

EFFECTIVE H^∞ INTERPOLATION

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ABSTRACT. Given a finite subset σ of the unit disc \mathbb{D} and a holomorphic function f in \mathbb{D} belonging to a class X , we are looking for a function g in another class Y which satisfies $g|_\sigma = f|_\sigma$ and is of minimal norm in Y . Then, we wish to compare $\|g\|_Y$ with $\|f\|_X$. More precisely, we consider the interpolation constant $c(\sigma, X, Y) = \sup_{f \in X, \|f\|_X \leq 1} \inf_{g|_\sigma = f|_\sigma} \|g\|_Y$. When $Y = H^\infty$, our interpolation problem includes those of Nevanlinna-Pick and Caratheodory-Schur. Moreover, Carleson's free interpolation problem can be interpreted in terms of the constant $c(\sigma, X, H^\infty)$. For $Y = H^\infty$, $X = H^p$ (the Hardy space) or $X = L_a^2$ (the Bergman space), we obtain an upper bound for the constant $c(\sigma, X, H^\infty)$ in terms of $n = \text{card } \sigma$ and $r = \max_{\lambda \in \sigma} |\lambda|$. Our upper estimates are shown to be sharp with respect to n and r .

1. INTRODUCTION

1.1. Statement and historical context of the problem. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc of the complex plane and let $\text{Hol}(\mathbb{D})$ be the space of holomorphic functions on \mathbb{D} . The problem considered is the following: given two Banach spaces X and Y of holomorphic functions on \mathbb{D} , $X, Y \subset \text{Hol}(\mathbb{D})$, and a finite subset $\sigma \subset \mathbb{D}$, find the least norm interpolation by functions of the space Y for the traces $f|_\sigma$ of functions of the space X , in the worst case of f . The case $X \subset Y$ is of no interest, and so one can suppose that either $Y \subset X$ or X, Y are incomparable.

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More precisely, our problem is to compute or estimate the following interpolation constant

$$c(\sigma, X, Y) = \sup_{f \in X, \|f\|_X \leq 1} \inf \{ \|g\|_Y : g|_\sigma = f|_\sigma \}.$$

If $r \in [0, 1)$ and $n \geq 1$, we also define

$$C_{n,r}(X, Y) = \sup \{ c(\sigma, X, Y) : \text{card } \sigma \leq n, |\lambda| \leq r, \forall \lambda \in \sigma \}.$$

Here and later on, H^∞ stands for the space (algebra) of bounded holomorphic functions on \mathbb{D} endowed with the norm $\|f\|_\infty = \sup_{z \in \mathbb{D}} |f(z)|$. The classical interpolation problems -those of Nevanlinna-Pick (1916) and Carathodory-Schur (1908) (see [15] p.231 for these two problems) on the one hand, and Carleson's free interpolation (1958) (see [16] p.158) on the other hand- are of this nature and correspond to the case $Y = H^\infty$. Two first are "individual", in the sense that one looks simply to compute the norms $\|f\|_{H^\infty_\sigma}$ or $\|f\|_{H^\infty/z^n H^\infty}$ for a given f . In the case of the third one, we consider infinite sets σ . Let $l^\infty(\sigma)$ be the space of bounded functions $(a_\lambda)_{\lambda \in \sigma}$ on σ endowed with the norm $\|a\|_{l^\infty(\sigma)} = \max_{\lambda \in \sigma} |a_\lambda|$. Carleson's free interpolation problem is to compare the norms $\|a\|_{l^\infty(\sigma)}$ and

$$\inf \{ \|g\|_\infty : g(\lambda) = a_\lambda, \lambda \in \sigma \}.$$

In other words, we want to estimate the interpolation constant defined as

$$c(\sigma, l^\infty(\sigma), H^\infty) = \sup_{a \in l^\infty(\sigma), \|a\|_{l^\infty} \leq 1} \inf \{ \|g\|_\infty : g(\lambda) = a_\lambda, \lambda \in \sigma \}.$$

Let us now explain why our problem includes those of Nevanlinna-Pick and Carathodory-Schur.

(i) Nevanlinna-Pick interpolation problem.

Given $\sigma = \{\lambda_1, \dots, \lambda_n\}$ a finite subset of \mathbb{D} and $\mathcal{W} = \{w_1, \dots, w_n\}$ a finite subset of \mathbb{C} , find

$$\text{NP}_{\sigma, \mathcal{W}} = \inf \{ \|f\|_\infty : f(\lambda_i) = w_i, i = 1, \dots, n \}.$$

The classical answer of Pick is the following:

$$\text{NP}_{\sigma, \mathcal{W}} = \inf \left\{ c > 0 : \left(\frac{c^2 - \overline{w_i} w_j}{1 - \overline{\lambda_i} \lambda_j} \right)_{1 \leq i, j \leq n} \gg 0 \right\},$$

where for any $n \times n$ matrix M , $M \gg 0$ means that M is positive definite.

(ii) Carathodory-Schur interpolation problem.

Given $\mathcal{W} = \{w_0, w_1, \dots, w_n\}$ a finite subset of \mathbb{C} , find

$$\text{CS}_{\mathcal{W}} = \inf \{ \|f\|_\infty : f(z) = w_0 + w_1 z + \dots + w_n z^n + \dots \}.$$

The classical answer of Schur is the following:

$$\text{CS}_{\mathcal{W}} = \|(T_\varphi)_n\|,$$

where T_φ is the Toeplitz operator associated with a symbol φ , $(T_\varphi)_n$ is the compression of T_φ on \mathcal{P}_n , the space of analytic polynomials of degree less or equal than n , and φ is the polynomial $\sum_{k=0}^n w_k z^k$. From a modern point of view, these two interpolation problems (i) and (ii) are included in the following mixed problem: given $\sigma = \{\lambda_1, \dots, \lambda_n\} \subset \mathbb{D}$ and $f \in \text{Hol}(\mathbb{D})$, compute or estimate

$$\|f\|_{H^\infty/B_\sigma H^\infty} = \inf \{\|g\|_\infty : f - g \in B_\sigma \text{Hol}(\mathbb{D})\}.$$

From now on, if $\sigma = \{\lambda_1, \dots, \lambda_n\} \subset \mathbb{D}$ is a finite subset of the unit disc, then

$$B_\sigma = \prod_{j=1}^n b_{\lambda_j}$$

is the corresponding finite Blaschke product where $b_\lambda = \frac{\lambda - z}{1 - \bar{\lambda}z}$, $\lambda \in \mathbb{D}$. The classical Nevanlinna-Pick problem corresponds to the case $X = \text{Hol}(\mathbb{D})$, $Y = H^\infty$, and the one of Carathodory-Schur to the case $\lambda_1 = \lambda_2 = \dots = \lambda_n = 0$ and $X = \text{Hol}(\mathbb{D})$, $Y = H^\infty$.

Looking at this problem in the form of computing or estimating the interpolation constant $c(\sigma, X, Y)$ which is nothing but the norm of the embedding operator $(X|_\sigma, \|\cdot\|_{X|_\sigma}) \rightarrow (Y|_\sigma, \|\cdot\|_{Y|_\sigma})$, one can think, of course, on passing (after) to the limit -in the case of an infinite sequence $\{\lambda_j\}$ and its finite sections $\{\lambda_j\}_{j=1}^n$ - in order to obtain a Carleson type interpolation theorem $X|_\sigma = Y|_\sigma$, but not necessarily. In particular, even the classical Nevanlinna-Pick theorem (giving a necessary and sufficient condition on a function a for the existence of $f \in H^\infty$ such that $\|f\|_\infty \leq 1$ and $f(\lambda) = a_\lambda$, $\lambda \in \sigma$), does not lead immediately to Carleson's criterion for $H|_\sigma^\infty = l^\infty(\sigma)$. (Finally, a direct deduction of Carleson's theorem from Pick's result was done by P. Koosis [12] in 1999 only). Similarly, the problem stated for $c(\sigma, X, Y)$ is of interest in its own. It is a kind of "effective interpolation" because we are looking for sharp estimates or a computation of $c(\sigma, X, Y)$ for a variety of norms $\|\cdot\|_X$, $\|\cdot\|_Y$.

1.2. Motivations.

a. As it is mentioned in Subsection 1.1, one of the most interesting cases is $Y = H^\infty$. In this case, the quantity $c(\sigma, X, H^\infty)$ has a meaning of an intermediate interpolation between the Carleson one (when $\|f\|_{X|_\sigma} \asymp \sup_{1 \leq i \leq n} |f(\lambda_i)|$) and the individual Nevanlinna-Pick interpolation (no conditions on f).

b. The following partial case was especially stimulating (which is a part of a more complicated question arising in an applied situation in [4, 5]): given a set $\sigma \subset \mathbb{D}$, how can one estimate $c(\sigma, H^2, H^\infty)$ in terms of $n = \text{card } \sigma$ and $\max_{\lambda \in \sigma} |\lambda| = r$ only? (Here, H^2 is the standard Hardy space of the disc \mathbb{D} and is defined below in Subsection 1.3).

c. There is a direct link between the constant $c(\sigma, X, Y)$ and numerical analysis. For example, in matrix analysis, it is of interest to bound the norm of an H^∞ -calculus $\|f(A)\| \leq c \|f\|_\infty$, $f \in H^\infty$, for a contraction A on an n -dimensional arbitrary Banach space, with a given spectrum $\sigma(A) \subset \sigma$. The best possible constant is $c = c(\sigma, H^\infty, W)$, so that

$$\begin{aligned} c(\sigma, H^\infty, W) &= \\ &= \sup_{\|f\|_\infty \leq 1} \sup \{ \|f(A)\| : A : (\mathbb{C}^n, |\cdot|) \rightarrow (\mathbb{C}^n, |\cdot|), \|A\| \leq 1, \sigma(A) \subset \sigma \}, \end{aligned}$$

where $W = \left\{ f = \sum_{k \geq 0} \hat{f}(k) z^k : \sum_{k \geq 0} |\hat{f}(k)| < \infty \right\}$ stands for the Wiener algebra, and the interior sup is taken over all contractions on n -dimensional Banach spaces. An estimate for $c(\sigma, H^\infty, W)$ is given in [16]. An interesting case occurs for f such that $f|_\sigma = \frac{1}{z}|_\sigma$ (estimates on condition numbers and the norm of inverses of $n \times n$ matrices) or $f|_\sigma = \frac{1}{\lambda - z}|_\sigma$ (for estimates on the norm of the resolvent of an $n \times n$ matrix). Notice that in the same spirit, the case $Y = B_{\infty,1}^0$ where $B_{\infty,1}^0$ is a Besov algebra presents an interesting case for the functional calculus of finite rank operators, in particular, those satisfying the so-called Ritt condition.

