

# Effective $H^\infty$ interpolation constrained by Hardy and Bergman weighted norms

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## Abstract

Given a finite set  $\sigma$  of the unit disc  $\mathbb{D}$  and a holomorphic function  $f$  in  $\mathbb{D}$  which belongs to a class  $X$  we are looking for a function  $g$  in another class  $Y$  which minimizes the norm  $\|g\|_Y$  among all functions  $g$  such that  $g|_\sigma = f|_\sigma$ . Generally speaking, the interpolation constant considered is  $c(\sigma, X, Y) = \sup_{f \in X, \|f\|_X \leq 1} \inf \{ \|g\|_Y : g|_\sigma = f|_\sigma \}$ . When  $Y = H^\infty$ , our interpolation problem includes those of Nevanlinna-Pick (1916), Caratheodory-Schur (1908). Moreover, Carleson's free interpolation (1958) has also an interpretation in terms of our constant  $c(\sigma, X, H^\infty)$ .

If  $X$  is a Hilbert space belonging to the scale of Hardy and Bergman weighted spaces, we show that  $c(\sigma, X, H^\infty) \leq a\varphi_X \left(1 - \frac{1-r}{n}\right)$  where  $n = \#\sigma$ ,  $r = \max_{\lambda \in \sigma} |\lambda|$  and where  $\varphi_X(t)$  stands for the norm of the evaluation functional  $f \mapsto f(t)$  on the space  $X$ . The upper bound is sharp over sets  $\sigma$  with given  $n$  and  $r$ .

If  $X$  is a general Hardy-Sobolev space or a general weighted Bergman space (not necessarily of Hilbert type), we also found upper and lower bounds for  $c(\sigma, X, H^\infty)$  (sometimes for special sets  $\sigma$ ) but with some gaps between these bounds.

This constrained interpolation is motivated by some applications in matrix analysis and in operator theory.

## Introduction

**1. Statement and historical context of the problem.** Let  $Hol(\mathbb{D})$  be the space of holomorphic functions on the unit disc  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ . We consider here the following problem : given two Banach spaces  $X$  and  $Y$  of holomorphic functions on the unit disc  $\mathbb{D}$ ,  $X, Y \subset Hol(\mathbb{D})$ , and a finite set  $\sigma \subset \mathbb{D}$ , what is the best possible interpolation by functions of the space  $Y$  for the traces  $f|_\sigma$  of functions of the space  $X$ , in the worst case? The case  $X \subset Y$  is of no interests, and so one can suppose that either  $Y \subset X$  or  $X, Y$  are incomparable. Here and everywhere below,  $H^\infty$  stands for the space (algebra) of bounded holomorphic functions in the unit disc  $\mathbb{D}$  endowed with the norm  $\|f\|_\infty = \sup_{z \in \mathbb{D}} |f(z)|$ .

Precisely, our problem is to compute or estimate the following interpolation constant

$$c(\sigma, X, Y) = \sup_{f \in X, \|f\|_X \leq 1} \inf \{ \|g\|_Y : g|_\sigma = f|_\sigma \} .$$

If  $r \in [0, 1)$  and  $n \geq 1$ , we also define

$$C_{n,r}(X, Y) = \sup \{ c(\sigma, X, Y) : \#\sigma \leq n, |\lambda| \leq r, \forall \lambda \in \sigma \} .$$

Let us first explain why the classical interpolation problems, those of Nevanlinna-Pick and Carathéodory-Schur (see [N2] p.231 for these two problems) on the one hand, and Carleson's free interpolation problem (1958) (see [N1] p.158) on the other hand, are of the nature of our interpolation problem.

**(i) The Carleson interpolation problem**

We suppose here that the sequence  $\sigma$  can be infinite. Let  $l^\infty(\sigma)$  be the space of complex sequences  $(a_\lambda)_{\lambda \in \sigma}$  of support  $\sigma$  endowed with the norm  $\|a\|_{l^\infty(\sigma)} = \max_{\lambda \in \sigma} |a_\lambda|$ . Carleson's free interpolation is to compare the norms  $\|a\|_{l^\infty(\sigma)}$  and

$$\inf \{ \|g\|_\infty : g(\lambda) = a_\lambda, \lambda \in \sigma \}.$$

In other words, it deals with the interpolation constant defined as

$$c(\sigma, l^\infty(\sigma), H^\infty) = \sup_{a \in l^\infty(\sigma), \|a\|_{l^\infty} \leq 1} \inf \{ \|g\|_\infty : g(\lambda) = a_\lambda, \lambda \in \sigma \} :$$

looking at our interpolation constant  $c(\sigma, X, H^\infty)$ , we have replaced the holomorphic functions space  $X$  by the space of sequences  $l^\infty(\sigma)$ .

Now on the contrary to Carleson's free interpolation, the Nevanlinna-Pick and Carathéodory-Schur interpolation problems are "individual", in the sense that they look simply to compute the norms  $\|f\|_{H^\infty_\sigma}$  or  $\|f\|_{H^\infty/z^n H^\infty}$  for a given  $f$ . We first recall those two classical problems (see (ii) and (iii) below) and then we explain in (iv) in which way they are included into our interpolation problem.

**(ii) The Nevanlinna-Pick interpolation problem**

Given  $\Lambda = (\lambda_1, \dots, \lambda_n)$  in  $\mathbb{D}^n$  and  $W = (w_1, \dots, w_n) \in \mathbb{C}^n$ , we are looking for

$$C(\Lambda, W) = \inf \{ \|f\|_\infty : f(\lambda_i) = w_i, i = 1..n \}.$$

The classical answer of Pick is the following :

$$C(\Lambda, W) = \inf \left\{ c > 0 : \left( \frac{c^2 - \overline{w_i} w_j}{1 - \overline{\lambda_i} \lambda_j} \right)_{1 \leq i, j \leq n} \gg 0 \right\},$$

where for any  $n \times n$  matrix  $M$ ,  $M \gg 0$  means that  $M$  is positive definite.

**(iii) The Carathéodory-Schur interpolation problem**

Given  $\mathcal{A} = (a_0, \dots, a_n) \in \mathbb{C}^{n+1}$ , we are looking for

$$C(\mathcal{A}) = \inf \{ \|f\|_\infty : f(z) = a_0 + a_1 z + \dots + a_n z^n + \dots \}.$$

The classical answer of Schur is the following :

$$C(\mathcal{A}) = \|(T_\varphi)_n\|,$$

where  $T_\varphi$  is the Toeplitz operator associated with a symbol  $\varphi$ ,  $(T_\varphi)_n$  is the compression of  $T_\varphi$  on  $\mathcal{P}_n$ , the space of analytic polynomials of degree less or equal than  $n$ , and  $\varphi$  is the polynomial  $\sum_{k=0}^n a_k z^k$ .

Notice that the Carathéodory-Schur interpolation theorem can be seen as a particular case of the famous commutant lifting theorem of Sarason and Sz-Nagy-Foias (1968) see [N2] p.230, Theorem 3.1.11.

**(iv) How are the Nevanlinna-Pick and the Caratheodory-Schur interpolation problems included into ours?**

From a modern point of view, the two interpolation problems (ii) and (iii) are included in the following mixed problem : given

- $\sigma = \{\lambda_1, \dots, \lambda_n\} \subset \mathbb{D}$ , the finite Blaschke product  $B_\sigma = \prod_j b_{\lambda_j}$ , where  $b_\lambda = \frac{\lambda-z}{1-\lambda z}$ ,  $\lambda \in \mathbb{D}$ ,
- $f \in Hol(\mathbb{D})$ ,

we would like to compute or estimate

$$\|f\|_{H^\infty/BH^\infty} = \inf \{ \|g\|_\infty : f - g \in B_\sigma Hol(\mathbb{D}) \}.$$

Now if we want to compare  $\|f\|_{H^\infty/BH^\infty}$  with the size of  $f$  (measured in a certain Banach space  $X$ ), then looking at our interpolation constant  $c(\sigma, X, Y)$  we have

$$c(\sigma, X, Y) = \sup_{f \in X, \|f\|_X \leq 1} \|f\|_{Y/BY}.$$

As a consequence, the classical Nevanlinna-Pick problem corresponds to the case  $Y = H^\infty$ , and the one of Carathéodory-Schur to the case where  $\lambda_1 = \lambda_2 = \dots = \lambda_n = 0$  and  $Y = H^\infty$ .

**Remarks. a.** Looking at this problem, say, in the form of computing/estimating the interpolation constant  $c(\sigma, X, Y)$  which is nothing but the norm of the embedding operator  $(X|_\sigma, \|\cdot\|_{X|_\sigma}) \rightarrow (Y|_\sigma, \|\cdot\|_{Y|_\sigma})$ , one can think, of course, on passing (after) to the limit - in the case of an infinite sequence  $\{\lambda_j\}$  and its finite sections  $\{\lambda_j\}_{j=1}^n$ , in order to obtain a Carleson type interpolation theorem  $X|_\sigma = Y|_\sigma$ . But not necessarily. In particular, even the classical Nevanlinna-Pick theorem (giving a necessary and sufficient condition on a function  $a$  for the existence of  $f \in H^\infty$  such that  $\|f\|_\infty \leq 1$  and  $f(\lambda) = a_\lambda$ ,  $\lambda \in \sigma$ ), does not lead immediately to Carleson's criterion for  $H^\infty|_\sigma = l^\infty(\sigma)$ . (In fact, a direct deduction of Carleson's theorem from Pick's result was done by P. Koosis [K] in 1999 only). Similarly, the problem stated for  $c(\sigma, X, Y)$  is of interest in its own. It is a kind of "effective interpolation" because we are looking for sharp estimations or a computation of  $c(\sigma, X, Y)$  for a variety of norms  $\|\cdot\|_X, \|\cdot\|_Y$ .

**b.** An interesting case occurs when  $X$  is larger than  $Y$ , and the sense of the issue lies in comparing  $\|\cdot\|_X$  and  $\|\cdot\|_Y$  when  $Y$  interpolates  $X$  on the set  $\sigma$ . For example, we can wonder what happens when  $X = H^p$ , the classical Hardy spaces of the disc or  $X = L_a^p$ , the Bergman spaces, etc..., and when  $Y = H^\infty$ , but also  $Y = W$  the Wiener algebra (of absolutely converging Fourier series) or  $Y = B_{\infty,1}^0$ , a Besov algebra (an interesting case for the functional calculus of finite rank operators, in particular, those satisfying the so-called Ritt condition).

## 2. Motivations.

We also can add some more motivations to our problem :

**a.** One of the most interesting cases is  $Y = H^\infty$ . As we have seen it in the paragraph 1 of the Introduction (above), in this case the quantity  $c(\sigma, X, H^\infty)$  has a meaning of an intermediate interpolation between the Carleson one (when  $\|f\|_{X|\sigma} \asymp \sup_{1 \leq i \leq n} |f(\lambda_i)|$ ) and the individual Nevanlinna-Pick interpolation (no conditions on  $f$ ).

