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S-STORAGE OPERATORS

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Abstract In 1990, J.L. Krivine introduced the notion of storage operator to simulate, for Church integers, the “call by value” in a context of a “call by name” strategy. In this present paper, we define, for every λ -term \underline{S} which realizes the successor function on Church integers, the notion of \underline{S} -storage operator. We prove that every storage operator is a \underline{S} -storage operator. But the converse is not always true.

Mathematics Subject Classification : 03B40, 68Q60

Keywords : Church integer ; Storage operator ; Call by value ; Call by name ; Head reduction ; Solvable ; Successor ; \underline{S} -storage operator.

1 Definitions and notations

- We denote by Λ the set of λ -terms modulo α -equivalence, and by \mathcal{V} the set of λ -variables.
- Let t, u, u_1, \dots, u_n be λ -terms, the application of t to u is denoted by $(t)u$. In the same way we write $(t)u_1\dots u_n$ instead of $(\dots((t)u_1)\dots)u_n$.
- The sequence of λ -terms u_1, \dots, u_n is denoted \bar{u} .
- If $\bar{u} = u_1, \dots, u_n$, we denote by $(t)\bar{u}$ the λ -term $(t)u_1\dots u_n$.
- The β -equivalence relation is denoted by $u \simeq_\beta v$.
- The notation $\sigma(t)$ represents the result of the simultaneous substitution σ to the free variables and the constants of t after a suitable renaming of the bounded variables of t .
- Let us recall that a λ -term t either has a head redex [i.e. $t = \lambda x_1 \dots \lambda x_n (\lambda x u) v \bar{w}$, the head redex being $(\lambda x u) v$], or is in head normal form [i.e. $t = \lambda x_1 \dots \lambda x_n (x) \bar{w}$].
- The notation $u \succ v$ means that v is obtained from u by some head reductions.
- If $u \succ v$, we denote by $h(u, v)$ the length of the head reduction between u and v .
- A λ -term t is said solvable iff the head reduction of t terminates.

The following results are well known (see [3]):

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–If u is β -equivalent to a solvable λ -term, then t is solvable.

–If $u \succ v$, then, for any substitution σ , $\sigma(u) \succ \sigma(v)$, and $h(\sigma(u), \sigma(v)) = h(u, v)$.
In particular, if for some substitution σ , $\sigma(t)$ is solvable, then t is solvable.

- We define $(u)^n v$ by induction : $(u)^0 v = v$ and $(u)^{n+1} v = (u)(u)^n v$.
- For each integer n , we define the Church integer $\underline{n} = \lambda f \lambda x (f)^n x$.
- A closed λ -term \underline{S} is called successor iff, for every $k \geq 0$, $(\underline{S})\underline{k} \simeq_\beta \underline{k+1}$.

Examples Let $\underline{S}_1 = \lambda n \lambda f \lambda x (f)((n)f)x$ and $\underline{S}_2 = \lambda n \lambda f \lambda x ((n)f)(f)x$.
It is easy to check that \underline{S}_1 and \underline{S}_2 are successors. \square

2 Introduction

In λ -calculus the left reduction strategy (iteration of the head reduction) has much advantages : it always terminates when applied to a normalizable λ -term and it seems more economic since we compute a λ -term only when we need it. But the major drawback of this strategy is that a function must compute its argument every time it uses it. In 1990 J-L. Krivine introduced the notion of storage operators in order to avoid this problem and to simulate call-by-value when necessary.

Let F be a λ -term (a function), and \underline{n} a Church integer. During the computation, by left reduction, of $(F)\theta_n$ (where $\theta_n \simeq_\beta \underline{n}$), θ_n may be computed several times (as many times as F uses it). We would like to transform $(F)\theta_n$ to $(F)\tau_n$ where τ_n is a fixed closed λ -term β -equivalent to \underline{n} . We also want this transformation depends only on θ_n (and not F).

Therefore the definition : A closed λ -term T is called storage operator if and only if for every $n \geq 0$, there is a closed λ -term $\tau_n \simeq_\beta \underline{n}$ such that for every $\theta_n \simeq_\beta \underline{n}$, $(T)\theta_n f \succ (f)\tau_n$ (where f is a new variable).

Let's analyse the head reduction $(T)\theta_n f \succ (f)\tau_n$, by replacing each λ -term which comes from θ_n by a new variable.

If $\theta_n \simeq_\beta \underline{n}$, then $\theta_n \succ \lambda g \lambda x (g)t_{n-1}$, $t_{n-k} \succ (g)t_{n-k-1}$ $1 \leq k \leq n-1$, $t_0 \succ x$, and $t_k \simeq_\beta (g)^k x$ $0 \leq k \leq n-1$.

Let x_n be a new variable (x_n represents θ_n). $(T)x_n f$ is solvable, and its head normal form does not begin by λ , therefore it is a variable applied to some arguments. The free variables of $(T)x_n f$ are x_n and f , we then have two possibilities for its head normal form : $(f)\delta_n$ (in this case we stop) or $(x_n)a_1 \dots a_m$.