1.3. The spaces X and Y considered here. We systematically use the following conditions for the spaces X and Y ,

$$(P_1) \quad \text{Hol}((1 + \epsilon)\mathbb{D}) \text{ is continuously embedded into } Y \text{ for every } \epsilon > 0,$$

$$(P_2) \quad \text{Pol}_+ \subset X \text{ and } \text{Pol}_+ \text{ is dense in } X,$$

where Pol_+ stands for the set of all complex polynomials p , $p = \sum_{k=0}^N a_k z^k$,

$$(P_3) \quad [f \in X] \Rightarrow \left[z^n f \in X, \forall n \geq 0 \text{ and } \overline{\lim} \|z^n f\|^{\frac{1}{n}} \leq 1 \right],$$

$$(P_4) \quad [f \in X, \lambda \in \mathbb{D} \text{ and } f(\lambda) = 0] \Rightarrow \left[\frac{f}{z - \lambda} \in X \right].$$

Assuming X satisfies property (P_4) and $Y \subset X$, then the quantity $c(\sigma, X, Y)$ can be written as follows

$$c(\sigma, X, Y) = \sup_{\|f\|_X \leq 1} \inf \{ \|g\|_Y : g \in Y, g - f \in B_\sigma X \}.$$

General spaces X and Y satisfying $(P_i)_{1 \leq i \leq 4}$ are studied in Section 3. Then, we study special cases of such spaces: from Section 4 to the end of this paper, $Y = H^\infty$, but X may change from one section to another. In particular, in Sections 4 and 5, $X = H^p = H^p(\mathbb{D})$, $1 \leq p \leq \infty$ which are the standard Hardy spaces on the disc \mathbb{D} (see [15] Chapter 2) of all $f \in \text{Hol}(\mathbb{D})$ satisfying

$$\sup_{0 \leq r < 1} \left(\int_{\mathbb{T}} |f(rz)|^p dm(z) \right)^{1/p} < \infty,$$

m being the Lebesgue normalized measure on \mathbb{T} . From now on, if $f \in \text{Hol}(\mathbb{D})$ and $k \in \mathbb{N}$,

$\hat{f}(k)$ stands for the k^{th} Taylor coefficient of f .

For $p = 2$, an equivalent description of H^2 is

$$H^2 = \left\{ f = \sum_{k \geq 0} \hat{f}(k) z^k : \sum_{k \geq 0} |\hat{f}(k)|^2 < \infty \right\}.$$

We also study (see Section 6) the case $X = l_a^2(1/\sqrt{k+1})$, which is the Bergman space of all $f = \sum_{k \geq 0} \hat{f}(k) z^k$ satisfying

$$\sum_{k \geq 0} |\hat{f}(k)|^2 \frac{1}{k+1} < \infty.$$

This space is also given by: $X = L_a^2$, the space of holomorphic functions f on \mathbb{D} such that

$$\int_{\mathbb{D}} |f(z)|^2 dA < \infty,$$

where dA stands for the area measure.

2. RESULTS

We start studying general Banach spaces X and Y and give some sufficient conditions under which $C_{n,r}(X, Y) < \infty$. In particular, we prove the following fact.

Theorem 2.1. *Let X, Y be Banach spaces satisfying properties (P_i) , $i = 1, \dots, 4$. Then*

$$C_{n,r}(X, Y) < \infty,$$

for every $n \geq 1$ and $r \in [0, 1)$.

Next, we add the condition that X is a Hilbert space, and give in this case a general upper bound for the quantity $C_{n,r}(X, Y)$.

Theorem 2.2. *Let Y be a Banach space satisfying property (P_1) and $X = (H, (\cdot, \cdot)_H)$ a Hilbert space satisfying properties (P_i) for $i = 2, 3, 4$. We moreover suppose that for every $0 < r < 1$ there exists $\epsilon > 0$ such that $k_\lambda \in \text{Hol}((1 + \epsilon)\mathbb{D})$ for all $|\lambda| < r$, where k_λ stands for the reproducing kernel of X at point λ , and $\bar{\lambda} \mapsto k_\lambda$ is holomorphic on $|\lambda| < r$ as a $\text{Hol}((1 + \epsilon)\mathbb{D})$ -valued function. Let $\sigma = \{\lambda_1, \dots, \lambda_1, \lambda_2, \dots, \lambda_2, \dots, \lambda_t, \dots, \lambda_t\}$ be a sequence in \mathbb{D} , where λ_s are repeated according to their multiplicity m_s , $\sum_{s=1}^t m_s = n$. Then we have,*

i)

$$c(\sigma, H, Y) \leq \left(\sum_{k=1}^n \|\mathcal{E}_k\|_Y^2 \right)^{\frac{1}{2}},$$

where $(\mathcal{E}_k)_{1 \leq k \leq n}$ stands for the Gram-Schmidt orthogonalization (in the space H) of the sequence

$$k_{\lambda_1,0}, k_{\lambda_1,1}, \dots, k_{\lambda_1,m_1-1}, k_{\lambda_2,0}, k_{\lambda_2,1}, \dots, k_{\lambda_2,m_2-1}, \dots, k_{\lambda_t,0}, k_{\lambda_t,1}, \dots, k_{\lambda_t,m_t-1},$$

and $k_{\lambda,i} = \left(\frac{d}{d\lambda}\right)^i k_\lambda$, $i \in \mathbb{N}$.

ii) For the case $Y = H^\infty$, we have

$$c(\sigma, H, H^\infty) \leq \sup_{\zeta \in \mathbb{D}} \|P_{B_\sigma}^H k_\zeta\|_H,$$

where $P_{B_\sigma}^H = \sum_{k=1}^n (\cdot, \mathcal{E}_k)_H \mathcal{E}_k$ stands for the orthogonal projection of H onto $K_{B_\sigma}(H)$,

$$K_{B_\sigma}(H) = \text{span} (k_{\lambda_j,i} : 1 \leq i < m_j, j = 1, \dots, t).$$

After that, we deal with H^∞ interpolation ($Y = H^\infty$). For general Banach spaces (of analytic functions in \mathbb{D}) of moderate growth X , we formulate the following conjecture:

$$c_1 \varphi_X \left(1 - \frac{1-r}{n} \right) \leq C_{n,r}(X, H^\infty) \leq c_2 \varphi_X \left(1 - \frac{1-r}{n} \right),$$

where $\varphi_X(t)$, $0 \leq t < 1$ stands for the norm of the evaluation functional $f \mapsto f(t)$ on the space X . We prove this conjecture for $X = H^p, L_a^2$, $p \in [1, \infty)$ (defined

in Subsection 1.3). More precisely, we prove the following Theorems 2.3 and 2.4. Here and later on,

$$\sigma_{n,\lambda} = \underbrace{\{\lambda, \lambda, \dots, \lambda\}}_n,$$

is the one-point set of multiplicity n corresponding to $\lambda \in \mathbb{D}$.

Theorem 2.3. *Let $1 \leq p \leq \infty$, $n \geq 1$, $r \in [0, 1)$, and $|\lambda| \leq r$. We have,*

$$\frac{1}{32^{\frac{1}{p}}} \left(\frac{n}{1-|\lambda|} \right)^{\frac{1}{p}} \leq c(\sigma_{n,\lambda}, H^p, H^\infty) \leq C_{n,r}(H^p, H^\infty) \leq A_p \left(\frac{n}{1-r} \right)^{\frac{1}{p}},$$

where A_p is a constant depending only on p and the left-hand side inequality is proved only for $p \in 2\mathbb{Z}_+$. For $p = 2$, we have $A_2 = \sqrt{2}$.

Theorem 2.4. *Let $1 \leq p \leq \infty$, $n \geq 1$, $r \in [0, 1)$, and $|\lambda| \leq r$. We have,*

$$\frac{1}{32^{\frac{1}{p}}} \left(\frac{n}{1-|\lambda|} \right)^{\frac{1}{p}} \leq c(\sigma_{n,\lambda}, H^p, H^\infty) \leq C_{n,r}(H^p, H^\infty) \leq A_p \left(\frac{n}{1-r} \right)^{\frac{1}{p}},$$

where A_p is a constant depending only on p and the left-hand side inequality is proved only for $p \in 2\mathbb{Z}_+$. For $p = 2$, we have $A_2 = \sqrt{2}$.

The above Theorems 2.1, 2.3 and 2.4 were already announced in the note [19].

In order to prove (Theorem 2.1, Theorem 2.2 and) the right-hand side inequality of Theorem 2.3 and Theorem 2.4, given $f \in X$ and σ a finite subset of \mathbb{D} , we first use a linear interpolation:

$$f \mapsto \sum_{k=1}^n \langle f, e_k \rangle e_k,$$

where $\langle \cdot, \cdot \rangle$ means the Cauchy sesquilinear form $\langle h, g \rangle = \sum_{k \geq 0} \hat{h}(k) \overline{\hat{g}(k)}$, and $(e_k)_{1 \leq k \leq n}$ is the Malmquist basis (effectively constructible) of the space $K_B = H^2 \ominus BH^2$, which $B = B_\sigma$ (see N. Nikolski, [16] p. 117). Next, we use the complex interpolation between Banach spaces, (see H. Triebel [17] Theorem 1.9.3-(a) p.59). Among the technical tools used in order to find an upper bound for $\|\sum_{k=1}^n \langle f, e_k \rangle e_k\|_\infty$ (in terms of $\|f\|_X$), the most important one is a Bernstein-type inequality $\|f'\|_p \leq c_p \|B'\|_\infty \|f\|_p$ for a (rational) function f in the star-invariant subspace $K_B^p := H^p \cap \overline{BzH^p}$, $1 \leq p \leq \infty$ (for $p = 2$, $K_B^2 = K_B$), generated by a (finite) Blaschke product B , (K. Dyakonov [9, 10]). For $p = 2$, we give an alternative proof of the Bernstein-type estimate we need and the constant c_2 we obtain is slightly better, see Section 6.

The lower bound problem of Theorems 2.3 and 2.4 is treated in Section 7 by using the ‘‘worst’’ interpolation n -tuple $\sigma = \sigma_{n,\lambda}$, (the Carathodory-Schur type

interpolation). The “worst” interpolation data comes from the Dirichlet kernels $\sum_{k=0}^{n-1} z^k$ transplanted from the origin to λ . We notice that spaces $X = H^p, L^2_a$, $p \in [1, \infty)$ satisfy the condition $X \circ b_\lambda \subset X$ which makes the problem of upper and lower bound easier.