**b.** There is a straight link between the constant  $c(\sigma, X, Y)$  and numerical analysis.

**b.1** For example, in matrix analysis, it is of interest to bound the norm of an  $H^\infty$ -calculus  $\|f(A)\| \leq c \|f\|_\infty$ ,  $\forall f \in H^\infty$ , for an arbitrary Banach space  $n$ -dimensional contraction  $A$  with a given spectrum  $\sigma(A) \subset \sigma$ . The best possible constant is  $c = c(\sigma, H^\infty, W)$ , so that

$$c(\sigma, H^\infty, W) = \sup_{\|f\|_\infty \leq 1} \sup \{ \|f(A)\| : A : (\mathbb{C}^n, |\cdot|) \rightarrow (\mathbb{C}^n, |\cdot|), \|A\| \leq 1, \sigma(A) \subset \sigma \},$$

where  $W = \left\{ f = \sum_{k \geq 0} \hat{f}(k) z^k : \sum_{k \geq 0} |\hat{f}(k)| < \infty \right\}$  stands for the Wiener algebra, and the interior sup is taken over all contractions on  $n$ -dimensional Banach spaces  $(\mathbb{C}^n, |\cdot|)$ .

**b.2** In the same spirit, our constant  $c(\sigma, X, H^\infty)$  is straightly linked with the well known Von-Neumann's inequality for contractions on Hilbert spaces, which asserts that if  $A$  is a contraction on a Hilbert space and  $f \in H^\infty$ , then the operator  $f(A)$  satisfies the inequality

$$\|f(A)\| \leq \|f\|_\infty.$$

Using this inequality we get the following interpretation of our interpolation constant  $c(\sigma, X, H^\infty)$  : it is the best possible constant  $c$  such that  $\|f(A)\| \leq c \|f\|_X$ ,  $\forall f \in X$ ,

$$c(\sigma, X, H^\infty) = \sup_{\|f\|_X \leq 1} \sup \{ \|f(A)\| : A : (\mathbb{C}^n, |\cdot|_2) \rightarrow (\mathbb{C}^n, |\cdot|_2), \|A\| \leq 1, \sigma(A) \subset \sigma \},$$

where the interior sup is taken over all contractions  $A$  on  $n$ -dimensional Hilbert spaces  $(\mathbb{C}^n, |\cdot|_2)$ , with a given spectrum  $\sigma(A) \subset \sigma$ .

**b.3** As in b.1 and b.2 we can choose  $Y = B_{\infty,1}^0$ , a Besov algebra instead of  $Y = H^\infty$  or  $Y = W$ . Our interpolation constant  $c(\sigma, X, B_{\infty,1}^0)$  is defined in the same way and presents an interesting case for the functional calculus of finite rank operators, in particular, those satisfying the so-called Ritt condition.

For b.1, b.2 and b.3, an interesting case occurs for  $f$  such that  $f|_\sigma = \frac{1}{z}|_\sigma$  (estimation of condition numbers and the norm of inverses of  $n \times n$  matrices) or  $f|_\sigma = \frac{1}{\lambda - z}|_\sigma$  (for estimation of the norm of the resolvent of an  $n \times n$  matrix), see for instance [Z4].

c. Let us notice that the following question (the special case  $X = H^2$ ,  $Y = H^\infty$ ) was especially stimulating (which is a part of a more complicated question arising in an applied situation in [BL1] and [BL2]) : given a set  $\sigma \subset \mathbb{D}$ , how to estimate  $c(\sigma, H^2, H^\infty)$  in terms of  $n = \text{card}(\sigma)$  and  $\max_{\lambda \in \sigma} |\lambda| = r$  only? Here and everywhere below,  $H^2$  is the standard Hardy space of the disc,

$$H^2 = \left\{ f = \sum_{k \geq 0} \hat{f}(k) z^k : \|f\|_{H^2} = \left( \sum_{k \geq 0} |\hat{f}(k)|^2 \right)^{1/2} < \infty \right\}.$$

The issue of estimating/computing  $c(\sigma, H^2, H^\infty)$  has been treated in [Z2], in which we also considered some others partial cases of the problem : more precisely, we obtained sharp estimations for the constant  $C_{n,r}(X, Y)$  (see Theorem A below) for the cases  $X = H^p$ ,  $L_a^2$ , and  $Y = H^\infty$ , where  $H^p$  ( $1 \leq p \leq \infty$ ) stand for the standard Hardy spaces of the unit disc (see [N2] for the definition) and  $L_a^2$  stands for the Bergman space of all holomorphic functions  $f$  such that

$$\int_{\mathbb{D}} |f(\omega)|^2 dA < \infty,$$

where  $dA$  stands for the area measure. Here is the main result of [Z2] :

**Theorem A.** *Let  $n \geq 1$ ,  $r \in [0, 1)$ ,  $p \in [1, +\infty]$  and  $\lambda$ ,  $|\lambda| \leq r$ . Then,*

$$(1) \quad \frac{1}{32^{\frac{1}{p}}} \left( \frac{n}{1-|\lambda|} \right)^{\frac{1}{p}} \leq c(\sigma_{n,\lambda}, H^p, H^\infty) \leq C_{n,r}(H^p, H^\infty) \leq A_p \left( \frac{n}{1-r} \right)^{\frac{1}{p}},$$

$$(2) \quad \frac{1}{32} \frac{n}{1-|\lambda|} \leq c(\sigma_{n,\lambda}, L_a^2, H^\infty) \leq C_{n,r}(L_a^2, H^\infty) \leq \sqrt{14} \frac{n}{1-r},$$

where  $\sigma_{n,\lambda} = \{\lambda, \dots, \lambda\}$  is the one-point set of multiplicity  $n$  corresponding to  $\lambda$ ,  $A_p$  is a constant depending only on  $p$  and the left-hand side inequality from (1) is valid only for  $p \in 2\mathbb{Z}_+$ . For  $p = 2$ , we have  $A_2 = \sqrt{2}$ .

**3. Main results.** Theorems B, C, D&E below in this paragraph, were already announced in the note [Z1]. In this paper, we extend the above results to the case where  $X$  is a weighted space  $X = l_a^p(w_k)$ ,

$$l_a^p(w_k) = \left\{ f = \sum_{k \geq 0} \hat{f}(k) z^k : \|f\|^p = \sum_{k \geq 0} |\hat{f}(k)|^p w_k^p < \infty \right\},$$

with a weight  $w$  satisfying  $w_k > 0$  for every  $k \geq 0$  and  $\overline{\lim}_k (1/w_k^{1/k}) = 1$ . The latter condition implies that  $l_a^p(w_k)$  is continuously embedded into the space of holomorphic functions  $Hol(\mathbb{D})$  on the unit disc  $\mathbb{D}$  (and not on a larger disc, i.e.  $l_a^p(w_k)$  does not contained in  $Hol(r\mathbb{D})$  for every  $r > 1$ ).

First, we study the special case  $p = 2$  and  $w_k = (k+1)^\alpha$ ,  $\alpha \leq 0$ . Then  $X = l_a^2((k+1)^\alpha)$ , are the spaces of all  $f(z) = \sum_{k \geq 0} \hat{f}(k) z^k$  satisfying

$$\sum_{k \geq 0} |\hat{f}(k)|^2 (k+1)^{2\alpha} < \infty.$$

Notice that  $H^2 = l_a^2(1)$  and  $L_a^2 = l_a^2\left(\frac{1}{(k+1)^{\frac{1}{2}}}\right)$ . An equivalent description of this scale of spaces is :

$$X = L_a^2\left((1 - |z|^2)^\beta dA\right), \quad \beta = -2\alpha - 1 > -1,$$

the weighted Bergman spaces of holomorphic functions such that

$$\int_{\mathbb{D}} |f(z)|^2 (1 - |z|^2)^\beta dA < \infty.$$

We shorten the notation and write

$$L_a^2(\beta) = L_a^2\left((1 - |z|^2)^\beta dA\right),$$

for every  $\beta > -1$ . For the case  $\beta = 0$ , we have  $L_a^2(0) = L_a^2$ .

Our principal result is the following.

**Theorem B.** *Let  $n \geq 1$ ,  $r \in [0, 1)$ ,  $\alpha \in ]-\infty, 0]$  and  $\lambda \in r\overline{\mathbb{D}}$ . Then,*

$$a\left(\frac{n}{1-|\lambda|}\right)^{\frac{1-2\alpha}{2}} \leq c(\sigma_{n,\lambda}, l_a^2((k+1)^\alpha), H^\infty) \leq C_{n,r}(l_a^2((k+1)^\alpha), H^\infty) \leq A\left(\frac{n}{1-r}\right)^{\frac{1-2\alpha}{2}}.$$

Furthermore if  $\beta \in (-1, +\infty)$ , then

$$a'\left(\frac{n}{1-|\lambda|}\right)^{\frac{\beta+2}{2}} \leq c(\sigma_{n,\lambda}, L_a^2(\beta), H^\infty) \leq C_{n,r}(L_a^2(\beta), H^\infty) \leq A'\left(\frac{n}{1-r}\right)^{\frac{\beta+2}{2}},$$

where  $A = A(\alpha)$  and  $a = a(\alpha)$  are constants depending only on  $\alpha$ ,  $A' = A'(\beta)$  and  $a' = a'(\beta)$  are constants depending only on  $\beta$ , and both of the two left-hand side inequalities are valid only for  $\alpha$  and  $\beta$  satisfying  $1 - 2\alpha \in \mathbb{N}$  and  $\frac{\beta+1}{2} \in \mathbb{N}$ .

Right-hand side inequalities from Theorem B are proved in Section 1 whereas the left-hand side ones are shown in Section 7.

**Remark.** *On the size of the constants  $a$ ,  $A$ ,  $a'$  and  $A'$  of Theorem B : if  $N = [1 - 2\alpha]$  stands for the integer part of  $1 - 2\alpha$ , then  $a$  and  $A$  are such that  $a \asymp \frac{1}{2^{3N}(2N)!}$  and  $A \asymp N^{2N}$ . In the same way, if  $N' = [2 + \beta]$  stands for the integer part of  $2 + \beta$ , then  $a'$  and  $A'$  are such that  $a' \asymp \frac{1}{2^{3N'}(2N')!}$  and  $A' \asymp N'^{2N'}$ . (The notation  $x \asymp y$  means that there exist numerical constants  $c_1, c_2 > 0$  such that  $c_1 y \leq x \leq c_2 y$ ).*

In Sections 2, 3, 4 and 5, we deal with an upper estimate for  $C_{n,r}(X, H^\infty)$  in the scale of spaces  $X = l_a^p((k+1)^\alpha)$ ,  $\alpha \leq 0$ ,  $1 \leq p \leq +\infty$ . We start giving a result for  $1 \leq p \leq 2$ .

**Theorem C.** *Let  $r \in [0, 1[$ ,  $n \geq 1$ ,  $1 \leq p \leq 2$ , and  $\alpha \leq 0$ . We have*

$$B\left(\frac{1}{1-r}\right)^{1-\alpha-\frac{1}{p}} \leq C_{n,r}(l_a^p((k+1)^\alpha), H^\infty) \leq A\left(\frac{n}{1-r}\right)^{\frac{1-2\alpha}{2}},$$

where  $A = A(\alpha, p)$  is a constant depending only on  $\alpha$  and  $p$  and  $B = B(p)$  is a constant depending only on  $p$ .

It is very likely that the bounds of Theorem C are not sharp. The sharp one should be probably  $\left(\frac{n}{1-r}\right)^{1-\alpha-\frac{1}{p}}$ . In the same way, for  $2 \leq p \leq \infty$ , we give the following theorem, in which we feel again that the upper bound  $\left(\frac{n}{1-r}\right)^{\frac{3}{2}-\alpha-\frac{2}{p}}$  is not sharp. The sharp one probably should be the lower bound  $\left(\frac{n}{1-r}\right)^{1-\alpha-\frac{1}{p}}$ .

