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# A CONJECTURE ON NUMERAL SYSTEMS

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## Abstract

A numeral system is an infinite sequence of different closed normal  $\lambda$ -terms intended to code the integers in  $\lambda$ -calculus. H. Barendregt has shown that if we can represent, for a numeral system, the functions : Successor, Predecessor, and Zero Test, then all total recursive functions can be represented. In this paper we prove the independancy of these particular three functions. We give at the end a conjecture on the number of unary functions necessary to represent all total recursive functions.

## 1 Introduction

A numeral system is an infinite sequence of different closed  $\beta\eta$ -normal  $\lambda$ -terms  $\mathbf{d} = d_0, d_1, \dots, d_n, \dots$  intended to code the integers in  $\lambda$ -calculus.

For each numeral system  $\mathbf{d}$ , we can represent total numeric functions as follows :

A total numeric function  $\phi : \mathbb{N}^p \rightarrow \mathbb{N}$  is  $\lambda$ -definable with respect to  $\mathbf{d}$  iff

$$\exists F_\phi \forall n_1, \dots, n_p \in \mathbb{N} (F_\phi d_{n_1} \dots d_{n_p}) \simeq_\beta d_{\phi(n_1, \dots, n_p)}$$

One of the differences between our numeral system definition and the H. Barendregt's definition given in [1] is the fact that the  $\lambda$ -terms  $d_i$  are normal and different. The last conditions allow with some fixed reduction strategies (for example the left reduction strategy) to find the exact value of a function computed on arguments.

H. Barendregt has shown that if we can represent, for a numeral system, the functions : Successor, Predecessor, and Zero Test, then all total recursive functions can be represented.

We prove in this paper that these three particular functions are independent. We think it is, at least, necessary to have three unary functions to represent all total recursive functions.

This paper is organized as follows:

- The section 2 is devoted to preliminaries.
- In section 3, we define the numeral systems, and we present the result of H. Barendregt.
- In section 4, we prove the independency of the functions: Successor, Predecessor, and Zero Test. We give at the end a conjecture on the number of unary functions necessary to represent all total recursive functions.

## 2 Notations and definitions

The notations are standard (see [1] and [2]).

- We denote by  $I$  (for Identity) the  $\lambda$ -term  $\lambda xx$ ,  $T$  (for True) the  $\lambda$ -term  $\lambda x\lambda yx$  and by  $F$  (for False) the  $\lambda$ -term  $\lambda x\lambda yy$ .
- The pair  $\langle M, N \rangle$  denotes the  $\lambda$ -term  $\lambda x(x M N)$ .
- The  $\beta$ -equivalence relation is denoted by  $M \simeq_\beta N$ .
- The notation  $\sigma(M)$  represents the result of the simultaneous substitution  $\sigma$  to the free variables of  $M$  after a suitable renaming of the bound variables of  $M$ .
- A  $\beta\eta$ -normal  $\lambda$ -term is a  $\lambda$ -term which does not contain neither a  $\beta$ -redex [i.e. a  $\lambda$ -term of the form  $(\lambda xM N)$ ] nor an  $\eta$ -redex [i.e. a  $\lambda$ -term of the form  $\lambda x(M x)$  where  $x$  does not appear in  $M$ ].

The following result is well known (Böhm Theorem):

*If  $U, V$  are two distinct closed  $\beta\eta$ -normal  $\lambda$ -terms then there is a closed  $\lambda$ -term  $W$  such that  $(W U) \simeq_\beta T$  and  $(W V) \simeq_\beta F$ .*

- Let us recall that a  $\lambda$ -term  $M$  either has a *head redex* [i.e.  $M = \lambda x_1 \dots \lambda x_n ((\lambda xU V) V_1 \dots V_m)$ , the head redex being  $(\lambda xU V)$ ], or is in *head normal form* [i.e.  $M = \lambda x_1 \dots \lambda x_n (x V_1 \dots V_m)$ ].
- The notation  $U \succ V$  means that  $V$  is obtained from  $U$  by some head reductions and we denote by  $h(U, V)$  the length of the head reduction between  $U$  and  $V$ .
- A  $\lambda$ -term is said *solvable* iff its head reduction terminates.