The paper is organized as follows. In Section 3 we prove Theorems 2.1 and 2.2. Sections 4 and 5 (resp. Section 6) are (resp. is) devoted to the proof of the upper estimate of Theorem 2.3 (resp. Theorem 2.4). In Section 7, we prove the lower bounds stated in Theorems 2.3 and 2.4. At the end of the paper, we shortly compare the method used in Sections 3, 4, 5 and 6 with those resulting from the Carleson free interpolation, see Section 8.

3. UPPER BOUNDS FOR $c(\sigma, X, Y)$

3.1. Banach spaces X, Y satisfying properties $(P_i)_{1 \leq i \leq 4}$. In this subsection, X and Y are Banach spaces which satisfy properties $(P_i)_{1 \leq i \leq 4}$. We prove Theorem 2.1 which shows that in this case our interpolation constant $c(\sigma, X, Y)$ is bounded by a quantity which depends only on $n = \text{card } \sigma$ and $r = \max_i |\lambda_i|$ (and of course on X and Y). In this generality, we cannot discuss the question of the sharpness of the bounds obtained. First, we prove the following lemma.

Lemma 3.1. *Under (P_2) , (P_3) and (P_4) , $B_\sigma X$ is a closed subspace of X and moreover if σ is a finite subset of \mathbb{D} ,*

$$B_\sigma X = \{f \in X : f(\lambda) = 0, \forall \lambda \in \sigma \text{ (including multiplicities)}\}.$$

PROOF. Since $X \subset \text{Hol}(\mathbb{D})$ continuously, and evaluation functionals $f \mapsto f(\lambda)$ and $f \mapsto f^{(k)}(\lambda)$, $k \in \mathbb{N}^*$, are continuous on $\text{Hol}(\mathbb{D})$, the subspace

$$M = \{f \in X : f(\lambda) = 0, \forall \lambda \in \sigma \text{ (including multiplicities)}\},$$

is closed in X .

On the other hand, $B_\sigma X \subset X$, and hence $B_\sigma X \subset M$. Indeed, properties (P_2) and (P_3) imply that $h.X \subset X$, for all $h \in \text{Hol}((1 + \epsilon)\mathbb{D})$ with $\epsilon > 0$. We can write $h = \sum_{k \geq 0} \widehat{h}(k)z^k$ with $|\widehat{h}(k)| \leq Cq^n$, $C > 0$ and $q < 1$. Then $\sum_{n \geq 0} \|\widehat{h}(k)z^k f\|_X < \infty$ for every $f \in X$. Since X is a Banach space we get $hf = \sum_{n \geq 0} \widehat{h}(k)z^k f \in X$.

In order to see that $M \subset B_\sigma X$, it suffices to justify that

$$[f \in X \text{ and } f(\lambda) = 0] \implies [f/b_\lambda = (1 - \bar{\lambda}z)f/(\lambda - z) \in X],$$

but this is obvious from (P_4) and the previous arguments. \square

In Definition 1 below, $\sigma = \{\lambda_1, \dots, \lambda_n\}$ is a sequence in the unit disc \mathbb{D} and B_σ is the corresponding Blaschke product.

Definition 1. *The model space K_{B_σ} .* We define K_{B_σ} to be the n -dimensional space:

$$(3.1.1) \quad K_{B_\sigma} = (B_\sigma H^2)^\perp = H^2 \ominus B_\sigma H^2.$$

Malmquist basis. For $k \in [1, n]$, we set $f_k(z) = \frac{1}{1-\lambda_k z}$, and define the family $(e_k)_{1 \leq k \leq n}$, (which is known as *Malmquist basis*, see [16] p.117), by

$$(3.1.2) \quad e_1 = \frac{f_1}{\|f_1\|_2} \quad \text{and} \quad e_k = \left(\prod_{j=1}^{k-1} b_{\lambda_j} \right) \frac{f_k}{\|f_k\|_2},$$

for $k \in [2, n]$, where $\|f_k\|_2 = (1 - |\lambda_k|^2)^{-1/2}$. The *Malmquist family* $(e_k)_{1 \leq k \leq n}$ corresponding to σ is an orthonormal basis of K_{B_σ} .

The orthogonal projection P_{B_σ} on K_{B_σ} . We define P_{B_σ} to be the orthogonal projection of H^2 on its n -dimensional subspace K_{B_σ} . In particular,

$$(3.1.3) \quad P_{B_\sigma} = \sum_{k=1}^n (\cdot, e_k)_{H^2} e_k,$$

where $(\cdot, \cdot)_{H^2}$ means the scalar product on H^2 .

Lemma 3.2. *Let $\sigma = \{\lambda_1, \dots, \lambda_n\}$ be a sequence in the unit disc \mathbb{D} , $(e_k)_{1 \leq k \leq n}$ be the Malmquist family corresponding to σ and $\langle \cdot, \cdot \rangle$ be the Cauchy sesquilinear form $\langle h, g \rangle = \sum_{j \geq 0} \hat{h}(j) \overline{\hat{g}(j)}$ for $h, g \in \text{Hol}(\mathbb{D})$. For all $f \in \text{Hol}(\mathbb{D})$ and $1 \leq k \leq n$, the series $\langle f, e_k \rangle$ are absolutely converging. Moreover, if Y is a Banach space satisfying (P_1) , the map $\Phi : \text{Hol}(\mathbb{D}) \rightarrow Y \subset \text{Hol}(\mathbb{D})$ given by*

$$\Phi : f \mapsto \sum_{k=1}^n \langle f, e_k \rangle e_k,$$

is well defined and has the following properties:

- (a) $\Phi|_{H^2} = P_{B_\sigma}$,
- (b) Φ is continuous on $\text{Hol}(\mathbb{D})$ with the topology of the uniform convergence on compact sets of \mathbb{D} ,
- (c) if X satisfies (P_2) , (P_3) , (P_4) and $Y \subset X$, and if $\Psi = \text{Id}|_X - \Phi|_X$, then $\text{Im}(\Psi) \subset B_\sigma X$,

(d) if $f \in \text{Hol}(\mathbb{D})$, then

$$|\Phi(f)(\zeta)| = |\langle f, P_{B_\sigma} k_\zeta \rangle| = \left| \left\langle f, \sum_{k=1}^n \overline{e_k(\zeta)} e_k \right\rangle \right|,$$

for all $\zeta \in \mathbb{D}$, where P_{B_σ} is defined in (3.1.2) and $k_\zeta = (1 - \bar{\zeta}z)^{-1}$.

PROOF. First of all we set $r = \max_{\lambda \in \sigma} |\lambda|$. If $f \in \text{Hol}(\mathbb{D})$ and $\rho \in]0, 1[$, then

$$\widehat{f}(j) = (2\pi)^{-1} \int_{\rho\mathbb{T}} f(w) w^{-j-1} dw,$$

for all $j \geq 0$. For a subset A of \mathbb{C} and for a bounded function h on A , we define $\|h\|_A := \sup_{z \in A} |h(z)|$. As a result,

$$(3.1.4) \quad |\langle f, e_k \rangle| \leq \sum_{j \geq 0} \left| \widehat{f}(j) \overline{\widehat{e}_k(j)} \right| \leq (2\pi\rho)^{-1} \|f\|_{\rho\mathbb{T}} \sum_{j \geq 0} |\widehat{e}_k(j)| \rho^{-j}.$$

Now if ρ is close enough to 1, it satisfies the inequality $1 \leq \rho^{-1} < r^{-1}$, which entails $\sum_{j \geq 0} |\widehat{e}_k(j)| \rho^{-j} < +\infty$ for each $k = 1, \dots, n$, and the series $\langle f, e_k \rangle$ are absolutely converging.

Now, the the point (a) is a direct consequence of (3.1.3). In order to check point (b), let $(f_i)_{i \in \mathbb{N}}$ be a sequence of $\text{Hol}(\mathbb{D})$ converging to 0 uniformly on compact sets of \mathbb{D} . We need to see that $(\Phi(f_i))_{i \in \mathbb{N}}$ converges to 0, for which it is sufficient to show that $\lim_l |\langle f_l, e_k \rangle| = 0$, for every $k = 1, 2, \dots, n$, which is clear applying (3.1.4) to $f = f_l$.

We now prove point (c). Using point (a), since $Pol_+ \subset H^2$, we get that $\text{Im}(\Psi|_{Pol_+}) \subset B_\sigma H^2$. Now, since $Pol_+ \subset Y$ and $\text{Im}(\Phi) \subset Y$, we deduce that

$$\text{Im}(\Psi|_{Pol_+}) \subset B_\sigma H^2 \cap Y \subset B_\sigma H^2 \cap X,$$

since $Y \subset X$. Now $\Psi(p) \in X$ and satisfies $(\Psi(p))|_\sigma = 0$ (that is to say $(\Psi(p))(\lambda) = 0, \forall \lambda \in \sigma$ (including multiplicities) for all $p \in Pol_+$). Using Lemma 3.1, we get that $\text{Im}(\Psi|_{Pol_+}) \subset B_\sigma X$. Now, Pol_+ being dense in X (property (P_2)), and Ψ being continuous on X (point (b)), we can conclude that $\text{Im}(\Psi) \subset B_\sigma X$.

In order to prove (d), we simply need to write that

$$\Phi(f)(\zeta) = \sum_{k=1}^n \langle f, e_k \rangle e_k(\zeta) = \left\langle f, \sum_{k=1}^n \overline{e_k(\zeta)} e_k \right\rangle,$$

$\forall f \in \text{Hol}(\mathbb{D}), \forall \zeta \in \mathbb{D}$ and to notice that $\sum_{k=1}^n \overline{e_k(\zeta)} e_k = \sum_{k=1}^n (k_\zeta, e_k)_{H^2} e_k = P_{B_\sigma} k_\zeta$. \square

PROOF OF THEOREM 2.1 . Let $\sigma = \{\lambda_1, \dots, \lambda_n\}$ be a sequence in the unit disc \mathbb{D} and $(e_k)_{1 \leq k \leq n}$ the Malmquist family (3.1.2) associated to σ . Taking $f \in X$, we set $g = \Phi(f) = \sum_{k=1}^n \langle f, e_k \rangle e_k$, where Φ is defined in Lemma 3.2. In the same spirit of that of (3.1.4), we notice that

$$\widehat{e}_k(j) = (2\pi i)^{-1} \int_{R\mathbb{T}} e_k(w) w^{-j-1} dw,$$

for all $j \geq 0$ and for all R , $1 < R < \frac{1}{r}$. As a result,

$$|\widehat{e}_k(j)| \leq (2\pi R^{j+1})^{-1} \|e_k\|_{R\mathbb{T}} \text{ and}$$

$$\sum_{j \geq 0} \left| \widehat{f}(j) \overline{\widehat{e}_k(j)} \right| \leq (2\pi R)^{-1} \|e_k\|_{R\mathbb{T}} \sum_{j \geq 0} \left| \widehat{f}(j) \right| R^{-j} < \infty,$$

since $R > 1$ and f is holomorphic in \mathbb{D} , (where $\|\cdot\|_A$ is defined above in the proof of Lemma 3.2).