**Theorem D.** *Let  $r \in [0, 1[$ ,  $n \geq 1$ ,  $2 \leq p \leq \infty$ , and  $\alpha \leq 0$ . We have*

$$B \left( \frac{1}{1-r} \right)^{1-\alpha-\frac{1}{p}} \leq C_{n,r} (l_a^p((k+1)^\alpha), H^\infty) \leq A \left( \frac{n}{1-r} \right)^{\frac{3}{2}-\alpha-\frac{2}{p}},$$

where  $A = A(\alpha, p)$  is a constant depending only on  $\alpha$  and  $p$  and  $B = B(p)$  is a constant depending only on  $p$ .

In Section 6, we suppose that  $X$  is equal to  $L_a^p(\beta)$ ,  $\beta > -1$ ,  $1 \leq p \leq 2$ . Our goal in this section is to give an estimate for the constant for a generalized Carathéodory-Schur interpolation, (a partial case of the Nevanlinna-Pick interpolation),

$$c(\sigma_{n,\lambda}, X, H^\infty) = \sup \{ \|f\|_{H^\infty/b_\lambda^n H^\infty} : f \in X, \|f\|_X \leq 1 \},$$

where  $\|f\|_{H^\infty/b_\lambda^n H^\infty} = \inf \{ \|f + b_\lambda^n g\|_\infty : g \in X \}$ ,  $\lambda \in \mathbb{D}$  and  $\sigma_{n,\lambda}$  is (again) the one-point set of multiplicity  $n$ :  $\sigma_{n,\lambda} = \{\lambda, \dots, \lambda\}$ . The corresponding interpolation problem is: given  $f \in X$ , to minimize  $\|h\|_\infty$  such that  $h^{(j)}(\lambda) = f^{(j)}(\lambda)$ ,  $0 \leq j < n$ .

For this partial case, we have the following generalization of the estimate from Theorem B.

**Theorem E.** *Let  $\lambda \in \mathbb{D}$ ,  $\beta \in (-1, +\infty)$  and  $p \in [1, 2]$ . Then,*

$$c(\sigma_{n,\lambda}, L_a^p(\beta), H^\infty) \leq A' \left( \frac{n}{1-|\lambda|} \right)^{\frac{\beta+2}{p}},$$

where  $A' = A'(\beta, p)$  is a constant depending only on  $\beta$  and  $p$ .

**4. Technical tools ( “ about the proofs “).** In order to find an upper bound for  $c(\sigma, X, H^\infty)$ , we first use a linear interpolation :

$$f \mapsto \sum_{k=1}^n \langle f, e_k \rangle e_k,$$

where  $\langle \cdot, \cdot \rangle$  means the Cauchy sesquilinear form  $\langle h, g \rangle = \sum_{k \geq 0} \hat{h}(k) \overline{\hat{g}(k)}$ , and  $(e_k)_{k=1}^n$  is the explicitly known Malmquist basis of the space  $K_B = H^2 \ominus B H^2$ ,  $\bar{B} = \prod_{i=1}^n b_{\lambda_i}$  being the corresponding Blaschke product,  $b_\lambda = \frac{\lambda-z}{1-\bar{\lambda}z}$  (see [N1] p. 117 or Definition 1.1 below). Next, we use the complex interpolation between Banach spaces, (see [Tr] Theorem 1.9.3-(a) p.59). Among the technical tools

used in order to find an upper bound for  $\|\sum_{k=1}^n \langle f, e_k \rangle e_k\|_\infty$  (in terms of  $\|f\|_X$ ), the most important is a Bernstein-type inequality (used by induction)  $\|f'\|_p \leq c_p \|B'\|_\infty \|f\|_p$  for a (rational) function  $f$  in the star-invariant subspace  $H^p \cap B\overline{z}H^p$  generated by a (finite) Blaschke product  $B$ , (K. Dyakonov [Dya1]&[Dya2]). For  $p = 2$ , it is given in [Z2] an alternative proof of the needed Bernstein-type estimate.

The lower bound problem (for  $C_{n,r}(X, H^\infty)$ ) is treated by using the “worst” interpolation  $n$ -tuple  $\sigma = \sigma_{n,\lambda} = \{\lambda, \dots, \lambda\}$ , a one-point set of multiplicity  $n$  (the Carathéodory-Schur type interpolation). The “worst” interpolation data comes from the Dirichlet kernels  $\sum_{k=0}^{n-1} z^k$  transplanted from the origin to  $\lambda$ . We notice that spaces  $X = l_a^p(w_k)$  satisfy the condition  $X \circ b_\lambda \subset X$  when  $p = 2$ , whereas this is not the case for  $p \neq 2$  and this makes the problem of upper/lower bound harder.

Before starting Section 1 and studying upper estimates for  $c(\sigma, X, H^\infty)$ , we give the following lemma which is going to be useful throughout this paper, in particular in view of applying interpolation between Banach spaces. Its proof is obvious.

**Lemma 0.** *Let  $X$  be a Banach space of holomorphic functions in the unit disc  $\mathbb{D}$  and  $\sigma = \{\lambda_1, \lambda_2, \dots, \lambda_n\} \subset \mathbb{D}$  a finite subset of the disc. We define the Blaschke product  $B_\sigma = \prod_{i=1}^n b_{\lambda_i}$  where  $b_\lambda = \frac{\lambda-z}{1-\lambda z}$ . Let  $T : X \rightarrow H^\infty/B_\sigma H^\infty$  be the restriction map defined by*

$$Tf = \{g \in H^\infty : f - g \in B_\sigma X\},$$

for every  $f \in X$ . Then,

$$\|T\|_{X \rightarrow H^\infty/B_\sigma H^\infty} = c(\sigma, X, H^\infty).$$

The paper is organized as follows. In Section 1 we prove the upper bounds from Theorem B. Sections 2&3 are devoted to the proof of Theorem C, and Sections 4&5 to the proof of Theorem D. Theorem E is proved in Section 6. Finally, in Section 7 we prove the lower bounds from Theorem B.

## 1. An upper bound for $c(\sigma, l_a^2(w_k), H^\infty)$

In this Section, we prove the right-hand side inequalities from Theorem B. That is to say that we give an upper bound for  $C_{n,r}(X, H^\infty)$  where  $X = l_a^2((k+1)^\alpha)$ ,  $\alpha \leq 0$ . We recall also that  $H^2 = l_a^2(1)$  and  $L_a^2 = l_a^2\left(\frac{1}{(k+1)^{\frac{1}{2}}}\right)$ . As it is mentioned in the paragraph 4 of the introduction, the main technical tool used in the proof of the upper estimate for

$$C_{n,r}(l_a^2((k+1)^\alpha), H^\infty),$$

is a Bernstein-type inequality applied to a rational function.

In Definitions 1.1, 1.2, 1.3 and in Remark 1.4 below,  $\sigma = \{\lambda_1, \dots, \lambda_n\}$  is a sequence in the unit disc  $\mathbb{D}$ ,  $B_\sigma = \prod_{i=1}^n b_{\lambda_i}$  is the finite Blaschke product corresponding to  $\sigma$ , where  $b_\lambda = \frac{\lambda-z}{1-\lambda z}$  is an elementary Blaschke factor for  $\lambda \in \mathbb{D}$ .

**Definition 1.1.** *Malmquist family.* For  $k \in [1, n]$ , we set  $f_k = \frac{1}{1-\lambda_k z}$ , and define the family  $(e_k)_{k=1}^n$ , (which is known as Malmquist basis, see [N1] p.117), by and



$$(1.1) \quad e_1 = \frac{f_1}{\|f_1\|_2} \quad \text{and} \quad e_k = \left( \prod_{j=1}^{k-1} b_{\lambda_j} \right) \frac{f_k}{\|f_k\|_2},$$

for  $k \in [2, n]$ , where  $\|f_k\|_2 = (1 - |\lambda_k|^2)^{-1/2}$ .

**Definition 1.2.** *The model space  $K_{B_\sigma}$ .* We define  $K_{B_\sigma}$  to be the  $n$ -dimensional space :

$$(1.2) \quad K_{B_\sigma} = (B_\sigma H^2)^\perp = H^2 \ominus B_\sigma H^2.$$

**Definition 1.3.** *The orthogonal projection  $P_{B_\sigma}$  on  $K_{B_\sigma}$ .* We define  $P_{B_\sigma}$  to be the orthogonal projection of  $H^2$  on its  $n$ -dimensional subspace  $K_{B_\sigma}$ .

**Remark 1.4.** The Malmquist family  $(e_k)_{k=1}^n$  corresponding to  $\sigma$  is an orthonormal basis of  $K_{B_\sigma}$ . In particular,

$$(1.4) \quad P_{B_\sigma} = \sum_{k=1}^n (\cdot, e_k)_{H^2} e_k,$$

where  $(\cdot, \cdot)_{H^2}$  means the scalar product on  $H^2$ .

## 1.1. Bernstein-type inequalities for rational functions

Bernstein-type inequalities for rational functions were the subject of a number of papers and monographs (see, for instance, [L], [BoEr], [DeLo], [B]). Perhaps, the stronger and closer to the one we need here (Lemma 1.1.1) of all known results are due to K.Dyakonov [Dya1]&[Dya2]. First, we recall the following lemma (proved in Proposition 4.1 of [Z2]).

**Lemma 1.1.1.** *Let  $B = \prod_{j=1}^n b_{\lambda_j}$ , be a finite Blaschke product (of order  $n$ ),  $r = \max_j |\lambda_j|$ , and  $f \in K_B$ . Then,*

$$\|f'\|_{H^2} \leq \frac{5}{2} \frac{n}{1-r} \|f\|_{H^2}.$$

Lemma 1.1.1 is in fact a partial case ( $p = 2$ ) of the following K. Dyakonov's result [Dya1] (which is, in turn, a generalization of M. Levin's inequality [L] corresponding to the case  $p = \infty$ ): it is proved in [Dya1] that the norm  $\|D\|_{K_B^p \rightarrow H^p}$  of the differentiation operator  $Df = f'$  on the star-invariant subspace of the Hardy space  $H^p$ ,  $K_B^p := H^p \cap \overline{BzH^p}$ , (where the bar denotes complex conjugation) satisfies the following inequalities

$$c'_p \left\| B' \right\|_\infty \leq \|D\|_{K_B^p \rightarrow H^p} \leq c_p \left\| B' \right\|_\infty,$$

for every  $p$ ,  $1 \leq p \leq \infty$  where  $c_p$  and  $c'_p$  are positives constants depending on  $p$  only,  $B$  is a finite Blaschke product and  $\|\cdot\|_\infty$  means the norm in  $L^\infty(\mathbb{T})$ . For the partial case considered in Lemma 1.1.1 above, the constant  $\frac{5}{2}$  is slightly better. More precisely, it is proved in [Dya1] that  $c'_2 = \frac{1}{36c}$ ,  $c_2 = \frac{36+c}{2\pi}$  and  $c = 2\sqrt{3\pi}$  (as one can check easily ( $c$  is not precised in [Dya1])). It implies the inequality of Lemma 1.1 with a constant about  $\frac{13}{2}$  instead of  $\frac{5}{2}$ .

The sharpness of the inequality from Lemma 1.1.1 is discussed in [Z3]. Here we use it by induction in order to get the following corollary.

