

# EFFECTIVE $H^\infty$ INTERPOLATION CONSTRAINED BY HARDY AND BERGMAN NORMS

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ABSTRACT. Given a finite subset  $\sigma$  of the unit disc  $\mathbb{D}$  and a holomorphic function  $f$  in  $\mathbb{D}$  belonging to a class  $X$ , we are looking for a function  $g$  in another class  $Y$  which satisfies  $g|_\sigma = f|_\sigma$  and is of minimal norm in  $Y$ . Then, we wish to compare  $\|g\|_Y$  with  $\|f\|_X$ . More precisely, we consider the interpolation constant  $c(\sigma, X, Y) = \sup_{f \in X, \|f\|_X \leq 1} \inf_{g|_\sigma = f|_\sigma} \|g\|_Y$ . When  $Y = H^\infty$ , our interpolation problem includes those of Nevanlinna-Pick (1916) and Caratheodory-Schur (1908). Moreover, Carleson's free interpolation problem (1958) can be interpreted in terms of the constant  $c(\sigma, X, H^\infty)$ . For  $Y = H^\infty$ ,  $X = H^p$  (the Hardy space) or  $X = L_a^2$  (the Bergman space), we obtain an upper bound for the constant  $c(\sigma, X, H^\infty)$  in terms of  $n = \text{card } \sigma$  and  $r = \max_{\lambda \in \sigma} |\lambda|$ . Our upper estimates are shown to be sharp with respect to  $n$  and  $r$ .

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## 1. INTRODUCTION

**1.1. Statement and historical context of the problem.** Let  $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$  be the unit disc of the complex plane and let  $\text{Hol}(\mathbb{D})$  be the space of holomorphic functions on  $\mathbb{D}$ . The problem considered is the following: given two Banach spaces  $X$  and  $Y$  of holomorphic functions on  $\mathbb{D}$ ,  $X, Y \subset \text{Hol}(\mathbb{D})$ , and a finite subset  $\sigma \subset \mathbb{D}$ , find the least norm interpolation by functions of the space  $Y$  for the traces  $f|_\sigma$  of functions of the space  $X$ , in the worst case of  $f$ . The case  $X \subset Y$  is of no interest, and so one can suppose that either  $Y \subset X$  or  $X, Y$  are incomparable.

More precisely, our problem is to compute or estimate the following interpolation constant

$$c(\sigma, X, Y) = \sup_{f \in X, \|f\|_X \leq 1} \inf \{ \|g\|_Y : g|_\sigma = f|_\sigma \}.$$

If  $r \in [0, 1)$  and  $n \geq 1$ , we also define

$$C_{n,r}(X, Y) = \sup \{ c(\sigma, X, Y) : \text{card } \sigma \leq n, |\lambda| \leq r, \forall \lambda \in \sigma \}.$$

Here and later on,  $H^\infty$  stands for the space (algebra) of bounded holomorphic functions on  $\mathbb{D}$  endowed with the norm  $\|f\|_\infty = \sup_{z \in \mathbb{D}} |f(z)|$ . The classical interpolation problems -those of Nevanlinna-Pick (1916) and Carathéodory-Schur (1908) (see [15] p.231 for these two problems) on the one hand, and Carleson's free interpolation (1958) (see [16] p.158) on the other hand- are of this nature and correspond to the case  $Y = H^\infty$ . Two first are "individual", in the sense that one looks simply to compute the norms  $\|f\|_{H^\infty|_\sigma}$  or  $\|f\|_{H^\infty/z^n H^\infty}$  for a given  $f$ , whereas the third one is to compare the norms  $\|a\|_{l^\infty(\sigma)} = \max_{\lambda \in \sigma} |a_\lambda|$  and

$$\inf (\|g\|_\infty, g(\lambda) = a_\lambda, \lambda \in \sigma),$$

or -in other words- to estimate the interpolation constant defined as

$$c(\sigma, l^\infty(\sigma), H^\infty) = \sup_{a \in l^\infty(\sigma), \|a\|_{l^\infty} \leq 1} \inf \{ \|g\|_\infty : g(\lambda) = a_\lambda, \lambda \in \sigma \},$$

where  $l^\infty(\sigma)$  is the space of bounded functions  $(a_\lambda)_{\lambda \in \sigma}$  on infinite sets  $\sigma$ , endowed with the norm  $\|\cdot\|_{l^\infty(\sigma)}$ .

Let us now explain why our problem includes those of Nevanlinna-Pick and Carathéodory-Schur.

(i) Nevanlinna-Pick interpolation problem.

Given  $\sigma = \{\lambda_1, \dots, \lambda_n\}$  a finite subset of  $\mathbb{D}$  and  $\mathcal{W} = \{w_1, \dots, w_n\}$  a finite subset of  $\mathbb{C}$ , find

$$\text{NP}_{\sigma, \mathcal{W}} = \inf \{ \|f\|_\infty : f(\lambda_i) = w_i, i = 1, \dots, n \}.$$

The classical answer of Pick is the following:

$$\text{NP}_{\sigma, \mathcal{W}} = \inf \left\{ c > 0 : \left( \frac{c^2 - \overline{w_i} w_j}{1 - \overline{\lambda_i} \lambda_j} \right)_{1 \leq i, j \leq n} \gg 0 \right\},$$

where for any  $n \times n$  matrix  $M$ ,  $M \gg 0$  means that  $M$  is positive definite.

(ii) Carathéodory-Schur interpolation problem.

Given  $\mathcal{W} = \{w_0, w_1, \dots, w_n\}$  a finite subset of  $\mathbb{C}$ , find

$$\text{CS}_{\mathcal{W}} = \inf \{ \|f\|_\infty : f(z) = w_0 + w_1 z + \dots + w_n z^n + \dots \}.$$

The classical answer of Schur is the following:

$$\text{CS}_{\mathcal{W}} = \|(T_\varphi)_n\|,$$

where  $T_\varphi$  is the Toeplitz operator associated with a symbol  $\varphi$ ,  $(T_\varphi)_n$  is the compression of  $T_\varphi$  on  $\mathcal{P}_n$ , the space of analytic polynomials of degree less or equal than  $n$ , and  $\varphi$  is the polynomial  $\sum_{k=0}^n w_k z^k$ .