Assume we obtain $(x_n)a_1\dots a_m$. The variable x_n represents θ_n , and $\theta_n \succ \lambda g \lambda x(g)t_{n-1}$, therefore $(\theta_n)a_1\dots a_m$ and $((a_1)t_{n-1}[a_1/x, a_2/g])a_3\dots a_m$ have the same head normal form. The λ -term $t_{n-1}[a_1/g, a_2/x]$ comes from θ_n . Let x_{n-1, a_1, a_2} be a new variable (x_{n-1, a_1, a_2} represents $t_{n-1}[a_1/g, a_2/x]$). The λ -term $((a_1)x_{n-1, a_1, a_2})a_3\dots a_m$ is solvable, and its head normal form does not begin by λ , therefore it is a variable applied to some arguments. The free variables of $((a_1)x_{n-1, a_1, a_2})a_3\dots a_m$ are among x_{n-1, a_1, a_2} , x_n , and f , we then have three possibilities for its head normal form : $(f)\delta_n$ (in this case we stop) or $(x_n)b_1\dots b_r$ or $(x_{n-1, a_1, a_2})b_1\dots b_r$.

Assume we obtain $(x_{n-1, a_1, a_2})b_1\dots b_r$. The variable x_{n-1, a_1, a_2} represents $t_{n-1}[a_1/g, a_2/x]$, and $t_{n-1} \succ (g)t_{n-2}$, therefore $(t_{n-1}[a_1/g, a_2/x])b_1\dots b_r$ and $((a_1)t_{n-2}[a_1/g, a_2/x])b_1\dots b_r$ have the same head normal form. The λ -term $t_{n-2}[a_1/g, a_2/x]$ comes from θ_n . Let x_{n-2, a_1, a_2} be a new variable (x_{n-2, a_1, a_2} represents $t_{n-2}[a_1/g, a_2/x]$). The λ -term $((a_1)x_{n-2, a_1, a_2})b_1\dots b_r$ is solvable, and its head normal form does not begin by λ , therefore it is a variable applied to arguments. The free variables of $((a_1)x_{n-2, a_1, a_2})b_1\dots b_r$ are among x_{n-2, a_1, a_2} , x_{n-1, a_1, a_2} , v_n , and f , therefore we have four possibilities for its head normal form : $(f)\delta_n$ (in this case we stop) or $(x_n)c_1\dots c_s$ or $(x_{n-1, a_1, a_2})c_1\dots c_s$ or $(x_{n-2, a_1, a_2})c_1\dots c_s$... and so on...

Assume we obtain $(x_{0, d_1, d_2})e_1\dots e_k$ during the construction. The variable x_{0, d_1, d_2} represents $t_0[d_1/g, d_2/x]$, and $t_0 \succ x$, therefore $(t_0[d_1/g, d_2/x])e_1\dots e_k$ and $(d_2)e_1\dots e_k$ have the same head normal form ; we then follow the construction with the λ -term $(d_2)e_1\dots e_k$. The λ -term $(T)\theta_n f$ is solvable, and has $(f)\tau_n$ as head normal form, so this construction always stops on $(f)\delta_n$. We can prove by a simple argument that $\delta_n \simeq_\beta \underline{n}$.

According to the previous construction, the reduction $(T)\theta_n f \succ (f)\tau_n$ can be divided into two parts :

- A reduction that does not depend on n :

$$\begin{aligned} (T)x_n f &\succ (x_n)a_1\dots a_m \\ ((a_1)x_{n-1, a_1, a_2})a_3\dots a_m &\succ (x_{n-1, a_1, a_2})b_1\dots b_r \\ ((a_1)x_{n-2, a_1, a_2})b_1\dots b_r &\succ (x_{n-2, a_1, a_2})b_1\dots b_r \\ &\cdot \\ &\cdot \\ &\cdot \end{aligned}$$

- A transformation that depends on n (and not on θ_n) :

$$\begin{aligned} (x_n)a_1\dots a_m &\rightsquigarrow ((a_1)x_{n-1, a_1, a_2})a_3\dots a_m \\ (x_{n-1, a_1, a_2})b_1\dots b_r &\rightsquigarrow ((a_1)x_{n-2, a_1, a_2})c_1\dots c_s \\ &\cdot \\ &\cdot \end{aligned}$$

$$(x_{0,d_1,d_2})e_1\dots e_k \rightsquigarrow (d_1)e_1\dots e_k$$

We add new constants x_i and $x_{i,a,b,\bar{c}}$ in λ -calculus, and we consider the following set of head reduction rules :

$$\begin{aligned} (\lambda x u)v\bar{w} &\succ (u[v/x])\bar{w} \\ (x_{i+1})ab\bar{c} &\succ ((a)x_{i,a,b,\bar{c}})\bar{c} \\ (x_0)ab\bar{c} &\succ (b)\bar{c} \\ (x_{i+1,a,b,\bar{c}})\bar{w} &\succ ((a)x_{i,a,b,\bar{w}})\bar{w} \\ (x_{0,a,b,\bar{c}})\bar{w} &\succ (b)\bar{w} \end{aligned}$$

We write $t \succ_x t'$ if t' is obtained from t by applying these rules finitely many times.