We now suppose that $\|f\|_X \leq 1$. Since $\text{Hol}(r^{-1}\mathbb{D}) \subset Y$, we have $g \in Y$ and using Lemma 3.2 point (c) we get

$$f - g = \Psi(f) \in B_\sigma X,$$

where Ψ is defined in Lemma 3.2, as Φ . Moreover,

$$\|g\|_Y \leq \sum_{k=1}^n |\langle f, e_k \rangle| \|e_k\|_Y.$$

In order to bound the right-hand side, recall that for all $j \geq 0$ and for $R = 2/(r+1) \in]1, 1/r[$,

$$\sum_{j \geq 0} \left| \widehat{f}(j) \overline{\widehat{e}_k(j)} \right| \leq (2\pi)^{-1} \|e_k\|_{2(r+1)^{-1}\mathbb{T}} \sum_{j \geq 0} \left| \widehat{f}(j) \right| (2^{-1}(r+1))^j.$$

Since the norm $f \mapsto \sum_{j \geq 0} \left| \widehat{f}(j) \right| (2^{-1}(r+1))^j$ is continuous on $\text{Hol}(\mathbb{D})$, and the inclusion $X \subset \text{Hol}(\mathbb{D})$ is also continuous, there exists $C_r > 0$ such that

$$\sum_{j \geq 0} \left| \widehat{f}(j) \right| (2^{-1}(r+1))^j \leq C_r \|f\|_X,$$

for every $f \in X$. On the other hand, $\text{Hol}(2(r+1)^{-1}\mathbb{D}) \subset Y$ (continuous inclusion again), and hence there exists $K_r > 0$ such that

$$\|e_k\|_Y \leq K_r \sup_{|z| < 2(r+1)^{-1}} |e_k(z)| = K_r \|e_k\|_{2(r+1)^{-1}\mathbb{T}}.$$

It is more or less clear that the right-hand side of the last inequality can be bounded in terms of r and n only. Let us give a proof to this fact. It is clear that it suffices to estimate

$$\sup_{1 < |z| < 2(r+1)^{-1}} |e_k(z)|.$$

In order to bound this quantity, notice that

$$(3.1.5) \quad |b_\lambda(z)|^2 \leq \left| \frac{\lambda - z}{1 - \bar{\lambda}z} \right|^2 = 1 + \frac{(|z|^2 - 1)(1 - |\lambda|^2)}{|1 - \bar{\lambda}z|^2},$$

for all $\lambda \in \mathbb{D}$ and all $z \in |\lambda|^{-1}\mathbb{D}$. Using the identity (3.1.5) for $\lambda = \lambda_j$, $1 \leq j \leq n$, and $z = \rho e^{it}$, $\rho = 2(1+r)^{-1}$, we get

$$\begin{aligned} |e_k(\rho e^{it})|^2 &\leq \left(\prod_{j=1}^{k-1} |b_{\lambda_j}(\rho e^{it})|^2 \right) \left| \frac{1}{1 - \bar{\lambda}_k \rho e^{it}} \right|^2 \leq \\ &\leq \left(\prod_{j=1}^{k-1} \left(1 + \frac{(\rho^2 - 1)(1 - |\lambda_j|^2)}{1 - |\lambda_j|^2 \rho^2} \right) \right) \left(\frac{1}{1 - |\lambda_k| \rho} \right)^2, \end{aligned}$$

for all $k = 2, \dots, n$. Expressing ρ in terms of r , we obtain

$$\|e_k\|_{2(r+1)^{-1}\mathbb{T}} \leq \frac{1}{1 - \frac{2r}{r+1}} \sqrt{2 \left(\prod_{j=1}^{n-1} \left(1 + \frac{2(\frac{1}{r^2} - 1)}{1 - r^2 \frac{4}{(r+1)^2}} \right) \right)} =: C_1(r, n),$$

and

$$\sum_{j \geq 0} \left| \hat{f}(j) \overline{\hat{e}_k(j)} \right| \leq (2\pi)^{-1} C_r \|e_k\|_{2(r+1)^{-1}\mathbb{T}} \|f\|_X \leq (2\pi)^{-1} C_r C_1(r, n) \|f\|_X.$$

On the other hand, since

$$\|e_k\|_Y \leq K_r \|e_k\|_{2(r+1)^{-1}\mathbb{T}} \leq K_r C_1(r, n),$$

we get

$$\|g\|_Y \leq \sum_{k=1}^n (2\pi)^{-1} C_r C_1(r, n) \|f\|_X K_r C_1(r, n) = (2\pi)^{-1} n C_r K_r (C_1(r, n))^2 \|f\|_X,$$

which proves that

$$c(\sigma, X, Y) \leq (2\pi)^{-1} n C_r K_r (C_1(r, n))^2,$$

and completes the proof of Theorem 2.1. \square

3.2. The case where X is a Hilbert space. We suppose in this subsection that X is a Hilbert space and that X, Y satisfy properties $(P_i)_{1 \leq i \leq 4}$. We prove Theorem 2.2 and obtain a better estimate for $c(\sigma, X, Y)$ than in Theorem 2.1 (see point (i) of Theorem 2.2). For the case $Y = H^\infty$, (point (ii) of Theorem 2.2), we can considerably improve this estimate. We omit an easy proof of the following lemma.

Lemma 3.3. *Let $\sigma = \{\lambda_1, \dots, \lambda_1, \lambda_2, \dots, \lambda_2, \dots, \lambda_t, \dots, \lambda_t\}$ be a finite sequence of \mathbb{D} where every λ_s is repeated according to its multiplicity m_s , $\sum_{s=1}^t m_s = n$. Let $(H, (\cdot, \cdot)_H)$ be a Hilbert space continuously embedded into $\text{Hol}(\mathbb{D})$ and satisfying properties (P_i) for $i = 2, 3, 4$. Then*

$$K_{B_\sigma}(H) =: H \ominus B_\sigma H = \text{span}(k_{\lambda_j, i} : 1 \leq j \leq t, 0 \leq i \leq m_j - 1),$$

where $k_{\lambda, i} = \left(\frac{d}{d\lambda}\right)^i k_\lambda$ and k_λ is the reproducing kernel of H at point λ for every $\lambda \in \mathbb{D}$, i.e. $k_\lambda \in H$ and $f(\lambda) = (f, k_\lambda)_H, \forall f \in H$.

PROOF OF THEOREM 2.2 . i). Let $f \in X, \|f\|_X \leq 1$. Lemma 3.3 shows that

$$g = P_{B_\sigma}^H f = \sum_{k=1}^n (f, \mathcal{E}_k)_H \mathcal{E}_k$$

is the orthogonal projection of f onto subspace K_{B_σ} . Function g belongs to Y because all $k_{\lambda_j, i}$ are in $\text{Hol}((1 + \epsilon)\mathbb{D})$ for a convenient $\epsilon > 0$, and Y satisfies (P_1) .

On the other hand, $g - f \in B_\sigma H$ (again by Lemma 3.3). Moreover, using Cauchy-Schwarz inequality,

$$\begin{aligned} \|g\|_Y &\leq \sum_{k=1}^n |(f, \mathcal{E}_k)_H| \|\mathcal{E}_k\|_Y \leq \left(\sum_{k=1}^n |(f, \mathcal{E}_k)_H|^2 \right)^{1/2} \left(\sum_{k=1}^n \|\mathcal{E}_k\|_Y^2 \right)^{1/2} \leq \\ &\leq \|f\|_H \left(\sum_{k=1}^n \|\mathcal{E}_k\|_Y^2 \right)^{1/2}, \end{aligned}$$

which proves i).

ii). If $Y = H^\infty$, then

$$|g(\zeta)| = |(P_{B_\sigma}^H f, k_\zeta)_H| = |(f, P_{B_\sigma}^H k_\zeta)_H| \leq \|f\|_H \|P_{B_\sigma}^H k_\zeta\|_H,$$

for all $\zeta \in \mathbb{D}$, which proves ii). \square

4. UPPER BOUNDS FOR $C_{n,r}(H^2, H^\infty)$

Here, we specialize the upper estimate obtained in point (ii) of Theorem 2.2 for the case $X = H^2$, the Hardy space of the disc. Later on, we will see that this estimate is sharp at least for some special sequences σ (see Section 7). We also develop a slightly different approach to the interpolation constant $c(\sigma, H^2, H^\infty)$ giving more estimates for individual sequences $\sigma = \{\lambda_1, \dots, \lambda_n\}$ of \mathbb{D} . We finally prove the right-hand side inequality of Theorem 2.3 for the particular case $p = 2$.

Proposition 4.1. *For every sequence $\sigma = \{\lambda_1, \dots, \lambda_n\}$ of \mathbb{D} we have*

$$(4.1.1) \quad c(\sigma, H^2, H^\infty) \leq \sup_{\zeta \in \mathbb{D}} \left(\frac{1 - |B_\sigma(\zeta)|^2}{1 - |\zeta|^2} \right)^{1/2},$$

and therefore

$$(4.1.2) \quad c(\sigma, H^2, H^\infty) \leq \sqrt{2} \sup_{|\zeta|=1} |B'(\zeta)|^{\frac{1}{2}} = \sqrt{2} \sup_{|\zeta|=1} \left| \sum_{i=1}^n \frac{1 - |\lambda_i|^2}{(1 - \bar{\lambda}_i \zeta)^2} \frac{B_\sigma(\zeta)}{b_{\lambda_i}(\zeta)} \right|^{1/2}.$$

PROOF. We prove (4.1.1). In order to simplify the notation, we set $B = B_\sigma$. Applying point (ii) of Theorem 2.2 for $X = H^2$ and $Y = H^\infty$, and using

$$k_\zeta(z) = \frac{1}{1 - \bar{\zeta}z} \quad \text{and} \quad (P_{B_\sigma} k_\zeta)(z) = \frac{1 - \overline{B_\sigma(\zeta)} B_\sigma(z)}{1 - \bar{\zeta}z},$$

(see [16] p.199), we obtain

$$\|P_{B_\sigma} k_\zeta\|_{H^2} = \left(\frac{1 - |B_\sigma(\zeta)|^2}{1 - |\zeta|^2} \right)^{1/2},$$

which gives the result.

