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Constrained Nevanlinna-Pick interpolation in Weighted Hardy and Bergman spaces

RACHID ZAROUF

Abstract

Given a finite set σ of the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and a holomorphic function f in \mathbb{D} which belongs to a class X , we are looking for a function g in another class Y (smaller than X) which minimizes the norm $\|g\|_Y$ among all functions g such that $g|_\sigma = f|_\sigma$. For $Y = H^\infty$, $X = l_a^p(w_k) = \left\{ f = \sum_{k \geq 0} \hat{f}(k)z^k : \|f\|^p = \sum_{k \geq 0} |\hat{f}(k)|^p w_k^p < \infty \right\}$, with a weight w satisfying $w_k > 0$ for every $k \geq 0$ and $\overline{\lim}_k (1/w_k^{1/k}) = 1$, and for the corresponding interpolation constant $c(\sigma, X, H^\infty)$, we show that if $p = 2$, $c(\sigma, X, H^\infty) \leq a\varphi_X \left(1 - \frac{1-r}{n}\right)$ where $n = \#\sigma$, $r = \max_{\lambda \in \sigma} |\lambda|$ and where $\varphi_X(t)$ stands for the norm of the evaluation functional $f \mapsto f(t)$ on the space X . The upper bound is sharp over sets σ with given n and r . For $X = l_a^p(w_k)$, $p \neq 2$ and $X = L_a^\beta \left((1 - |z|^2)^\beta dx dy \right)$ (the weighted Bergman space), $\beta > -1$, $1 \leq p < 2$, we also found upper and lower bounds for $c(\sigma, X, H^\infty)$ (sometimes for special sets σ) but with some gaps between these bounds.

0. Introduction

We recall that the problem considered in [Z] is the following: given two Banach spaces X and Y of holomorphic functions on the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, $X \supset Y$, and a finite set $\sigma \subset \mathbb{D}$, what is the best possible interpolation by functions of the space Y for the traces $f|_\sigma$ of functions of the space X , in the worst case?

Looking at this problem, we are lead to define and compute/estimate the interpolation constant

$$c(\sigma, X, Y) = \sup_{f \in X, \|f\|_X \leq 1} \inf \{ \|g\|_Y : g|_\sigma = f|_\sigma \},$$

(which is nothing but the norm of the embedding operator $(X|_\sigma, \|\cdot\|_{X|_\sigma}) \rightarrow (Y|_\sigma, \|\cdot\|_{Y|_\sigma})$). Let us notice that the following question was especially stimulating (which is a part of a more complicated question arising in an applied situation in [BL1] and [BL2]): given a set $\sigma \subset \mathbb{D}$, how to estimate $c(\sigma, H^2, H^\infty)$ in terms of $n = \text{card}(\sigma)$ and $\max_{\lambda \in \sigma} |\lambda| = r$ only? Here and everywhere below, H^2 is the standard Hardy space of the disc,

$$H^2 = \left\{ f = \sum_{k \geq 0} \hat{f}(k)z^k : \|f\|_{H^2} = \left(\sum_{k \geq 0} |\hat{f}(k)|^2 \right)^{\frac{1}{2}} < \infty \right\},$$

and H^∞ stands for the space (algebra) of bounded holomorphic functions in the unit disc \mathbb{D} endowed with the norm $\|f\|_\infty = \sup_{z \in \mathbb{D}} |f(z)|$. The issue of estimating/computing $c(\sigma, H^2, H^\infty)$ has been treated in [Z].

More precisely, we have treated in [Z] the cases $X = H^p, L_a^2$, where H^p ($1 \leq p \leq \infty$) stands for the standard Hardy space of the unit disc and where L_a^2 stands for the Bergman space of all holomorphic functions f such that

$$\int_{\mathbb{D}} |f(z)|^2 dA < \infty,$$

where dA stands for the area measure, and proved the following result, through **Theorems A, B&C** (see the **Introduction** of [Z]).

Theorems A, B&C.

(1)

$$\frac{1}{4\sqrt{2}} \frac{\sqrt{n}}{\sqrt{1-r}} \leq c(\sigma_{r,n}, H^2, H^\infty) \leq C_{n,r}(H^2, H^\infty) \leq \sqrt{2} \frac{\sqrt{n}}{\sqrt{1-r}},$$

$$\frac{1}{32} \frac{n}{1-r} \leq c(\sigma_{r,n}, L_a^2, H^\infty) \leq C_{n,r}(L_a^2, H^\infty) \leq \sqrt{14} \frac{n}{1-r},$$

for all $n \geq 1$, $0 \leq r < 1$, where $\sigma_{r,n}$ is the one set point $\{r, r, \dots, r\}$ (n times).

(2) Let $1 \leq p \leq \infty$. Then

$$\frac{1}{32^{\frac{1}{p}}} \left(\frac{n}{1-|r|} \right)^{\frac{1}{p}} \leq c(\sigma_{r,n}, H^p, H^\infty) \leq C_{n,r}(H^p, H^\infty) \leq A_p \left(\frac{n}{1-r} \right)^{\frac{1}{p}},$$

for all $n \geq 1$, $0 \leq r < 1$, where A_p is a constant depending only on p and the left hand side inequality is proved only for $p \in 2\mathbb{Z}_+$.

In this paper, we extend those results to the case where X is a weighted space $X = l_a^p(w_k)$,

$$l_a^p(w_k) = \left\{ f = \sum_{k \geq 0} \hat{f}(k) z^k : \|f\|^p = \sum_{k \geq 0} |\hat{f}(k)|^p w_k^p < \infty \right\},$$

with a weight w satisfying $w_k > 0$ for every $k \geq 0$ and $\overline{\lim}_k (1/w_k^{1/k}) = 1$. The latter condition implies that $l_a^p(w_k)$ is continuously embedded into the space of holomorphic functions $Hol(\mathbb{D})$ on the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ (and not on a larger disc, i.e. $l_a^p(w_k)$ does not contained in $Hol(r\mathbb{D})$ for every $r > 1$).

As in [Z], in order to find an upper bound for $c(\sigma, X, H^\infty)$, we first use a linear interpolation:

$$f \mapsto \sum_{k=1}^n \langle f, e_k \rangle e_k,$$

where $\langle \cdot, \cdot \rangle$ means the Cauchy sesquilinear form $\langle h, g \rangle = \sum_{k \geq 0} \hat{h}(k) \overline{\hat{g}(k)}$, and $(e_k)_{k=1}^n$ is the explicitly known Malmquist basis of the space $K_B = H^2 \Theta B H^2$, $B = \prod_{i=1}^n b_{\lambda_i}$ being the corresponding Blaschke product, $b_\lambda = \frac{\lambda - z}{1 - \bar{\lambda}z}$ (see N. Nikolski, [N1] p. 117)). Next, we use the complex interpolation between Banach spaces, (see H. Triebel [Tr] Theorem 1.9.3 p.59). Among the technical tools used in order to find an upper bound for $\|\sum_{k=1}^n \langle f, e_k \rangle e_k\|_\infty$ (in terms of $\|f\|_X$), the most important is a Bernstein-type inequality $\|f'\|_p \leq c_p \|B'\|_\infty \|f\|_p$ for a (rational) function f in the star-invariant subspace $H^p \cap B \overline{H_0^p}$ generated by a (finite) Blaschke product B , (K. Dyakonov [Dy]). For $p = 2$, we give an alternative proof of the Bernstein-type estimate we need.

The lower bound problem (for $C_{n,r}(X, H^\infty)$) is treated by using the “worst” interpolation n -tuple $\sigma = \sigma_{\lambda,n} = \{\lambda, \dots, \lambda\}$, a one-point set of multiplicity n (the Carathéodory-Schur type interpolation). The “worst” interpolation data comes from the Dirichlet kernels $\sum_{k=0}^{n-1} z^k$ transplanted from the origin to λ . We notice that spaces $X = l_a^p(w_k)$ satisfy the condition $X \circ b_\lambda \subset X$ when $p = 2$, whereas this is not the case for $p \neq 2$ and this makes the problem of upper/lower bound harder.

Our principal case is $p = 2$, where $l_a^2(w_k)$ is a reproducing kernel Hilbert space on the disc \mathbb{D} . It is important to recall that

$$l_a^2 \left(\frac{1}{(k+1)^{\alpha-1}} \right) = L_a^2 \left((1-|z|^2)^{2\alpha-3} dA \right), \quad \alpha > 1,$$

where $L_a^2 \left((1-|z|^2)^\beta dA \right)$, $\beta > -1$, stand for the Bergman weighted spaces of all holomorphic functions f such that

$$\int_{\mathbb{D}} |f(z)|^2 (1-|z|^2)^\beta dA < \infty.$$

Theorem. 1.0 *Let σ be a sequence in \mathbb{D} . Then*

$$c \left(\sigma, l_a^2 \left(\frac{1}{(k+1)^{\alpha-1}} \right), H^\infty \right) \leq A \left(\frac{n}{1-r} \right)^{\frac{2\alpha-1}{2}}.$$

Otherwise,

$$C_{n,r} \left(l_a^2 \left(\frac{1}{(k+1)^{\alpha-1}} \right), H^\infty \right) \leq A \left(\frac{n}{1-r} \right)^{\frac{2\alpha-1}{2}},$$

$$C_{n,r} \left(L_a^2 \left((1-|z|^2)^\beta dA \right), H^\infty \right) \leq A' \left(\frac{n}{1-r} \right)^{\frac{\beta+2}{2}},$$

for all $n \geq 1$, $0 \leq r < 1$, $\alpha \geq 1$, $\beta > -1$, where $A = A(\alpha - 1)$ is a constant depending only on α and $A' = A'(\beta)$ is a constant depending only on β .