With this formalisme we have the following result (see [1] and [4]):

A closed λ -term T is a storage operator iff for every $n \geq 0$, $(T)x_n f \succ_x (f)\tau_n$ and where τ_n is a closed λ -term β -equivalent to \underline{n} .

The constants x_i and $x_{i,a,b,\bar{c}}$ represent intuitively the λ -terms which come from a non calculated Church integer. The uniform shape of Church integers allows to describe the behaviour of these constants when they are in the head position. However, another method to describe a Church integer is simply to say that it is zero or a successor.

Formally, we add new constants X_i et $X_{i,a,b,\bar{c}}$ in λ -calculus, and we consider, for every successor \underline{S} , the following set of head reduction rules :

$$\begin{aligned} (\lambda x u)v\bar{w} &\succ (u[v/x])\bar{w} \\ (X_{i+1})ab\bar{c} &\succ ((\underline{S})X_{i,a,b,\bar{c}})ab\bar{c} \\ (X_0)ab\bar{c} &\succ (\underline{Q})ab\bar{c} \\ (X_{i+1,a,b,\bar{c}})uv\bar{w} &\succ ((\underline{S})X_{i,u,v,\bar{w}})uv\bar{w} \\ (X_{0,a,b,\bar{c}})uv\bar{w} &\succ (\underline{Q})uv\bar{w} \end{aligned}$$

We write $t \succ_X t'$ if t' is obtained from t by applying these rules finitely many times.

A \underline{S} -storage operator is defined as follows :

A closed λ -term T is a \underline{S} -storage operator iff for every $n \geq 0$, $(T)X_n f \succ_X (f)\tau_n$ where τ_n is a closed λ -term β -equivalent to \underline{n} .

This paper studies the link between the storage operators and the \underline{S} -storage operators. We prove that every storage operator is a \underline{S} -storage operator. But the converse is not always true.

3 Storage operators and \underline{S} -storage operators

Definition Let T be a closed λ -term. We say that T is a storage operator iff for every $n \geq 0$, there is a closed ³ λ -term $\tau_n \simeq_\beta \underline{n}$, such that for every $\theta_n \simeq_\beta \underline{n}$, $(T)\theta_n f \succ (f)\tau_n$.

Remark Let F be any λ -term (for a function), and θ_n a λ -term β -equivalent to \underline{n} . During the computation of $(F)\theta_n$, θ_n may be computed each time it comes in head position. Instead of computing $(F)\theta_n$, let us look at the head reduction of $(T)\theta_n F$. Since it is $\{(T)\theta_n f\}[F/f]$, we shall first reduce $(T)\theta_n f$ to its head normal form, which is $(f)\tau_n$, and then compute $(F)\tau_n$. The computation has been decomposed into two parts, the first being independent of F . This first part is essentially a computation of θ_n , the result being τ_n , which is a kind of normal form of θ_n . So, in the computation of $(T)\theta_n F$, θ_n is computed first, and the result is given to F as an argument, T has stored the result, before giving it, as many times as needed, to any function. \square

Examples Let \underline{S} be a successor. If we take :

$T_1 = \lambda n((n)G)\delta$ where $G = \lambda x \lambda y(x) \lambda z(y)(\underline{S})z$ and $\delta = \lambda f(f)\underline{0}$

$T_2 = \lambda n \lambda f(((n)F)f)\underline{0}$ where $F = \lambda x \lambda y(x)(\underline{S})y$,

then it is easy to check that (see [1] and [3]):

for every $\theta_n \simeq_\beta \underline{n}$, $(T_i)\theta_n f \succ (f)(\underline{S})^n \underline{0}$ ($i = 1$ or 2).

Therefore T_1 and T_2 are storage operators. \square

Let $\{x_i\}_{i \geq 0}$ be a set of different constants. We define a set of terms (denoted by Λ_x) in the following way :

- If $x \in \mathcal{V} \cup \{x_i\}_{i \geq 0}$, then $x \in \Lambda_x$;
- If $x \in \mathcal{V}$, and $u \in \Lambda_x$, then $\lambda x u \in \Lambda_x$;
- If $u \in \Lambda_x$, and $v \in \Lambda_x$, then $(u)v \in \Lambda_x$;
- If $n \in \mathbb{N}$, and $a, b, \bar{c} \in \Lambda_x$, then $x_{n,a,b,\bar{c}} \in \Lambda_x$.

$x_{n,a,b,\bar{c}}$ is considered as a constant which does not appear in a, b, \bar{c} .

The terms of the set Λ_x are called λx -terms.

We have the following result (see [1] and [4]) :

A closed λ -term T is a storage operator iff for every $n \geq 0$, there is a finite sequence of head reduction $\{U_i \succ V_i\}_{1 \leq i \leq r}$ such that :

- 1) U_i and V_i are λx -terms ;

³In his definition of storage operator, J.L. Krivine autorizes the τ_n to contain free variables which are replaced by terms depend of θ_n . The results of this paper remain valid with this definition but the proofs will be too technical.