We now prove (4.1.2) using (4.1.1). The map

$$\zeta \mapsto \|P_B(k_\zeta)\| = \sup \{|f(\zeta)| : f \in K_B, \|f\| \leq 1\},$$

and hence the map

$$\zeta \mapsto \left(\frac{1 - |B(\zeta)|^2}{1 - |\zeta|^2} \right)^{1/2},$$

is a subharmonic function so

$$\sup_{|\zeta| < 1} \left(\frac{1 - |B(\zeta)|^2}{1 - |\zeta|^2} \right)^{1/2} \leq \sup_{|w|=1} \lim_{r \rightarrow 1} \left(\frac{1 - |B(rw)|^2}{1 - |rw|^2} \right)^{1/2}.$$

Now applying Taylor's Formula of order 1 at points $w \in \mathbb{T}$ and $u = rw$, $0 < r < 1$ (it is applicable because B is holomorphic at every point of \mathbb{T}), we get

$$(B(u) - B(w))(u - w)^{-1} = B'(w) + o(1),$$

and since $|u - w| = 1 - |u|$,

$$|(B(u) - B(w))(u - w)^{-1}| = |B(u) - B(w)|(1 - |u|)^{-1} = |B'(w) + o(1)|.$$

Then we have

$$|B(u) - B(w)| \geq |B(w)| - |B(u)| = 1 - |B(u)|,$$

$$(1 - |B(u)|)(1 - |u|)^{-1} \leq (1 - |u|)^{-1} |B(u) - B(w)| = |B'(w) + o(1)|,$$

and

$$\lim_{r \rightarrow 1} ((1 - |B(rw)|)(1 - |rw|)^{-1})^{\frac{1}{2}} \leq \sqrt{|B'(w)|}.$$

Moreover,

$$B'(w) = - \sum_{i=1}^n (1 - |\lambda_i|^2) (1 - \bar{\lambda}_i w)^{-2} \prod_{j=1, j \neq i}^n b_{\lambda_j}(w),$$

for all $w \in \mathbb{T}$. This completes the proof since

$$\frac{1 - |B(rw)|^2}{1 - |rw|^2} = \frac{(1 - |B(rw)|)(1 + |B(rw)|)}{(1 - |rw|)(1 + |rw|)} \leq 2 \frac{1 - |B(rw)|}{1 - |rw|}.$$

□

Corollary 4.2. *Let $n \geq 1$ and $r \in [0, 1[$. Then,*

$$C_{n,r}(H^2, H^\infty) \leq 2(n(1-r)^{-1})^{\frac{1}{2}}.$$

PROOF. Indeed, applying Proposition 4.1 we obtain

$$|B'(w)| \leq \left| \sum_{i=1}^n \frac{1 - |\lambda_i|^2}{(1 - |\lambda_i|)^2} \right| \leq n \frac{1+r}{1-r} \leq \frac{2n}{1-r}.$$

□

Now, we develop a slightly different approach to the interpolation constant $c(\sigma, H^2, H^\infty)$.

Corollary 4.3. *For every sequence $\sigma = \{\lambda_1, \dots, \lambda_n\}$ of \mathbb{D} ,*

$$c(\sigma, H^2, H^\infty) \leq \sup_{z \in \mathbb{T}} \left(\sum_{k=1}^n \frac{(1 - |\lambda_k|^2)}{|z - \lambda_k|^2} \right)^{1/2}.$$

PROOF. In order to simplify the notation, we set $B = B_\sigma$. We consider K_B (see Definition 1) and the Malmquist family $(e_k)_{1 \leq k \leq n}$ corresponding to σ (see Definition 1). Now, let $f \in H^2$ and

$$g = P_B f = \sum_{k=1}^n (f, e_k)_{H^2} e_k,$$

(see (3.1.3)). The function g belongs to H^∞ (it is a finite sum of H^∞ functions) and satisfies $g - f \in BH^2$. Applying Cauchy-Schwarz inequality we get

$$|g(\zeta)| \leq \sum_{k=1}^n |(f, e_k)_{H^2}| |e_k(\zeta)| \leq \left(\sum_{k=1}^n |(f, e_k)_{H^2}|^2 \right)^{1/2} \left(\sum_{k=1}^n \frac{(1 - |\lambda_k|^2)}{|1 - \lambda_k \zeta|^2} \right)^{1/2},$$

for all $\zeta \in \mathbb{D}$. As a result, since f is an arbitrary H^2 function, we obtain

$$c(\sigma, H^2, H^\infty) \leq \sup_{\zeta \in \mathbb{T}} \left(\sum_{k=1}^n \frac{(1 - |\lambda_k|^2)}{|\zeta - \lambda_k|^2} \right)^{1/2},$$

which completes the proof. \square

Corollary 4.4. *For any sequence $\sigma = \{\lambda_1, \dots, \lambda_n\}$ in \mathbb{D} ,*

$$c(\sigma, H^2, H^\infty) \leq \left(\sum_{j=1}^n \frac{1 + |\lambda_j|}{1 - |\lambda_j|} \right)^{1/2}.$$

PROOF. Indeed,

$$\sum_{k=1}^n \frac{(1 - |\lambda_k|^2)}{|\zeta - \lambda_k|^2} \leq \left(\sum_{k=1}^n \frac{(1 - |\lambda_k|^2)}{(1 - |\lambda_k|)^2} \right)^{1/2}$$

and the result follows using Theorem 4.3. \square

Now we prove the right-hand side inequality of Theorem 2.3 for the special case $p = 2$.

PROOF OF THEOREM 2.3 ($p = 2$, THE RIGHT-HAND SIDE INEQUALITY ONLY).

Since $1 + |\lambda_j| \leq 2$ and $1 - |\lambda_j| \geq 1 - r$ for all $j \in [1, n]$, applying Corollary 4.4 we get

$$C_{n,r}(H^2, H^\infty) \leq \sqrt{2n}^{1/2} (1 - r)^{-1/2}.$$

\square

Remark. As a result, we get once more the same estimate for $C_{n,r}(H^2, H^\infty)$ as in Corollary 4.2, with the constant $\sqrt{2}$ instead of 2.

It is natural to wonder if it is possible to improve the bound $\sqrt{2}n^{1/2}(1-r)^{-1/2}$. We return to this question in Section 7 below.

5. UPPER BOUNDS FOR $C_{n,r}(H^p, H^\infty)$, $p \geq 1$

In this section we extend Corollary 4.2 to all Hardy spaces H^p : we prove the right-hand side inequality of Theorem 2.3, $p \neq 2$. We first prove the following lemma.

Lemma 5.1. *Let $n \geq 1$ and $0 \leq r < 1$. Then,*

$$C_{n,r}(H^1, H^\infty) \leq 2n(1-r)^{-1}.$$

PROOF. Let σ be a finite subset of \mathbb{D} , $(e_k)_{1 \leq k \leq n}$ be the Malmquist basis corresponding to σ (see Definition 1), and $f \in \bar{H}^1$ such that $\|f\|_{H^1} \leq 1$. Let also $g = \Phi(f)$ where Φ is defined in Lemma 3.2. Applying point (d) of Lemma 3.2, we get

$$|g(\zeta)| \leq \|f\|_{H^1} \left\| \sum_{k=1}^n e_k \overline{e_k(\zeta)} \right\|_{H^\infty} \leq \left\| \sum_{k=1}^n e_k \overline{e_k(\zeta)} \right\|_{H^\infty}.$$

Since Blaschke factors have modulus 1 on the unit circle,

$$\|e_k\|_{H^\infty} \leq (1 + |\lambda_k|)^{1/2} (1 - |\lambda_k|)^{-1/2}.$$

As a consequence,

$$|g(\zeta)| \leq \sum_{k=1}^n \|e_k\|_{H^\infty} \left| \overline{e_k(\zeta)} \right| \leq \sum_{k=1}^n \|e_k\|_{H^\infty}^2 \leq \sum_{k=1}^n (1 + |\lambda_k|) (1 - |\lambda_k|)^{-1} \leq 2n(1-r)^{-1},$$

for all $\zeta \in \mathbb{D}$, which completes the proof. \square

Before proving the upper bound in Theorem 2.3, we give the following general observation which is a direct consequence of the classical complex interpolation between Banach spaces, see [6, 17]. In particular, we use the notation of [6], Chapter 4.

Lemma 5.2. *Let X_1 and X_2 be two Banach spaces of holomorphic functions in the unit disc \mathbb{D} . Let also $\theta \in [0, 1]$ and $(X_1, X_2)_{[\theta]}$ be the corresponding intermediate Banach space resulting from the classical complex interpolation method applied between X_1 and X_2 . Then,*

$$C_{n,r} \left((X_1, X_2)_{[\theta]}, H^\infty \right) \leq C_{n,r}(X_1, H^\infty)^{1-\theta} C_{n,r}(X_2, H^\infty)^\theta,$$

for all $n \geq 1$, $r \in [0, 1)$.

PROOF. Let X be a Banach space of holomorphic functions in the unit disc \mathbb{D} and let $\sigma = \{\lambda_1, \lambda_2, \dots, \lambda_n\} \subset \mathbb{D}$ be a finite subset of the disc. Let $T : X \rightarrow H^\infty/B_\sigma H^\infty$ be the restriction map defined by

$$Tf = \{g \in H^\infty : f - g \in B_\sigma X\},$$

for every $f \in X$. Then,

$$\|T\|_{X \rightarrow H^\infty/B_\sigma H^\infty} = c(\sigma, X, H^\infty).$$

Now, since $(X_1, X_2)_{[\theta]}$ is an exact interpolation space of exponent θ (see [6] or [17] Theorem 1.9.3-(a), p.59), we can complete the proof. \square

Now we prove the right-hand side inequality of Theorem 2.3 for the remaining case $p \neq 2$.

PROOF OF THEOREM 2.3 ($p \neq 2$, THE RIGHT-HAND SIDE INEQUALITY ONLY).