From a modern point of view, these two interpolation problems (i) and (ii) are included in the following mixed problem: given  $\sigma = \{\lambda_1, \dots, \lambda_n\} \subset \mathbb{D}$  and  $f \in \text{Hol}(\mathbb{D})$ , compute or estimate

$$\|f\|_{H^\infty/B_\sigma H^\infty} = \inf \{ \|g\|_\infty : f - g \in B_\sigma \text{Hol}(\mathbb{D}) \}.$$

From now on, if  $\sigma = \{\lambda_1, \dots, \lambda_n\} \subset \mathbb{D}$  is a finite subset of the unit disc, then

$$B_\sigma = \prod_{j=1}^n b_{\lambda_j}$$

is the corresponding finite Blaschke product where  $b_\lambda = \frac{\lambda - z}{1 - \overline{\lambda} z}$ ,  $\lambda \in \mathbb{D}$ . The classical Nevanlinna-Pick problem corresponds to the case  $X = \text{Hol}(\mathbb{D})$ ,  $Y = H^\infty$ , and the one of Carathéodory-Schur to the case  $\lambda_1 = \lambda_2 = \dots = \lambda_n = 0$  and  $X = \text{Hol}(\mathbb{D})$ ,  $Y = H^\infty$ .

Looking at this problem in the form of computing or estimating the interpolation constant  $c(\sigma, X, Y)$  which is nothing but the norm of the embedding operator  $(X|_\sigma, \|\cdot\|_{X|_\sigma}) \rightarrow (Y|_\sigma, \|\cdot\|_{Y|_\sigma})$ , one can think, of course, on passing (after) to the limit -in the case of an infinite sequence  $\{\lambda_j\}$  and its finite sections  $\{\lambda_j\}_{j=1}^n$ - in order to obtain a Carleson type interpolation theorem  $X|_\sigma = Y|_\sigma$ , but not necessarily. In particular, even the classical Nevanlinna-Pick theorem (giving a necessary and sufficient condition on a function  $a$  for the existence of  $f \in H^\infty$  such that  $\|f\|_\infty \leq 1$  and  $f(\lambda) = a_\lambda$ ,  $\lambda \in \sigma$ ), does not lead immediately to Carleson's criterion for  $H|_\sigma = l^\infty(\sigma)$ . (Finally, a direct deduction of Carleson's theorem from Pick's result was done by P. Koosis [12] in 1999 only). Similarly, the problem stated for  $c(\sigma, X, Y)$  is of interest in its own. It is a kind of "effective interpolation" because we are looking for sharp estimates or a computation of  $c(\sigma, X, Y)$  for a variety of norms  $\|\cdot\|_X$ ,  $\|\cdot\|_Y$ .

## 1.2. Motivations.

**a.** As it is mentioned in Subsection 1.1, one of the most interesting cases is  $Y = H^\infty$ . In this case, the quantity  $c(\sigma, X, H^\infty)$  has a meaning of an intermediate interpolation between the Carleson one (when  $\|f\|_{X|\sigma} \asymp \sup_{1 \leq i \leq n} |f(\lambda_i)|$ ) and the individual Nevanlinna-Pick interpolation (no conditions on  $f$ ).

**b.** The following partial case was especially stimulating (which is a part of a more complicated question arising in an applied situation in [4, 5]): given a set  $\sigma \subset \mathbb{D}$ , how to estimate  $c(\sigma, H^2, H^\infty)$  in terms of  $n = \text{card } \sigma$  and  $\max_{\lambda \in \sigma} |\lambda| = r$  only? (Here,  $H^2$  is the standard Hardy space of the disc  $\mathbb{D}$  and is defined below in Subsection 1.3).

**c.** There is a straight link between the constant  $c(\sigma, X, Y)$  and numerical analysis. For example, in matrix analysis, it is of interest to bound the norm of an  $H^\infty$ -calculus  $\|f(A)\| \leq c \|f\|_\infty$ ,  $f \in H^\infty$ , for a contraction  $A$  on an  $n$ -dimensional arbitrary Banach space, with a given spectrum  $\sigma(A) \subset \sigma$ . The best possible constant is  $c = c(\sigma, H^\infty, W)$ , so that

$$c(\sigma, H^\infty, W) = \sup_{\|f\|_\infty \leq 1} \sup \{ \|f(A)\| : A : (\mathbb{C}^n, |\cdot|) \rightarrow (\mathbb{C}^n, |\cdot|), \|A\| \leq 1, \sigma(A) \subset \sigma \},$$

where  $W = \left\{ f = \sum_{k \geq 0} \hat{f}(k) z^k : \sum_{k \geq 0} |\hat{f}(k)| < \infty \right\}$  stands for the Wiener algebra, and the interior sup is taken over all contractions on  $n$ -dimensional Banach spaces. An estimate for  $c(\sigma, H^\infty, W)$  is given in [16]. An interesting case occurs for  $f \in H^\infty$  such that  $f|_\sigma = (1/z)|_\sigma$  (estimates of condition numbers and the norm of inverses of  $n \times n$  matrices) or  $f|_\sigma = [1/(\lambda - z)]|_\sigma$  (for estimates of the norm of the resolvent of an  $n \times n$  matrix). Notice that in the same spirit, the case  $Y = B_{\infty,1}^0$  where  $B_{\infty,1}^0$  is a Besov algebra presents an interesting case for the functional calculus of finite rank operators, in particular, those satisfying the so-called Ritt condition.

## 1.3. The spaces $X$ and $Y$ considered here.

We systematically use the following conditions for the spaces  $X$  and  $Y$ ,

$$(P_1) \quad \text{Hol}((1 + \epsilon)\mathbb{D}) \text{ is continuously embedded into } Y \text{ for every } \epsilon > 0,$$

$$(P_2) \quad \text{Pol}_+ \subset X \text{ and } \text{Pol}_+ \text{ is dense in } X,$$

where  $\text{Pol}_+$  stands for the set of all complex polynomials  $p$ ,  $p = \sum_{k=0}^N a_k z^k$ ,

$$(P_3) \quad [f \in X] \Rightarrow \left[ z^n f \in X, \forall n \geq 0 \text{ and } \overline{\lim} \|z^n f\|^{1/n} \leq 1 \right],$$

$$(P_4) \quad [f \in X, \lambda \in \mathbb{D} \text{ and } f(\lambda) = 0] \Rightarrow \left[ \frac{f}{z - \lambda} \in X \right].$$

Assuming  $X$  satisfies property  $(P_4)$ , then the quantity  $c(\sigma, X, Y)$  can be written as follows

$$c(\sigma, X, Y) = \sup_{\|f\|_X \leq 1} \inf \{ \|g\|_Y : g \in Y, g - f \in B_\sigma X \}.$$