- 2) $U_1 = (T)x_n f$ and $V_r = (f)\tau_n$ where τ_n is closed λ -term β -equivalent to \underline{n} ;
- 3) $V_i = (x_n)ab\bar{c}$ or $V_i = (x_{l,a,b,\bar{c}})\bar{d}$ $0 \leq l \leq n - 1$;
- 4) If $V_i = (x_n)ab\bar{c}$, then $U_{i+1} = (b)\bar{c}$ if $n = 0$ and $U_{i+1} = ((a)x_{n-1,a,b,\bar{c}})\bar{c}$ if $n \neq 0$;
- 5) If $V_i = (x_{l,a,b,\bar{c}})\bar{d}$ $0 \leq l \leq n-1$, then $U_{i+1} = (b)\bar{d}$ if $l = 0$ and $U_{i+1} = ((a)x_{l-1,a,b,\bar{d}})\bar{d}$ if $l \neq 0$.

Definitions

1) Let $\{X_i\}_{i \geq 0}$ be a set of different constants. We define a set of terms (denoted by Λ_X) in the following way :

- If $x \in \mathcal{V} \cup \{X_i\}_{i \geq 0}$, then $x \in \Lambda_X$;
- If $x \in \mathcal{V}$, and $u \in \Lambda_X$, then $\lambda x u \in \Lambda_X$;
- If $u \in \Lambda_X$, and $v \in \Lambda_X$, then $(u)v \in \Lambda_X$;
- If $n \in \mathbb{N}$, and $a, b, \bar{c} \in \Lambda_X$, then $X_{n,a,b,\bar{c}} \in \Lambda_X$.

$X_{n,a,b,\bar{c}}$ is considered as a constant which does not appear in a, b, \bar{c} .

The terms of the set Λ_X are called λX -terms.

2) Let \underline{S} be a successor. A closed λ -term T is called a \underline{S} -storage operator iff for every $n \geq 0$, there is a finite sequence of head reduction $\{U_i \succ V_i\}_{1 \leq i \leq r}$ such that :

- 1) U_i and V_i are λX -terms ;
- 2) $U_1 = (T)X_n f$ and $V_r = (f)\tau_n$ where τ_n is closed λ -term β -equivalent to \underline{n} ;
- 3) $V_i = (X_n)ab\bar{c}$ or $V_i = (X_{l,a,b,\bar{c}})uv\bar{w}$ $0 \leq l \leq n - 1$;
- 4) If $V_i = (X_n)ab\bar{c}$, then $U_{i+1} = (\underline{0})ab\bar{c}$ if $n = 0$ and $U_{i+1} = ((\underline{S})X_{n-1,a,b,\bar{c}})ab\bar{c}$ if $n \neq 0$;
- 5) If $V_i = (X_{l,a,b,\bar{c}})uv\bar{w}$ $0 \leq l \leq n - 1$, then $U_{i+1} = (\underline{0})uv\bar{w}$ if $l = 0$ and $U_{i+1} = ((\underline{S})X_{l-1,u,v,\bar{w}})uv\bar{w}$ if $l \neq 0$.

Examples It is easy to check that, for $1 \leq i, j \leq 2$, the above operator T_i is an \underline{S}_j -storage operator. We check here (for example) that T_2 is an \underline{S}_2 -storage operator:

Let n be an integer.

If $n = 0$, then we check that $(T_2)X_n f \succ (X_n)F f \underline{0}$ and $(\underline{0}) F f \underline{0} \succ (f)\underline{0}$.

If $n \neq 0$, then we check that:

$$\begin{aligned}
(T)X_n f &\succ (X_n) F f \underline{0} \\
((\underline{S_2})X_{n-1, F, f, \underline{0}}) F f \underline{0} &\succ (X_{n-1, F, f, \underline{0}}) F (F) f \underline{0} \\
&\vdots \\
&\vdots \\
&\vdots \\
((\underline{S_2})X_{0, F, (F)^{n-1} f, \underline{0}}) F (F)^{n-1} f \underline{0} &\succ (X_{0, F, (F)^{n-1} f, \underline{0}}) F (F)^n f \underline{0} \\
(\underline{0}) F (F)^n f \underline{0} &\succ (F)^n f \underline{0}
\end{aligned}$$

We prove (by induction on k) that, for every λ -term u , and for every $0 \leq k \leq n$, we have $(F)^k f u \succ (f)(\underline{S_2})^k u$.

- For $k = 0$, it is true.

- Assume that is true for k , and prove it for $k + 1$.

$(F)^{k+1} f u = (F)(F)^k f u \succ (F)^k f (\underline{S_2})u$. By induction hypothesis we have that for every λ -term v , $(F)^k f v \succ (f)(\underline{S_2})^k v$, then $(F)^{k+1} f u \succ (f)(\underline{S_2})(\underline{S_2})^k u = (f)(\underline{S_2})^{k+1} u$.