In the case $X = H^p$, there exists $0 \leq \theta \leq 1$ such that $1/p = 1 - \theta$, and $[H^1, H^\infty]_\theta = H^p$ (a topological identity: the spaces are the same and the norms are equivalent (up to constants depending on p only), see [11]). As a consequence, applying Lemma 5.2 with $X_1 = H^1$ and $X_2 = H^\infty$, we get

$$\begin{aligned} C_{n,r}(H^p, H^\infty) &\leq \gamma_p (C_{n,r}(H^1, H^\infty))^{1-\theta} (C_{n,r}(H^\infty, H^\infty))^\theta \leq \\ &\leq \gamma_p (2n(1-r)^{-1})^{1-\theta} (1)^\theta, \end{aligned}$$

where $\theta = 1 - 1/p$, and γ_p is a constant depending only on p . Using Lemma 5.1 and the fact that $C_{n,r}(H^\infty, H^\infty) \leq 1$, we get

$$C_{n,r}(H^p, H^\infty) \leq \gamma_p (2n(1-r)^{-1})^{1/p},$$

which completes the proof. \square

6. UPPER BOUNDS FOR $C_{n,r}(L_a^2, H^\infty)$

Our aim here is to generalize Corollary 4.2 to the case $X = l_a^2((k+1)^\alpha)$, $\alpha \in [-1, 0]$, the Hardy weighted spaces of all $f = \sum_{k \geq 0} \hat{f}(k)z^k$ satisfying

$$\|f\|_X^2 := \sum_{k \geq 0} |\hat{f}(k)|^2 (k+1)^{2\alpha} < \infty.$$

Note that $H^2 = l_a^2(1)$ and $L_a^2 = l_a^2\left((k+1)^{-\frac{1}{2}}\right)$. We prove the upper bound of Theorem 2.4. The main technical tool used here is a Bernstein-type inequality for rational functions.

6.1. Bernstein-type inequalities for rational functions. Bernstein-type inequalities for rational functions were the subject of a number of papers and monographs (see, for instance, [2-3, 7-8, 13]). Perhaps, the stronger and closer to ours (Proposition 6.1) of all known results are due to K. Dyakonov [9, 10]. First, we prove Proposition 6.1 below, which tells that if $\sigma = \{\lambda_1, \dots, \lambda_n\} \subset \mathbb{D}$, $r = \max_j |\lambda_j|$, $B = B_\sigma$ and $f \in K_B$, then

$$(\star) \quad \|f'\|_{H^2} \leq \alpha_{n,r} \|f\|_{H^2},$$

where $\alpha_{n,r}$ is a constant (explicitly given in Proposition 6.1) depending on n and r only such that $0 < \alpha_{n,r} \leq 3 \frac{n}{1-r}$. Proposition 6.1 is in fact a special case ($p = 2$) of a K. Dyakonov's result [9] for $p = 2$. It is important to recall that this result is in its turn, a generalization of M. Levin's inequality [13] corresponding to the case $p = \infty$. More precisely, it is proved in [9] that the norm $\|D\|_{K_B^p \rightarrow H^p}$ of the differentiation operator $Df = f'$ on the star-invariant subspace of the Hardy space H^p , $K_B^p := H^p \cap Bz\overline{H^p}$, (where the bar denotes complex conjugation) satisfies the following inequalities

$$c'_p \|B'\|_\infty \leq \|D\|_{K_B^p \rightarrow H^p} \leq c_p \|B'\|_\infty,$$

for every p , $1 \leq p \leq \infty$ where c_p and c'_p are positives constants depending on p only, B is a finite Blaschke product and $\|\cdot\|_\infty$ means the norm in $L^\infty(\mathbb{T})$. For the partial case considered in Proposition 6.1 below, our proof is different from [9, 10]. It is based on an elementary Hilbert space construction for an orthonormal basis in K_B and the constant c_2 obtained is slightly better. More precisely, it is proved in [9] that $c'_2 = \frac{1}{36c}$, $c_2 = \frac{36+c}{2\pi}$ and $c = 2\sqrt{3\pi}$ (as one can check easily (c is not evaluated in [9])). It implies an inequality of type (\star) (with a constant about 6.5 instead of 3).

In [18], we discuss the ‘‘asymptotic sharpness’’ of our constant $\alpha_{n,r}$. We find an inequality for

$$C_{n,r} = \sup \|D\|_{K_B \rightarrow H^2},$$

(sup is over all B with given $n = \deg B$ and $r = \max_{\lambda \in \sigma} |\lambda|$), which is asymptotically sharp as $n \rightarrow \infty$. Our result in [18] is that $\lim_{n \rightarrow \infty} \frac{C_{n,r}}{n} = \frac{1+r}{1-r}$ for every r , $0 \leq r < 1$.

Proposition 6.1. *Let $B = \prod_{j=1}^n b_{\lambda_j}$, be a finite Blaschke product (of order n), $r = \max_j |\lambda_j|$, and $f \in K_B = H^2 \ominus BH^2$. Then for every $n \geq 2$ and $r \in [0, 1)$,*

$$\|f'\|_{H^2} \leq \alpha_{n,r} \|f\|_{H^2},$$

where $\alpha_{n,r} = [1 + (1+r)(n-1) + \sqrt{n-2}] (1-r)^{-1}$, and in particular

$$\|f'\|_{H^2} \leq 3 \frac{n}{1-r} \|f\|_{H^2},$$

for all $n \geq 1$ and $r \in [0, 1)$.

PROOF. Using (3.1.3), $f = P_B f = \sum_{k=1}^n (f, e_k)_{H^2} e_k$, for all $f \in K_B$. Noticing that, e'_k

$$e'_k = \sum_{i=1}^{k-1} \frac{b'_{\lambda_i}}{b_{\lambda_i}} e_k + \overline{\lambda_k} \frac{1}{(1 - \overline{\lambda_k} z)} e_k,$$

for $k \in [2, n]$, we get

$$\begin{aligned} f' &= \sum_{k=2}^n (f, e_k)_{H^2} \sum_{i=1}^{k-1} \frac{b'_{\lambda_i}}{b_{\lambda_i}} e_k + \sum_{k=1}^n (f, e_k)_{H^2} \overline{\lambda_k} \frac{1}{(1 - \overline{\lambda_k} z)} e_k, \\ &= \sum_{i=1}^n \frac{b'_{\lambda_i}}{b_{\lambda_i}} \sum_{k=i+1}^{n-1} (f, e_k)_{H^2} e_k + \sum_{k=1}^n (f, e_k)_{H^2} \overline{\lambda_k} \frac{1}{(1 - \overline{\lambda_k} z)} e_k. \end{aligned}$$

Now using Cauchy-Schwarz inequality and the fact that e_k is a vector of norm 1 in H^2 for $k = 1, \dots, n$, we get

$$\begin{aligned} \left\| \sum_{k=2}^n \overline{\lambda_k} (f, e_k)_{H^2} \frac{1}{(1 - \overline{\lambda_k} z)} e_k \right\|_{H^2} &\leq \sum_{k=1}^n |(f, e_k)_{H^2}| \left\| \overline{\lambda_k} \frac{1}{(1 - \overline{\lambda_k} z)} \right\|_{\infty} \|e_k\|_{H^2} \leq \\ &\leq \frac{1}{1-r} \sum_{k=1}^n |(f, e_k)_{H^2}| \leq \frac{1}{1-r} \left(\sum_{k=1}^n |(f, e_k)_{H^2}|^2 \right)^{\frac{1}{2}} \sqrt{n} \leq \frac{1}{1-r} \|f\|_{H^2} \sqrt{n}. \end{aligned}$$

Further,

$$\begin{aligned} \left\| \sum_{i=1}^{n-1} \frac{b'_{\lambda_i}}{b_{\lambda_i}} \sum_{k=i+1}^n e_k (f, e_k)_{H^2} \right\|_{H^2} &\leq \sum_{i=1}^{n-1} \left\| \frac{b'_{\lambda_i}}{b_{\lambda_i}} \right\|_{\infty} \left\| \sum_{k=i+1}^n (f, e_k)_{H^2} e_k \right\|_{H^2} = \\ &= \left(\max_{1 \leq i \leq n-1} \left\| \frac{b'_{\lambda_i}}{b_{\lambda_i}} \right\|_{\infty} \right) \sum_{i=1}^{n-1} \left(\sum_{k=i+1}^n |(f, e_k)_{H^2}|^2 \right)^{\frac{1}{2}} \leq \max_i \left\| \frac{b'_{\lambda_i}}{b_{\lambda_i}} \right\|_{\infty} \sum_{i=1}^{n-1} \|f\|_{H^2}. \end{aligned}$$

Now, using

$$\left\| \frac{b'_{\lambda_i}}{b_{\lambda_i}} \right\|_{\infty} = \left\| \frac{|\lambda_i|^2 - 1}{(1 - \overline{\lambda_i} z)(\lambda_i - z)} \right\|_{\infty} \leq \frac{1 + |\lambda_i|}{1 - |\lambda_i|} \leq \frac{1+r}{1-r},$$

we get

$$\left\| \sum_{i=1}^{n-1} \frac{b'_{\lambda_i}}{b_{\lambda_i}} \sum_{k=i+1}^n (f, e_k)_{H^2} e_k \right\|_{H^2} \leq (1+r) \frac{n-1}{1-r} \|f\|_{H^2}.$$

Finally,

$$\|f'\|_{H^2} \leq [1 + (1+r)(n-1) + \sqrt{n-2}] (1-r)^{-1} \|f\|_{H^2}.$$

In particular,

$$\|f'\|_{H^2} \leq (2n-2 + \sqrt{n}) (1-r)^{-1} \|f\|_{H^2} \leq 3n(1-r)^{-1} \|f\|_{H^2},$$

for all $n \geq 1$ and for every $f \in K_B$. \square

6.2. An upper bound for $c(\sigma, L_a^2, H^\infty)$. Here, we apply Proposition 6.1 to prove the right-hand side inequality of Theorem 2.4. We first prove the following corollary.

Corollary 6.2. *Let σ be a sequence in \mathbb{D} . Then,*

$$c(\sigma, l_a^2((k+1)^{-1}), H^\infty) \leq 2\sqrt{10} (n(1-r)^{-1})^{3/2}.$$

PROOF. Indeed, let $X = l_a^2(-1)$, σ a finite subset of \mathbb{D} and $B = B_\sigma$. If $f \in X$, then using part (c) of Lemma 3.2, $f|_\sigma$ we get that $\Phi(f)|_\sigma = f|_\sigma$. Now, denoting X^* the dual of X with respect to the Cauchy pairing $\langle \cdot, \cdot \rangle$ (defined in Lemma 3.2) and applying point (d) of the same lemma, we obtain $X^* = l_a^2(1)$ and

$$|\Phi(f)(\zeta)| \leq \|f\|_X \|P_B k_\zeta\|_{X^*} \leq \|f\|_X K \left(\|P_B k_\zeta\|_{H^2}^2 + \|(P_B k_\zeta)'\|_{H^2}^2 \right)^{\frac{1}{2}},$$

where

$$K = \max \left\{ 1, \sup_{k \geq 1} (k+1)k^{-1} \right\} = 2.$$

Since $P_B k_\zeta \in K_B$, Proposition 6.1 implies

$$\begin{aligned} |\Phi(f)(\zeta)| &\leq \|f\|_X K \left(\|P_B k_\zeta\|_{H^2}^2 + 9(n(1-r)^{-1})^2 \|P_B k_\zeta\|_{H^2}^2 \right)^{\frac{1}{2}} \leq \\ &\leq A (n(1-r)^{-1})^{3/2} \|f\|_X, \end{aligned}$$

where $A = K(2/2 + 9)^{\frac{1}{2}} = 2\sqrt{10}$, since $\|P_B k_\zeta\|_2 \leq \sqrt{2} (n(1-r)^{-1})^{1/2}$, and since we can suppose $n \geq 2$, (the case $n = 1$ being obvious). \square

Now let us give the proof of the right-hand side inequality of Theorem 2.4.