General spaces  $X$  and  $Y$  satisfying  $(P_i)_{1 \leq i \leq 4}$  are studied in Section 3. Then, we study special cases of such spaces: from Section 4 to the end of this paper,  $Y = H^\infty$ , but  $X$  may change from a section to another. In particular, in Sections 4 and 5,  $X = H^p = H^p(\mathbb{D})$ ,  $1 \leq p \leq \infty$  which are the standard Hardy spaces on the disc  $\mathbb{D}$  (see [15] Chapter 2) of all  $f \in \text{Hol}(\mathbb{D})$  satisfying

$$\sup_{0 \leq r < 1} \left( \int_{\mathbb{T}} |f(rz)|^p dm(z) \right)^{1/p} < \infty,$$

$m$  being the Lebesgue normalized measure on  $\mathbb{T}$ . For  $p = 2$ , an equivalent description of  $H^2$  is

$$H^2 = \left\{ f = \sum_{k \geq 0} \hat{f}(k) z^k : \sum_{k \geq 0} |\hat{f}(k)|^2 < \infty \right\}.$$

We also study (see Section 6) the case  $X = l_a^2(1/\sqrt{k+1})$ , which is the Bergman space of all  $f = \sum_{k \geq 0} \hat{f}(k) z^k$  satisfying

$$\sum_{k \geq 0} |\hat{f}(k)|^2 \frac{1}{k+1} < \infty.$$

This space is also given by:  $X = L_a^2$ , the space of holomorphic functions  $f$  on  $\mathbb{D}$  such that

$$\int_{\mathbb{D}} |f(z)|^2 dx dy < \infty.$$

## 2. RESULTS

We start studying general Banach spaces  $X$  and  $Y$  and give some sufficient conditions under which  $C_{n,r}(X, Y) < \infty$ . In particular, we prove the following fact.

**Theorem A.** *Let  $X, Y$  be Banach spaces satisfying properties  $(P_i)$ ,  $i = 1, \dots, 4$ . Then*

$$C_{n,r}(X, Y) < \infty,$$

for every  $n \geq 1$  and  $r \in [0, 1)$ .

Next, we add the condition that  $X$  is a Hilbert space, and give in this case a general upper bound for the quantity  $C_{n,r}(X, Y)$ .

**Theorem B.** *Let  $Y$  be a Banach space satisfying property  $(P_1)$  and  $X = (H, (\cdot)_H)$  a Hilbert space satisfying properties  $(P_i)$  for  $i = 2, 3, 4$ . We moreover suppose that for every  $0 < r < 1$  there exists  $\epsilon > 0$  such that  $k_\lambda \in \text{Hol}((1 + \epsilon)\mathbb{D})$  for all  $|\lambda| < r$ , where  $k_\lambda$  stands for the reproducing kernel of  $X$  at point  $\lambda$ , and  $\bar{\lambda} \mapsto k_\lambda$  is holomorphic on  $|\lambda| < r$  as a  $\text{Hol}((1 + \epsilon)\mathbb{D})$ -valued function. Let  $\sigma = \{\lambda_1, \dots, \lambda_1, \lambda_2, \dots, \lambda_2, \dots, \lambda_t, \dots, \lambda_t\}$  be a sequence in  $\mathbb{D}$ , where  $\lambda_s$  are repeated according to their multiplicity  $m_s$ ,  $\sum_{s=1}^t m_s = n$ . Then we have,*

i)

$$c(\sigma, H, Y) \leq \left( \sum_{k=1}^n \|e_k\|_Y^2 \right)^{\frac{1}{2}},$$

where  $(e_k)_{1 \leq k \leq n}$  stands for the Gram-Schmidt orthogonalization (in the space  $H$ ) of the sequence

$$k_{\lambda_1,0}, k_{\lambda_1,1}, k_{\lambda_1,2}, \dots, k_{\lambda_1,m_1-1}, k_{\lambda_2,0}, k_{\lambda_2,1}, k_{\lambda_2,2}, \dots, k_{\lambda_2,m_2-1}, \dots, k_{\lambda_t,0}, k_{\lambda_t,1}, k_{\lambda_t,2}, \dots, k_{\lambda_t,m_t-1},$$

and  $k_{\lambda,i} = \left( \frac{d}{d\lambda} \right)^i k_\lambda$ ,  $i \in \mathbb{N}$ .

ii) For the case  $Y = H^\infty$ , we have

$$c(\sigma, H, H^\infty) \leq \sup_{z \in \mathbb{D}} \|P_{B_\sigma}^H k_z\|_H,$$

where  $P_{B_\sigma}^H = \sum_{k=1}^n (\cdot, e_k)_H e_k$  stands for the orthogonal projection of  $H$  onto  $K_{B_\sigma}(H)$ ,

$$K_{B_\sigma}(H) = \text{span} (k_{\lambda_j, i} : 1 \leq i < m_j, j = 1, \dots, t).$$

After that, we deal with  $H^\infty$  interpolation ( $Y = H^\infty$ ). For general Banach spaces (of analytic functions in  $\mathbb{D}$ ) of moderate growth  $X$ , we formulate the following conjecture:

$$c_1 \varphi_X \left( 1 - \frac{1-r}{n} \right) \leq C_{n,r}(X, H^\infty) \leq c_2 \varphi_X \left( 1 - \frac{1-r}{n} \right),$$

where  $\varphi_X(t)$ ,  $0 \leq t < 1$  stands for the norm of the evaluation functional  $f \mapsto f(t)$  on the space  $X$ . We show this conjecture for  $X = H^p, L_a^2$ ,  $p \in [1, \infty)$  (defined in Subsection 1.3). More precisely, we prove the following Theorems C and D. Here and later on,

$$\sigma_{n,\lambda} = \{\lambda, \dots, \lambda\}, \text{ (} n \text{ times)},$$

is the one-point set of multiplicity  $n$  corresponding to  $\lambda \in \mathbb{D}$ .

**Theorem C.** *Let  $1 \leq p \leq \infty$ ,  $n \geq 1$ ,  $r \in [0, 1)$ , and  $|\lambda| \leq r$ . We have,*

$$\frac{1}{32^{\frac{1}{p}}} \left( \frac{n}{1-|\lambda|} \right)^{\frac{1}{p}} \leq c(\sigma_{n,\lambda}, H^p, H^\infty) \leq C_{n,r}(H^p, H^\infty) \leq A_p \left( \frac{n}{1-r} \right)^{\frac{1}{p}},$$

where  $A_p$  is a constant depending only on  $p$  and the left-hand side inequality is proved only for  $p \in 2\mathbb{Z}_+$ . For  $p = 2$ , we have  $A_2 = \sqrt{2}$ .