Later on, in **Section 7** we show that for $\alpha = \frac{N+1}{2}$, where $N \geq 1$ is an integer, the latter estimate is sharp.

Theorem. 7.0 *Let $N \geq 1$ be an integer and $\sigma_{\lambda,n} = \{\lambda, \dots, \lambda\}$ (n times). Then,*

$$c \left(\sigma_{\lambda,n}, l_a^2 \left(\frac{1}{(k+1)^{\frac{N-1}{2}}} \right), H^\infty \right) \geq a \left(\frac{n}{1-|\lambda|} \right)^{\frac{N}{2}}$$

for a positive constant $a = a_N$ depending on N only. In particular,

$$a \left(\frac{n}{1-r} \right)^{\frac{N}{2}} \leq C_{n,r} \left(l_a^2 \left(\frac{1}{(k+1)^{\frac{N-1}{2}}} \right), H^\infty \right) \leq A \left(\frac{n}{1-r} \right)^{\frac{N}{2}},$$

for all $n \geq 1$, $0 \leq r < 1$, where $A = A\left(\frac{N-1}{2}\right)$ is a constant defined in Theorem 1.0. Moreover, a and A are such that $a \asymp \frac{1}{2^{3N}(2N)!}$ and $A \asymp N^{2N}$, N standing for the integer part of α . (The notation $x \asymp y$ means that there exists numerical constants $c_1, c_2 > 0$ such that $c_1 y \leq x \leq c_2 y$).

In **Sections 2, 3 and 4**, we deal with an upper estimate for $C_{n,r}(X, H^\infty)$ in the scale of spaces $X = l_A^p \left(\frac{1}{(k+1)^{\alpha-1}} \right)$, $\alpha \geq 1$, $1 \leq p \leq +\infty$. (The case $p = 2$ is solved in **Section 1** (for the upper bound) **and in Section 7** (for sharpness)). We start giving a result for $1 \leq p \leq 2$.

Theorem. 3.0 *Let $1 \leq p \leq 2$, $\alpha \geq 1$. Then*

$$B \left(\frac{1}{1-r} \right)^{\alpha - \frac{1}{p}} \leq C_{n,r} \left(l_a^p \left(\frac{1}{(k+1)^{\alpha-1}} \right), H^\infty \right) \leq A \left(\frac{n}{1-r} \right)^{\alpha - \frac{1}{2}},$$

for all $r \in [0, 1[$, $n \geq 1$, where $A = A(\alpha - 1, p)$ is a constant depending only on α and p and $B = B(p)$ is a constant depending only on p .

It is very likely that the bounds of **Theorem 3.0** are not sharp. The sharp one should be probably $\left(\frac{n}{1-r}\right)^{\alpha - \frac{1}{p}}$. In the same way, for $2 \leq p \leq \infty$, we give the following theorem, in which we feel again that the upper bound $\left(\frac{n}{1-r}\right)^{\alpha + \frac{1}{2} - \frac{2}{p}}$ is not sharp. The sharp one probably should be the lower bound $\left(\frac{n}{1-r}\right)^{\alpha - \frac{1}{p}}$.

Theorem. 5.0 *Let $2 \leq p \leq \infty$, $\alpha \geq 1$. Then*

$$B \left(\frac{1}{1-r} \right)^{\alpha - \frac{1}{p}} \leq C_{n,r} \left(l_a^p \left(\frac{1}{(k+1)^{\alpha-1}} \right), H^\infty \right) \leq A \left(\frac{n}{1-r} \right)^{\alpha + \frac{1}{2} - \frac{2}{p}},$$

for all $r \in [0, 1[$, $n \geq 1$, where $A = A(\alpha - 1, p)$ is a constant depending only on α and p and $B = B(p)$ is a constant depending only on p .

In **Section 6**, we suppose that X is equal to $L_a^p \left((1 - |z|^2)^\beta dA \right)$, $\beta > -1$, $1 \leq p \leq 2$, where dA stands for the area measure, the Bergman weighted spaces of all holomorphic functions f such that

$$\int_{\mathbb{D}} |f(z)|^p (1 - |z|^2)^\beta dA < \infty.$$

Notice that for $p = 2$, the two latter series of spaces coincide:

$$l_a^2 \left(\frac{1}{(k+1)^{\alpha-1}} \right) = L_a^2 \left((1 - |z|^2)^{2\alpha-3} dA \right), \alpha > 1.$$

Our goal in this section is to give an estimate for the constant for a generalized Carathéodory-Schur interpolation, (a partial case of the Nevanlinna-Pick interpolation),

$$c(\sigma_{\lambda,n}, X, H^\infty) = \sup \{ \|f\|_{H^\infty/b_\lambda^n H^\infty} : f \in X, \|f\|_X \leq 1 \},$$

where $\|f\|_{H^\infty/b_\lambda^n H^\infty} = \inf \{ \|f + b_\lambda^n g\|_\infty : g \in X \}$, and $\sigma_{\lambda,n} = \{\lambda, \lambda, \dots, \lambda\}$, $\lambda \in \mathbb{D}$. The corresponding interpolation problem is: given $f \in X$, to minimize $\|h\|_\infty$ such that $h^{(j)}(\lambda) = f^{(j)}(\lambda)$, $0 \leq j < n$.

For this partial case, we have the following generalization of the estimate from **Theorem 1.0**.

Theorem. 6.0 *Let $\lambda \in \mathbb{D}$, $\beta > -1$ and $1 \leq p \leq 2$. Then,*

$$c\left(\sigma_{\lambda,n}, L_a^p\left((1-|z|^2)^\beta dA\right), H^\infty\right) \leq A' \left(\frac{n}{1-|\lambda|}\right)^{\frac{\beta+2}{p}},$$

where $A' = A'(\beta, p)$ is a constant depending only on β and p .

Before starting **Section 1** and studying upper estimates for $c(\sigma, X, H^\infty)$, we give the following lemma which is going to be useful throughout this paper, in particular in view of applying interpolation between Banach spaces.

Lemma 0. *Let X be a Banach space of holomorphic functions in the unit disc \mathbb{D} and $\sigma = \{\lambda_1, \lambda_2, \dots, \lambda_n\} \subset \mathbb{D}$ a finite subset of the disc. We define the Blaschke product $B_\sigma = \prod_{i=1}^n b_{\lambda_i}$ where $b_\lambda = \frac{\lambda-z}{1-\bar{\lambda}z}$. Let $T : X \rightarrow H^\infty/B_\sigma H^\infty$ be the restriction map defined by*

$$Tf = \{g \in H^\infty : f - g \in B_\sigma X\},$$

for every $f \in X$. Then,

$$\|T\|_{X \rightarrow H^\infty/B_\sigma H^\infty} = c(\sigma, X, H^\infty).$$

Proof. The proof is obvious. \square

1. An upper bound for $c(\sigma, l_a^2(w_k), H^\infty)$

In this section, we generalize the upper bound obtained in [Z] for $C_{n,r}(X, H^\infty)$ where $X = H^2, L_a^2$ to the case of spaces X which contain H^2 : $X = l_a^2\left(\frac{1}{(k+1)^{\alpha-1}}\right)$, $\alpha \geq 1$, the Hardy weighted spaces of all $f(z) = \sum_{k \geq 0} \hat{f}(k)z^k$ satisfying

$$\sum_{k \geq 0} \left| \hat{f}(k) \right|^2 \frac{1}{(k+1)^{2(\alpha-1)}} < \infty.$$

It is also important to recall that

$$l_a^2\left(\frac{1}{(k+1)^{\alpha-1}}\right) = L_a^2\left((1-|z|^2)^{2\alpha-3} dA\right), \alpha > 1,$$

where $L_a^2 \left((1 - |z|^2)^\beta dA \right)$, $\beta > -1$, stand for the Bergman weighted spaces of all holomorphic functions f such that

$$\int_{\mathbb{D}} |f(z)|^2 (1 - |z|^2)^\beta dA < \infty.$$

Notice also that $H^2 = l_a^2(1)$ and $L_a^2(\mathbb{D}) = l_a^2 \left(\frac{1}{(k+1)^{\frac{1}{2}}} \right)$, where $L_a^2(\mathbb{D})$ stands for the Bergman space of the unit disc \mathbb{D} .

Theorem. 1.0 *Let σ be a sequence in \mathbb{D} . Then*

$$c \left(\sigma, l_a^2 \left(\frac{1}{(k+1)^{\alpha-1}} \right), H^\infty \right) \leq A \left(\frac{n}{1-r} \right)^{\frac{2\alpha-1}{2}}.$$

Otherwise,

$$C_{n,r} \left(l_a^2 \left(\frac{1}{(k+1)^{\alpha-1}} \right), H^\infty \right) \leq A \left(\frac{n}{1-r} \right)^{\frac{2\alpha-1}{2}},$$

$$C_{n,r} \left(L_a^2 \left((1 - |z|^2)^\beta dA \right), H^\infty \right) \leq A' \left(\frac{n}{1-r} \right)^{\frac{\beta+2}{2}},$$

for all $n \geq 1$, $0 \leq r < 1$, $\alpha \geq 1$, $\beta > -1$, where $A = A(\alpha - 1)$ is a constant depending only on α and $A' = A'(\beta)$ is a constant depending only on α .