In particular, for $u = \underline{0}$ and $k = n$, we have $(F)^n f \underline{0} \succ (f)(\underline{S_2})^n \underline{0}$.

Therefore T_2 is a $\underline{S_2}$ -storage operator. \square

A question arises : **Is there a link between the storage operators and the \underline{S} -storage operators ?**

4 Link between the storage operators and the \underline{S} -storage operators

Theorem 1 *If T is a storage operator, then, for every successor \underline{S} , T is a \underline{S} -storage operator.*

Proof Let \underline{S} be a successor and T a storage operator.

Then for every $n \geq 0$, there is a closed λ -term $\tau_n \simeq_\beta \underline{n}$ such that for every $\theta_n \simeq_\beta \underline{n}$, $(T)\theta_n f \succ (f)\tau_n$. In particular $((T)(\underline{S})^n \underline{0}) f \succ (f)\tau_n$.

Let $\sigma : \Lambda_X \rightarrow \Lambda$ the simultaneous substitution defined by :

$$\begin{aligned}
\sigma(X_n) &= (\underline{S})^n \underline{0} \\
\text{for every } 0 \leq k \leq n-1, \sigma(X_{k, a, b, \bar{c}}) &= (\underline{S})^k \underline{0} \\
\sigma(x) &= x \text{ if } x \neq X_n, X_{k, a, b, \bar{c}}
\end{aligned}$$

For every $n \geq 0$, we construct a set of head equation $\{U_i \succ V_i\}_{1 \leq i \leq r}$ such that :

1) U_i and V_i are λX -terms ;

- 2) $V_r = (f)\delta_n$;
- 3) for every $1 \leq i \leq r - 1$, $V_i = (X_n)ab\bar{c}$ or $V_i = (X_{l,a,b,\bar{c}})uv\bar{w}$;
- 4) $\sigma(V_i)$ is solvable.

Let $U_1 = (T)X_n f$. We have $\sigma(U_1) = ((T)(\underline{S})^n \underline{Q})f$ is solvable, then U_1 is solvable and $U_1 \succ V_1$ where $V_1 = (f)\delta_n$ or $V_1 = (X_n)ab\bar{c}$. It is clear that $\sigma(V_1)$ is solvable.

Assume that we have the head reduction $U_k \succ V_k$ and $V_k \neq (f)\delta_n$.

- If $V_k = (X_n)ab\bar{c}$, then, by induction hypothesis, $\sigma(V_k) = ((\underline{S})^n \underline{Q})\sigma(a)\sigma(b)\overline{\sigma(c)}$ is solvable.

- If $n = 0$, let $U_{k+1} = (\underline{Q})ab\bar{c}$. Then $\sigma(U_{k+1}) = (\underline{Q})\sigma(b)\sigma(b)\overline{\sigma(c)}$ is solvable.

- If $n \neq 0$, let $U_{k+1} = ((\underline{S})X_{n-1,a,b,\bar{c}})ab\bar{c}$. Then $\sigma(U_{k+1}) = ((\underline{S})(\underline{S})^{n-1} \underline{Q})\sigma(a)\sigma(b)\overline{\sigma(c)} = \sigma(V_k)$ is solvable.

- If $V_k = (X_{l,a,b,\bar{c}})uv\bar{w}$, then, by induction hypothesis, $\sigma(V_k) = ((\underline{S})^l \underline{Q})\sigma(u)\sigma(v)\overline{\sigma(w)}$ is solvable.

- If $l = 0$, let $U_{k+1} = (\underline{Q})uv\bar{w}$. Then $\sigma(U_{k+1}) = (\underline{Q})\sigma(u)\sigma(v)\overline{\sigma(w)}$ is solvable.

- If $l \neq 0$, let $U_{k+1} = ((\underline{S})X_{l-1,u,v,\bar{w}})uv\bar{w}$. Then $\sigma(U_{k+1}) = ((\underline{S})(\underline{S})^{l-1} \underline{Q})\sigma(u)\sigma(v)\overline{\sigma(w)} = \sigma(V_k)$ is solvable.

Therefore U_{k+1} is solvable and $U_{k+1} \succ V_{k+1}$ where $V_{k+1} = (f)\delta_n$ or $V_{k+1} = (X_n)a'b'\bar{c}'$ or $V_{k+1} = (X_{r,a',b',\bar{c}'})a''b''\bar{c}''$. Since $\sigma(U_{k+1})$ is solvable, then $\sigma(V_{k+1})$ is also solvable.

This construction always terminates (i.e there is a $r \geq 0$ such that $V_r = (f)\delta_n$). Indeed, if not, we check easily that the λ -term $((T)(\underline{S})^n \underline{Q})f$ is not solvable.

Let y be a variable, $\hat{\underline{S}} = (\lambda x \underline{S})y$, and $\hat{\underline{Q}} = (\lambda x \underline{Q})y$.