**Theorem D.** *Let  $n \geq 1$ ,  $r \in [0, 1)$ , and  $|\lambda| \leq r$ . We have,*

$$\frac{1}{32} \frac{n}{1-|\lambda|} \leq c(\sigma_{n,\lambda}, L_a^2, H^\infty) \leq C_{n,r}(L_a^2, H^\infty) \leq 10^{\frac{1}{4}} \sqrt{2} \frac{n}{1-r}.$$

The above Theorems A, C and D were already announced in the note [19].

In order to prove (Theorems A, B and) the right-hand side inequality of Theorems C and D, given  $f \in X$  and  $\sigma$  a finite subset of  $\mathbb{D}$ , we first use a linear interpolation:

$$f \mapsto \sum_{k=1}^n \langle f, e_k \rangle e_k,$$

where  $\langle \cdot, \cdot \rangle$  means the Cauchy sesquilinear form  $\langle h, g \rangle = \sum_{k \geq 0} \hat{h}(k) \overline{\hat{g}(k)}$ , and  $(e_k)_{1 \leq k \leq n}$  is the Malmquist basis (effectively constructible) of the space  $K_B = H^2 \ominus B H^2$ , which  $B = B_\sigma$  (see N. Nikolski, [16] p. 117)). Next, we use the complex interpolation between Banach spaces, (see H. Triebel [17] Theorem 1.9.3-(a) p.59). Among the technical tools used in order to find an upper bound for  $\|\sum_{k=1}^n \langle f, e_k \rangle e_k\|_\infty$  (in terms of  $\|f\|_X$ ), the most important one is a Bernstein-type inequality  $\|f'\|_p \leq c_p \|B'\|_\infty \|f\|_p$  for a (rational) function  $f$  in the star-invariant subspace  $K_B^p := H^p \cap \overline{BzH^p}$ ,  $1 \leq p \leq \infty$  (for  $p = 2$ ,  $K_B^2 = K_B$ ), generated by a (finite) Blaschke product  $B$ , (K. Dyakonov [9, 10]). For  $p = 2$ , we give an alternative proof of the Bernstein-type estimate we need and the constant  $c_2$  we obtain is slightly better, see Section 6.

The lower bound problem of Theorems C and D is treated in Section 7 by using the “worst” interpolation  $n$ -tuple  $\sigma = \sigma_{n, \lambda}$ , (the Carathéodory-Schur type interpolation). The “worst” interpolation data comes from the Dirichlet kernels  $\sum_{k=0}^{n-1} z^k$  transplanted from the origin to  $\lambda$ . We notice that spaces  $X = H^p, L_a^2, p \in [1, \infty)$  satisfy the condition  $X \circ b_\lambda \subset X$  which makes the problem of upper and lower bound easier.

The paper is organized as follows. In Section 3 we prove Theorems A and B. Sections 4 and 5 (resp. Section 6) are (resp. is) devoted to the proof of the upper estimate of Theorem C (resp. Theorem D). In Section 7, we prove the lower bounds stated in Theorems C and D. At the end of the paper, we shortly compare the method used in Sections 3, 4, 5 and 6 with those resulting from the Carleson free interpolation, see Section 8.

### 3. UPPER BOUNDS FOR $c(\sigma, X, Y)$

**3.1. Banach spaces  $X, Y$  satisfying properties  $(P_i)_{1 \leq i \leq 4}$ .** In this subsection,  $X$  and  $Y$  are Banach spaces which satisfy properties  $(P_i)_{1 \leq i \leq 4}$ . We prove Theorem A which shows that in this case our interpolation constant  $c(\sigma, X, Y)$  is bounded by a quantity which depends only on  $n = \text{card } \sigma$  and  $r = \max_i |\lambda_i|$  (and of course on  $X$  and  $Y$ ). In this generality, we cannot discuss the question of the sharpness of the bounds obtained. First, we prove the following lemma.

**Lemma. 3.1.1.** *Under  $(P_2), (P_3)$  and  $(P_4)$ ,  $B_\sigma X$  is a closed subspace of  $X$  and moreover if  $\sigma$  is a finite subset of  $\mathbb{D}$ ,*

$$B_\sigma X = \{f \in X : f(\lambda) = 0, \forall \lambda \in \sigma \text{ (including multiplicities)}\}.$$

*Proof.* Since  $X \subset \text{Hol}(\mathbb{D})$  continuously, and evaluation functionals  $f \mapsto f(\lambda)$  and  $f \mapsto f^{(k)}(\lambda), k \in \mathbb{N}^*$ , are continuous on  $\text{Hol}(\mathbb{D})$ , the subspace

$$M = \{f \in X : f(\lambda) = 0, \forall \lambda \in \sigma \text{ (including multiplicities)}\},$$

is closed in  $X$ .

On the other hand,  $B_\sigma X \subset X$ , and hence  $B_\sigma X \subset M$ . Indeed, properties  $(P_2)$  and  $(P_3)$  imply that  $h.X \subset X$ , for all  $h \in \text{Hol}((1 + \epsilon)\mathbb{D})$  with  $\epsilon > 0$ ; we can write  $h = \sum_{k \geq 0} \widehat{h}(k)z^k$  with  $|\widehat{h}(k)| \leq Cq^n$ ,  $C > 0$  and  $q < 1$ . Then  $\sum_{n \geq 0} \|\widehat{h}(k)z^k f\|_X < \infty$  for every  $f \in X$ . Since  $X$  is a Banach space we get  $hf = \sum_{n \geq 0} \widehat{h}(k)z^k f \in X$ .

In order to see that  $M \subset B_\sigma X$ , it suffices to justify that

$$[f \in X \text{ and } f(\lambda) = 0] \implies [f/b_\lambda = (1 - \bar{\lambda}z)f/(\lambda - z) \in X],$$

but this is obvious from  $(P_4)$  and the previous arguments. □

In Definitions 3.1.2, 3.1.3, 3.1.4 and in Remark 3.1.5 below,  $\sigma = \{\lambda_1, \dots, \lambda_n\}$  is a sequence in the unit disc  $\mathbb{D}$  and  $B_\sigma$  is its corresponding Blaschke product.

**Definition 3.1.2.** *Malmquist family.* For  $k \in [1, n]$ , we set  $f_k(z) = \frac{1}{1 - \bar{\lambda}_k z}$ , and define the family  $(e_k)_{1 \leq k \leq n}$ , (which is known as Malmquist basis, see [16] p.117), by

$$(3.1.2) \quad e_1 = \frac{f_1}{\|f_1\|_2} \text{ and } e_k = \left( \prod_{j=1}^{k-1} b_{\lambda_j} \right) \frac{f_k}{\|f_k\|_2},$$

for  $k \in [2, n]$ , where  $\|f_k\|_2 = (1 - |\lambda_k|^2)^{-1/2}$ .