First, we recall the following lemma (see [Z]). In fact, **Lemma 1.1** below is a partial case ($p = 2$) of the following K. Dyakonov's result [Dy] (which is, in turn, a generalization of M. Levin's inequality [L] corresponding to the case $p = \infty$): for every p , $1 < p \leq \infty$ there exists a constant $c_p > 0$ such that

$$\|f'\|_{H^p} \leq c_p \|B'\|_\infty \|f\|_{H^p}$$

for all $f \in K_B$, where B is a finite Blaschke product (of order n) and $\|\cdot\|_\infty$ means the norm in $L^\infty(\mathbb{T})$. For our partial case, our proof (in [Z]) is different and the constant is slightly better. We notice that in general, Bernstein type inequalities have already been the subject of a lot of papers. Among others, Chapter 7 of P. Borwein and T. Erdélyi's book, see [BoEr], is devoted to such inequalities. This is also the case of A. Baranov's work, see [B1], [B2] and [B3], and also of R. A. DeVore and G. G. Lorentz's book, see [DeLo].

Lemma. 1.1 *Let $B = \prod_{j=1}^n b_{\lambda_j}$, be a finite Blaschke product (of order n), $r = \max_j |\lambda_j|$, and $f \in K_B =: H^2 \ominus BH^2$. Then,*

$$\|f'\|_{H^2} \leq \frac{5}{2} \frac{n}{1-r} \|f\|_{H^2}.$$

Corollary. 1.2 *Let $B = \prod_{j=1}^n b_{\lambda_j}$, be a finite Blaschke product (of order n), $r = \max_j |\lambda_j|$, and $f \in K_B =: H^2 \ominus BH^2$. Then,*

$$\|f^{(k)}\|_{H^2} \leq k! \left(\frac{5}{2} \right)^k \left(\frac{n}{1-r} \right)^k \|f\|_{H^2},$$

for every $k = 0, 1, \dots$

Indeed, since $f^{(k-1)} \in K_{B^k}$, we obtain applying Lemma 1.1 for B^k instead of B ,

$$\|f^{(k)}\|_{H^2} \leq \frac{5}{2} \frac{kn}{1-r} \|f^{(k-1)}\|_{H^2},$$

and by induction,

$$\|f^{(k)}\|_{H^2} \leq k! \left(\frac{5}{2} \frac{n}{1-r} \right)^k \|f\|_{H^2}. \quad \square$$

Corollary. 1.3 *Let $N \geq 0$ be an integer and σ a sequence in \mathbb{D} . Then,*

$$c \left(\sigma, l_a^2 \left(\frac{1}{(k+1)^N} \right), H^\infty \right) \leq A \left(\frac{n}{1-r} \right)^{\frac{2N+1}{2}},$$

where $A = A(N)$ is a constant depending on N (of order N^{2N} from the proof below).

Indeed, if $f \in l_a^2 \left(\frac{1}{(k+1)^N} \right) = H$ then $|P_B f(\zeta)| = |\langle P_B f, k_\zeta \rangle| = |\langle f, P_B k_\zeta \rangle|$, where $\langle \cdot, \cdot \rangle$ means the Cauchy pairing and $k_\zeta = (1 - \bar{\zeta}z)^{-1}$. Denoting H^* the dual of H with respect to this pairing, $H^* = l_a^2((k+1)^N)$, we get

$$|P_B f(\zeta)| \leq \|f\|_H \|P_B k_\zeta\|_{H^*} \leq \|f\|_H K_N \left(\|P_B k_\zeta\|_{H^2} + \left\| (P_B k_\zeta)^{(N)} \right\|_{H^2} \right),$$

where

$$\begin{aligned} K_N &= \max \left\{ N^N, \sup_{k \geq N} \frac{(k+1)^N}{k(k-1)\dots(k-N+1)} \right\} = \\ &= \max \left\{ N^N, \frac{(N+1)^N}{N!} \right\} = \begin{cases} N^N, & \text{if } N \geq 3 \\ \frac{(N+1)^N}{N!}, & \text{if } N = 1, 2 \end{cases}. \end{aligned}$$

(Indeed, the sequence $\left(\frac{(k+1)^N}{k(k-1)\dots(k-N+1)} \right)_{k \geq N}$ is decreasing since $(1+x)^{-N} \geq 1-Nx$ for all $x \in [0, 1]$, and $\left[N^N > \frac{(N+1)^N}{N!} \right] \iff N \geq 3$). Since $P_B k_\zeta \in K_B$, Corollary 1.2 implies

$$\begin{aligned} |P_B f(\zeta)| &\leq \|f\|_H \|P_B k_\zeta\|_{H^*} \leq \|f\|_H K_N \left(\|P_B k_\zeta\|_{H^2} + N! \left(\frac{5}{2} \frac{n}{1-r} \right)^N \|P_B k_\zeta\|_{H^2} \right) \leq \\ &\leq A(N) \left(\frac{n}{1-r} \right)^{N+\frac{1}{2}} \|f\|_H, \end{aligned}$$

where $A(N) = \sqrt{2} K_N \left(1 + N! \left(\frac{5}{2} \right)^N \right)$, since $\|P_B k_\zeta\|_2 \leq \frac{\sqrt{2n}}{\sqrt{1-r}}$. \square

Proof of Theorem 1.0. Applying Lemma 0 with $X = l_a^2 \left(\frac{1}{(k+1)^{\alpha-1}} \right)$, we get

$$\|T\|_{l_a^2 \left(\frac{1}{(k+1)^{\alpha-1}} \right) \rightarrow H^\infty / B_\sigma H^\infty} = c \left(\sigma, l_a^2 \left(\frac{1}{(k+1)^{\alpha-1}} \right), H^\infty \right),$$

where T and B_σ are defined in Lemma 0. Moreover, there exists an integer N such that $N \leq \alpha \leq N+1$. In particular, there exists $0 \leq \theta \leq 1$ such that $\alpha - 1 = (1 - \theta)(N - 1) + \theta N$. And since (as

in Theorem 4.0 of [Z]), we use the notation of the interpolation theory between Banach spaces see [Tr] or [Be])

$$\begin{aligned} \left[l_a^2 \left(\frac{1}{(k+1)^{N-1}} \right), l_a^2 \left(\frac{1}{(k+1)^N} \right) \right]_{\theta, 2} &= l_a^2 \left(\left(\frac{1}{(k+1)^{N-1}} \right)^{2\frac{1-\theta}{2}} \left(\frac{1}{(k+1)^N} \right)^{2\frac{\theta}{2}} \right) = \\ &= l_a^2 \left(\frac{1}{(k+1)^{(1-\theta)(N-1)+\theta N}} \right) = l_A^2 \left(\frac{1}{(k+1)^{\alpha-1}} \right), \end{aligned}$$

this gives, using Corollary 1.3 and (again) [Tr] Theorem 1.9.3 p.59,

$$\begin{aligned} &\| T \|_{l_a^2 \left(\frac{1}{(k+1)^{\alpha-1}} \right) \rightarrow H^\infty / B_\sigma H^\infty} \leq \\ &\leq \left(c \left(\sigma, l_a^2 \left(\frac{1}{(k+1)^{N-1}} \right), H^\infty \right) \right)^{1-\theta} \left(c \left(\sigma, l_a^2 \left(\frac{1}{(k+1)^N} \right), H^\infty \right) \right)^\theta \leq \\ &\leq \left(A(N-1) \left(\frac{n}{1-r} \right)^{\frac{2N-1}{2}} \right)^{1-\theta} \left(A(N) \left(\frac{n}{1-r} \right)^{\frac{2N+1}{2}} \right)^\theta = \\ &= A(N-1)^{1-\theta} A(N)^\theta \left(\frac{n}{1-r} \right)^{\frac{(2N-1)(1-\theta)}{2} + \frac{(2N+1)\theta}{2}}. \end{aligned}$$

It remains to use $\theta = \alpha - N$ and set $A(\alpha - 1) = A(N - 1)^{1-\theta} A(N)^\theta$.

□

2. An upper bound for $c \left(\sigma, l_a^1(w_k), H^\infty \right)$

The aim of this section is to prove the following theorem, in which the upper bound $\left(\frac{n}{1-r} \right)^{\alpha-\frac{1}{2}}$ is not as sharp as in **Section 1**. We suspect $\left(\frac{n}{1-r} \right)^{\alpha-1}$ is the sharp bound for the quantity $C_{n,r} \left(l_a^1 \left(\frac{1}{(k+1)^{\alpha-1}} \right), H^\infty \right)$.

Theorem. 2.0 *Let $\alpha \geq 1$. Then,*

$$C_{n,r} \left(l_a^1 \left(\frac{1}{(k+1)^{\alpha-1}} \right), H^\infty \right) \leq A_1 \left(\frac{n}{1-r} \right)^{\alpha-\frac{1}{2}},$$

for all $r \in [0, 1[$, $n \geq 1$, where $A_1 = A_1(\alpha - 1)$ is a constant depending only on α .

First, we prove the following lemma.

Lemma. 2.1 *Let $B = \prod_{j=1}^n b_{\lambda_j}$, be a finite Blaschke product (of order n), $r = \max_j |\lambda_j|$, and $f \in K_B$. Then,*

$$\| f^{(k)} \|_{H^1} \leq k! \left(\frac{2n}{1-r} \right)^k \| f \|_{H^1}$$

for every $k = 0, 1, \dots$

Proof. By A. Baranov (see [B1] Theorem 5.1 p.50),

$$\|f'\|_{H^1} \leq \|B'\|_{\infty} \|f\|_{H^1}$$

for every $f \in K_B$. (A private communication with A. Baranov shows that Theorem 5.1 of [B1] is also true for the Hardy spaces of the unit disc \mathbb{D} ; see also [B2] Corollary 1.4, and [B3]). Since $f^{(k-1)} \in K_{B^k}$, we obtain, applying Baranov's inequality for B^k instead of B ,

$$\|f^{(k)}\|_{H^1} \leq \|kB' B^{k-1}\|_{\infty} \|f^{(k-1)}\|_{H^1},$$

and by induction,

$$\|f^{(k)}\|_{H^1} \leq k! \|B'\|_{\infty}^k \|f\|_{H^1}.$$

On the other hand, $|B'| = \left| -\sum_j \frac{1-|\lambda_j|^2}{(1-\bar{\lambda}_j z)^2} \cdot \frac{B}{b_{\lambda_j}} \right| \leq \sum_j \frac{1+|\lambda_j|}{1-|\lambda_j|} \leq \frac{2n}{1-r}$, which completes the proof. \square

Corollary. 2.2 *Let $N \geq 0$ be an integer. Then,*

$$C_{n,r} \left(l_a^1 \left(\frac{1}{(k+1)^N} \right), H^{\infty} \right) \leq A_1 \left(\frac{n}{1-r} \right)^{N+\frac{1}{2}},$$

for all $r \in [0, 1[$, $n \geq 1$, where $A_1 = A_1(N)$ is a constant depending only on N (of order N^{2N} from the proof below).