**Corollary 1.1.2.** *Let  $B = \prod_{j=1}^n b_{\lambda_j}$ , be a finite Blaschke product (of order  $n$ ),  $r = \max_j |\lambda_j|$ , and  $f \in K_B$ . Then,*

$$\|f^{(k)}\|_{H^2} \leq k! \left(\frac{7}{2}\right)^k \left(\frac{n}{1-r}\right)^k \|f\|_{H^2},$$

for every  $k = 0, 1, \dots$

Indeed, since  $z^{k-1}f^{(k-1)} \in K_{B^k}$ , we obtain applying Lemma 1.1.1 for  $B^k$  instead of  $B$ ,

$$\|z^{k-1}f^{(k)} + (k-1)z^{k-2}f^{(k-1)}\|_{H^2} \leq \frac{5}{2} \frac{kn}{1-r} \|z^{k-1}f^{(k-1)}\|_{H^2} = \frac{5}{2} \frac{kn}{1-r} \|f^{(k-1)}\|_{H^2},$$

since  $|z^{k-1}| = 1, \forall z \in \mathbb{T}$ . In particular,

$$\left| \|z^{k-1}f^{(k)}\|_{H^2} - \|(k-1)z^{k-2}f^{(k-1)}\|_{H^2} \right| \leq \frac{5}{2} \frac{kn}{1-r} \|f^{(k-1)}\|_{H^2},$$

which gives, since  $|z^{k-1}| = |z^{k-2}| = 1, \forall z \in \mathbb{T}$ ,

$$\begin{aligned} \|f^{(k)}\|_{H^2} &\leq \frac{5}{2} \frac{kn}{1-r} \|f^{(k-1)}\|_{H^2} + (k-1) \|f^{(k-1)}\|_{H^2} \leq \\ &\leq \left(\frac{5}{2} \frac{n}{1-r} + 1\right) k \|f^{(k-1)}\|_{H^2} = \frac{5}{2} \frac{kn}{1-r} \left(1 + \frac{2}{5} \frac{1-r}{n}\right) \|f^{(k-1)}\|_{H^2} \leq \\ &\leq \frac{5}{2} \frac{kn}{1-r} \left(1 + \frac{2}{5}\right) \|f^{(k-1)}\|_{H^2} = \frac{7}{2} \frac{kn}{1-r} \|f^{(k-1)}\|_{H^2}. \end{aligned}$$

By induction,

$$\|f^{(k)}\|_{H^2} \leq k! \left(\frac{7}{2} \frac{n}{1-r}\right)^k \|f\|_{H^2}. \quad \square$$

## 1.2. The proof of Theorem B (the upper bound only)

The first consequence of Corollary 1.1.2 in the following one.

**Corollary 1.2.1.** *Let  $N \geq 0$  be an integer. Then,*

$$C_{n,r} (l_a^2((k+1)^{-N}), H^\infty) \leq A \left(\frac{n}{1-r}\right)^{\frac{2N+1}{2}},$$

for all  $r \in [0, 1], n \geq 1$ , where  $A = A(N)$  is a constant depending on  $N$  (of order  $N^{2N}$  from the proof below).

Indeed, let  $H = l_a^2((k+1)^{-N})$  and  $B = B_\sigma$  the finite Blaschke product corresponding to  $\sigma$ . Let  $\widetilde{P}_B$  be the orthogonal projection of  $H$  onto  $K_B = K_B(H^2)$ . Then  $\widetilde{P}_B|_{H^2} = P_B$ , where  $P_B$  is defined in 4.3. We notice that  $\widetilde{P}_B : H \rightarrow H$  is a bounded operator and the adjoint  $\widetilde{P}_B^* : H^* \rightarrow H^*$  of  $\widetilde{P}_B$  relatively to the Cauchy pairing  $\langle \cdot, \cdot \rangle$  satisfies  $\widetilde{P}_B^* \varphi = \widetilde{P}_B \varphi = P_B \varphi, \forall \varphi \in H^* \subset H^2$ , where  $H^* = l_a^2((k+1)^N)$  is the dual of  $H$  with respect to this pairing. If  $f \in H$ , then  $\left| \widetilde{P}_B f(\zeta) \right| = \left| \langle \widetilde{P}_B f, k_\zeta \rangle \right| = \left| \langle f, \widetilde{P}_B^* k_\zeta \rangle \right|$ , with  $k_\zeta = (1 - \bar{\zeta}z)^{-1} \in H^2$  and

$$\left| \widetilde{P}_B f(\zeta) \right| \leq \|f\|_H \|P_B k_\zeta\|_{H^*} \leq \|f\|_H K_N \left( \|P_B k_\zeta\|_{H^2} + \left\| (P_B k_\zeta)^{(N)} \right\|_{H^2} \right),$$

where

$$\begin{aligned} K_N &= \max \left\{ N^N, \sup_{k \geq N} \frac{(k+1)^N}{k(k-1)\dots(k-N+1)} \right\} = \\ &= \max \left\{ N^N, \frac{(N+1)^N}{N!} \right\} = \begin{cases} N^N, & \text{if } N \geq 3 \\ \frac{(N+1)^N}{N!}, & \text{if } N = 1, 2 \end{cases} . \end{aligned}$$

(Indeed, the sequence  $\left(\frac{(k+1)^N}{k(k-1)\dots(k-N+1)}\right)_{k \geq N}$  is decreasing since  $(1+x)^{-N} \geq 1-Nx$  for all  $x \in [0, 1]$ , and  $\left[N^N > \frac{(N+1)^N}{N!}\right] \iff N \geq 3$ ). Since  $P_B k_\zeta \in K_B$ , Corollary 1.1.2 implies

$$\begin{aligned} \left| \widetilde{P}_B f(\zeta) \right| &\leq \|f\|_H K_N \left( \|P_B k_\zeta\|_{H^2} + N! \left(\frac{7}{2} \frac{n}{1-r}\right)^N \|P_B k_\zeta\|_{H^2} \right) \leq \\ &\leq A(N) \left(\frac{n}{1-r}\right)^{N+\frac{1}{2}} \|f\|_H, \end{aligned}$$

where  $A(N) = \sqrt{2} K_N \left(1 + N! \left(\frac{7}{2}\right)^N\right)$ , since  $\|P_B k_\zeta\|_2 \leq \frac{\sqrt{2n}}{\sqrt{1-r}}$ .  $\square$

*Proof of Theorem B (the right-hand side inequality only).* Applying Lemma 0 with  $X = l_a^2((k+1)^\alpha)$ , we get

$$\|T\|_{l_a^2((k+1)^\alpha) \rightarrow H^\infty / B_\sigma H^\infty} = c \left( \sigma, l_a^2((k+1)^\alpha), H^\infty \right),$$

where  $T$  and  $B_\sigma$  are defined in Lemma 0. Moreover, there exists an integer  $N$  such that  $N-1 \leq -\alpha \leq N$ . In particular, there exists  $0 \leq \theta \leq 1$  such that  $-\alpha = (1-\theta)(N-1) + \theta N$ . And since (as in the proof of Theorem C of [Z2], we use the notation of the interpolation theory between Banach spaces see [Tr] or [Be])

$$\begin{aligned} \left[ l_a^2 \left( \frac{1}{(k+1)^{N-1}} \right), l_a^2 \left( \frac{1}{(k+1)^N} \right) \right]_{\theta, 2} &= l_a^2 \left( \left( \frac{1}{(k+1)^{N-1}} \right)^{2\frac{1-\theta}{2}} \left( \frac{1}{(k+1)^N} \right)^{2\frac{\theta}{2}} \right) = \\ &= l_a^2 \left( \frac{1}{(k+1)^{(1-\theta)(N-1) + \theta N}} \right) = l_A^2((k+1)^\alpha), \end{aligned}$$

this gives, using Corollary 1.2.1 and [Tr] Theorem 1.9.3-(a) p.59,

$$\begin{aligned} &\|T\|_{l_a^2((k+1)^\alpha) \rightarrow H^\infty / B_\sigma H^\infty} \leq \\ &\leq \left( c \left( \sigma, l_a^2 \left( \frac{1}{(k+1)^{N-1}} \right), H^\infty \right) \right)^{1-\theta} \left( c \left( \sigma, l_a^2 \left( \frac{1}{(k+1)^N} \right), H^\infty \right) \right)^\theta \leq \\ &\leq \left( A(N-1) \left( \frac{n}{1-r} \right)^{\frac{2N-1}{2}} \right)^{1-\theta} \left( A(N) \left( \frac{n}{1-r} \right)^{\frac{2N+1}{2}} \right)^\theta = \\ &= A(N-1)^{1-\theta} A(N)^\theta \left( \frac{n}{1-r} \right)^{\frac{(2N-1)(1-\theta)}{2} + \frac{(2N+1)\theta}{2}} . \end{aligned}$$

It remains to use  $\theta = 1 - \alpha - N$  and set  $A(\alpha) = A(N-1)^{1-\theta} A(N)^\theta$ .

$\square$

## 2. An upper bound for $c(\sigma, l_a^1(w_k), H^\infty)$

The aim of this Section is to prove the right-hand side inequality of Theorem C for the partial case  $p = 1$ , in which the upper bound  $\left(\frac{n}{1-r}\right)^{\frac{1-2\alpha}{2}}$  is not as sharp as in Section 1. We suspect  $\left(\frac{n}{1-r}\right)^{-\alpha}$  is the sharp bound for the quantity  $C_{n,r}(l_a^1((k+1)^\alpha), H^\infty)$ . First, we recall the following lemma already proved in [Z2] (see Lemma 1.1.5 of [Z2]).

**Lemma 2.0.** *Let  $\sigma = \{\lambda_1, \dots, \lambda_n\}$  be a sequence in the unit disc  $\mathbb{D}$  and  $(e_k)_{k=1}^n$  the Malmquist family (see 4.1) corresponding to  $\sigma$ . The map  $\Phi : \text{Hol}(\mathbb{D}) \rightarrow Y \subset \text{Hol}(\mathbb{D})$  defined by*

$$\Phi : f \mapsto \sum_{k=1}^n \left( \sum_{j \geq 0} \hat{f}(j) \overline{\hat{e}_k(j)} \right) e_k,$$

is well defined (if  $h \in \text{Hol}(\mathbb{D})$  and  $k \in \mathbb{N}$ ,  $\hat{h}(k)$  stands for the  $k^{\text{th}}$  Taylor coefficient of  $h$ ) and has the following properties :

- (a)  $\Phi|_{H^2} = P_{B_\sigma}$ ,
- (b)  $\Phi$  is continuous on  $\text{Hol}(\mathbb{D})$  for the uniform convergence on compact sets of  $\mathbb{D}$ ,
- (c) Let  $\Psi = \text{Id}_X - \Phi|_X$ , then  $\text{Im}(\Psi) \subset B_\sigma X$ ,

where  $B = B_\sigma$  is the finite Blaschke product corresponding to  $\sigma$  and  $P_{B_\sigma}$  is defined in 1.3.

Now we prove the following partial case of Theorem C.

**Lemma 2.1.** *Let  $N \geq 0$  be an integer. Then,*

$$C_{n,r}(l_a^1((k+1)^{-N}), H^\infty) \leq A_1 \left( \frac{n}{1-r} \right)^{N+\frac{1}{2}},$$

for all  $r \in [0, 1[$ ,  $n \geq 1$ , where  $A_1 = A_1(N)$  is a constant depending only on  $N$  (of order  $N^{2N}$  from the proof below).