**Definition 3.1.3.** *The model space  $K_{B_\sigma}$ .* We define  $K_{B_\sigma}$  to be the  $n$ -dimensional space:

$$(3.1.3) \quad K_{B_\sigma} = (B_\sigma H^2)^\perp = H^2 \ominus B_\sigma H^2.$$

**Definition 3.1.4.** *The orthogonal projection  $P_{B_\sigma}$  on  $K_{B_\sigma}$ . We define  $P_{B_\sigma}$  to be the orthogonal projection of  $H^2$  on its  $n$ -dimensional subspace  $K_{B_\sigma}$ .*

**Remark 3.1.5.** The Malmquist family  $(e_k)_{1 \leq k \leq n}$  corresponding to  $\sigma$  is an orthonormal basis of  $K_{B_\sigma}$ . In particular,

$$(3.1.5) \quad P_{B_\sigma} = \sum_{k=1}^n (\cdot, e_k)_{H^2} e_k,$$

where  $(\cdot, \cdot)_{H^2}$  means the scalar product on  $H^2$ .

**Lemma 3.1.6.** *Let  $\sigma = \{\lambda_1, \dots, \lambda_n\}$  be a sequence in the unit disc  $\mathbb{D}$  and let  $(e_k)_{1 \leq k \leq n}$  be the Malmquist family (see 3.1.2) corresponding to  $\sigma$ . The map  $\Phi : \text{Hol}(\mathbb{D}) \rightarrow Y \subset \text{Hol}(\mathbb{D})$  defined by*

$$\Phi : f \mapsto \sum_{k=1}^n \left( \sum_{j \geq 0} \hat{f}(j) \overline{\hat{e}_k(j)} \right) e_k,$$

is well defined and has the following properties.

- (a)  $\Phi|_{H^2} = P_{B_\sigma}$ ,
- (b)  $\Phi$  is continuous on  $\text{Hol}(\mathbb{D})$  for the uniform convergence on compact sets of  $\mathbb{D}$ ,
- (c) Let  $\Psi = \text{Id}|_X - \Phi|_X$ , then  $\text{Im}(\Psi) \subset B_\sigma X$ .

*Proof.* Indeed, the point (a) is obvious since  $(e_k)_{1 \leq k \leq n}$  is an orthonormal basis of  $K_{B_\sigma}$  and

$$\sum_{j \geq 0} \hat{f}(j) \overline{\hat{e}_k(j)} = \langle f, e_k \rangle,$$

where  $\langle \cdot, \cdot \rangle$  means the Cauchy sesquilinear form  $\langle h, g \rangle = \sum_{k \geq 0} \hat{h}(k) \overline{\hat{g}(k)}$ . In order to check point (b), let  $(f_l)_{l \in \mathbb{N}}$  be a sequence of  $\text{Hol}(\mathbb{D})$  converging to 0 uniformly on compact sets of  $\mathbb{D}$ . We need to see that  $(\Phi(f_l))_{l \in \mathbb{N}}$  converges to 0, for which it is sufficient to show that  $\lim_l \left| \sum_{j \geq 0} \hat{f}_l(j) \overline{\hat{e}_k(j)} \right| = 0$ , for every  $k = 1, 2, \dots, n$ . Let  $\rho \in ]0, 1[$ , then  $\hat{f}_l(j) = (2\pi)^{-1} \int_{\rho\mathbb{T}} f_l(w) w^{-j-1} dw$ , for all  $j, l \geq 0$ . As a result,

$$\left| \sum_{j \geq 0} \hat{f}_l(j) \overline{\hat{e}_k(j)} \right| \leq \sum_{j \geq 0} \left| \hat{f}_l(j) \overline{\hat{e}_k(j)} \right| \leq (2\pi\rho)^{-1} \|f_l\|_{\rho\mathbb{T}} \sum_{j \geq 0} |\hat{e}_k(j)| \rho^{-j}.$$

Now if  $\rho$  is close enough to 1, it satisfies the inequality  $1 \leq \rho^{-1} < r^{-1}$ , which entails  $\sum_{j \geq 0} |\hat{e}_k(j)| \rho^{-j} < +\infty$  for each  $k = 1, \dots, n$ . The result follows.

We now prove point (c). Using point (a), since  $Pol_+ \subset H^2$  (see Subsection 1.3, property  $(P_2)$  for the definition of  $Pol_+$ ), we get that  $\text{Im}(\Psi|_{Pol_+}) \subset B_\sigma H^2$ . Now, since  $Pol_+ \subset Y$  and  $\text{Im}(\Phi) \subset Y$ , we deduce that

$$\text{Im}(\Psi|_{Pol_+}) \subset B_\sigma H^2 \cap Y \subset B_\sigma H^2 \cap X,$$

since  $Y \subset X$ . Now  $\Psi(p) \in X$  and satisfies  $(\Psi(p))|_\sigma = 0$  (that is to say  $(\Psi(p))(\lambda) = 0, \forall \lambda \in \sigma$  (including multiplicities) for all  $p \in Pol_+$ ). Using Lemma 3.1.1, we get that  $\text{Im}(\Psi|_{Pol_+}) \subset B_\sigma X$ . Now,

$Pol_+$  being dense in  $X$  (property  $(P_2)$ ), and  $\Psi$  being continuous on  $X$  (point (b)), we can conclude that  $\text{Im}(\Psi) \subset B_\sigma X$ .

□

*Proof of Theorem A.* Let  $\sigma = \{\lambda_1, \dots, \lambda_n\}$  be a sequence in the unit disc  $\mathbb{D}$  and  $(e_k)_{1 \leq k \leq n}$  the Malmquist family (3.1.2) associated to  $\sigma$ . Taking  $f \in X$ , we set

$$g = \sum_{k=1}^n \left( \sum_{j \geq 0} \hat{f}(j) \overline{\hat{e}_k(j)} \right) e_k,$$

where the series  $\sum_{j \geq 0} \hat{f}(j) \overline{\hat{e}_k(j)}$  are absolutely convergent. Indeed,

$$\hat{e}_k(j) = (2\pi i)^{-1} \int_{R\mathbb{T}} e_k(w) w^{-j-1} dw,$$

for all  $j \geq 0$  and for all  $R$ ,  $1 < R < \frac{1}{r}$ . For a subset  $A$  of  $\mathbb{C}$  and for a bounded function  $h$  on  $A$ , we define  $\|h\|_A := \sup_{z \in A} |h(z)|$ . As a result,

$$|\hat{e}_k(j)| \leq (2\pi R^{j+1})^{-1} \|e_k\|_{R\mathbb{T}} \quad \text{and} \quad \sum_{j \geq 0} \left| \hat{f}(j) \overline{\hat{e}_k(j)} \right| \leq (2\pi R)^{-1} \|e_k\|_{R\mathbb{T}} \sum_{j \geq 0} |\hat{f}(j)| R^{-j} < \infty,$$

since  $R > 1$  and  $f$  is holomorphic in  $\mathbb{D}$ .