PROOF OF THEOREM 2.4 (THE RIGHT-HAND SIDE INEQUALITY ONLY). The case $\alpha = 0$ corresponds to $X = H^2$ and has already been studied in Section 3 (we can choose $A(0) = \sqrt{2}$). We now suppose $\alpha \in [-1, 0)$ and we set $\theta = -\alpha$, $0 \leq \theta \leq 1$, so that $\alpha = (1-\theta)(1-N) + \theta(-N)$. Since

$$(l_a^2(0), l_a^2(-1))_{[\theta]} = l_a^2(\alpha),$$

(see [6, 17]), this gives, using Lemma 5.2 with $X_1 = l_a^2(0) = H^2$ and $X_2 = l_a^2(-1)$, Corollary 6.2, that

$$\begin{aligned} C_{n,r}(l_a^2(\alpha), H^\infty) &\leq \left(\sqrt{2}(n(1-r)^{-1})^{\frac{1}{2}}\right)^{1-\theta} \left(2\sqrt{10}(n(1-r)^{-1})^{\frac{3}{2}}\right)^\theta = \\ &= A(0)^{1-\theta} A(1)^\theta (n(1-r)^{-1})^{\frac{1-\theta}{2} + \frac{3\theta}{2}}. \end{aligned}$$

It remains to use $\theta = -\alpha$, set $A(\alpha) = A(0)^{1-\theta} A(1)^\theta$ and apply (5.1.2) with $X = l_a^2((k+1)^\alpha)$. In particular, for $\alpha = -1/2$ we get $(1-\theta)/2 + 3\theta/2 = 1$ and

$$A(-1/2) = A(0)^{(1-1/2)} A(1)^{1/2} = \sqrt{2}^{1/2} (2\sqrt{10})^{1/2} = \sqrt{2} \cdot 10^{\frac{1}{4}}.$$

□

7. LOWER BOUNDS FOR $C_{n,r}(X, H^\infty)$

7.1. The cases $X = H^2$ and $X = L_a^2$. Here, we consider the standard Hardy and Bergman spaces, $X = H^2 = l_a^2(1)$ and $X = L_a^2 = l_a^2((k+1)^{-1/2})$, where the spaces $l_a^2((k+1)^\alpha)$ are defined in Section 6, and the problem of lower estimates for the one-point special case $\sigma_{n,\lambda} = \underbrace{\{\lambda, \lambda, \dots, \lambda\}}_n$, $\lambda \in \mathbb{D}$. Recall the definition of our constrained interpolation constant for this case

$$c(\sigma_{n,\lambda}, H, H^\infty) = \sup \left\{ \|f\|_{H^\infty/b_\lambda^n H^\infty} : f \in H, \|f\|_H \leq 1 \right\},$$

where $\|f\|_{H^\infty/b_\lambda^n H^\infty} = \inf \{ \|f + b_\lambda^n g\|_\infty : g \in H \}$. Our goal in this subsection is to prove the sharpness of the upper estimate stated in Theorem 2.3 ($p = 2$) and in Theorem 2.4 for the quantities $C_{n,r}(H^2, H^\infty)$ and $C_{n,r}(L_a^2, H^\infty)$, that is to say, to get the lower bounds of Theorem 2.3 ($p = 2$) and Theorem 2.4.

In the proof, we use properties of reproducing kernel Hilbert space on the disc \mathbb{D} , see for example [15]. Let us recall some of them adapting the general setting to special cases $X = l_a^2((k+1)^\alpha)$. The reproducing kernel of $l_a^2((k+1)^\alpha)$, by definition, is a $l_a^2((k+1)^\alpha)$ -valued function $\lambda \mapsto k_\lambda^\alpha$, $\lambda \in \mathbb{D}$, such that $(f, k_\lambda^\alpha) = f(\lambda)$ for every $f \in l_a^2((k+1)^\alpha)$, where (\cdot, \cdot) means the scalar product $(h, g) = \sum_{k \geq 0} \hat{h}(k) \overline{\hat{g}(k)} (k+1)^{-2\alpha}$. Since one has $f(\lambda) = \sum_{k \geq 0} \hat{f}(k) \lambda^k (k+1)^{2\alpha} (k+1)^{-2\alpha}$ ($\lambda \in \mathbb{D}$), it follows that

$$k_\lambda^\alpha(z) = \sum_{k \geq 0} (k+1)^{2\alpha} \overline{\lambda^k} z^k, \quad z \in \mathbb{D}.$$

In particular, for the Hardy space $H^2 = l_a^2(1)$ ($\alpha = 0$), we get the Szeg kernel $k_\lambda(z) = (1 - \overline{\lambda}z)^{-1}$ and for the Bergman space $L_a^2 = l_a^2((k+1)^{-1/2})$ ($\alpha = -1/2$), the Bergman kernel $k_\lambda^{-1/2}(z) = (1 - \overline{\lambda}z)^{-2}$.

We will use the previous observations for the following composed reproducing kernels (Aronszajn-deBranges, see [1, 15]): given the reproducing kernel k of H^2 and $\varphi \in \{z^N : N = 1, 2\}$, the function $\varphi \circ k$ is also positive definite and the corresponding Hilbert space is

$$H_\varphi = \varphi(H^2) = l_a^2 \left((k+1)^{\frac{1-N}{2}} \right).$$

It satisfies the following property: for every $f \in H^2$, $\varphi \circ f \in \varphi(H^2)$ and $\|\varphi \circ f\|_{\varphi(H^2)}^2 \leq \varphi(\|f\|_{H^2}^2)$ (see [15], page 320).

We notice in particular that

$$(7.1.1) \quad H_z = H^2 \text{ and } H_{z^2} = L_a^2.$$

The above relation between the spaces H^2 , L_a^2 and the spaces $\varphi(H^2) = H_\varphi$ leads to establish the proof of the left-hand side inequalities stated in Theorem 2.3 (for $p = 2$ only) and in Theorem 2.4 .

PROOF OF THEOREM 2.3 ($p = 2$) AND THEOREM 2.4. (left-hand side inequalities only)

0) For $N = 1, 2$ we set $\varphi_N(z) = z^N$.

1) We set

$$Q_n = \sum_{k=0}^{n-1} (1 - |\lambda|^2)^{1/2} b_\lambda^k (1 - \bar{\lambda}z)^{-1}, \quad H_{n,N} = \varphi_N \circ Q_n \quad \text{and} \quad \Psi = bH_{n,N}, \quad b > 0.$$

Then $\|Q_n\|_2^2 = n$, and hence by the above Aronszajn-deBranges inequality,

$$\|\Psi\|_{H_\varphi}^2 \leq b^2 \varphi_N (\|Q_n\|_2^2) = b^2 \varphi_N(n).$$

Let $b > 0$ such that $b^2 \varphi_N(n) = 1$.

2) Since the spaces H_{φ_N} and H^∞ are rotation invariant, we have

$$c(\sigma_{n,\lambda}, H_{\varphi_N}, H^\infty) = c(\sigma_{\mu,n}, H_{\varphi_N}, H^\infty)$$

for every λ, μ with $|\lambda| = |\mu| = r$. Let $\lambda = -r$. To get a lower estimate for $\|\Psi\|_{H_{\varphi_N}/b_\lambda^n H_{\varphi_N}}$ consider G such that $\Psi - G \in b_\lambda^n \text{Hol}(\mathbb{D})$, i.e. such that $bH_{n,N} \circ b_\lambda - G \circ b_\lambda \in z^n \text{Hol}(\mathbb{D})$.

3) First, we show that

$$\psi =: \Psi \circ b_\lambda = bH_{n,N} \circ b_\lambda$$

is a polynomial (of degree n if $\varphi = z$ and $2n$ if $\varphi = z^2$) with positive coefficients. Note that

$$\begin{aligned} Q_n \circ b_\lambda &= \sum_{k=0}^{n-1} z^k \frac{(1 - |\lambda|^2)^{1/2}}{1 - \bar{\lambda} b_\lambda(z)} = (1 - |\lambda|^2)^{-\frac{1}{2}} \left(1 + (1 - \bar{\lambda}) \sum_{k=1}^{n-1} z^k - \bar{\lambda} z^n \right) = \\ &= (1 - r^2)^{-1/2} \left(1 + (1 + r) \sum_{k=1}^{n-1} z^k + r z^n \right) =: (1 - r^2)^{-1/2} \psi_1. \end{aligned}$$

Hence, $\psi = \Psi \circ b_\lambda = bH_{n,N} \circ b_\lambda = b\varphi_N \circ \left((1 - r^2)^{-\frac{1}{2}} \psi_1 \right)$ and

$$\varphi_N \circ \psi_1 = \psi_1^N(z), \quad N = 1, 2.$$

4) Next, we show that

$$\sum_{j=0}^m (\psi) =: \sum_{j=0}^m \hat{\psi}(j) \geq \begin{cases} (2\sqrt{2})^{-1} \sqrt{n(1-r)^{-1}} & \text{if } N = 1 \\ 16^{-1} n (1-r)^{-1} & \text{if } N = 2 \end{cases},$$

where $m = n/2$ if n is even and $m = (n+1)/2$ if n is odd.