Indeed, the proof is exactly the same as in Corollary 1.2.1 : if  $\sigma$  is a sequence of  $\mathbb{D}$  with  $\#\sigma \leq n$ , and  $f \in l_a^1\left(\frac{1}{(k+1)^N}\right) = X$ , then

$$\Phi(f)(\zeta) = \sum_{k=1}^n \langle f, e_k \rangle e_k(\zeta) = \left\langle f, \sum_{k=1}^n \overline{e_k(\zeta)} e_k \right\rangle,$$

where  $\langle \cdot, \cdot \rangle$  means the Cauchy sesquilinear form  $\langle h, g \rangle = \sum_{k \geq 0} \hat{h}(k) \overline{\hat{g}(k)}$ . In particular,

$$|\Phi(f)(\zeta)| = |\langle f, P_B k_\zeta \rangle|,$$

where  $B = B_\sigma$  is the finite Blaschke product corresponding to  $\sigma$ ,  $P_B$  is defined in 1.3 and  $k_\zeta = (1 - \bar{\zeta}z)^{-1}$ . Denoting  $X^*$  the dual of  $X$  with respect to this pairing,  $X^* = l_a^\infty((k+1)^N)$ , we get,

$$\begin{aligned} |\Phi(f)(\zeta)| &\leq \|f\|_X \|P_B^* k_\zeta\|_{X^*} = \|f\|_X \|P_B k_\zeta\|_{X^*} \leq \\ &\leq \|f\|_X K_N \max \left\{ \sup_{0 \leq k \leq N-1} \left| \widehat{P_B k_\zeta}(k) \right|, \sup_{k \geq N} \left| \widehat{(P_B k_\zeta)^{(N)}}(k-N) \right| \right\} \leq \\ &\leq \|f\|_X K_N \max \left\{ \|P_B k_\zeta\|_{H^2}, \left\| (P_B k_\zeta)^{(N)} \right\|_{H^2} \right\}, \end{aligned}$$

where  $K_N$  is defined in the the proof of Corollary 1.2.1. Since  $P_B k_\zeta \in K_B$ , Corollary 1.1.2 implies

$$\begin{aligned} |\Phi(f)f(\zeta)| &\leq \|f\|_X K_N \left( \|P_B k_\zeta\|_{H^2} + N! \left(\frac{7}{2}\right)^N \left(\frac{n}{1-r}\right)^N \|P_B k_\zeta\|_{H^2} \right) \leq \\ &\leq K_N \|f\|_X \|P_B k_\zeta\|_2 \left( 1 + N! \left(\frac{7}{2}\right)^N \left(\frac{n}{1-r}\right)^N \right) \leq K_N \|f\|_H \left(\frac{2n}{1-r}\right)^{\frac{1}{2}} \left( 1 + N! \left(\frac{7}{2}\right)^N \left(\frac{n}{1-r}\right)^N \right), \end{aligned}$$

which completes the proof setting  $A_1(N) = 2\sqrt{2}N! \left(\frac{7}{2}\right)^N K_N$ .  $\square$

*Proof of Theorem C for  $p = 1$  only (the right-hand inequality only).* This is the same reasoning as in Theorem B. Applying Lemma 0 with  $X = l_a^1((k+1)^\alpha)$ , we get

$$\|T\|_{l_a^1((k+1)^\alpha) \rightarrow H^\infty / B_\sigma H^\infty} = c(\sigma, l_a^1((k+1)^\alpha), H^\infty),$$

where  $T$  and  $B_\sigma$  are defined in Lemma 0. It remains to use Lemma 2.1 and (again) [Tr] Theorem 1.9.3-(a) p.59 to complete the proof.  $\square$

### 3. An upper bound for $c(\sigma, l_a^p(w_k), H^\infty)$ , $1 \leq p \leq 2$

The aim of this section is to generalize the result of Section 2 to the case  $p \in [1, 2]$ . In other words we prove Theorem C, in which again, the upper bound  $\left(\frac{n}{1-r}\right)^{\frac{1-2\alpha}{2}}$  is not sharp as sharp as in Section 1. We suppose that the sharp upper (and lower) bound here should be of the order of  $\left(\frac{n}{1-r}\right)^{1-\alpha-\frac{1}{p}}$ .

*Proof of Theorem C.* We first prove the right hand side inequality. The scheme of the proof is completely the same as in Theorem B, but we simply use interpolation between  $l^1$  and  $l^2$  (the classical Riesz-Thorin theorem). Applying Lemma 0 with  $X = l_a^p((k+1)^\alpha)$ , we get

$$\|T\|_{l_a^p((k+1)^\alpha) \rightarrow H^\infty / B_\sigma H^\infty} = c(\sigma, l_a^p((k+1)^\alpha), H^\infty),$$

where  $T$  and  $B_\sigma$  are defined in Lemma 0. It remains to use both Theorem B, Theorem C for the special case  $p = 1$  (already proved in Section 2), and (again) [Tr] Theorem 1.9.3-(a) p.59 to complete the proof of the right hand side inequality.

Now, we prove the left hand side one. First, it is clear that

$$\begin{aligned} C_{n,r}(l_a^p((k+1)^\alpha), H^\infty) &\geq \|\varphi_r\|_{l_a^{p'}((k+1)^{-\alpha})} = \\ &= \left( \sum_{k \geq 0} (k+1)^{(\alpha-1)p'} (r^{p'})^k \right)^{\frac{1}{p'}}, \end{aligned}$$

where  $\varphi_r$  is the evaluation functional

$$\varphi_r(f) = f(r), \quad f \in X,$$

and  $p'$  is the conjugate of  $p$  :  $\frac{1}{p} + \frac{1}{p'} = 1$ . Now, since

$$\sum_{k \geq 1} k^s x^k \sim \int_1^\infty t^s x^t dt \sim \Gamma(s+1)(1-x)^{-s-1}, \text{ as } x \rightarrow 1,$$

for all  $s > -1$ , where  $\Gamma$  stands for the usual Gamma function,  $\Gamma(z) = \int_0^{+\infty} e^{-s} s^{z-1} ds$ , we get

$$\sum_{k \geq 0} (k+1)^{(\alpha-1)p'} (r^{p'})^k \sim \Gamma((\alpha-1)p'+1) (1-r^{p'})^{-(\alpha-1)p'-1}, \text{ as } r \rightarrow 1.$$

But,

$$(1-r^{p'})^{-(\alpha-1)p'-1} = \left( \frac{r-1}{r^{p'}-1} \frac{1}{1-r} \right)^{(\alpha-1)p'+1} \sim \left( \frac{1}{p'} \frac{1}{1-r} \right)^{(\alpha-1)p'+1}, \text{ as } r \rightarrow 1.$$

As a result,

$$\sum_{k \geq 0} (k+1)^{(\alpha-1)p'} (r^{p'})^k \sim \frac{\Gamma((\alpha-1)p'+1)}{p'^{(\alpha-1)p'+1}} \left( \frac{1}{1-r} \right)^{(\alpha-1)p'+1}, \text{ as } r \rightarrow 1,$$

and

$$\left( \sum_{k \geq 0} (k+1)^{(\alpha-1)p'} (r^{p'})^k \right)^{\frac{1}{p'}} \sim \left( \frac{1}{p'} \right)^{\frac{1}{p'} + (\alpha-1)} \left( \Gamma((\alpha-1)p'+1) \right)^{\frac{1}{p'}} (1-r)^{-(\alpha-1) - \frac{1}{p'}}, \text{ as } r \rightarrow 1.$$

This completes the proof since  $\frac{1}{p'} = 1 - \frac{1}{p}$ .  $\square$

#### 4. An upper bound for $c(\sigma, l_a^\infty(w_k), H^\infty)$

In this Section, we prove the right-hand side inequality from Theorem D for  $p = \infty$  only, in which -again- the upper bound  $\left(\frac{n}{1-r}\right)^{\frac{3-2\alpha}{2}}$  is not as sharp as in Section 1. We can suppose here that the constant  $\left(\frac{n}{1-r}\right)^{1-\alpha}$  is the sharp bound for the quantity  $C_{n,r}(l_a^\infty((k+1)^\alpha), H^\infty)$ . First we prove the following partial case of Theorem D.

**Corollary 4.1** *Let  $N \geq 0$  be an integer. Then,*

$$C_{n,r}(l_a^\infty((k+1)^{-N}), H^\infty) \leq A_\infty \left( \frac{n}{1-r} \right)^{N+\frac{3}{2}},$$

for all  $r \in [0, 1[$ ,  $n \geq 1$ , where  $A_\infty = A_\infty(N)$  is a constant depending on  $N$  (of order  $N^{2N}$  from the proof below).

*Proof.* We use literally the same method as in Corollary 1.2.1 and Lemma 2.1. Indeed, if  $f \in l_a^\infty\left(\frac{1}{(k+1)^N}\right) = X$ , then

$$|\Phi(f)(\zeta)| = |\langle f, P_B k_\zeta \rangle|,$$

where  $\Phi$  is defined in Lemma 2.0,  $B = B_\sigma$  is the finite Blaschke product corresponding to  $\sigma$ ,  $P_B$  is defined in 1.3,  $k_\zeta = (1 - \bar{\zeta}z)^{-1}$  and  $\langle \cdot, \cdot \rangle$  means the Cauchy pairing. Denoting  $X^*$  the dual of  $X$

with respect to this pairing,  $X^* = l_a^1((k+1)^N)$ , we get

$$|\Phi(f)(\zeta)| \leq \|f\|_X \|P_B k_\zeta\|_{X^*} \leq \|f\|_H K_N \left( \|P_B k_\zeta\|_W + \left\| (P_B k_\zeta)^{(N)} \right\|_W \right),$$

where  $W = \left\{ f = \sum_{k \geq 0} \hat{f}(k) z^k : \|f\|_W := \sum_{k \geq 0} |\hat{f}(k)| < \infty \right\}$  stands for the Wiener algebra, and  $K_N$  is defined in Corollary 1.3. Now, applying Hardy's inequality (see [N2] p.370, 8.7.4 (c)),

$$\begin{aligned} |\Phi(f)(\zeta)| &\leq \|f\|_X K_N \left( \pi \left\| (P_B k_\zeta)' \right\|_{H^1} + |(P_B k_\zeta)(0)| + \pi \left\| (P_B k_\zeta)^{(N+1)} \right\|_{H^1} + \left| (P_B k_\zeta)^{(N)}(0) \right| \right) \leq \\ &\leq \|f\|_X K_N \left( \pi \left\| (P_B k_\zeta)' \right\|_{H^2} + |(P_B k_\zeta)(0)| + \pi \left\| (P_B k_\zeta)^{(N+1)} \right\|_{H^2} + \left| (P_B k_\zeta)^{(N)}(0) \right| \right), \end{aligned}$$

which gives using both Lemma 1.1 and Corollary 1.2,

$$\begin{aligned} |\Phi(f)(\zeta)| &\leq \\ &\leq \|f\|_X K_N \pi \left( \left( \frac{5}{2} \frac{n}{1-r} \right) \|P_B k_\zeta\|_{H^2} + |(P_B k_\zeta)(0)| + \right. \\ &+ (N+1)! \left( \frac{7}{2} \frac{n}{1-r} \right)^{N+1} \|P_B k_\zeta\|_{H^2} + \left. \left| (P_B k_\zeta)^{(N)}(0) \right| \right) \leq \\ &\leq \|f\|_X K_N \pi \left( \left( \frac{5}{2} \frac{n}{1-r} \right) \|P_B k_\zeta\|_{H^2} + \|P_B k_\zeta\|_{H^2} + \right. \\ &+ (N+1)! \left( \frac{7}{2} \frac{n}{1-r} \right)^{N+1} \|P_B k_\zeta\|_{H^2} + N! \|P_B k_\zeta\|_{H^2} \left. \right). \end{aligned}$$

This completes the proof since  $\|P_B k_\zeta\|_{H^2} \leq \left( \frac{2n}{1-r} \right)^{\frac{1}{2}}$ .  $\square$

*Proof of Theorem D for  $p = \infty$  (the right-hand side inequality only).* This is the same application of interpolation between Banach spaces, as before (Theorem 1.0&2.0) excepted that this time we apply Lemma 0 with  $X = l_a^\infty((k+1)^\alpha)$  to get

$$\|T\|_{l_a^\infty((k+1)^\alpha) \rightarrow H^\infty / B_\sigma H^\infty} = c(\sigma, l_a^\infty((k+1)^\alpha), H^\infty),$$

where  $T$  and  $B_\sigma$  are defined in Lemma 0.