Indeed, the proof is exactly the same as in Corollary 1.3: if $f \in l_a^1 \left(\frac{1}{(k+1)^N} \right) = H$ then $|P_B f(\zeta)| = |\langle P_B f, k_{\zeta} \rangle| = |\langle f, P_B k_{\zeta} \rangle|$, where $\langle \cdot, \cdot \rangle$ means the Cauchy pairing and $k_{\zeta} = (1 - \bar{\zeta}z)^{-1}$. Denoting H^* the dual of H with respect to this pairing, $H^* = l_a^{\infty}((k+1)^N)$, we get,

$$\begin{aligned} |P_B f(\zeta)| &\leq \|f\|_H \|P_B k_{\zeta}\|_{H^*} \leq \\ &\leq \|f\|_H K_N \max \left\{ \sup_{0 \leq k \leq N-1} \left| \widehat{P_B k_{\zeta}}(k) \right|, \sup_{k \geq N} \left| \widehat{(P_B k_{\zeta})^{(N)}}(k-N) \right| \right\} \leq \\ &\leq \|f\|_H K_N \max \left\{ \|P_B k_{\zeta}\|_{H^1}, \left\| (P_B k_{\zeta})^{(N)} \right\|_{H^1} \right\}, \end{aligned}$$

where K_N is defined in the the proof of Corollary 1.3. Since $P_B k_{\zeta} \in K_B$, Lemma 2.1 implies

$$\begin{aligned} |P_B f(\zeta)| &\leq \|f\|_H \|P_B k_{\zeta}\|_{H^*} \leq \|f\|_H K_N \left(\|P_B k_{\zeta}\|_{H^1} + N! 2^N \left(\frac{n}{1-r} \right)^N \|P_B k_{\zeta}\|_{H^1} \right) \leq \\ &\leq K_N \|f\|_H \|P_B k_{\zeta}\|_2 \left(1 + N! 2^N \left(\frac{n}{1-|\lambda|} \right)^N \right) \leq K_N \|f\|_H \left(\frac{2n}{1-r} \right)^{\frac{1}{2}} \left(1 + N! 2^N \left(\frac{n}{1-r} \right)^N \right), \end{aligned}$$

which completes the proof setting $A_1(N) = \sqrt{2} (1 + N! 2^N) K_N$. \square

Proof of Theorem 2.0. This is the same reasoning as in Theorem 1.0. Applying Lemma 0 with $X = l_a^1\left(\frac{1}{(k+1)^{\alpha-1}}\right)$, we get

$$\|T\|_{l_a^1\left(\frac{1}{(k+1)^{\alpha-1}}\right) \rightarrow H^\infty/B_\sigma H^\infty} = c\left(\sigma, l_a^1\left(\frac{1}{(k+1)^{\alpha-1}}\right), H^\infty\right),$$

where T and B_σ are defined in Lemma 0. It remains to use Corollary 2.2 and (again) [Tr] Theorem 1.9.3 p.59 to complete the proof. \square

3. An upper bound for $c(\sigma, l_a^p(w_k), H^\infty)$, $1 \leq p \leq 2$

The aim of this section is to prove the following theorem, in which the upper bound $\left(\frac{n}{1-r}\right)^{\alpha-\frac{1}{2}}$ is not sharp as sharp as in **Section 1**. We suppose that the sharp upper (and lower) bound here should be of the order of $\left(\frac{n}{1-r}\right)^{\alpha-\frac{1}{p}}$.

Theorem. 3.0 *Let $1 \leq p \leq 2$, $\alpha \geq 1$. Then*

$$B\left(\frac{1}{1-r}\right)^{\alpha-\frac{1}{p}} \leq C_{n,r}\left(l_a^p\left(\frac{1}{(k+1)^{\alpha-1}}\right), H^\infty\right) \leq A\left(\frac{n}{1-r}\right)^{\alpha-\frac{1}{2}},$$

for all $r \in [0, 1[$, $n \geq 1$, where $A = A(\alpha - 1, p)$ is constant depending only on α and p and $B = B(p)$ is a constant depending only on p .

Proof. We first prove the right hand side inequality. The scheme of the proof is completely the same as in Theorem 1.0 and Theorem 2.0, but we simply use interpolation between l^1 and l^2 (the classical Riesz-Thorin theorem). Applying Lemma 0 with $X = l_a^p\left(\frac{1}{(k+1)^{\alpha-1}}\right)$, we get

$$\|T\|_{l_a^p\left(\frac{1}{(k+1)^{\alpha-1}}\right) \rightarrow H^\infty/B_\sigma H^\infty} = c\left(\sigma, l_a^p\left(\frac{1}{(k+1)^{\alpha-1}}\right), H^\infty\right),$$

where T and B_σ are defined in Lemma 0. It remains to use both Theorem 1.0&2.0 and (again) [Tr] Theorem 1.9.3 p.59 to complete the proof of the right hand side inequality.

Now, we prove the left hand side one. Firstly, it is clear that

$$\begin{aligned} C_{n,r}\left(l_a^p\left(\frac{1}{(k+1)^{\alpha-1}}\right), H^\infty\right) &\geq \|\varphi_r\|_{l_a^{p'}\left(\frac{1}{(k+1)^{\alpha-1}}\right)} = \\ &= \left(\sum_{k \geq 0} (k+1)^{(\alpha-1)p'} \left(r^{p'}\right)^k\right)^{\frac{1}{p'}}, \end{aligned}$$

where φ_r is the evaluation functional

$$\varphi_r(f) = f(r), \quad f \in X,$$

and p' is the conjugate of p : $\frac{1}{p} + \frac{1}{p'} = 1$. Now, since

$$\sum_{k \geq 1} k^s x^k \sim \int_1^\infty t^s x^t dt \sim \Gamma(s+1)(1-x)^{-s-1}, \quad \text{as } x \rightarrow 1,$$

for all $s > -1$, we get

$$\sum_{k \geq 0} (k+1)^{(\alpha-1)p'} \left(r^{p'}\right)^k \sim \int_1^\infty t^{(\alpha-1)p'} r^{p't} dt, \text{ as } r \rightarrow 1.$$

But

$$\begin{aligned} \int_1^\infty t^{(\alpha-1)p'} r^{p't} dt &= \left(\frac{1}{p'}\right)^{1+(\alpha-1)p'} \int_{p'}^\infty t^{(\alpha-1)p'} r^t dt \sim \\ &\sim \left(\frac{1}{p'}\right)^{1+(\alpha-1)p'} \int_1^\infty t^{(\alpha-1)p'} r^t dt \sim \left(\frac{1}{p'}\right)^{1+(\alpha-1)p'} \Gamma\left((\alpha-1)p' + 1\right) (1-r)^{-(\alpha-1)p'-1}, \text{ as } r \rightarrow 1. \end{aligned}$$

This gives

$$\left(\sum_{k \geq 0} (k+1)^{(\alpha-1)p'} \left(r^{p'}\right)^k\right)^{\frac{1}{p'}} \sim \left(\frac{1}{p'}\right)^{\frac{1}{p'}+(\alpha-1)} \left(\Gamma\left((\alpha-1)p' + 1\right)\right)^{\frac{1}{p'}} (1-r)^{-(\alpha-1)\frac{1}{p'}}, \text{ as } r \rightarrow 1.$$

This completes the proof since $\frac{1}{p'} = 1 - \frac{1}{p}$. \square

4. An upper bound for $c(\sigma, l_a^\infty(w_k), H^\infty)$

The aim of this section is the following theorem, in which -again- the upper bound $\left(\frac{n}{1-r}\right)^{\alpha+\frac{1}{2}}$ is not as sharp as in **Section 1**. We can suppose here that the constant $\left(\frac{n}{1-r}\right)^\alpha$ is the sharp bound for the quantity $C_{n,r}\left(l_a^\infty\left(\frac{1}{(k+1)^{\alpha-1}}\right), H^\infty\right)$.

Theorem. 4.0 *Let $\alpha \geq 1$. Then*

$$C_{n,r}\left(l_a^\infty\left(\frac{1}{(k+1)^{\alpha-1}}\right), H^\infty\right) \leq A_\infty \left(\frac{n}{1-r}\right)^{\alpha+\frac{1}{2}},$$

for all $r \in [0, 1[$, $n \geq 1$, where $A_\infty = A_\infty(\alpha-1)$ is a constant depending only on α .

First, we prove the following partial case of Theorem 4.0.

Lemma. 4.1 *Let $N \geq 0$ be an integer. Then,*

$$C_{n,r}\left(l_a^\infty\left(\frac{1}{(k+1)^N}\right), H^\infty\right) \leq A_\infty \left(\frac{n}{1-r}\right)^{N+\frac{3}{2}},$$

for all $r \in [0, 1[$, $n \geq 1$, where $A_\infty = A_\infty(N)$ is a constant depending on N (of order N^{2N} from the proof below).

