  $C_p$  such that  $\mathbb{E}|y_t - y_s|^{2p} \leq C_p |t-s|^{2\alpha p}$  for every  $s, t \in \bar{\Pi}^+$ .

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