Indeed, setting $S_n = \sum_{j=0}^n z^j$, we have both for $N = 1$ and $N = 2$

$$\sum_{j=0}^m (\psi_1^N) = \sum_{j=0}^m \left(\left(1 + (1+r) \sum_{t=1}^{n-1} z^t + r z^n \right)^N \right) \geq \sum_{j=0}^m (S_{n-1}^N).$$

Next, we obtain

$$\begin{aligned} \sum_{j=0}^m (S_{n-1}^N) &= \sum_{j=0}^m \left(\left(\frac{1 - z^n}{1 - z} \right)^N \right) = \\ &= \sum_{j=0}^m ((1-z)^{-N}) = \frac{1}{(N-1)!} \sum_{j=0}^m \left(\frac{d^{N-1}}{dz^{N-1}} \frac{1}{1-z} \right) = \sum_{j=0}^m C_{N+j-1}^j. \end{aligned}$$

Now if $N = 1$, then

$$\sum_{j=0}^m C_{N+j-1}^j = m+1 \geq \frac{n}{2},$$

whereas if $N = 2$ then

$$\sum_{j=0}^m C_{N+j-1}^j = \frac{(m+1)(m+2)}{2} \geq \frac{(n+2)(n+4)}{8} \geq \frac{n^2}{8}.$$

Finally, since $\sum^m (\psi) = b \sum^m (\varphi_N \circ \psi_1) = b (1 - r^2)^{-N/2} \sum^m (\psi_1^N)$ we get

$$\sum^m (\psi) \geq \begin{cases} (2(1-r))^{-1/2} nb/2 & \text{if } N = 1 \\ (2(1-r))^{-1} n^2 b/8 & \text{if } N = 2 \end{cases},$$

with $b = \varphi_N(n) = \begin{cases} n^{-1/2} & \text{if } N = 1 \\ n^{-1} & \text{if } N = 2 \end{cases}$. This gives the result claimed. 5)

Let $F_n = \Phi_m + z^m \Phi_m$, where Φ_k stands for the k -th Fejer kernel. We have $\|g\|_\infty \|F_n\|_{L^1} \geq \|g * F_n\|_\infty$ for every $g \in L^\infty(\mathbb{T})$, and taking the infimum over all $g \in H^\infty$ satisfying $\hat{g}(k) = \hat{\psi}(k)$, $\forall k \in [0, n-1]$, we obtain

$$\|\psi\|_{H^\infty/z^n H^\infty} \geq \frac{1}{2} \|\psi * F_n\|_\infty,$$

where $*$ stands for the usual convolution product. Now using part 4),

$$\begin{aligned} \|\Psi\|_{H^\infty/b_\lambda^2 H^\infty} &= \|\psi\|_{H^\infty/z^n H^\infty} \geq \frac{1}{2} \|\psi * F_n\|_\infty \geq \\ &\geq \frac{1}{2} |(\psi * F_n)(1)| \geq \frac{1}{2} \sum_{j=0}^m \hat{\psi}(j) \geq \\ &\geq \begin{cases} (4\sqrt{2})^{-1} \sqrt{n(1-r)^{-1}} & \text{if } N = 1 \\ 32^{-1} n(1-r)^{-1} & \text{if } N = 2 \end{cases}. \end{aligned}$$

6) In order to conclude, it remains to use (7.1.1). \square

7.2. The case $X = H^p$, $1 \leq p \leq +\infty$. Here we prove the sharpness (for even p) of the upper estimate found in Theorem 2.3. We first prove the following lemma.

Lemma 7.1. *Let p, q such that $\frac{p}{q} \in \mathbb{Z}_+$, then $c(\sigma, H^p, H^\infty) \geq c(\sigma, H^q, H^\infty)^{\frac{q}{p}}$ for every sequence σ of \mathbb{D} .*

PROOF. Step 1. Recalling that

$$c(\sigma, H^p, H^\infty) = \sup_{\|f\|_p \leq 1} \inf \{ \|g\|_\infty : g \in Y, g|_\sigma = f|_\sigma \},$$

we first prove that

$$c(\sigma, H^p, H^\infty) = \sup_{\|f\|_p \leq 1, f \text{ outer}} \inf \{ \|g\|_\infty : g \in Y, g|_\sigma = f|_\sigma \}.$$

Indeed, we clearly have the inequality

$$\sup_{\|f\|_p \leq 1, f \text{ outer}} \inf \{ \|g\|_\infty : g \in Y, g|_\sigma = f|_\sigma \} \leq c(\sigma, H^p, H^\infty),$$

and if the inequality were strict, that is to say

$$\sup_{\|f\|_p \leq 1, f \text{ outer}} \inf \{ \|g\|_\infty : g \in Y, g|_\sigma = f|_\sigma \} < \sup_{\|f\|_p \leq 1} \inf \{ \|g\|_\infty : g \in Y, g|_\sigma = f|_\sigma \},$$

then we could write that there exists $\epsilon > 0$ such that for every $f = f_i \cdot f_o \in H^p$ (where f_i stands for the inner function corresponding to f and f_o to the outer one) with $\|f\|_p \leq 1$ (which also implies that $\|f_o\|_p \leq 1$, since $\|f_o\|_p = \|f\|_p$),

there exists a function $g \in H^\infty$ verifying both $\|g\|_\infty \leq (1 - \epsilon)c(\sigma, H^p, H^\infty)$ and $g|_\sigma = f_o|_\sigma$. This entails that $f|_\sigma = (f_i g)|_\sigma$ and since $\|f_i g\|_\infty = \|g\|_\infty \leq (1 - \epsilon)c(\sigma, H^p, H^\infty)$, we get that $c(\sigma, H^p, H^\infty) \leq (1 - \epsilon)c(\sigma, H^p, H^\infty)$, which is a contradiction and proves the equality of Step 1.

Step 2. Using the result of Step 1, we get that $\forall \epsilon > 0$ there exists an outer function $f_o \in H^q$ with $\|f_o\|_q \leq 1$ and such that

$$\inf \{ \|g\|_\infty : g \in Y, g|_\sigma = f_o|_\sigma \} \geq c(\sigma, H^q, H^\infty) - \epsilon.$$

Now let $F = f_o^{\frac{q}{p}} \in H^p$, then $\|F\|_p^p = \|f_o\|_q^q \leq 1$. We suppose that there exists $g \in H^\infty$ such that $g|_\sigma = F|_\sigma$ with

$$\|g\|_\infty < (c(\sigma, H^q, H^\infty) - \epsilon)^{\frac{q}{p}}.$$

Then, since $g(\lambda_i) = F(\lambda_i) = f_o(\lambda_i)^{\frac{q}{p}}$ for all $i = 1, \dots, n$, we have $g(\lambda_i)^{\frac{p}{q}} = f_o(\lambda_i)$ and $g^{\frac{p}{q}} \in H^\infty$ since $\frac{p}{q} \in \mathbb{Z}_+$. We also have

$$\left\| g^{\frac{p}{q}} \right\|_\infty = \|g\|_\infty^{\frac{p}{q}} < (c(\sigma, H^q, H^\infty) - \epsilon)^{\frac{q}{p}},$$

which is a contradiction. As a result, we have

$$\|g\|_\infty \geq (c(\sigma, H^q, H^\infty) - \epsilon)^{\frac{q}{p}},$$

for all $g \in H^\infty$ such that $g|_\sigma = F|_\sigma$, which gives

$$c(\sigma, H^p, H^\infty) \geq (c(\sigma, H^q, H^\infty) - \epsilon)^{\frac{q}{p}},$$

and since that inequality is true for every $\epsilon > 0$, we get the result. \square

PROOF OF THEOREM 2.3 (THE LEFT-HAND SIDE INEQUALITY FOR $p \in 2\mathbb{N}$, $p > 2$ ONLY).

We first prove the lower estimate for $c(\sigma_{n, \lambda}, H^p, H^\infty)$. Writing $p = 2(p/2)$, we apply Lemma 7.1 with $q = 2$ and this gives

$$c(\sigma_{n, \lambda}, H^p, H^\infty) \geq c(\sigma_{n, \lambda}, H^2, H^\infty)^{\frac{2}{p}} \geq 32^{-\frac{1}{p}} (n(1 - |\lambda|)^{-1})^{\frac{2}{p}}$$

for all integer $n \geq 1$. The last inequality is a consequence of Theorem 2.3 (left-hand side inequality) for the particular case $p = 2$ which has been proved in Subsection 7.1. \square

8. COMPARING OUR RESULTS WITH CARLESON INTERPOLATION

Recall that given a (finite) set $\sigma = \{\lambda_1, \dots, \lambda_n\} \subset \mathbb{D}$, the Carleson interpolation constant $C_I(\sigma)$ is defined by

$$C_I(\sigma) = \sup_{\|a\|_{l^\infty} \leq 1} \inf (\|g\|_\infty : g \in H^\infty, g|_\sigma = a).$$

We introduce the evaluation functionals φ_λ for $\lambda \in \mathbb{D}$, as well as the evaluation of the derivatives $\varphi_{\lambda,s}$ ($s = 0, 1, \dots$)

$$\varphi_\lambda(f) = f(\lambda), \quad f \in X, \quad \text{and} \quad \varphi_{\lambda,s}(f) = f^{(s)}(\lambda), \quad f \in X.$$

Theorem 8.1. *Let X be a Banach space, $X \subset \text{Hol}(\mathbb{D})$, and $\sigma = \{\lambda_1, \dots, \lambda_n\}$ be a sequence of distinct points in the unit disc \mathbb{D} . We have,*

$$\max_{1 \leq i \leq n} \|\varphi_{\lambda_i}\| \leq c(\sigma, X, H^\infty) \leq C_I(\sigma) \max_{1 \leq i \leq n} \|\varphi_{\lambda_i}\|,$$

where $C_I(\sigma)$ stands for the Carleson interpolation constant.

Theorem 8.1 (see [20] for its proof) tells us that, for σ with a “reasonable” interpolation constant $C_I(\sigma)$, the quantity $c(\sigma, X, H^\infty)$ behaves as $\max_i \|\varphi_{\lambda_i}\|$. However, for “tight” sequences σ , the constant $C_I(\sigma)$ is so large that the estimate in question contains almost no information. On the other hand, an advantage of the estimate of Theorem 8.1 is that it does not contain $\text{card} \sigma = n$ explicitly. Therefore, for well-separated sequences σ , Theorem 8.1 should give a better estimate than those of Theorem 2.3 and Theorem 2.4 .

Now, how does the interpolation constant $C_I(\sigma)$ behave in terms of the characteristics r and n of σ ? We answer this question in [20] for some particular sequences σ . More precisely, we compare these quantities for the cases $X = H^2$, $X = L_a^2$ and for three geometrically simple configurations: *two-points sets* σ , *circular* and *radial* sequences σ .

Let us recall that our specific upper bounds in Theorems C and D are sharp over all n elements sequences σ . However, we give in [20] some very special *radial* and *circular* sequences σ such that the estimate of $c(\sigma, H^2, H^\infty)$ via the Carleson constant $C_I(\sigma)$ (using Theorem 8.1) is comparable with or better than the estimates of Theorem 2.3 (for $X = H^2$) and Theorem 2.4 (for $X = L_a^2$) . We also give some examples of *radial* and *circular* sequences but also of *two-points sets*, such that it is worse (i.e. for which our estimate is better). More specific *radial sequences* are studied in [20]: *sparse sequences*, *condensed sequences* and *long sequences*.

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