Proof. We use literally the same method as in Corollary 1.3&2.2. Indeed, if $f \in l_a^\infty\left(\frac{1}{(k+1)^N}\right) = H$ then $|P_B f(\zeta)| = |\langle P_B f, k_\zeta \rangle| = |\langle f, P_B k_\zeta \rangle|$, where $\langle \cdot, \cdot \rangle$ means the Cauchy pairing and $k_\zeta = (1 - \bar{\zeta}z)^{-1}$. Denoting H^* the dual of H with respect to this pairing, $H^* = l_a^1((k+1)^N)$, we get

$$|P_B f(\zeta)| \leq \|f\|_H \|P_B k_\zeta\|_{H^*} \leq \|f\|_H K_N \left(\|P_B k_\zeta\|_W + \left\| (P_B k_\zeta)^{(N)} \right\|_W \right),$$

where $W = \left\{ f = \sum_{k \geq 0} \hat{f}(k) z^k : \|f\|_W := \sum_{k \geq 0} |\hat{f}(k)| < \infty \right\}$ stands for the Wiener algebra, and K_N is defined in Corollary 1.3. Now, applying Hardy's inequality (see [N2] p.370, 8.7.4 (c)),

$$|P_B f(\zeta)| \leq \|f\|_H K_N \left(\pi \left\| (P_B k_\zeta)' \right\|_{H^1} + |(P_B k_\zeta)(0)| + \pi \left\| (P_B k_\zeta)^{(N+1)} \right\|_{H^1} + \left| (P_B k_\zeta)^{(N)}(0) \right| \right),$$

which gives using Lemma 2.1,

$$\begin{aligned} |P_B f(\zeta)| &\leq \\ &\leq \|f\|_H K_N \pi \left(\left(\frac{2n}{1-r} \right) \|P_B k_\zeta\|_{H^1} + |(P_B k_\zeta)(0)| + \right. \\ &+ (N+1)! \left(\frac{2n}{1-r} \right)^{N+1} \|P_B k_\zeta\|_{H^1} + \left. \left| (P_B k_\zeta)^{(N)}(0) \right| \right) \leq \\ &\leq \|f\|_H K_N \pi \left(\left(\frac{2n}{1-r} \right) \|P_B k_\zeta\|_{H^2} + \|P_B k_\zeta\|_{H^2} + \right. \\ &+ (N+1)! \left(\frac{2n}{1-r} \right)^{N+1} \|P_B k_\zeta\|_{H^2} + N! \|P_B k_\zeta\|_{H^2} \left. \right). \end{aligned}$$

This completes the proof since $\|P_B k_\zeta\|_{H^2} \leq \left(\frac{2n}{1-r} \right)^{\frac{1}{2}}$. \square

Proof of Theorem 4.0. This is the same application of interpolation between Banach spaces, as before (Theorem 1.0&2.0) excepted that this time we apply Lemma 0 with $X = l_a^\infty \left(\frac{1}{(k+1)^{\alpha-1}} \right)$ to get

$$\|T\|_{l_a^\infty \left(\frac{1}{(k+1)^{\alpha-1}} \right) \rightarrow H^\infty / B_\sigma H^\infty} = c \left(\sigma, l_a^\infty \left(\frac{1}{(k+1)^{\alpha-1}} \right), H^\infty \right),$$

where T and B_σ are defined in Lemma 0.

Applying Lemma 4.1 and using (again) [Tr] Theorem 1.9.3 p.59, we can complete the proof. \square

5. An upper bound for $c \left(\sigma, l_a^p(w_k), H^\infty \right)$, $2 \leq p \leq \infty$

The aim of this section is to prove the following theorem.

Theorem. 5.0 *Let $2 \leq p \leq \infty$, $\alpha \geq 1$. Then*

$$B \left(\frac{1}{1-r} \right)^{\alpha - \frac{1}{p}} \leq C_{n,r} \left(l_a^p \left(\frac{1}{(k+1)^{\alpha-1}} \right), H^\infty \right) \leq A \left(\frac{n}{1-r} \right)^{\alpha + \frac{1}{2} - \frac{2}{p}},$$

for all $r \in [0, 1[$, $n \geq 1$, where $A = A(\alpha - 1, p)$ is a constant depending only on α and p and $B = B(p)$ is a constant depending only on p .

Remark. As before, the upper bound $\left(\frac{n}{1-r} \right)^{\alpha + \frac{1}{2} - \frac{2}{p}}$ is not as sharp as in **Section 1**. We can suppose here the constant $\left(\frac{n}{1-r} \right)^{\alpha - \frac{1}{p}}$ should be a sharp upper (and lower) bound for the quantity $C_{n,r} \left(l_a^p \left(\frac{1}{(k+1)^{\alpha-1}} \right), H^\infty \right)$, $2 \leq p \leq +\infty$.

Proof. We first prove the right hand side inequality. The proof repeats the scheme from previous theorems and from Theorem 3.0 in particular. We have already seen (in Theorem 3.0) that

$$\|T\|_{l_a^p\left(\frac{1}{(k+1)^{\alpha-1}}\right) \rightarrow H^\infty/B_\sigma H^\infty} = c\left(\sigma, l_a^p\left(\frac{1}{(k+1)^{\alpha-1}}\right), H^\infty\right),$$

where T and B_σ are defined in Lemma 0. Now, using both Theorems 1.0&4.0, and [Tr] Theorem 1.9.3 p.59, we complete the proof. The proof of the left hand side inequality is exactly the same as in Theorem 3.0. \square

6. Carathéodory-Schur Interpolation in weighted Bergman spaces

We suppose that $X = L_a^p\left((1-|z|^2)^\beta dA\right)$, $\beta > -1$ and $1 \leq p \leq 2$. Our aim in this section is to give an estimate for the constant for a generalized Carathéodory-Schur interpolation, (a partial case of the Nevanlinna-Pick interpolation),

$$c(\sigma_{\lambda,n}, X, H^\infty) = \sup\{\|f\|_{H^\infty/b_\lambda^n H^\infty} : f \in X, \|f\|_X \leq 1\},$$

where $\|f\|_{H^\infty/b_\lambda^n H^\infty} = \inf\{\|f + b_\lambda^n g\|_\infty : g \in X\}$, and $\sigma_{\lambda,n} = \{\lambda, \lambda, \dots, \lambda\}$, $\lambda \in \mathbb{D}$. The corresponding interpolation problem is: given $f \in X$, to minimize $\|h\|_\infty$ such that $h^{(j)}(\lambda) = f^{(j)}(\lambda)$, $0 \leq j < n$.

For this partial case, we have the following generalization of the estimate from Theorem 1.0.

Theorem. 6.0 *Let $\lambda \in \mathbb{D}$, $\beta > -1$ and $1 \leq p \leq 2$. Then,*

$$c\left(\sigma_{\lambda,n}, L_a^p\left((1-|z|^2)^\beta dA\right), H^\infty\right) \leq A' \left(\frac{n}{1-|\lambda|}\right)^{\frac{\beta+2}{p}},$$

where $A' = A'(\beta, p)$ is a constant depending only on β and p .

We first need a simple equivalent to $I_k(\beta) = \int_0^1 r^{2k+1}(1-r^2)^\beta dr$, $\beta > -1$.

Lemma. 6.1 *Let $k \geq 0$, $\beta > -1$ and $I_k(\beta) = \int_0^1 r^{2k+1}(1-r^2)^\beta dr$. Then,*

$$I_k(\beta) \sim \frac{1}{2} \frac{\Gamma(\beta+1)}{k^{\beta+1}},$$

for $k \rightarrow \infty$, where Γ stands for the usual Gamma function, $\Gamma(z) = \int_0^{+\infty} e^{-s} s^{z-1} ds$.

Proof. Let $a = \frac{1}{\sqrt{k+1}}$, $b = \max(1, a^\beta)$. Since $1 - e^{-u} \sim u$ as $u \rightarrow 0$, we have

$$\begin{aligned} I_k(\beta) &= \int_0^1 r^{2k+1}(1-r^2)^\beta dr = \int_0^\infty e^{-(2k+1)t}(1-e^{-2t})^\beta e^{-t} dt = \\ &= \int_0^a e^{-2(k+1)t}(1-e^{-2t})^\beta dt + \int_a^\infty e^{-2(k+1)t}(1-e^{-2t})^\beta dt = \end{aligned}$$

$$\begin{aligned}
&= \int_0^a e^{-2(k+1)t} (1 - e^{-2t})^\beta dt + O\left(\frac{b}{k+1} e^{-2a(k+1)}\right) = \\
&= (1 + o(1)) \int_0^a e^{-2(k+1)t} (2t)^\beta dt + O\left(\frac{b}{k+1} e^{-2a(k+1)}\right) = \\
&= (1 + o(1)) \int_0^{2(k+1)a} e^{-s} \left(\frac{s}{k+1}\right)^\beta \frac{ds}{2(k+1)} + O\left(\frac{b}{k+1} e^{-2a(k+1)}\right) = \\
&= \frac{1}{2} \frac{1}{(k+1)^{\beta+1}} (1 + o(1)) \int_0^{2(k+1)a} e^{-s} s^\beta ds + O\left(\frac{b}{k+1} e^{-2a(k+1)}\right) = \\
&= \frac{1}{2} \frac{\Gamma(\beta+1)}{(k+1)^{\beta+1}} (1 + o(1)) + O\left(\frac{b}{k+1} e^{-2a(k+1)}\right) = \\
&= \frac{1}{2} \frac{\Gamma(\beta+1)}{(k+1)^{\beta+1}} (1 + o(1)) \sim \frac{1}{2} \frac{\Gamma(\beta+1)}{k^{\beta+1}},
\end{aligned}$$

which completes the proof. \square

Proof of Theorem 6.0. Step 1. We start to prove the Theorem for $p = 1$.