Let $\hat{\sigma} : \Lambda_X \rightarrow \Lambda$ the simultaneous substitution defined by :

$$\begin{aligned} \hat{\sigma}(X_n) &= (\hat{\underline{S}})^n \hat{\underline{Q}} \\ \text{for every } 0 \leq k \leq n-1, \hat{\sigma}(X_{k,a,b,\bar{c}}) &= (\hat{\underline{S}})^k \hat{\underline{Q}} \\ \hat{\sigma}(x) &= x \text{ if } x \neq X_n, X_{k,a,b,\bar{c}} \end{aligned}$$

Since $(\hat{\underline{S}})t \succ (\underline{S})t$ and $\hat{\underline{Q}} \succ \underline{Q}$, we check easily that $((T)(\hat{\underline{S}})^n \hat{\underline{Q}})f \succ (f)\hat{\sigma}(\delta_n)$. But $(\hat{\underline{S}})^n \hat{\underline{Q}} \simeq_\beta \underline{n}$, then $((T)(\hat{\underline{S}})^n \hat{\underline{Q}})f \succ (f)\tau_n$. Therefore $\hat{\sigma}(\delta_n) = \tau_n$. Since τ_n is closed, then δ_n is also closed and $\delta_n = \tau_n \simeq_\beta \underline{n}$.

Therefore T is a \underline{S} -storage operator. \square

Definition We say that a λX -term U satisfies the property (P) iff for each constant $X_{l,a,b,\bar{c}}$ of U we have :

- a, b, \bar{c} satisfy (P)
- $X_{l,a,b,\bar{c}}$ is applied to a and b ;
- a, b do not contain free variables which are bounded in U .

Lemma 1 *Let U, V be λX -terms which do not begin by λ . If U satisfies (P) and $U \succ V$, then V satisfies (P) .*

Proof It is enough to do the proof for one step of head reduction. We have $U = (\lambda x u)v\bar{w}$ and $V = (u[v/x])\bar{w}$. Since U satisfies (P) , then u, v, \bar{w} satisfy (P) and x is not free in a, b if the constant $X_{l,a,b,\bar{a}}$ appears in u . Therefore $u[v/x], u_1, \dots, u_m$ satisfy (P) and V satisfies (P) . \square

Let $\Delta : \Lambda_x \rightarrow \Lambda_X$ the simultaneous substitution defined by :

$$\begin{aligned} \Delta(x_n) &= X_n \\ \text{for every } 0 \leq k \leq n-1, \Delta(x_{k,a,b,\bar{c}}) &= (X_{k,\Delta(a),\Delta(b),\overline{\Delta(c)}})\Delta(a)\Delta(b) \\ \sigma(x) &= x \text{ if } x \neq x_n, x_{k,a,b,\bar{c}} \end{aligned}$$

Lemma 2 *If U is a λX -term satisfies (P) , then there is a λx -term U' such that $\Delta(U') = U$.*

Proof By induction on U .

- For $U = x$, it is true.
- If $U = \lambda x V$, then V satisfies (P) , and, by induction hypothesis, there is a λx -term V' such that $\Delta(V') = V$. We put $U' = \lambda x V'$. We have $\Delta(U') = U$.
- If $U = (U_1)U_2$ (where U_1 does not begin by a constant), then U_1, U_2 satisfy (P) , and, by induction hypothesis, there are λx -terms U'_1, U'_2 such that $\Delta(U'_1) = U_1$ and $\Delta(U'_2) = U_2$. We put $U' = (U'_1)U'_2$. We have $\Delta(U') = U$.
- If $U = (X_{k,a,b,\bar{c}})ab\bar{V}$, then a, b, \bar{c}, \bar{V} satisfy (P) , and, by induction hypothesis, there are λx -terms $a', b', \bar{c}', \bar{V}'$ such that $\Delta(a') = a, \Delta(b') = b, \Delta(\bar{c}') = \bar{c}$, and $\Delta(\bar{V}') = \bar{V}$. We put $U' = (X_{k,a',b',\bar{c}'}\bar{V}')$. We have $\Delta(U') = U$. \square

Theorem 2 *T is a \underline{S}_1 -storage operator iff T is a storage operator.*

Proof Let $n \geq 0$. If T is a \underline{S}_1 -storage operator, then there is a finite sequence of head reduction $\{U_i \succ V_i\}_{1 \leq i \leq r}$ such that :

- 1) U_i and V_i are λX -terms ;
- 2) $U_1 = (T)X_n f$ and $V_r = (f)\tau_n$ where τ_n is closed λ -term β -equivalent to \underline{n} ;

- 3) $V_i = (X_n)ab\bar{c}$ or $V_i = (X_{l,a,b,\bar{c}})uv\bar{w}$ $0 \leq l \leq n - 1$;
- 4) If $V_i = (X_n)ab\bar{c}$, then $U_{i+1} = (\underline{0})ab\bar{c}$ if $n = 0$ and $U_{i+1} = ((\underline{S}_1)X_{n-1,a,b,\bar{c}})ab\bar{c}$ if $n \neq 0$;
- 5) If $V_i = (X_{l,a,b,\bar{c}})uv\bar{w}$ $0 \leq l \leq n - 1$, then $U_{i+1} = (\underline{0})uv\bar{w}$ if $l = 0$ and $U_{i+1} = ((\underline{S}_1)X_{l-1,u,v,\bar{w}})uv\bar{w}$ if $l \neq 0$.

