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► **To cite this version:**

René David, Karim Nour. A syntactical proof of the operational equivalence of two λ -terms. Theoretical Computer Science, 1997, 180, pp.371-375. hal-00381603

HAL Id: hal-00381603

<https://hal.science/hal-00381603v1>

Submitted on 6 May 2009

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A syntactical proof of the operational equivalence of two λ -terms

René DAVID and Karim NOUR

Abstract *In this paper we present a purely syntactical proof of the operational equivalence of $I = \lambda xx$ and the λ -term J that is the η -infinite expansion of I .*

1 Introduction

Two λ -terms M and N are operationnely equivalent ($M \simeq_{oper} N$) iff for all context $C : C[M]$ is solvable iff $C[N]$ is solvable.

Let $I = \lambda xx$ and $J = (Y G)$ where Y is the Turing's fixed point operator and $G = \lambda x \lambda y \lambda z (y (x z))$. J is the η -infinite expansion of I . His Bóhm tree is in fact $\lambda x \lambda x_1 (x \lambda x_2 (x_1 \lambda x_3 (x_2 \lambda x_4 (x_3 \dots)))$. The following Theorem is well known (see [1],[3]).

Theorem $I \simeq_{oper} J$.

The usual proof is semantic : two λ -terms are operationnely equivalent iff they have the same interpretation in the modele D_∞ .

We give below an elementary and a purely syntactical proof of this result. This proof analyses in a fine way the reductions of $C[I]$ and $C[J]$ by distinguant the "real" β -redex of ceux which come of the η -expansion.

This proof may be generalize to prove (this result is also well known) the operationnely equivalence of two λ -terms where the Bóhm tree are equal à η -infinite expansion près. The necessary technical tool is the directed λ -calculus (see [2]).

2 Definitions and notations

- $\lambda \bar{x} U$ represents a sequence of abstractions.
- Let T, U, U_1, \dots, U_n be λ -terms, the application of T to U is denoted by $(T U)$ or TU . In the same way we write $TU_1 \dots U_n$ or $T\bar{U}$ instead of $(\dots(T U_1) \dots U_n)$.
- Let us recall that a λ -term T either has a head redex [i.e. $t = \lambda \bar{x} (\lambda x U V) \bar{V}$, the head redex being $(\lambda x U V)$], or is in head normal form [i.e. $t = \lambda \bar{x} x \bar{V}$].

- The notation $U \rightarrow_t V$ (resp. $U \rightarrow_{t^*} V$) means that V is obtained from U by one head reduction (resp. some head reductions).
- A λ -term T is said solvable iff the head reduction of T terminates.

The following Lemma is well known.

Lemma 2.1 *($U V$) is solvable iff U is solvable (and has U' as head normal form) and $(U' V)$ is solvable.*

3 Proof of the Theorem

The idea of the proof is the following : we prove that, if we assimilate the reductions where I (resp J) are in head position, $C[I]$ and $C[J]$ reduce, by head reduction in the same way. For this we add a constante H (which represente either I or J). We define on those terms the I (resp J) head reduction, corresponding to the case where $H = I$ (resp J). To prove that the reductions are equivalent we prove that the terms obtained by "removing" the constante H are equal. This is the role of the extraction fonction E .

3.1 λH -calculus and the application E

- We add a new constante H to the λ -calculus and we call λH -terms the terms which we obtain.
- We define (by induction) on the set of λH -terms the application E :

$$\begin{aligned} E(x) &= x ; E(H) = H ; E(\lambda x U) = \lambda x E(U) ; \\ E(UV) &= E(U)E(V) \text{ if } U \neq HU_1U_2\dots U_n ; \\ E(HU_1U_2\dots U_n) &= E(U_1U_2\dots U_n) . \end{aligned}$$

- A λH -term is in head normal form if it is of the forme : $\lambda \bar{x} H$ or $\lambda \bar{x} x \bar{V}$.

Lemma 3.1 *If T is a λH -term, then $E(T)$ is of the forme $\lambda \bar{x} H$ or $\lambda \bar{x} x \bar{V}$ or $\lambda \bar{x} (\lambda x U V) \bar{V}$.*

Proof By induction on T . \square

Lemma 3.2 *If T is a λH -term, then $E(E(T)) = E(T)$.*

Proof By induction on T . \square

Lemma 3.3 *Let T, \bar{U} be λH -terms. $E(T\bar{U}) = E(E(T)\overline{E(\bar{U})})$.*

Proof By induction on T . We distinguish the cases: $T \neq H\bar{V}$ and $T = H\bar{V}$. \square

Lemma 3.4 *Let U, V be λH -terms and x a variable, $E(U[V/x]) = E(E(U)[E(V)/x])$.*

Proof By induction on U . The only interesting case is $U = x\bar{U}$. By Lemma 3.3, $E(U[V/x]) = E(E(V)\overline{E(U[V/x])})$. Therefore, by induction hypothesis and Lemma 3.3, $E(U[V/x]) = E(E(V)\overline{E(E(U)[E(V)/x])}) = E(E(U)[E(V)/x])$. \square

Lemma 3.5 *Let U_1, U_2, V_1, V_2 be λH -terms such that $E(U_1) = E(U_2)$ and $E(V_1) = E(V_2)$. $E(U_1[V_1/x]) = E(U_2[V_2/x])$.*

Proof By Lemma 3.4. \square

Lemma 3.6 *Let U_1, U_2, V_1, V_2 be λH -terms. If $U_1 \rightarrow_t V_1$, $U_2 \rightarrow_t V_2$, and $E(U_1) = E(U_2)$, then $E(V_1) = E(V_2)$.*

Proof By Lemmas 3.3 and 3.5. \square

3.2 The I -reduction

- We define on the λH -terms a new head reduction :

$$HU_1\dots U_n \rightarrow_I U_1U_2\dots U_n$$

- We denote by \rightarrow_{I^*} the reflexive and transitive closure of \rightarrow_I .
- A λH -term U is I - t -solvable iff a finite sequence of I -reductions and t -reductions of U gives a head normal form.