Let $f \in X = L_a^1\left((1 - |z|^2)^\beta dA\right)$ such that $\|f\|_X \leq 1$. Since $X \circ b_\lambda = X$, we have

$f \circ b_\lambda = \sum_{k \geq 0} a_k z^k \in X$. Let $p_n = \sum_{k=0}^{n-1} a_k z^k$ and $g = p_n \circ b_\lambda$. Then, $f \circ b_\lambda - p_n \in z^n X$ and $f - p_n \circ b_\lambda \in (z^n X) \circ b_\lambda = b_\lambda^n X$. Now, $p_n \circ b_\lambda = \sum_{k=0}^{n-1} a_k b_\lambda^k$ and

$$\|p_n \circ b_\lambda\|_\infty = \|p_n\|_\infty \leq A_n \|f \circ b_\lambda\|_X,$$

where $A_n = \left\| \sum_{k \geq 0} a_k z^k \mapsto \sum_{k=0}^{n-1} a_k z^k \right\|_{X \rightarrow H^\infty}$. Now,

$$\begin{aligned}
\|f \circ b_\lambda\|_X &\leq \int_{\mathbb{D}} |f(b_\lambda(z))| (1 - |z|^2)^\beta dA = \int_{\mathbb{D}} |f(w)| (1 - |b_\lambda(w)|^2)^\beta \left| b'_\lambda(w) \right|^2 dA = \\
&\leq 2^\beta \int_{\mathbb{D}} |f(w)| \left(\frac{(1 - |\lambda|^2)(1 - |w|^2)}{|1 - \bar{\lambda}w|^2} \right)^\beta \left(\frac{(1 - |\lambda|^2)}{|1 - \bar{\lambda}w|^2} \right)^2 dA = \\
&= \int_{\mathbb{D}} |f(w)| (1 - |w|^2)^\beta \left(\frac{(1 - |\lambda|^2)}{|1 - \bar{\lambda}w|^2} \right)^{2+\beta} dA \leq \\
&\leq \sup_{w \in \mathbb{D}} \left(\frac{(1 - |\lambda|^2)}{|1 - \bar{\lambda}w|^2} \right)^{2+\beta} \int_{\mathbb{D}} |f(w)| (1 - |w|^2)^\beta dA \leq \left(\frac{(1 - |\lambda|^2)}{(1 - |\lambda|^2)^2} \right)^{2+\beta} \|f\|_X,
\end{aligned}$$

which gives,

$$\|f \circ b_\lambda\|_X \leq \left(\frac{1 + |\lambda|}{1 - |\lambda|} \right)^{2+\beta} \|f\|_X.$$

We now give an estimation for A_n . Let $g(z) = \sum_{k \geq 0} \hat{g}(k) z^k \in X$, then

$$\left\| \sum_{k=0}^{n-1} \hat{g}(k) z^k \right\|_{\infty} \leq \sum_{k=0}^{n-1} |\hat{g}(k)|.$$

Now, noticing that

$$\begin{aligned} \int_{\mathbb{D}} g(w) \bar{w}^k (1 - |w|^2)^{\beta} dA &= \int_0^1 \int_0^{2\pi} f(re^{it}) r^k e^{-ikt} (1 - r^2)^{\beta} r dt dr = \\ &= \int_0^1 (1 - r^2)^{\beta} r^{k+1} \int_0^{2\pi} f(re^{it}) e^{-ikt} dt dr = \int_0^1 \hat{g}_r(k) r^{k+1} (1 - r^2)^{\beta} dr, \end{aligned}$$

where $g_r(z) = g(rz)$, $\hat{g}_r(k) = r^k \hat{g}(k)$. Setting $I_k(\beta) = \int_0^1 r^{2k+1} (1 - r^2)^{\beta} dr$, we get

$$\hat{g}(k) = \frac{1}{I_k(\beta)} \int_{\mathbb{D}} g(w) \bar{w}^k (1 - |w|^2)^{\beta} dA.$$

Then,

$$|\hat{g}(k)| = \frac{1}{I_k(\beta)} \left| \int_{\mathbb{D}} g(w) \bar{w}^k (1 - |w|^2)^{\beta} dA \right| \leq \frac{1}{I_k(\beta)} \|g\|_X,$$

which gives

$$\left\| \sum_{k=0}^{n-1} \hat{g}(k) z^k \right\|_{\infty} \leq \left(\sum_{k=0}^{n-1} \frac{1}{I_k(\beta)} \right) \|g\|_X.$$

Now using Lemma 6.1,

$$\sum_{k=0}^{n-1} \frac{1}{I_k(\beta)} \sim_{n \rightarrow \infty} \frac{2}{\Gamma(\beta + 1)} \sum_{k=0}^{n-1} k^{\beta+1} \sim \frac{2c_{\beta}}{\Gamma(\beta + 1)} n^{\beta+2},$$

where c_{β} is a constant depending on β only. This gives

$$\left\| \sum_{k=0}^{n-1} \hat{g}(k) z^k \right\|_{\infty} \leq C_{\beta} n^{\alpha+2} \|g\|_X,$$

where C_{β} is also a constant depending on β only. Finally, we conclude that $A_n \leq C_{\beta} n^{\beta+2}$, and as a result,

$$\|p_n \circ b_{\lambda}\|_{\infty} \leq C_{\beta} n^{\beta+2} \left(\frac{1 + |\lambda|}{1 - |\lambda|} \right)^{2+\beta} \|f\|_X,$$

which proves the Theorem for $p = 1$.

Step 2. This step of the proof repetes the scheme from Theorems 3.0&5.0. Let $T : L_a^p \left((1 - |z|^2)^{\beta} dA \right) \longrightarrow H^{\infty} / b_{\lambda}^n H^{\infty}$ be the restriction map defined by

$$Tf = \left\{ g \in H^{\infty} : f - g \in b_{\lambda}^n L_a^p \left((1 - |z|^2)^{\beta} dA \right) \right\},$$

for every f . Then,

$$\|T\|_{L_a^p \left((1 - |z|^2)^{\beta} dA \right) \rightarrow H^{\infty} / b_{\lambda}^n H^{\infty}} = c \left(\sigma, L_a^p \left((1 - |z|^2)^{\beta} dA \right), H^{\infty} \right).$$

Now, let $\gamma > \beta$ and $P_\gamma : L^p \left((1 - |z|^2)^\beta dA \right) \longrightarrow L^p_a \left((1 - |z|^2)^\beta dA \right)$ be the Bergman projection, (see [H], p.6), defined by

$$P_\gamma f = (\gamma + 1) \int_{\mathbb{D}} \frac{(1 - |w|^2)^\gamma}{(1 - z\bar{w})^{2+\gamma}} f(w) dA(w),$$

for every f . P_γ is a bounded projection from $L^p \left((1 - |z|^2)^\beta dA \right)$ onto $L^p_a \left((1 - |z|^2)^\beta dA \right)$ (see [H], Theorem 1.10 p.12), (since $1 \leq p \leq 2$). Moreover, since $L^p_a \left((1 - |z|^2)^\beta dA \right) \subset L^p_a \left((1 - |z|^2)^\gamma dA \right)$, we have $P_\gamma f = f$ for all $f \in L^p_a \left((1 - |z|^2)^\beta dA \right)$, (see [H], Corollary 1.5 p.6). As a result,

$$\|T\|_{L^p_a \left((1 - |z|^2)^\beta dA \right) \rightarrow H^\infty / b_\lambda^n H^\infty} \leq \|TP_\gamma\|_{L^p \left((1 - |z|^2)^\beta dA \right) \rightarrow H^\infty / b_\lambda^n H^\infty},$$

for all $1 \leq p \leq 2$. We set

$$c_i(\beta) = \|P_\gamma\|_{L^i \left((1 - |z|^2)^\beta dA \right) \rightarrow L^i_a \left((1 - |z|^2)^\beta dA \right)},$$

for $i = 1, 2$. Then,

$$\begin{aligned} & \|TP_\gamma\|_{L^1 \left((1 - |z|^2)^\beta dA \right) \rightarrow H^\infty / b_\lambda^n H^\infty} \leq \\ & \leq \|T\|_{L^1_a \left((1 - |z|^2)^\beta dA \right) \rightarrow H^\infty / b_\lambda^n H^\infty} \|P_\gamma\|_{L^1 \left((1 - |z|^2)^\beta dA \right) \rightarrow L^1_a \left((1 - |z|^2)^\beta dA \right)} = \\ & = c \left(\sigma, L^1_a \left((1 - |z|^2)^\beta dA \right), H^\infty \right) c_1(\beta) \leq \\ & \leq A'(\beta, 1) \left(\frac{n}{1 - |\lambda|} \right)^{\beta+2} c_1(\beta), \end{aligned}$$

using Step 1. In the same way,

$$\|TP_\gamma\|_{L^2 \left((1 - |z|^2)^\beta dA \right) \rightarrow H^\infty / b_\lambda^n H^\infty} \leq \|T\|_{L^2_a \left((1 - |z|^2)^\beta dA \right) \rightarrow H^\infty / b_\lambda^n H^\infty} c_2(\beta).$$

Now, we recall that

$$L^2_a \left((1 - |z|^2)^\beta dA \right) = l^2_a \left(\frac{1}{(k+1)^{\frac{\beta+1}{2}}} \right), \quad \beta > -1.$$