We prove (by induction on i) that, for every $1 \leq i \leq r$, V_i satisfies (P) .

- For $i = 1$, it is true.
- Assume that is true for i , and prove it for $i + 1$.

If $V_i = (X_n)ab\bar{c}$, we have two cases :

- if $n = 0$, then $U_{i+1} = (\underline{0})ab\bar{c}$. By induction hypothesis V_i satisfies (P) , then a, b, \bar{c} satisfy (P) , therefore U_{i+1} and V_{i+1} satisfy (P) .
- if $n \neq 0$, then $U_{i+1} = ((\underline{S})X_{n-1,a,b,\bar{c}})ab\bar{c}$. By induction hypothesis V_i satisfies (P) , then a, b, \bar{c} satisfy (P) . Since $U_{i+1} \succ ((a)(X_{n-1,a,b,\bar{c}})ab)\bar{c}$, then V_{i+1} satisfies (P) .

If $V_i = (X_{l,a,b,\bar{c}})uv\bar{w}$ $0 \leq l \leq n - 1$, then $u = a$, $v = b$, and $\bar{w} = \bar{c}$ since, by induction hypothesis, V_i satisfies (P) . We have two cases :

- if $n = 0$, then $U_{i+1} = (\underline{0})ab\bar{c}$. By induction hypothesis V_i satisfies (P) , then a, b, \bar{c} satisfy (P) , therefore U_{i+1} and V_{i+1} satisfy (P) .
- if $n \neq 0$, then $U_{i+1} = ((\underline{S})X_{l-1,a,b,\bar{c}})ab\bar{c}$. By induction hypothesis V_i satisfies (P) , then a, b, \bar{c} satisfy (P) . Since $U_{i+1} \succ ((a)(X_{l-1,a,b,\bar{c}})ab)\bar{c}$, then V_{i+1} satisfies (P) .

Therefore there is a finite sequence of head reduction $\{M_i \succ N_i\}_{1 \leq i \leq r}$ such that :

- 1) M_i and N_i are λX -terms ;
- 2) $M_1 = (T)X_n f$ and $N_r = (f)\tau_n$ where τ_n is closed λ -term β -equivalent to \underline{n} ;
- 3) $N_i = (X_n)ab\bar{c}$ or $N_i = (X_{l,a,b,\bar{c}})ab\bar{d}$ $0 \leq l \leq n - 1$;
- 4) If $N_i = (X_n)ab\bar{c}$, then $M_{i+1} = (b)\bar{c}$ if $n = 0$ and $M_{i+1} = ((a)(X_{n-1,a,b,\bar{c}})ab)\bar{c}$ if $n \neq 0$;
- 5) If $N_i = (X_{l,a,b,\bar{c}})ab\bar{d}$ $0 \leq l \leq n - 1$, then $M_{i+1} = (b)\bar{d}$ if $l = 0$ and $M_{i+1} = ((a)(X_{l-1,a,b,\bar{d}})ab)\bar{d}$ if $l \neq 0$.

Since, for every $1 \leq i \leq r$, M_i and N_i satisfy (P) , let M'_i and N'_i the λx -terms such that : $\Delta(M'_i) = M_i$ and $\Delta(N'_i) = N_i$. We have :

- 1) M'_i and N'_i are λx -terms ;

- 2) $M'_1 = (T)x_n f$ and $N'_r = (f)\tau'_n$ where τ'_n is closed λ -term β -equivalent to \underline{n} ;
- 3) $N'_i = (x_n)a'b'\overline{c'}$ or $N'_i = (x_{l,a',b',\overline{c'}})\overline{d'}$ $0 \leq l \leq n-1$;
- 4) If $N'_i = (x_n)a'b'\overline{c'}$, then $M'_{i+1} = (b')\overline{c'}$ if $n = 0$ and $M'_{i+1} = ((a')x_{n-1,a',b',\overline{c'}})\overline{c'}$ if $n \neq 0$;
- 5) If $N'_i = (x_{l,a',b',\overline{c'}})\overline{d'}$ $0 \leq l \leq n-1$, then $M'_{i+1} = (b')\overline{d'}$ if $l = 0$ and $M'_{i+1} = ((a')x_{l-1,a',b',\overline{d'}})\overline{d'}$ if $l \neq 0$.

Therefore T is a storage operator. \square

Theorem 3 *There is a \underline{S}_2 -storage operator which is a no storage operator.*

Proof Let $T = \lambda x(x) a b \underline{0} \underline{S}$ where
 $a = \lambda x \lambda y \lambda z((x)(z)(x)II\lambda x \underline{0})\lambda x(\underline{S})(z)x$,
 $b = \lambda x \lambda y \lambda z(z)x$,
and \underline{S} a successor.

Let n be an integer.