We now suppose that  $\|f\|_X \leq 1$  and  $g = \Phi(f)$ , where  $\Phi$  is defined in Lemma 3.1.6. Since  $\text{Hol}(r^{-1}\mathbb{D}) \subset Y$ , we have  $g \in Y$  and using Lemma 3.1.6 point (c) we get

$$f - g = \Psi(f) \in B_\sigma X,$$

where  $\Psi$  is defined in Lemma 3.1.6, as  $\Phi$ . Moreover,

$$\|g\|_Y \leq \sum_{k=1}^n |\langle f, e_k \rangle| \|e_k\|_Y.$$

In order to bound the right-hand side, recall that for all  $j \geq 0$  and for  $R = 2/(r+1) \in ]1, 1/r[$ ,

$$\sum_{j \geq 0} \left| \hat{f}(j) \overline{\hat{e}_k(j)} \right| \leq (2\pi)^{-1} \|e_k\|_{2(r+1)^{-1}\mathbb{T}} \sum_{j \geq 0} |\hat{f}(j)| (2^{-1}(r+1))^j.$$

Since the norm  $f \mapsto \sum_{j \geq 0} |\hat{f}(j)| (2^{-1}(r+1))^j$  is continuous on  $\text{Hol}(\mathbb{D})$ , and the inclusion  $X \subset \text{Hol}(\mathbb{D})$  is also continuous, there exists  $C_r > 0$  such that

$$\sum_{j \geq 0} |\hat{f}(j)| (2^{-1}(r+1))^j \leq C_r \|f\|_X,$$

for every  $f \in X$ . On the other hand,  $\text{Hol}(2(r+1)^{-1}\mathbb{D}) \subset Y$  (continuous inclusion again), and hence there exists  $K_r > 0$  such that

$$\|e_k\|_Y \leq K_r \sup_{|z| < 2(r+1)^{-1}} |e_k(z)| = K_r \|e_k\|_{2(r+1)^{-1}\mathbb{T}}.$$

It is more or less clear that the right-hand side of the last inequality can be bounded in terms of  $r$  and  $n$  only. Let us give a proof to this fact. It is clear that it suffices to estimate



$$\sup_{1 < |z| < 2(r+1)^{-1}} |e_k(z)|.$$

In order to bound this quantity, notice that

$$(3.1.7) \quad |b_\lambda(z)|^2 \leq \left| \frac{\lambda - z}{1 - \bar{\lambda}z} \right|^2 = 1 + \frac{(|z|^2 - 1)(1 - |\lambda|^2)}{|1 - \bar{\lambda}z|^2},$$

for all  $\lambda \in \mathbb{D}$  and all  $z \in |\lambda|^{-1}\mathbb{D}$ . Using the identity (3.1.7) for  $\lambda = \lambda_j$ ,  $1 \leq j \leq n$ , and  $z = \rho e^{it}$ ,  $\rho = 2(1+r)^{-1}$ , we get

$$|e_k(\rho e^{it})|^2 \leq \left( \prod_{j=1}^{k-1} |b_{\lambda_j}(\rho e^{it})|^2 \right) \left| \frac{1}{1 - \bar{\lambda}_k \rho e^{it}} \right|^2 \leq \left( \prod_{j=1}^{k-1} \left( 1 + \frac{(\rho^2 - 1)(1 - |\lambda_j|^2)}{1 - |\lambda_j|^2 \rho^2} \right) \right) \left( \frac{1}{1 - |\lambda_k| \rho} \right)^2,$$

for all  $k = 2, \dots, n$ . Expressing  $\rho$  in terms of  $r$ , we obtain

$$\|e_k\|_{2(r+1)^{-1}\mathbb{T}} \leq \frac{1}{1 - \frac{2r}{r+1}} \sqrt{2 \left( \prod_{j=1}^{n-1} \left( 1 + \frac{2(\frac{1}{r^2} - 1)}{1 - r^2 \frac{4}{(r+1)^2}} \right) \right)} =: C_1(r, n),$$

and

$$\sum_{j \geq 0} \left| \hat{f}(j) \overline{\hat{e}_k(j)} \right| \leq (2\pi)^{-1} C_r \|e_k\|_{2(r+1)^{-1}\mathbb{T}} \|f\|_X \leq (2\pi)^{-1} C_r C_1(r, n) \|f\|_X.$$

On the other hand, since

$$\|e_k\|_Y \leq K_r \|e_k\|_{2(r+1)^{-1}\mathbb{T}} \leq K_r C_1(r, n),$$

we get

$$\|g\|_Y \leq \sum_{k=1}^n (2\pi)^{-1} C_r C_1(r, n) \|f\|_X K_r C_1(r, n) = (2\pi)^{-1} n C_r K_r (C_1(r, n))^2 \|f\|_X,$$

which proves that

$$c(\sigma, X, Y) \leq (2\pi)^{-1} n C_r K_r (C_1(r, n))^2,$$

and completes the proof of Theorem A. □

**3.2. The case where  $X$  is a Hilbert space.** We suppose in this subsection that  $X$  is a Hilbert space and that  $X, Y$  satisfy properties  $(P_i)_{1 \leq i \leq 4}$ . We prove Theorem B and obtain a better estimate for  $c(\sigma, X, Y)$  than in Theorem A (see point (i) of Theorem B). For the case  $Y = H^\infty$ , (point (ii) of Theorem B), we can considerably improve this estimate. We omit an easy proof of the following lemma.