As a consequence,

$$\|T\|_{L^2_a \left((1 - |z|^2)^\beta dA \right) \rightarrow H^\infty / b_\lambda^n H^\infty} = c \left(\sigma, l^2_a \left(\frac{1}{(k+1)^{\frac{\beta+1}{2}}} \right), H^\infty \right),$$

and, applying Theorem 1.0,

$$\begin{aligned} \|TP_\gamma\|_{L^2((1-|z|^2)^\beta dA) \rightarrow H^\infty/b_\lambda^n H^\infty} &\leq c_2(\beta)A' \left(\frac{\beta+1}{2}, 2 \right) \left(\frac{n}{1-|\lambda|} \right)^{\frac{2\frac{\beta+1}{2}+1}{2}} = \\ &= c_2(\beta)A' \left(\frac{\beta+1}{2}, 2 \right) \left(\frac{n}{1-|\lambda|} \right)^{\frac{\beta+2}{2}}. \end{aligned}$$

We finish the reasoning applying Riesz-Thorin Theorem, (see [Tr] for example), to the operator TP_γ . If $p \in [1, 2]$, there exists $0 \leq \theta \leq 1$ such that

$$\frac{1}{p} = (1-\theta)\frac{1}{1} + \theta\frac{1}{2} = 1 - \frac{\theta}{2},$$

and then,

$$\left[L_a^1 \left((1-|z|^2)^\beta dA \right), L_a^2 \left((1-|z|^2)^\beta dA \right) \right]_\theta = L_a^p \left((1-|z|^2)^\beta dA \right),$$

and

$$\begin{aligned} \|TP_\gamma\|_{L^p((1-|z|^2)^\beta dA) \rightarrow H^\infty/b_\lambda^n H^\infty} &\leq \\ &\leq \left(\|TP_\gamma\|_{L^1((1-|z|^2)^\beta dA) \rightarrow H^\infty/b_\lambda^n H^\infty} \right)^{1-\theta} \left(\|TP_\gamma\|_{L^2((1-|z|^2)^\beta dA) \rightarrow H^\infty/b_\lambda^n H^\infty} \right)^\theta \leq \\ &\leq \left(c_1(\beta)A'(\beta, 1) \left(\frac{n}{1-|\lambda|} \right)^{\beta+2} \right)^{1-\theta} \left(c_2(\beta)A' \left(\frac{\beta+1}{2}, 2 \right) \left(\frac{n}{1-|\lambda|} \right)^{\frac{\beta+2}{2}} \right)^\theta = \\ &= \left(c_1(\beta)A'(\beta, 1) \right)^{1-\theta} \left(c_2(\beta)A' \left(\frac{\beta+1}{2}, 2 \right) \right)^\theta \left(\frac{n}{1-|\lambda|} \right)^{(\beta+2)(1-\theta) + \theta\frac{\beta+2}{2}}. \end{aligned}$$

Now, since $\theta = 2(1 - \frac{1}{p})$, $(\beta+2)(1-\theta) + \theta\frac{\beta+2}{2} = \beta - (1 - \frac{1}{p})\beta + 2 - 2 + \frac{2}{p} = \frac{\beta+2}{p}$, and

$$\|T\|_{L_a^p((1-|z|^2)^\beta dA) \rightarrow H^\infty/b_\lambda^n H^\infty} \leq \|TP_\gamma\|_{L^p((1-|z|^2)^\beta dA) \rightarrow H^\infty/b_\lambda^n H^\infty},$$

we complete the proof. \square

7. A lower bound for $C_{n,r} (l_a^2(w_k), H^\infty)$

Here, we consider the weighted spaces $l_a^2(w_k)$ of polynomial growth and the problem of lower estimates for the one point special case $\sigma_{\lambda,n} = \{\lambda, \lambda, \dots, \lambda\}$, (n times) $\lambda \in \mathbb{D}$. Recall the definition of the semi-free interpolation constant

$$c(\sigma_{\lambda,n}, H, H^\infty) = \sup \{ \|f\|_{H^\infty/b_\lambda^n H^\infty} : f \in H, \|f\|_H \leq 1 \},$$

where $\|f\|_{H^\infty/b_\lambda^n H^\infty} = \inf \{ \|f + b_\lambda^n g\|_\infty : g \in H \}$. In particular, our aim is to prove the sharpness of the upper estimate for the quantity

$$C_{n,r} \left(l_a^2 \left(\frac{1}{(k+1)^{\frac{N-1}{2}}} \right), H^\infty \right),$$

(where $N \geq 1$ is an integer), in **Theorem 1.0**.

Theorem. 7.0 *Let $N \geq 1$ be an integer. Then,*

$$c \left(\sigma_{\lambda,n}, l_A^2 \left(\frac{1}{(k+1)^{\frac{N-1}{2}}} \right), H^\infty \right) \geq a_N \left(\frac{n}{1-|\lambda|} \right)^{\frac{N}{2}}$$

for a positive constant a_N depending on N only. In particular,

$$a_N \left(\frac{n}{1-r} \right)^{\frac{N}{2}} \leq C_{n,r} \left(l_a^2 \left(\frac{1}{(k+1)^{\frac{N-1}{2}}} \right), H^\infty \right) \leq A \left(\frac{n}{1-r} \right)^{\frac{N}{2}},$$

for all $n \geq 1$, $0 \leq r < 1$, where $A = A \left(\frac{N-1}{2} \right)$ is a constant defined in Theorem 1.0.

(1) We first recall some properties of spaces $X = l_a^p(w_k)$. As it is mentioned in the Introduction,

$$l_a^p(w_k) = \left\{ f = \sum_{k \geq 0} \hat{f}(k) z^k : \|f\|^p = \sum_{k \geq 0} |\hat{f}(k)|^p w_k^p < \infty \right\},$$

with a weight w satisfying $w_k > 0$ for every $k \geq 0$ and $\overline{\lim}_k (1/w_k^{1/k}) = 1$. The latter condition implies that $l_a^p(w_k)$ is continuously embedded into the space of holomorphic functions $Hol(\mathbb{D})$ on the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ (and not on a larger disc, i.e. $l_a^p(w_k)$ does not contained in $Hol(r\mathbb{D})$ for every $r > 1$). In this section, we study the case $p = 2$, so that $l_a^2(w_k)$ is a reproducing kernel Hilbert space on the disc \mathbb{D} . The reproducing kernel of $l_a^2(w_k)$, by definition, is a $l_a^2(w_k)$ -valued function $\lambda \mapsto k_\lambda^w$, $\lambda \in \mathbb{D}$, such that $(f, k_\lambda^w) = f(\lambda)$ for every $f \in l_a^2(w_k)$, where

(\cdot, \cdot) means the scalar product $(f, g) = \sum_{k \geq 0} \hat{h}(k) \overline{\hat{g}(k)} w_k^2$. Since one has $f(\lambda) = \sum_{k \geq 0} \hat{f}(k) \lambda^k \frac{1}{w_k^2} w_k^2$ ($\lambda \in \mathbb{D}$), it follows that

$$k_\lambda^w(z) = \sum_{k \geq 0} \frac{\overline{\lambda}^k z^k}{w_k^2}, \quad z \in \mathbb{D}.$$

In particular, for the Hardy space $H^2 = l_a^2(1)$, we get the Szegő kernel

$$k_\lambda(z) = (1 - \overline{\lambda}z)^{-1},$$

for the Bergman space $L_a^2(\mathbb{D}) = l_a^2\left(\frac{1}{(k+1)^2}\right)$ - the Bergman kernel $k_\lambda(z) = (1 - \overline{\lambda}z)^{-2}$.

(2) Conversely, following the Aronszajn theory of RKHS (see, for example [A] or [N2] p.317), given a positive definit function $(\lambda, z) \mapsto k(\lambda, z)$ on $\mathbb{D} \times \mathbb{D}$ (i.e. such that $\sum_{i,j} \overline{a_i} a_j k(\lambda_i, \lambda_j) > 0$ for all finite subsets $(\lambda_i) \subset \mathbb{D}$ and all non-zero families of complex numbers (a_i)) one can define the corresponding Hilbert spaces $H(k)$ as the completion of finite linear combinations $\sum_i \overline{a_i} k(\lambda_i, \cdot)$ endowed with the norm

$$\left\| \sum_i \overline{a_i} k(\lambda_i, \cdot) \right\|^2 = \sum_{i,j} \overline{a_i} a_j k(\lambda_i, \lambda_j).$$

When k is holomorphic with respect to the second variable and antiholomorphic with respect to the first one, we obtain a RKHS of holomorphic functions $H(k)$ embedded into $Hol(\mathbb{D})$.

For functions k of the form $k(\lambda, z) = K(\overline{\lambda}z)$, where $K \in Hol(\mathbb{D})$, the positive definitness is equivalent to $\hat{K}(j) > 0$ for every $j \geq 0$, where $\hat{K}(j)$ stands for Taylor coefficients, and in this case we have $H(k) = l_a^2(w_j)$, where $w_j = 1/\sqrt{\hat{K}(j)}$, $j \geq 0$. In particular, for $K(w) = (1-w)^{-\beta}$, $k_\lambda(z) = (1 - \overline{\lambda}z)^{-\beta}$, $\beta > 0$, we have $\hat{K}(j) = \binom{\beta+j-1}{\beta-1}$ (binomial coefficients), and hence $w_j = \left(\frac{j!}{\beta(\beta+1)\dots(\beta+j-1)}\right)^{\frac{1}{2}}$. Indeed, deriving $\frac{1}{1-z}$, we get by induction

$$(1-z)^{-\beta} = \frac{1}{(\beta-1)!} \sum_{j \geq 0} (j+\beta-1)\dots(j+1) z^j = \sum_{j \geq 0} \binom{\beta+j-1}{\beta-1} z^j.$$

Clearly, $w_j \simeq 1/j^{\frac{\beta-1}{2}}$, where $a_j \simeq b_j$ means that there exist constants $c_1 > 0$, $c_2 > 0$ such that $c_1 a_j \leq b_j \leq c_2 a_j$ for every j . Therefore, $H(k) = l_a^2\left(\frac{1}{(k+1)^{\frac{\beta-1}{2}}}\right)$ (a topological identity: the spaces are the same and the norms are equivalent).