The following results are well known :

- If  $M$  is  $\beta$ -equivalent to a head normal form then  $M$  is solvable.
- If  $U \succ V$ , then, for any substitution  $\sigma$ ,  $\sigma(U) \succ \sigma(V)$ , and  $h(\sigma(U), \sigma(V)) = h(U, V)$ .  
In particular, if for some substitution  $\sigma$ ,  $\sigma(M)$  is solvable, then  $M$  is solvable.

### 3 Numeral systems

- A *numeral system* is an infinite sequence of different closed  $\beta\eta$ -normal  $\lambda$ -terms  $\mathbf{d} = d_0, d_1, \dots, d_n, \dots$
- Let  $\mathbf{d}$  be a numeral system.

- A closed  $\lambda$ -term  $S_d$  is called *Successor* for  $\mathbf{d}$  iff :

$$(S_d d_n) \simeq_{\beta} d_{n+1} \text{ for all } n \in \mathbb{N}.$$

- A closed  $\lambda$ -term  $P_d$  is called *Predecessor* for  $\mathbf{d}$  iff :

$$(P_d d_{n+1}) \simeq_{\beta} d_n \text{ for all } n \in \mathbb{N}.$$

- A closed  $\lambda$ -term  $Z_d$  is called *Zero Test* for  $\mathbf{d}$  iff :

$$(Z_d d_0) \simeq_{\beta} T$$

and

$$(Z_d d_{n+1}) \simeq_{\beta} F \text{ for all } n \in \mathbb{N}.$$

- A numeral system is called *adequate* iff it possesses closed  $\lambda$ -terms for Successor, Predecessor, and Zero Test.

#### Examples of adequate numeral systems

##### 1) The Barendregt numeral system

For each  $n \in \mathbb{N}$ , we define the Barendregt integer  $\bar{n}$  by :  $\bar{0} = I$  and  $\overline{n+1} = \langle F, \bar{n} \rangle$ .

It is easy to check that

$$\bar{S} = \lambda x \langle F, x \rangle,$$

$$\bar{P} = \lambda x (x F),$$

$$\bar{Z} = \lambda x (x T).$$

are respectively  $\lambda$ -terms for Successor, Predecessor, and Zero Test.  $\square$

##### 2) The Church numeral system

For each  $n \in \mathbb{N}$ , we define the Church integer  $\underline{n} = \lambda f \lambda x (f(f \dots (f x) \dots))$  ( $f$  occurs  $n$  times).

It is easy to check that

$$\underline{S} = \lambda n \lambda f \lambda x (f (n f x)),$$

$$\underline{P} = \lambda n (n U < \underline{0}, \underline{0} > T) \text{ where } U = \lambda a < (\underline{s} (a T)), (a F) >,$$

$$\underline{Z} = \lambda n (n \lambda x F T).$$

are respectively  $\lambda$ -terms for Successor, Predecessor, and Zero Test.  $\square$

Each numeral system can be naturally considered as a coding of integers into  $\lambda$ -calculus and then we can represent total numeric functions as follows.

- A total numeric function  $\phi : \mathbb{N}^p \rightarrow \mathbb{N}$  is  $\lambda$ -definable with respect to a numeral system  $\mathbf{d}$  iff

$$\exists F_\phi \forall n_1, \dots, n_p \in \mathbb{N} (F_\phi d_{n_1} \dots d_{n_p}) \simeq_\beta d_{\phi(n_1, \dots, n_p)}$$

The Zero Test can be considered as a function on integers. Indeed :

**Lemma 1** *A numeral system  $\mathbf{d}$  has a  $\lambda$ -term for Zero Test iff the function  $\phi$  defined by :  $\phi(0) = 0$  and  $\phi(n) = 1$  for every  $n \geq 1$  is  $\lambda$ -definable with respect to  $\mathbf{d}$ .*