If $n = 0$, then we check that :

$$\begin{aligned} (T) X_n f &\succ (X_n) a b \underline{0} \underline{S} f \\ (\underline{0}) a b \underline{0} \underline{S} f &\succ (f)\underline{0} \end{aligned}$$

If $n \neq 0$, then we check that :

$$\begin{aligned} (T) X_n f &\succ (X_n) a b \underline{0} \underline{S} f \\ ((\underline{S}_2)X_{n-1,a,b,\underline{0},\underline{S},f}) a b \underline{0} \underline{S} f &\succ (X_{n-1,a,b,\underline{0},\underline{S},f}) a (a)b \underline{0} \underline{S} f \\ &\vdots \\ ((\underline{S}_2)X_{0,a,(a)^{n-1}b,\underline{0},\underline{S},f}) a (a)^{n-1}b \underline{0} \underline{S} f &\succ (X_{0,a,(a)^{n-1}b,\underline{0},\underline{S},f}) a (a)^n b \underline{0} \underline{S} f \\ (\underline{0}) a (a)^n b \underline{0} \underline{S} f &\succ (a)^n b \underline{0} \underline{S} f \end{aligned}$$

We define two sequences of λ -terms $(P_i)_{0 \leq i \leq n}$ and $(Q_i)_{0 \leq i \leq n}$ by :

$$\begin{aligned} Q_0 &= \underline{S}, \text{ and, for every } 0 \leq k \leq n-1, \text{ we put } Q_{k+1} = \lambda x(\underline{S})(Q_k)x \\ P_0 &= \underline{0}, \text{ and, for every } 0 \leq k \leq n-1, \text{ we put } P_{k+1} = (Q_k)((a)^{n-k-1}b)II\lambda x \underline{0} \end{aligned}$$

It is easy to check that, for every $1 \leq k \leq n$, $Q_k \simeq_\beta \lambda x(\underline{S})^{k+1}x$.

We prove (by induction on k) that, for every $0 \leq k \leq n$, we have $(a)^n b \underline{0} \underline{S} f \succ (a)^{n-k} b P_k Q_k f$.

- For $k = 0$, it is true.

- Assume that is true for k , and prove it for $k + 1$.

$$\begin{aligned} (a)^{n-k} b P_k Q_k f &= (a) (a)^{n-k-1} b P_k Q_k f \succ \\ ((a)^{n-k-1} b) (Q_k) ((a)^{n-k-1} b) II \lambda x \underline{0} \lambda x (\underline{S})(Q_k) x f &= (a)^{n-k+1} b P_{k+1} Q_{k+1} f. \end{aligned}$$

In particular, for $k = n$, we have $(a)^n b \underline{0} \underline{S} f \succ (b) P_n Q_n f \succ (f) P_n$.
 $P_n = (Q_{n-1})(b) II \lambda x \underline{0} \simeq_{\beta} (\lambda x (\underline{S})^n x)(b) II \lambda x \underline{0} \simeq_{\beta} (\underline{S})^n (\lambda x \underline{0}) I \simeq_{\beta} (\underline{S})^n \underline{0} \simeq_{\beta} \underline{n}$.

Therefore T is a \underline{S}_2 -storage operator.

We define a sequence of λ -terms $(P'_i)_{0 \leq i \leq n}$ by :

$$P'_0 = \underline{0}, \text{ and for every } 0 \leq k \leq n-1, \text{ we put } P'_{k+1} = (Q_k)(x_{n-k-1, a, b, P'_{n-k}, Q_{n-k}, f}) II J$$

We check (as before) that :

$$\begin{aligned} (T) x_n f &\succ (x_n) a b P'_0 Q_0 f \\ (a) x_{n-1, a, b, P'_0, Q_0, f} P'_0 Q_0 f &\succ (x_{n-1, a, b, P'_0, Q_0, f}) P'_1 Q_1 f \\ &\vdots \\ &\vdots \\ (a) x_{0, a, b, P'_{n-1}, Q_{n-1}, f} P'_{n-1} Q_{n-1} f &\succ (x_{0, a, b, P'_{n-1}, Q_{n-1}, f}) P'_n Q_n f \\ (b) P'_n Q_n f &\succ (f) P'_n \end{aligned}$$

But $P'_n = (Q_{n-1})(x_{0, a, b, P'_{n-1}, Q_{n-1}, f}) II \lambda x \underline{0}$ is not closed.

Note that $P'_n \simeq_{\beta} (\underline{S})^n (X_{0, a, b, P'_{n-1}, Q_{n-1}, f}) II \lambda x \underline{0} \not\simeq_{\beta} \underline{n}$. Indeed, if $(\underline{S})^n (X_{0, a, b, P'_{n-1}, Q_{n-1}, f}) II \lambda x \underline{0} \simeq_{\beta} \underline{n}$, then $(\underline{S})^n (\lambda x_1 \lambda x_2 \lambda x_3 (\underline{S}) \underline{0}) II \lambda x \underline{0} \simeq_{\beta} \underline{n}$, therefore $\underline{n+1} \simeq_{\beta} \underline{n}$. A contradiction.

Therefore T is a no storage operator. \square

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