(3) Reproducing kernel Hilbert spaces containing H^2 . We will use the previous observations for the following composed reproducing kernels (Aronszajn-deBranges, see [N2] p.320): given a reproducing kernel k and an entire function $\varphi = \sum_{j \geq 0} \hat{\varphi}(j) z^j$ with $\hat{\varphi}(j) \geq 0$ for every $j \geq 0$, the function $\varphi \circ k$ is also positive definit and the corresponding RKHS

$$H(\varphi \circ k) =: \varphi(H(k))$$

satisfies the following. For every $f \in H(k)$ we have $\varphi \circ f \in \varphi(H(k))$ and $\|\varphi \circ f\|_{\varphi(H(k))}^2 \leq \varphi(\|f\|_{H(k)}^2)$ (see [N2] p.320). In particular, if φ is a polynomial of degree N and k is the Szegő kernel then

$\varphi \circ k_\lambda(z) = \sum_{j \geq 0} c_j \bar{\lambda}^j z^j$ with $c_k \simeq (k+1)^{N-1}$, and hence

$$\varphi(H^2) = l_a^2 \left(\frac{1}{(k+1)^{\frac{N-1}{2}}} \right)$$

(a topological identity: the spaces are the same and the norms are equivalent). The link between spaces of type $l_a^2 \left(\frac{1}{(k+1)^{\frac{N-1}{2}}} \right)$ (already mentionned in **Section 1**) and of type $\varphi(H^2) = H_\varphi$ being established, we give the following result.

Lemma 7.1 *Let $\varphi(z) = \sum_{k=0}^N a_k z^k$, $a_k \geq 0$ ($a_N > 0$), and $H_\varphi = \varphi(H^2)$ be the reproducing kernel Hilbert space corresponding to the kernel $\varphi \left(\frac{1}{1-\bar{\lambda}z} \right)$. Then, there exists a constant $a(\varphi) > 0$ such that*

$$c(\sigma_{\lambda,n}, H_\varphi, H^\infty) \geq a(\varphi) \left(\frac{n}{1-|\lambda|} \right)^{\frac{N}{2}}.$$

Proof. 1) We set

$$Q_n = \sum_{k=0}^{n-1} b_\lambda^k \frac{(1-|\lambda|^2)^{1/2}}{1-\bar{\lambda}z}, \quad H_n = \varphi \circ Q_n,$$

$$\Psi = bH_n.$$

Then $\|Q_n\|_2^2 = n$, and hence by the Aronszajn-deBranges inequality, see [N2] p.320, point (k) of Exercise 6.5.2, with $\varphi(z) = z^N$ and $K(\lambda, z) = k_\lambda(z) = \frac{1}{1-\bar{\lambda}z}$, and noticing that $H(\varphi \circ K) = H_\varphi$,

$$\|\Psi\|_{H_\varphi}^2 \leq b^2 \varphi(\|Q_n\|_2^2) = b^2 \varphi(n).$$

Let $b > 0$ such that $b^2 \varphi(n) = 1$.

2) Since the spaces H_φ and H^∞ are rotation invariant, we have $c(\sigma_{\lambda,n}, H_\varphi, H^\infty) = c(\sigma_{\mu,n}, H_\varphi, H^\infty)$ for every λ, μ with $|\lambda| = |\mu| = r$. Let $\lambda = -r$. To get a lower estimate for $\|\Psi\|_{H_\varphi/b_\lambda^n H_\varphi}$ consider G such that $\Psi - G \in b_\lambda^n \text{Hol}(\mathbb{D})$, i.e. such that $bH_n \circ b_\lambda - G \circ b_\lambda \in z^n \text{Hol}(\mathbb{D})$.

3) First, we show that

$$\psi =: \Psi \circ b_\lambda = bH_n \circ b_\lambda$$

is a polynomial (of degree nN) with positive coefficients. Note that

$$\begin{aligned}
Q_n \circ b_\lambda &= \sum_{k=0}^{n-1} z^k \frac{(1 - |\lambda|^2)^{1/2}}{1 - \bar{\lambda} b_\lambda(z)} = \\
&= (1 - |\lambda|^2)^{-\frac{1}{2}} \left(1 + (1 - \bar{\lambda}) \sum_{k=1}^{n-1} z^k - \bar{\lambda} z^n \right) = \\
&= (1 - r^2)^{-1/2} \left(1 + (1 + r) \sum_{k=1}^{n-1} z^k + r z^n \right) =: (1 - r^2)^{-1/2} \psi_1.
\end{aligned}$$

Hence, $\psi = \Psi \circ b_\lambda = bH_n \circ b_\lambda = b\varphi \circ \left((1 - r^2)^{-\frac{1}{2}} \psi_1 \right)$ and

$$\varphi \circ \psi_1 = \sum_{k=0}^N a_k \psi_1^k(z).$$

(In fact, we can simply assume that $\varphi \circ \psi_1 = \psi_1^N(z)$ since $H_\varphi = l_a^2 \left(\frac{1}{(k+1)^{\frac{N-1}{2}}} \right) = H_{z^N}$). Now, it is clear that ψ is a polynomial of degree Nn such that

$$\psi(1) = \sum_{j=0}^{Nn} \hat{\psi}(j) = b\varphi \left((1 - r^2)^{-1/2} (1 + r)n \right) = b\varphi \left(\sqrt{\frac{1+r}{1-r}} n \right) > 0.$$

4) Next, we show that there exists a constant $c = c(\varphi) > 0$ (for example, $c = \alpha/2^{2N}(N-1)!$, α is a numerical constant) such that

$$\sum_{j=0}^m (\psi) =: \sum_{j=0}^m \hat{\psi}(j) \geq c \sum_{j=0}^{Nn} \hat{\psi}(j) = c\psi(1),$$

where $m \geq 1$ is such that $2m = n$ if n is even and $2m - 1 = n$ if n is odd.

Indeed, setting

$$S_n = \sum_{j=0}^n z^j,$$

we have

$$\sum_{j=0}^m (\psi_1^k) = \sum_{j=0}^m \left(\left(1 + (1+r) \sum_{k=1}^{n-1} z^k + r z^n \right)^k \right) \geq \sum_{j=0}^m (S_{n-1}^k).$$

Next, we obtain

$$\sum_{j=0}^m (S_{n-1}^k) = \sum_{j=0}^m \left(\left(\frac{1 - z^n}{1 - z} \right)^k \right) =$$

$$\begin{aligned}
&= \sum^m \left(\sum_{j=0}^k C_k^j \frac{1}{(1-z)^j} \cdot \left(\frac{-z^n}{1-z} \right)^{k-j} \right) = \sum^m \left(\frac{1}{(1-z)^k} \right) = \\
&= \sum^m \left(\sum_{j \geq 0} C_{k+j-1}^j z^j \right) = \sum_{j=0}^m C_{k+j-1}^j \geq \sum_{j=0}^m \frac{(j+1)^{k-1}}{(k-1)!} \geq \\
&\geq \alpha \frac{m^k}{(k-1)!},
\end{aligned}$$

where $\alpha > 0$ is a numerical constant. Finally,

$$\begin{aligned}
\sum^m (\psi_1^k) &\geq \alpha \frac{m^k}{(k-1)!} \geq \alpha \frac{(n/2)^k}{(k-1)!} = \\
&= \frac{\alpha}{2^k (k-1)!} \cdot \frac{((1+r)n)^k}{(1+r)^k} = \frac{\alpha}{2^k (1+r)^k (k-1)!} \cdot (\psi_1(1))^k \geq \\
&\geq \frac{\alpha}{2^N (1+r)^N (N-1)!} \cdot (\psi_1(1))^k.
\end{aligned}$$

Summing up these inequalities in $\sum^m(\psi) = b \sum^m(\varphi \circ \psi_1) = b \sum_{k=0}^N a_k (1-r^2)^{-k/2} \sum^m(\psi_1^k)$ (or simply taking $k = N$, if we already supposed $\varphi = z^N$), we obtain the result claimed.

5) Now, using point 4) and the preceding Fejer kernel argument and denoting $F_n = \Phi_m + z^m \Phi_m$, where Φ_k stands for the k -th Fejer kernel, we get

$$\begin{aligned}
\|\Psi\|_{H^\infty/b_\lambda^n H^\infty} &= \|\psi\|_{H^\infty/z^n H^\infty} \geq \frac{1}{2} \|\psi * F_n\|_\infty \geq \frac{1}{2} \sum_{j=0}^m \hat{\psi}(j) \geq \\
&\geq \frac{c}{2} \psi(1) = \frac{c}{2} b \varphi \left(\sqrt{\frac{1+r}{1-r}} n \right) = \frac{c}{2} \cdot \frac{\varphi \left(\sqrt{\frac{1+r}{1-r}} n \right)}{(\varphi(n))^{1/2}} \geq
\end{aligned}$$

(assuming that $\varphi = z^N$)

$$\geq a(\varphi) \left(\frac{n}{1-r} \right)^{\frac{N}{2}}.$$

□

Proof of Theorem 7.0. In order to prove the left hand side inequality, it suffices to apply Lemma 7.1 with $\varphi(z) = z^N$. Indeed, in this case $H_\varphi = l_a^2 \left(\frac{1}{(k+1)^{\frac{N-1}{2}}} \right) = H_{z^N}$. The right hand side inequality is a straightforward consequence of Theorem 1.0.

□

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