Applying Lemma 4.1 and using (again) [Tr] Theorem 1.9.3-(a) p.59, we can complete the proof.

$\square$

## 5. An upper bound for $c(\sigma, l_a^p(w_k), H^\infty)$ , $2 \leq p \leq \infty$

The aim of this section is to prove Theorem D. As before, the upper bound  $\left( \frac{n}{1-r} \right)^{\frac{3}{2}-\alpha-\frac{2}{p}}$  is not as sharp as in Section 1. We can suppose here the constant  $\left( \frac{n}{1-r} \right)^{1-\alpha-\frac{1}{p}}$  should be a sharp upper (and lower) bound for the quantity  $C_{n,r}(l_a^p((k+1)^\alpha), H^\infty)$ ,  $2 \leq p \leq +\infty$ .

*Proof of Theorem D.* We first prove the right hand side inequality. The proof repeats the scheme from previous theorems and from Theorem C in particular. We have already seen (in Theorem C) that

$$\|T\|_{l_a^p((k+1)^\alpha) \rightarrow H^\infty/B_\sigma H^\infty} = c(\sigma, l_a^p((k+1)^\alpha), H^\infty),$$

where  $T$  and  $B_\sigma$  are defined in Lemma 0. Now, using both Theorem B, Theorem D for the particular case  $p = \infty$  (already proved in Section 4), and [Tr] Theorem 1.9.3-(a) p.59, we complete the proof. The proof of the left hand side inequality is exactly the same as in Theorem C.  $\square$

## 6. Carathéodory-Schur Interpolation in weighted Bergman spaces

We suppose that  $X = L_a^p\left((1 - |z|^2)^\beta dA\right)$ ,  $\beta > -1$  and  $1 \leq p \leq 2$ . Our aim in this section is to give an estimate for the constant for a generalized Carathéodory-Schur interpolation, (a partial case of the Nevanlinna-Pick interpolation),

$$c(\sigma_{n,\lambda}, X, H^\infty) = \sup \{ \|f\|_{H^\infty/b_\lambda^n H^\infty} : f \in X, \|f\|_X \leq 1 \},$$

where  $\|f\|_{H^\infty/b_\lambda^n H^\infty} = \inf \{ \|f + b_\lambda^n g\|_\infty : g \in X \}$ , and  $\sigma_{n,\lambda} = \{\lambda, \lambda, \dots, \lambda\}$ ,  $\lambda \in \mathbb{D}$ . The corresponding interpolation problem is: given  $f \in X$ , to minimize  $\|h\|_\infty$  such that  $h^{(j)}(\lambda) = f^{(j)}(\lambda)$ ,  $0 \leq j < n$ .

For this partial case, we prove Theorem E which is a generalization of the estimate from Theorem B.

We first need a simple equivalent to  $I_k(\beta) = \int_0^1 r^{2k+1}(1 - r^2)^\beta dr$ ,  $\beta > -1$ .

**Lemma 6.1.** *Let  $k \geq 0$ ,  $\beta > -1$  and  $I_k(\beta) = \int_0^1 r^{2k+1}(1 - r^2)^\beta dr$ . Then,*

$$I_k(\beta) \sim \frac{1}{2} \frac{\Gamma(\beta + 1)}{k^{\beta+1}},$$

for  $k \rightarrow \infty$ , where  $\Gamma$  stands for the usual Gamma function,  $\Gamma(z) = \int_0^{+\infty} e^{-s} s^{z-1} ds$ .

*Proof.* Let  $a = \frac{1}{\sqrt{k+1}}$ ,  $b = \max(1, a^\beta)$ . Since  $1 - e^{-u} \sim u$  as  $u \rightarrow 0$ , we have

$$\begin{aligned} I_k(\beta) &= \int_0^1 r^{2k+1}(1 - r^2)^\beta dr = \int_0^\infty e^{-(2k+1)t}(1 - e^{-2t})^\beta e^{-t} dt = \\ &= \int_0^a e^{-2(k+1)t}(1 - e^{-2t})^\beta dt + \int_a^\infty e^{-2(k+1)t}(1 - e^{-2t})^\beta dt = \\ &= \int_0^a e^{-2(k+1)t}(1 - e^{-2t})^\beta dt + O\left(\frac{b}{k+1} e^{-2a(k+1)}\right) = \\ &= (1 + o(1)) \int_0^a e^{-2(k+1)t}(2t)^\beta dt + O\left(\frac{b}{k+1} e^{-2a(k+1)}\right) = \\ &= (1 + o(1)) \int_0^{2(k+1)a} e^{-s} \left(\frac{s}{k+1}\right)^\beta \frac{ds}{2(k+1)} + O\left(\frac{b}{k+1} e^{-2a(k+1)}\right) = \\ &= \frac{1}{2} \frac{1}{(k+1)^{\beta+1}} (1 + o(1)) \int_0^{2(k+1)a} e^{-s} s^\beta ds + O\left(\frac{b}{k+1} e^{-2a(k+1)}\right) = \end{aligned}$$



$$\begin{aligned}
&= \frac{1}{2} \frac{\Gamma(\beta+1)}{(k+1)^{\beta+1}} (1+o(1)) + O\left(\frac{b}{k+1} e^{-2a(k+1)}\right) = \\
&= \frac{1}{2} \frac{\Gamma(\beta+1)}{(k+1)^{\beta+1}} (1+o(1)) \sim \frac{1}{2} \frac{\Gamma(\beta+1)}{k^{\beta+1}},
\end{aligned}$$

which completes the proof.  $\square$

*Proof of Theorem E. Step 1.* We start to prove the Theorem for  $p = 1$ .

Let  $f \in X = L_a^1\left(\left(1 - |z|^2\right)^\beta dA\right)$  such that  $\|f\|_X \leq 1$ . Since  $X \circ b_\lambda = X$ , we have  $f \circ b_\lambda = \sum_{k \geq 0} a_k z^k \in X$ . Let  $p_n = \sum_{k=0}^{n-1} a_k z^k$  and  $g = p_n \circ b_\lambda$ . Then,  $f \circ b_\lambda - p_n \in z^n X$  and  $f - p_n \circ b_\lambda \in (z^n X) \circ b_\lambda = b_\lambda^n X$ . Now,  $p_n \circ b_\lambda = \sum_{k=0}^{n-1} a_k b_\lambda^k$  and

$$\|p_n \circ b_\lambda\|_\infty = \|p_n\|_\infty \leq A_n \|f \circ b_\lambda\|_X,$$

where  $A_n = \left\| \sum_{k \geq 0} a_k z^k \mapsto \sum_{k=0}^{n-1} a_k z^k \right\|_{X \rightarrow H^\infty}$ . Now,

$$\begin{aligned}
\|f \circ b_\lambda\|_X &\leq \int_{\mathbb{D}} |f(b_\lambda(z))| (1 - |z|^2)^\beta dA = \int_{\mathbb{D}} |f(w)| (1 - |b_\lambda(w)|^2)^\beta \left| b'_\lambda(w) \right|^2 dA = \\
&\leq 2^\beta \int_{\mathbb{D}} |f(w)| \left( \frac{(1 - |\lambda|^2)(1 - |w|^2)}{|1 - \bar{\lambda}w|^2} \right)^\beta \left( \frac{(1 - |\lambda|^2)}{|1 - \bar{\lambda}w|^2} \right)^2 dA = \\
&= \int_{\mathbb{D}} |f(w)| (1 - |w|^2)^\beta \left( \frac{(1 - |\lambda|^2)}{|1 - \bar{\lambda}w|^2} \right)^{2+\beta} dA \leq \\
&\leq \sup_{w \in \mathbb{D}} \left( \frac{(1 - |\lambda|^2)}{|1 - \bar{\lambda}w|^2} \right)^{2+\beta} \int_{\mathbb{D}} |f(w)| (1 - |w|^2)^\beta dA \leq \left( \frac{(1 - |\lambda|^2)}{(1 - |\lambda|^2)} \right)^{2+\beta} \|f\|_X,
\end{aligned}$$

which gives,

$$\|f \circ b_\lambda\|_X \leq \left( \frac{1 + |\lambda|}{1 - |\lambda|} \right)^{2+\beta} \|f\|_X.$$

We now give an estimation for  $A_n$ . Let  $g(z) = \sum_{k \geq 0} \hat{g}(k) z^k \in X$ , then

$$\left\| \sum_{k=0}^{n-1} \hat{g}(k) z^k \right\|_\infty \leq \sum_{k=0}^{n-1} |\hat{g}(k)|.$$

Now, noticing that

$$\begin{aligned}
\int_{\mathbb{D}} g(w) \bar{w}^k (1 - |w|^2)^\beta dA &= \int_0^1 \int_0^{2\pi} f(re^{it}) r^k e^{-ikt} (1 - r^2)^\beta r dt dr = \\
&= \int_0^1 (1 - r^2)^\beta r^{k+1} \int_0^{2\pi} f(re^{it}) e^{-ikt} dt dr = \int_0^1 \hat{g}_r(k) r^{k+1} (1 - r^2)^\beta dr,
\end{aligned}$$

where  $g_r(z) = g(rz)$ ,  $\widehat{g}_r(k) = r^k \widehat{g}(k)$ . Setting  $I_k(\beta) = \int_0^1 r^{2k+1} (1-r^2)^\beta dr$ , we get

$$\widehat{g}(k) = \frac{1}{I_k(\beta)} \int_{\mathbb{D}} g(w) \overline{w}^k (1-|w|^2)^\beta dA.$$

Then,

$$|\widehat{g}(k)| = \frac{1}{I_k(\beta)} \left| \int_{\mathbb{D}} g(w) \overline{w}^k (1-|w|^2)^\beta dA \right| \leq \frac{1}{I_k(\beta)} \|g\|_X,$$

which gives

$$\left\| \sum_{k=0}^{n-1} \widehat{g}(k) z^k \right\|_{\infty} \leq \left( \sum_{k=0}^{n-1} \frac{1}{I_k(\beta)} \right) \|g\|_X.$$

Now using Lemma 6.1,

$$\sum_{k=0}^{n-1} \frac{1}{I_k(\beta)} \sim_{n \rightarrow \infty} \frac{2}{\Gamma(\beta+1)} \sum_{k=0}^{n-1} k^{\beta+1} \sim \frac{2c_\beta}{\Gamma(\beta+1)} n^{\beta+2},$$

where  $c_\beta$  is a constant depending on  $\beta$  only. This gives

$$\left\| \sum_{k=0}^{n-1} \widehat{g}(k) z^k \right\|_{\infty} \leq C_\beta n^{\alpha+2} \|g\|_X,$$

where  $C_\beta$  is also a constant depending on  $\beta$  only. Finally, we conclude that  $A_n \leq C_\beta n^{\beta+2}$ , and as a result,

$$\|p_n \circ b_\lambda\|_{\infty} \leq C_\beta n^{\beta+2} \left( \frac{1+|\lambda|}{1-|\lambda|} \right)^{2+\beta} \|f\|_X,$$

which proves the Theorem for  $p = 1$ .