**Proof** It suffices to see that  $d_0$  and  $d_1$  are distinct  $\beta\eta$ -normal  $\lambda$ -terms.  $\square$

H. Barendregt has shown in [1] that :

**Theorem 1** *A numeral system  $\mathbf{d}$  is adequate iff all total recursive functions are  $\lambda$ -definable with respect to  $\mathbf{d}$ .*

## 4 Some results on numeral systems

**Theorem 2**<sup>1</sup> *There is a numeral system with Successor and Predecessor but without Zero Test.*

**Proof** For every  $n \in \mathbb{N}$ , let  $a_n = \lambda x_1 \dots \lambda x_n I$ .

It is easy to check that the  $\lambda$ -terms  $S_a = \lambda n \lambda x n$  and  $P_a = \lambda n (n I)$  are  $\lambda$ -terms for Successor and Predecessor for  $\mathbf{a}$ .

Let  $\nu, x, y$  be different variables.

If  $\mathbf{a}$  possesses a closed  $\lambda$ -term  $Z_a$  for Zero Test, then :

$$(Z_a a_n x y) \simeq_\beta \begin{cases} x & \text{if } n = 0 \\ y & \text{if } n \geq 1 \end{cases}$$

---

<sup>1</sup>This Theorem is the exercise 6.8.21 of Barendregt's book (see [1]). We give here a proof based on the techniques developed by J.-L. Krivine in [3].

and

$$(Z_a a_n x y) \succ \begin{cases} x & \text{if } n = 0 \\ y & \text{if } n \geq 1 \end{cases}$$

Therefore  $(Z_a \nu x y)$  is solvable and its head normal form does not begin with  $\lambda$ .

We have three cases to look at :

- $(Z_a \nu x y) \succ (x u_1 \dots u_k)$ , then  $(Z_a a_1 x y) \neq y$ .
- $(Z_a \nu x y) \succ (y u_1 \dots u_k)$ , then  $(Z_a a_0 x y) \neq x$ .
- $(Z_a \nu x y) \succ (\nu u_1 \dots u_k)$ , then  $(Z_a a_{k+2} x y) \neq y$ .

Each case is impossible.  $\square$

**Theorem 3** *There is a numeral system with Successor and Zero Test but without Predecessor.*

**Proof** Let  $b_0 = \langle T, I \rangle$  and for every  $n \geq 1$ ,  $b_n = \langle F, a_{n-1} \rangle$ .

It is easy to check that the  $\lambda$ -terms  $S_b = \lambda n \langle F, ((n T) a_0 \lambda x(n F)) \rangle$  and  $Z_b = \lambda n(n T)$  are  $\lambda$ -terms for Successor and Zero Test for  $\mathbf{b}$ .

If  $\mathbf{b}$  possesses a closed  $\lambda$ -term  $P_b$  for Predecessor, then the  $\lambda$ -term  $P'_b = \lambda n(P_b \langle F, n \rangle T)$  is a  $\lambda$ -term for Zero Test for  $\mathbf{a}$ . A contradiction.  $\square$

### Remarks

1) Let  $b'_0 = b_1$ ,  $b'_1 = b_0$ , and for every  $n \geq 2$ ,  $b'_n = b_n$ . It is easy to check that the numeral system  $\mathbf{b}'$  does not have  $\lambda$ -terms for Successor, Predecessor, and Test for Zero.

2) The proofs of Theorems 1 and 2 rest on the fact that we are considering sequences of  $\lambda$ -terms with a strictly increasing order (number of abstractions). Considering sequences of  $\lambda$ -terms with a strictly increasing degree (number of arguments) does not work as well. See the following example.

We define  $\tilde{0} = I$  and for each  $n \geq 1$ ,  $\tilde{n} = \lambda x(x x \dots x)$  ( $x$  occurs  $n + 1$  times).