**Lemma. 3.2.1.** *Let  $\sigma = \{\lambda_1, \dots, \lambda_1, \lambda_2, \dots, \lambda_2, \dots, \lambda_t, \dots, \lambda_t\}$  be a finite sequence of  $\mathbb{D}$  where every  $\lambda_s$  is repeated according to its multiplicity  $m_s$ ,  $\sum_{s=1}^t m_s = n$ . Let  $(H, (\cdot)_H)$  be a Hilbert space continuously embedded into  $\text{Hol}(\mathbb{D})$  and satisfying properties  $(P_i)$  for  $i = 2, 3, 4$ . Then*

$$K_{B_\sigma}(H) =: H \Theta B_\sigma H = \text{span} (k_{\lambda_j, i} : 1 \leq j \leq t, 0 \leq i \leq m_j - 1),$$

where  $k_{\lambda, i} = \left( \frac{d}{d\lambda} \right)^i k_\lambda$  and  $k_\lambda$  is the reproducing kernel of  $H$  at point  $\lambda$  for every  $\lambda \in \mathbb{D}$ , i.e.  $k_\lambda \in H$  and  $f(\lambda) = (f, k_\lambda)_H$ ,  $\forall f \in H$ .

*Proof of Theorem B. i).* Let  $f \in X$ ,  $\|f\|_X \leq 1$ . Lemma 3.2.1 shows that

$$g = P_{B_\sigma}^H f = \sum_{k=1}^n (f, e_k)_H e_k$$

is the orthogonal projection of  $f$  onto subspace  $K_{B_\sigma}$ . Function  $g$  belongs to  $Y$  because all  $k_{\lambda_j, i}$  are in  $\text{Hol}((1 + \epsilon)\mathbb{D})$  for a convenient  $\epsilon > 0$ , and  $Y$  satisfies  $(P_1)$ .

On the other hand,  $g - f \in B_\sigma H$  (again by Lemma 3.2.1). Moreover, using Cauchy-Schwarz inequality,

$$\|g\|_Y \leq \sum_{k=1}^n |(f, e_k)_H| \|e_k\|_Y \leq \left( \sum_{k=1}^n |(f, e_k)_H|^2 \right)^{1/2} \left( \sum_{k=1}^n \|e_k\|_Y^2 \right)^{1/2} \leq \|f\|_H \left( \sum_{k=1}^n \|e_k\|_Y^2 \right)^{1/2},$$

which proves i).

ii). If  $Y = H^\infty$ , then

$$|g(z)| = |(P_{B_\sigma}^H f, k_z)_H| = |(f, P_{B_\sigma}^H k_z)_H| \leq \|f\|_H \|P_{B_\sigma}^H k_z\|_H,$$

for all  $z \in \mathbb{D}$ , which proves ii). □

#### 4. UPPER BOUNDS FOR $C_{n,r}(H^2, H^\infty)$

Here, we specialize the upper estimate obtained in point (ii) of Theorem B for the case  $X = H^2$ , the Hardy space of the disc. Later on, we will see that this estimate is sharp at least for some special sequences  $\sigma$  (see Section 7). We also develop a slightly different approach to the interpolation constant  $c(\sigma, H^2, H^\infty)$  giving more estimates for individual sequences  $\sigma = \{\lambda_1, \dots, \lambda_n\}$  of  $\mathbb{D}$ . We finally prove the right-hand side inequality of Theorem C for the particular case  $p = 2$ .

**Proposition. 4.1.** *For every sequence  $\sigma = \{\lambda_1, \dots, \lambda_n\}$  of  $\mathbb{D}$  we have*

$$(4.1.1) \quad c(\sigma, H^2, H^\infty) \leq \sup_{z \in \mathbb{D}} \left( \frac{1 - |B_\sigma(z)|^2}{1 - |z|^2} \right)^{1/2},$$

$$(4.1.2) \quad c(\sigma, H^2, H^\infty) \leq \sqrt{2} \sup_{|\zeta|=1} |B'(\zeta)|^{1/2} = \sqrt{2} \sup_{|\zeta|=1} \left| \sum_{i=1}^n \frac{1 - |\lambda_i|^2}{(1 - \bar{\lambda}_i \zeta)^2} \frac{B_\sigma(\zeta)}{b_{\lambda_i}(\zeta)} \right|^{1/2}.$$

*Proof.* We prove (4.1.1). In order to simplify the notation, we set  $B = B_\sigma$ . Applying point (ii) of Theorem B for  $X = H^2$  and  $Y = H^\infty$ , and using

$$k_z(\zeta) = \frac{1}{1 - \bar{z}\zeta} \quad \text{and} \quad (P_{B_\sigma} k_z)(\zeta) = \frac{1 - \overline{B_\sigma(z)} B_\sigma(\zeta)}{1 - \bar{z}\zeta},$$

(see [14] p.199), we obtain

$$\|P_{B_\sigma} k_z\|_{H^2} = \left( \frac{1 - |B_\sigma(z)|^2}{1 - |z|^2} \right)^{1/2},$$

which gives the result.

We now prove (4.1.2) using (4.1.1). The map  $\zeta \mapsto \|P_B(k_\zeta)\| = \sup \{|f(\zeta)| : f \in K_B, \|f\| \leq 1\}$ , and hence the map

$$\zeta \mapsto \left( \frac{1 - |B(\zeta)|^2}{1 - |\zeta|^2} \right)^{1/2},$$

is a subharmonic function so

$$\sup_{|\zeta| < 1} \left( \frac{1 - |B(\zeta)|^2}{1 - |\zeta|^2} \right)^{1/2} \leq \sup_{|w|=1} \lim_{r \rightarrow 1} \left( \frac{1 - |B(rw)|^2}{1 - |rw|^2} \right)^{1/2}.$$

Now applying Taylor's Formula of order 1 at points  $w \in \mathbb{T}$  and  $u = rw$ ,  $0 < r < 1$  (it is applicable because  $B$  is holomorphic at every point of  $\mathbb{T}$ ), we get

$$(B(u) - B(w))(u - w)^{-1} = B'(w) + o(1),$$

and since  $|u - w| = 1 - |u|$ ,

$$|(B(u) - B(w))(u - w)^{-1}| = |B(u) - B(w)|(1 - |u|)^{-1} = |B'(w) + o(1)|.$$

Then we have

$$|B(u) - B(w)| \geq |B(w)| - |B(u)| = 1 - |B(u)|,$$

$$(1 - |B(u)|)(1 - |u|)^{-1} \leq (1 - |u|)^{-1} |B(u) - B(w)| = |B'(w) + o(1)|,$$

and

$$\lim_{r \rightarrow 1} ((1 - |B(rw)|)(1 - |rw|)^{-1})^{\frac{1}{2}} \leq \sqrt{|B'(w)|}.$$

Moreover,

$$B'(w) = - \sum_{i=1}^n (1 - |\lambda_i|^2) (1 - \bar{\lambda}_i w)^{-2} \prod_{j=1, j \neq i}^n b_{\lambda_j}(w),$$

for all  $w \in \mathbb{T}$ . This completes the proof since

$$\frac{1 - |B(rw)|^2}{1 - |rw|^2} = \frac{(1 - |B(rw)|)(1 + |B(rw)|)}{(1 - |rw|)(1 + |rw|)} \leq 2 \frac{1 - |B(rw)|}{1 - |rw|}.$$

□

**Corollary. 4.2.** *Let  $n \geq 1$  and  $r \in [0, 1[$ . Then,*

$$C_{n,r}(H^2, H^\infty) \leq 2 (n(1-r)^{-1})^{\frac{1}{2}}.$$

*Proof.* Indeed, applying Proposition 4.1 we obtain

$$|B'(w)| \leq \left| \sum_{i=1}^n \frac{1 - |\lambda_i|^2}{(1 - |\lambda_i|)^2} \right| \leq n \frac{1+r}{1-r} \leq \frac{2n}{1-r}.$$

□

Now, we develop a slightly different approach to the interpolation constant  $c(\sigma, H^2, H^\infty)$ .