Lemma 3.7 *Let U, V be λH -terms. If $U \rightarrow_{I^*} V$, then $E(U) = E(V)$.*

Proof By induction on the reduction of U . \square

Lemma 3.8 *Each I -reduction is finite.*

Proof The I -reduction decreases the complexity of a λH -term. \square

Lemma 3.9 *Let U be λH -term. U is I - t -solvable iff $U[I/H]$ is solvable.*

Proof Immediate. \square

3.3 The J -reduction

- We define on the λH -terms a new head reduction :

$$HU_1 \dots U_n \rightarrow_J U_1(H U_2)U_3 \dots U_n$$

- We denote by \rightarrow_{J^*} the reflexive and transitive closure of \rightarrow_J .
- A λH -term U is J - t -solvable iff a finite sequence of J -reductions and t -reductions of U gives a head normal form.

Lemma 3.10 *Let U, V be λH -terms. If $U \rightarrow_{J^*} V$, then $E(U) = E(V)$.*

Proof It is enough to do the proof for one step of J -reduction. The only interesting case is $U = (H)U_1U_2\bar{U}$. In this case $U \rightarrow_J U_1(H U_2)\bar{U}$, and, by induction hypothesis, $E((U_1(H U_2)\bar{U})) = E(V)$, therefore -by Lemma 3.3- $E(U) = E(V)$. \square

Lemma 3.11 *Let U, V be λH -terms. If $U \rightarrow_{J^*} V$, then, for each sequence $\bar{W} = W_1 \dots W_n$, there is a sequence $\bar{W}' = W'_1 \dots W'_n$ such that $U\bar{W} \rightarrow_{J^*} V\bar{W}'$ and for, all $1 \leq k \leq n$, $W'_k \rightarrow_{J^*} W_k$.*

Proof By induction on the reduction of U . It enough to do the proof for one step of J -reduction. The only interesting case is $U = HU'$ and $\bar{W} = W_1\bar{W}'$. In this case $V = U'$, $UW_1\bar{W}' \rightarrow_J V(H W_1)\bar{W}'$ and $HW_1 \rightarrow_J W_1$. \square

Lemma 3.12 *Each J -reduction is finite.*

Proof By induction on U . The only interesting case is $U = HV_1 \dots V_n$ ($n \geq 2$). We prove, by recurrence on n , that if the reductions of V_1, \dots, V_n are finite, then so is for $U = HV_1 \dots V_n$. $U \rightarrow_J V_1(H V_2) V_3 \dots V_n$ and $V_1 \rightarrow_{J^*} V'_1$. By Lemma 3.11, $U \rightarrow_J V'_1W_2W_3 \dots W_n$ where $W_2 \rightarrow_J H V_2 \rightarrow_J V_2$ and $W_i \rightarrow_J V_i$, therefore the reductions of W_i are finite.

- If $E(V_1) \neq H$. V'_1 begin soit by λ , soit by a β -redex, soit by a variable. Therefore, by Lemma 3.11, the J -reduction of U is finite.

- If $E(V_1) = H$. By Lemma 3.11, $U \rightarrow_{J^*} HW_2 \dots W_n$ and the recurrence hypothesis allows to conclude. \square

Lemma 3.13 *Let U be a λH -term. U is J - t -solvable iff $U[J/H]$ is solvable.*

Proof The only difficulty is to prove that : if U is J - t -solvable, then $U[J/H]$ is solvable.

We prove that by induction on the reduction of U . The only interesting case is $U = \lambda\bar{x} HV$. In this case, $U \rightarrow_J \lambda\bar{x} V$ and $U[J/H] \rightarrow_t \lambda\bar{x} \lambda y V[J/H] (J y)$. By induction hypothesis $V[J/H]$ is solvable, and, by Lemma 2.1, we may begin to reduce $V[J/H]$ in $\lambda\bar{x} \lambda y V[J/H] (J y)$. If the head normal form of $V[J/H]$ is not of the forme $\lambda x \lambda \bar{z} x \bar{W}$, the result is true. If not the head reduction of $U[J/H]$ gives $\lambda\bar{x} \lambda \bar{z} (J y) \bar{W}$ which is solvable. \square

3.4 The proof of the Theorem

$U \rightarrow_{(I^*,k)} V$ (resp. $U \rightarrow_{(J^*,k)} V$) means that V is obtained from U by I -reductions (resp. J -reductions) and k t -reductions.

Lemma 3.14 *Let U_1, U_2, V_1, V_2 be λH -terms. If $U_1 \rightarrow_{(I^*,k)} V_1$, $U_2 \rightarrow_{(J^*,k)} V_2$, and $E(U_1) = E(U_2)$, then $E(V_1) = E(V_2)$.*

Proof Consequence of Lemmas 3.6, 3.7 and 3.10. \square

Lemma 3.15 *Let U be a λH -term. U is I - t -solvable iff U is J - t -solvable.*

Proof Consequence of Lemmas 3.8, 3.12 and 3.14. \square

Proof of the Theorem Consequence of Lemmas 3.9, 3.13 and 3.15. \square

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