Let

$$\tilde{S} = \lambda n \lambda x(n x x),$$

$$\tilde{Z} = \lambda n(n A I I T) \text{ where } A = \lambda x \lambda y(y x),$$

$$\tilde{P} = \lambda n \lambda x(n U F) \text{ where } U = \lambda y(y V I) \text{ and } V = \lambda a \lambda b \lambda c \lambda d(d a (c x))$$

It is easy to check that  $\tilde{S}$ ,  $\tilde{Z}$ , and  $\tilde{P}$  are respectively  $\lambda$ -terms for Successor, Zero Test, and Predecessor.  $\square$

### Definitions

- We denote by  $\Lambda^0$  the set of closed  $\lambda$ -terms and by  $\Lambda^1$  the set of the infinite sequences of closed normal  $\lambda$ -terms. It is easy to see that  $\Lambda^0$  is countable but  $\Lambda^1$  is not countable.
- For every finite sequence of  $\lambda$ -terms  $U_1, U_2, \dots, U_n$  we denote by  $\langle U_1, U_2, \dots, U_n \rangle$  the  $\lambda$ -term  $\langle \dots \langle \langle I, U_1 \rangle, U_2 \rangle, \dots, U_n \rangle$ .
- Let  $\mathbf{U} = U_1, U_2, \dots$  be a sequence of normal closed  $\lambda$ -terms. A closed  $\lambda$ -term  $A$  is called *generator* for  $\mathbf{U}$  iff :

$$(A I) \simeq_{\beta} U_1$$

and

$$(A \langle U_1, U_2, \dots, U_n \rangle) \simeq_{\beta} U_{n+1} \text{ for every } n \geq 1$$

**Lemma 2** *There is a sequence of normal closed  $\lambda$ -terms without generator.*

**Proof** If not, let  $\phi$  be a bijection between  $\Lambda^0$  and  $\mathbb{N}$  and  $\Phi$  the function from  $\Lambda^1$  into  $\Lambda^0$  defined by:  $\Phi(\mathbf{U})$  is the generator  $G_{\mathbf{U}}$  such that  $\phi(G_{\mathbf{U}})$  is minimum. It is easy to check that  $\Phi$  is a one-to-one mapping. A Contradiction.  $\square$

**Theorem 4** *There is a numeral system with Predecessor and Zero Test but without Successor.*

**Proof** Let  $\mathbf{e}$  be a sequence of normal closed  $\lambda$ -terms without generator.

Let  $c_0 = I$  and for every  $n \geq 1$ ,  $c_n = \langle c_{n-1}, e_n \rangle$ .

It is easy to check that the  $\lambda$ -terms  $P_c = \lambda n(n T)$  and  $Z_c = \lambda n(n \lambda x \lambda y I T F T)$  are  $\lambda$ -terms for Predecessor and Zero Test for  $\mathbf{c}$ .

If  $\mathbf{c}$  possesses a closed  $\lambda$ -term  $S_c$  for Successor, then the  $\lambda$ -term  $S'_c = \lambda n(S_c n F)$  is a generator for  $\mathbf{e}$ . A Contradiction.  $\square$

The result of H. Barendregt (Theorem 1) means that, for a numeral system, it suffices to represent three particular functions in order to represent all total recursive functions. We have proved that these three particular functions are independent. We think it is, at least, necessary to have three functions as is mentioned below :

**Conjecture** *There are no total recursive functions  $f, g : \mathbb{N} \rightarrow \mathbb{N}$  such that : for all numeral systems  $\mathbf{d}$ ,  $f, g$  are  $\lambda$ -definable iff all total recursive functions are  $\lambda$ -definable with respect to  $\mathbf{d}$ .*

If we authorize the binary functions we obtain the following result :

**Theorem 5** *There is a binary total function  $k$  such that for all numeral systems  $\mathbf{d}$ ,  $k$  is  $\lambda$ -definable iff all total recursive functions are  $\lambda$ -definable with respect to  $\mathbf{d}$ .*

**Proof** Let  $k$  the total binary function defined by :

$$k(n, m) = \begin{cases} n + 1 & \text{if } m = 0 \\ |n - m| & \text{if } m \neq 0 \end{cases}$$

It suffices to see that :

$$k(n, n) = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases},$$

$$k(n, 0) = n + 1,$$

$$k(n, 1) = n - 1 \text{ if } n \neq 0. \quad \square$$

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