**Step 2.** This step of the proof repeats the scheme from Theorems C&D. Let  $T : L_a^p \left( (1-|z|^2)^\beta dA \right) \rightarrow H^\infty / b_\lambda^n H^\infty$  be the restriction map defined by

$$Tf = \left\{ g \in H^\infty : f - g \in b_\lambda^n L_a^p \left( (1-|z|^2)^\beta dA \right) \right\},$$

for every  $f$ . Then applying Lemma 0,

$$\|T\|_{L_a^p \left( (1-|z|^2)^\beta dA \right) \rightarrow H^\infty / b_\lambda^n H^\infty} = c \left( \sigma, L_a^p \left( (1-|z|^2)^\beta dA \right), H^\infty \right).$$

Now, let  $\gamma > \beta$  and  $P_\gamma : L^p \left( (1-|z|^2)^\beta dA \right) \rightarrow L_a^p \left( (1-|z|^2)^\beta dA \right)$  be the Bergman projection, (see [H], p.6), defined by

$$P_\gamma f = (\gamma+1) \int_{\mathbb{D}} \frac{(1-|w|^2)^\gamma}{(1-z\overline{w})^{2+\gamma}} f(w) dA(w),$$

for every  $f$ .  $P_\gamma$  is a bounded projection from  $L^p \left( (1 - |z|^2)^\beta dA \right)$  onto  $L_a^p \left( (1 - |z|^2)^\beta dA \right)$  (see [H], Theorem 1.10 p.12), (since  $1 \leq p \leq 2$ ). Moreover, since  $L_a^p \left( (1 - |z|^2)^\beta dA \right) \subset L_a^p \left( (1 - |z|^2)^\gamma dA \right)$ , we have  $P_\gamma f = f$  for all  $f \in L_a^p \left( (1 - |z|^2)^\beta dA \right)$ , (see [H], Corollary 1.5 p.6). As a result,

$$\|T\|_{L_a^p \left( (1 - |z|^2)^\beta dA \right) \rightarrow H^\infty / b_\lambda^n H^\infty} \leq \|TP_\gamma\|_{L^p \left( (1 - |z|^2)^\beta dA \right) \rightarrow H^\infty / b_\lambda^n H^\infty},$$

for all  $1 \leq p \leq 2$ . We set

$$c_i(\beta) = \|P_\gamma\|_{L^i \left( (1 - |z|^2)^\beta dA \right) \rightarrow L_a^i \left( (1 - |z|^2)^\beta dA \right)},$$

for  $i = 1, 2$ . Then,

$$\begin{aligned} & \|TP_\gamma\|_{L^1 \left( (1 - |z|^2)^\beta dA \right) \rightarrow H^\infty / b_\lambda^n H^\infty} \leq \\ & \leq \|T\|_{L_a^1 \left( (1 - |z|^2)^\beta dA \right) \rightarrow H^\infty / b_\lambda^n H^\infty} \|P_\gamma\|_{L^1 \left( (1 - |z|^2)^\beta dA \right) \rightarrow L_a^1 \left( (1 - |z|^2)^\beta dA \right)} = \\ & = c \left( \sigma, L_a^1 \left( (1 - |z|^2)^\beta dA \right), H^\infty \right) c_1(\beta) \leq \\ & \leq A'(\beta, 1) \left( \frac{n}{1 - |\lambda|} \right)^{\beta+2} c_1(\beta), \end{aligned}$$

using Step 1. In the same way,

$$\|TP_\gamma\|_{L^2 \left( (1 - |z|^2)^\beta dA \right) \rightarrow H^\infty / b_\lambda^n H^\infty} \leq \|T\|_{L_a^2 \left( (1 - |z|^2)^\beta dA \right) \rightarrow H^\infty / b_\lambda^n H^\infty} c_2(\beta).$$

Now, we recall that

$$L_a^2 \left( (1 - |z|^2)^\beta dA \right) = l_a^2 \left( (k+1)^{-\frac{\beta+1}{2}} \right), \quad \beta > -1.$$

As a consequence,

$$\|T\|_{L_a^2 \left( (1 - |z|^2)^\beta dA \right) \rightarrow H^\infty / b_\lambda^n H^\infty} = c \left( \sigma, l_a^2 \left( (k+1)^{-\frac{\beta+1}{2}} \right), H^\infty \right),$$

and, applying Theorem B,

$$\begin{aligned} \|TP_\gamma\|_{L^2 \left( (1 - |z|^2)^\beta dA \right) \rightarrow H^\infty / b_\lambda^n H^\infty} & \leq c_2(\beta) A' \left( \frac{\beta+1}{2}, 2 \right) \left( \frac{n}{1 - |\lambda|} \right)^{\frac{2\frac{\beta+1}{2}+1}{2}} = \\ & = c_2(\beta) A' \left( \frac{\beta+1}{2}, 2 \right) \left( \frac{n}{1 - |\lambda|} \right)^{\frac{\beta+2}{2}}. \end{aligned}$$

We finish the reasoning applying Riesz-Thorin Theorem, (see [Tr] for example), to the operator  $TP_\gamma$ . If  $p \in [1, 2]$ , there exists  $0 \leq \theta \leq 1$  such that

$$\frac{1}{p} = (1 - \theta)\frac{1}{1} + \theta\frac{1}{2} = 1 - \frac{\theta}{2},$$

and then,

$$\left[ L_a^1 \left( (1 - |z|^2)^\beta dA \right), L_a^2 \left( (1 - |z|^2)^\beta dA \right) \right]_\theta = L_a^p \left( (1 - |z|^2)^\beta dA \right),$$

and

$$\begin{aligned} & \|TP_\gamma\|_{L^p((1-|z|^2)^\beta dA) \rightarrow H^\infty/b_\lambda^n H^\infty} \leq \\ & \leq \left( \|TP_\gamma\|_{L^1((1-|z|^2)^\beta dA) \rightarrow H^\infty/b_\lambda^n H^\infty} \right)^{1-\theta} \left( \|TP_\gamma\|_{L^2((1-|z|^2)^\beta dA) \rightarrow H^\infty/b_\lambda^n H^\infty} \right)^\theta \leq \\ & \leq \left( c_1(\beta)A'(\beta, 1) \left( \frac{n}{1-|\lambda|} \right)^{\beta+2} \right)^{1-\theta} \left( c_2(\beta)A' \left( \frac{\beta+1}{2}, 2 \right) \left( \frac{n}{1-|\lambda|} \right)^{\frac{\beta+2}{2}} \right)^\theta = \\ & = \left( c_1(\beta)A'(\beta, 1) \right)^{1-\theta} \left( c_2(\beta)A' \left( \frac{\beta+1}{2}, 2 \right) \right)^\theta \left( \frac{n}{1-|\lambda|} \right)^{(\beta+2)(1-\theta) + \theta \frac{\beta+2}{2}}. \end{aligned}$$

Now, since  $\theta = 2(1 - \frac{1}{p})$ ,  $(\beta + 2)(1 - \theta) + \theta \frac{\beta+2}{2} = \beta - (1 - \frac{1}{p})\beta + 2 - 2 + \frac{2}{p} = \frac{\beta+2}{p}$ , and

$$\|T\|_{L_a^p((1-|z|^2)^\beta dA) \rightarrow H^\infty/b_\lambda^n H^\infty} \leq \|TP_\gamma\|_{L^p((1-|z|^2)^\beta dA) \rightarrow H^\infty/b_\lambda^n H^\infty},$$

we complete the proof.  $\square$

## 7. A lower bound for $C_{n,r} (l_a^2(w_k), H^\infty)$

Here, we consider the weighted spaces  $H = l_a^2(w_k)$  of polynomial growth and the problem of lower estimates for the one point special case  $\sigma_{n,\lambda} = \{\lambda, \lambda, \dots, \lambda\}$ , ( $n$  times)  $\lambda \in \mathbb{D}$ . Recall the definition of the interpolation constant

$$c(\sigma_{n,\lambda}, H, H^\infty) = \sup \left\{ \|f\|_{H^\infty/b_\lambda^n H^\infty} : f \in H, \|f\|_H \leq 1 \right\},$$

where  $\|f\|_{H^\infty/b_\lambda^n H^\infty} = \inf \{ \|f + b_\lambda^n g\|_\infty : g \in H \}$ . In particular, our aim is to prove the sharpness of the upper estimate for the quantity

$$C_{n,r} \left( l_a^2 \left( \frac{1}{(k+1)^{\frac{N-1}{2}}} \right), H^\infty \right),$$

(where  $N \geq 1$  is an integer), in Theorem B.

In the proof, we use properties of spaces  $X = l_a^p(w_k)$ . As it is mentioned in the Introduction,

$$l_a^p(w_k) = \left\{ f = \sum_{k \geq 0} \hat{f}(k) z^k : \|f\|^p = \sum_{k \geq 0} |\hat{f}(k)|^p w_k^p < \infty \right\},$$

with a weight  $w$  satisfying  $w_k > 0$  for every  $k \geq 0$  and  $\overline{\lim}_k (1/w_k^{1/k}) = 1$ . In this section, we study the case  $p = 2$ , so that  $l_a^2(w_k)$  is a reproducing kernel Hilbert space on the disc  $\mathbb{D}$ . The reproducing kernel of  $l_a^2(w_k)$ , by definition, is a  $l_a^2(w_k)$ -valued function  $\lambda \mapsto k_\lambda^w$ ,  $\lambda \in \mathbb{D}$ , such that  $(f, k_\lambda^w) = f(\lambda)$  for every  $f \in l_a^2(w_k)$ , where  $(\cdot, \cdot)$  means the scalar product  $(f, g) = \sum_{k \geq 0} \hat{f}(k) \overline{\hat{g}(k)} w_k^2$ . Since one has  $f(\lambda) = \sum_{k \geq 0} \hat{f}(k) \lambda^k \frac{1}{w_k^2} w_k^2$  ( $\lambda \in \mathbb{D}$ ), it follows that

$$k_\lambda^w(z) = \sum_{k \geq 0} \frac{\overline{\lambda}^k z^k}{w_k^2}, \quad z \in \mathbb{D}.$$

In particular, for the Hardy space  $H^2 = l_a^2(1)$ , we get the Szegő kernel

$$k_\lambda(z) = (1 - \overline{\lambda}z)^{-1},$$

for the Bergman space  $L_a^2 = l_a^2\left(\frac{1}{(k+1)^2}\right)$  - the Bergman kernel  $k_\lambda(z) = (1 - \overline{\lambda}z)^{-2}$ .

Conversely, following the Aronszajn theory of RKHS (see, for example [A] or [N2] p.317), given a positive definit function  $(\lambda, z) \mapsto k(\lambda, z)$  on  $\mathbb{D} \times \mathbb{D}$  (i.e. such that  $\sum_{i,j} \overline{a_i} a_j k(\lambda_i, \lambda_j) > 0$  for all finite subsets  $(\lambda_i) \subset \mathbb{D}$  and all non-zero families of complex numbers  $(a_i)$ ) one can define the corresponding Hilbert spaces  $H(k)$  as the completion of finite linear combinations  $\sum_i \overline{a_i} k(\lambda_i, \cdot)$  endowed with the norm

$$\left\| \sum_i \overline{a_i} k(\lambda_i, \cdot) \right\|^2 = \sum_{i,j} \overline{a_i} a_j k(\lambda_i, \lambda_j).$$

When  $k$  is holomorphic with respect to the second variable and antiholomorphic with respect to the first one, we obtain a RKHS of holomorphic functions  $H(k)$  embedded into  $Hol(\mathbb{D})$ .

For functions  $k$  of the form  $k(\lambda, z) = K(\overline{\lambda}z)$ , where  $K \in Hol(\mathbb{D})$ , the positive definitness is equivalent to  $\hat{K}(j) > 0$  for every  $j \geq 0$ , where  $\hat{K}(j)$  stands for Taylor coefficients, and in this case we have  $H(k) = l_a^2(w_j)$ , where  $w_j = 1/\sqrt{\hat{K}(j)}$ ,  $j \geq 0$ . In particular, for  $K(w) = (1-w)^{-\beta}$ ,

$k_\lambda(z) = (1 - \bar{\lambda}z)^{-\beta}$ ,  $\beta > 0$ , we have  $\hat{K}(j) = \binom{\beta+j-1}{\beta-1}$  (binomial coefficients), and hence  $w_j = \left(\frac{j!}{\beta(\beta+1)\dots(\beta+j-1)}\right)^{\frac{1}{2}}$ . Indeed, deriving  $\frac{1}{1-z}$ , we get by induction

$$(1-z)^{-\beta} = \frac{1}{(\beta-1)!} \sum_{j \geq 0} (j+\beta-1)\dots(j+1)z^j = \sum_{j \geq 0} \binom{\beta+j-1}{\beta-1} z^j.$$

Clearly,  $w_j \simeq 1/j^{\frac{\beta-1}{2}}$ , where  $a_j \simeq b_j$  means that there exist constants  $c_1 > 0$ ,  $c_2 > 0$  such that  $c_1 a_j \leq b_j \leq c_2 a_j$  for every  $j$ . Therefore,  $H(k) = l_a^2\left(\frac{1}{(k+1)^{\frac{\beta-1}{2}}}\right)$  (a topological identity: the spaces are the same and the norms are equivalent).

We will use the previous observations for the following composed reproducing kernels (Aronszajn-deBranges, see [N2] p.320): given the reproducing kernel  $k$  of  $H^2$  and  $\varphi \in \{z^N : N \in \mathbb{N} \setminus \{0\}\}$ , the function  $\varphi \circ k$  is also positive definit and the corresponding Hilbert space is

$$H_\varphi = \varphi(H^2) = l_a^2\left((k+1)^{\frac{1-N}{2}}\right).$$

It satisfies the following property : for every  $f \in H^2$ ,  $\varphi \circ f \in \varphi(H^2)$  and  $\|\varphi \circ f\|_{\varphi(H^2)}^2 \leq \varphi(\|f\|_{H^2}^2)$  (see [N2] p.320). In particular, if  $\varphi$  is a polynomial of degree  $N$  and  $k$  is the Szegő kernel then  $\varphi \circ k_\lambda(z) = \sum_{j \geq 0} c_j \bar{\lambda}^j z^j$  with  $c_k \simeq (k+1)^{N-1}$ , and hence

$$(7.0) \quad H_\varphi = l_a^2\left(\frac{1}{(k+1)^{\frac{N-1}{2}}}\right),$$

(a topological identity: the spaces are the same and the norms are equivalent). In particular

$$(7.1) \quad H_{z^N} = l_a^2\left(\frac{1}{(k+1)^{\frac{N-1}{2}}}\right),$$

for all  $N \in \mathbb{N} \setminus \{0\}$ . The link between spaces of type  $l_a^2\left(\frac{1}{(k+1)^{\frac{N-1}{2}}}\right)$  and of type  $\varphi(H^2) = H_\varphi$  being established, we give the proof of the left-hand side from Theorem F.

*Proof of Theorem B (the left-hand side inequalities only).*

0) Setting  $N = 1 - 2\alpha \in \mathbb{N} \setminus \{0\}$ ,  $\alpha = \frac{1-N}{2}$  and

$$l_a^2((k+1)^\alpha) = l_a^2\left(\frac{1}{(k+1)^{\frac{N-1}{2}}}\right).$$

1) We set

$$Q_n = \sum_{k=0}^{n-1} b_\lambda^k \frac{(1-|\lambda|^2)^{1/2}}{1-\bar{\lambda}z}, \quad H_n = \varphi \circ Q_n,$$

$$\Psi = bH_n.$$

Then  $\|Q_n\|_2^2 = n$ , and hence by the Aronszajn-deBranges inequality, see [N2] p.320, point (k) of Exercise 6.5.2, with  $\varphi(z) = z^N$  and  $K(\lambda, z) = k_\lambda(z) = \frac{1}{1-\lambda z}$ , and noticing that  $H(\varphi \circ K) = H_\varphi$ ,

$$\|\Psi\|_{H_\varphi}^2 \leq b^2 \varphi (\|Q_n\|_2^2) = b^2 \varphi(n).$$

Let  $b > 0$  such that  $b^2 \varphi(n) = 1$ .

2) Since the spaces  $H_\varphi$  and  $H^\infty$  are rotation invariant, we have  $c(\sigma_{n,\lambda}, H_\varphi, H^\infty) = c(\sigma_{n,\mu}, H_\varphi, H^\infty)$  for every  $\lambda, \mu$  with  $|\lambda| = |\mu| = r$ . Let  $\lambda = -r$ . To get a lower estimate for  $\|\Psi\|_{H_\varphi/b_\lambda^n H_\varphi}$  consider  $G$  such that  $\Psi - G \in b_\lambda^n \text{Hol}(\mathbb{D})$ , i.e. such that  $bH_n \circ b_\lambda - G \circ b_\lambda \in z^n \text{Hol}(\mathbb{D})$ .

3) First, we show that

$$\psi =: \Psi \circ b_\lambda = bH_n \circ b_\lambda$$

is a polynomial (of degree  $nN$ ) with positive coefficients. Note that

$$\begin{aligned} Q_n \circ b_\lambda &= \sum_{k=0}^{n-1} z^k \frac{(1 - |\lambda|^2)^{1/2}}{1 - \bar{\lambda} b_\lambda(z)} = \\ &= (1 - |\lambda|^2)^{-\frac{1}{2}} \left( 1 + (1 - \bar{\lambda}) \sum_{k=1}^{n-1} z^k - \bar{\lambda} z^n \right) = \\ &= (1 - r^2)^{-1/2} \left( 1 + (1 + r) \sum_{k=1}^{n-1} z^k + r z^n \right) =: (1 - r^2)^{-1/2} \psi_1. \end{aligned}$$

Hence,  $\psi = \Psi \circ b_\lambda = bH_n \circ b_\lambda = b\varphi \circ \left( (1 - r^2)^{-\frac{1}{2}} \psi_1 \right)$  and

$$\varphi \circ \psi_1 = \sum_{k=0}^N a_k \psi_1^k(z).$$

(In fact, we can simply assume that  $\varphi \circ \psi_1 = \psi_1^N(z)$  since  $H_\varphi = l_a^2 \left( \frac{1}{(k+1)^{\frac{N-1}{2}}} \right) = H_{z^N}$ ). Now, it is clear that  $\psi$  is a polynomial of degree  $Nn$  such that

$$\psi(1) = \sum_{j=0}^{Nn} \hat{\psi}(j) = b\varphi \left( (1 - r^2)^{-1/2} (1 + r)n \right) = b\varphi \left( \sqrt{\frac{1+r}{1-r}} n \right) > 0.$$

4) Next, we show that there exists a constant  $c = c(\varphi) > 0$  (for example,  $c = \alpha/2^{2N}(N-1)!$ ,  $\alpha$  is a numerical constant) such that

$$\sum^m(\psi) =: \sum_{j=0}^m \hat{\psi}(j) \geq c \sum_{j=0}^{Nn} \hat{\psi}(j) = c\psi(1),$$

where  $m \geq 1$  is such that  $2m = n$  if  $n$  is even and  $2m - 1 = n$  if  $n$  is odd.

Indeed, setting

$$S_n = \sum_{j=0}^n z^j,$$

we have

$$\sum^m(\psi_1^k) = \sum^m \left( \left( 1 + (1+r) \sum_{k=1}^{n-1} z^k + rz^n \right)^k \right) \geq \sum^m (S_{n-1}^k).$$

Next, we obtain

$$\begin{aligned} \sum^m (S_{n-1}^k) &= \sum^m \left( \left( \frac{1-z^n}{1-z} \right)^k \right) = \\ &= \sum^m \left( \sum_{j=0}^k C_k^j \frac{1}{(1-z)^j} \cdot \left( \frac{-z^n}{1-z} \right)^{k-j} \right) = \sum^m \left( \frac{1}{(1-z)^k} \right) = \\ &= \sum^m \left( \sum_{j \geq 0} C_{k+j-1}^j z^j \right) = \sum_{j=0}^m C_{k+j-1}^j \geq \sum_{j=0}^m \frac{(j+1)^{k-1}}{(k-1)!} \geq \\ &\geq \alpha \frac{m^k}{(k-1)!}, \end{aligned}$$

where  $\alpha > 0$  is a numerical constant. Finally,

$$\begin{aligned} \sum^m(\psi_1^k) &\geq \alpha \frac{m^k}{(k-1)!} \geq \alpha \frac{(n/2)^k}{(k-1)!} = \\ &= \frac{\alpha}{2^k(k-1)!} \cdot \frac{((1+r)n)^k}{(1+r)^k} = \frac{\alpha}{2^k(1+r)^k(k-1)!} \cdot (\psi_1(1))^k \geq \\ &\geq \frac{\alpha}{2^N(1+r)^N(N-1)!} \cdot (\psi_1(1))^k. \end{aligned}$$

Summing up these inequalities in  $\sum^m(\psi) = b \sum^m(\varphi \circ \psi_1) = b \sum_{k=0}^N a_k (1-r^2)^{-k/2} \sum^m(\psi_1^k)$  (or simply taking  $k = N$ , if we already supposed  $\varphi = z^N$ ), we obtain the result claimed.

5) Now, using point 4) and the preceding Fejer kernel argument and denoting  $F_n = \Phi_m + z^m \Phi_m$ , where  $\Phi_k$  stands for the  $k$ -th Fejer kernel, we get



$$\begin{aligned} \|\Psi\|_{H^\infty/b_\lambda^n H^\infty} &= \|\psi\|_{H^\infty/z^n H^\infty} \geq \frac{1}{2} \|\psi * F_n\|_\infty \geq \frac{1}{2} \sum_{j=0}^m \hat{\psi}(j) \geq \\ &\geq \frac{c}{2} \psi(1) = \frac{c}{2} b \varphi \left( \sqrt{\frac{1+r}{1-r}} n \right) = \frac{c}{2} \cdot \frac{\varphi \left( \sqrt{\frac{1+r}{1-r}} n \right)}{(\varphi(n))^{1/2}} \geq \end{aligned}$$

(assuming that  $\varphi = z^N$ )

$$\geq a(\varphi) \left( \frac{n}{1-r} \right)^{\frac{N}{2}}.$$

6) In order to conclude, it remains to use (7.1). □

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