**Theorem. 4.3.** *For every sequence  $\sigma = \{\lambda_1, \dots, \lambda_n\}$  of  $\mathbb{D}$ ,*

$$c(\sigma, H^2, H^\infty) \leq \sup_{z \in \mathbb{T}} \left( \sum_{k=1}^n \frac{(1 - |\lambda_k|^2)}{|z - \lambda_k|^2} \right)^{1/2}.$$

*Proof.* In order to simplify the notation, we set  $B = B_\sigma$ . We consider  $K_B$  (see Definition 3.1.3) and the Malmquist family  $(e_k)_{1 \leq k \leq n}$  corresponding to  $\sigma$  (see Definition 3.1.2). Now, let  $f \in H^2$  and

$$g = P_B f = \sum_{k=1}^n (f, e_k)_{H^2} e_k,$$

(see Definition 3.1.4 and Remark 3.1.5). Function  $g$  belongs to  $H^\infty$  (it is a finite sum of  $H^\infty$  functions) and satisfies  $g - f \in BH^2$ . Applying Cauchy-Schwarz inequality we get

$$|g(\zeta)| \leq \sum_{k=1}^n |(f, e_k)_{H^2}| |e_k(\zeta)| \leq \left( \sum_{k=1}^n |(f, e_k)_{H^2}|^2 \right)^{1/2} \left( \sum_{k=1}^n \frac{(1 - |\lambda_k|^2)}{|1 - \lambda_k \zeta|^2} \right)^{1/2},$$

for all  $\zeta \in \mathbb{D}$ . As a result, since  $f$  is an arbitrary  $H^2$  function, we obtain

$$c(\sigma, H^2, H^\infty) \leq \sup_{\zeta \in \mathbb{T}} \left( \sum_{k=1}^n \frac{(1 - |\lambda_k|^2)}{|\zeta - \lambda_k|^2} \right)^{1/2},$$

which completes the proof. □

**Corollary. 4.4.** *For any sequence  $\sigma = \{\lambda_1, \dots, \lambda_n\}$  in  $\mathbb{D}$ ,*

$$c(\sigma, H^2, H^\infty) \leq \left( \sum_{j=1}^n \frac{1 + |\lambda_j|}{1 - |\lambda_j|} \right)^{1/2}.$$

*Proof.* Indeed,

$$\sum_{k=1}^n \frac{(1 - |\lambda_k|^2)}{|\zeta - \lambda_k|^2} \leq \left( \sum_{k=1}^n \frac{(1 - |\lambda_k|^2)}{(1 - |\lambda_k|)^2} \right)^{1/2}$$

and the result follows using Theorem 4.3. □

*Proof of Theorem C ( $p = 2$ , the right-hand side inequality only).* Since  $1 + |\lambda_j| \leq 2$  and  $1 - |\lambda_j| \geq 1 - r$  for all  $j \in [1, n]$ , applying Corollary 4.4 we get

$$C_{n,r}(H^2, H^\infty) \leq \sqrt{2} n^{1/2} (1 - r)^{-1/2}.$$

□

*Remark.* As a result, we get once more the same estimate for  $C_{n,r}(H^2, H^\infty)$  as in Corollary 4.2, with the constant  $\sqrt{2}$  instead of 2.

It is natural to wonder if it is possible to improve the bound  $\sqrt{2} n^{1/2} (1 - r)^{-1/2}$ . We return to this question in Section 7 below.

5. UPPER BOUNDS FOR  $C_{n,r}(H^p, H^\infty)$ ,  $p \geq 1$ 

In this section we extend Corollary 4.2 to all Hardy spaces  $H^p$ : we prove the right-hand side inequality of Theorem C,  $p \neq 2$ . We first prove the following lemma.

**Lemma 5.1.** *Let  $n \geq 1$  and  $0 \leq r < 1$ . Then,*

$$C_{n,r}(H^1, H^\infty) \leq 2n(1-r)^{-1}.$$

*Proof.* Let  $\sigma$  be a finite subset of  $\mathbb{D}$  and  $f \in H^1$  such that  $\|f\|_{H^1} \leq 1$ . Let also  $g = \Phi(f)$  where  $\Phi$  is defined in Lemma 3.1.6, and where  $\langle \cdot, \cdot \rangle$  means the Cauchy sesquilinear form  $\langle f, g \rangle = \sum_{k \geq 0} \hat{h}(k) \overline{\hat{g}(k)}$ . That is to say that,

$$g(\zeta) = \sum_{k=1}^n \langle f, e_k \rangle e_k(\zeta) = \left\langle f, \sum_{k=1}^n e_k \overline{e_k(\zeta)} \right\rangle,$$

for all  $\zeta \in \mathbb{D}$ , where as always,  $(e_k)_{1 \leq k \leq n}$  is the Malmquist basis corresponding to  $\sigma$  (see 3.1.2), which gives,

$$|g(\zeta)| \leq \|f\|_{H^1} \left\| \sum_{k=1}^n e_k \overline{e_k(\zeta)} \right\|_{H^\infty} \leq \left\| \sum_{k=1}^n e_k \overline{e_k(\zeta)} \right\|_{H^\infty}.$$

Since Blaschke factors have modulus 1 on the unit circle,

$$\|e_k\|_{H^\infty} \leq (1 + |\lambda_k|)^{1/2} (1 - |\lambda_k|)^{-1/2}.$$

As a consequence,

$$|g(\zeta)| \leq \sum_{k=1}^n \|e_k\|_{H^\infty} \left| \overline{e_k(\zeta)} \right| \leq \sum_{k=1}^n \|e_k\|_{H^\infty}^2 \leq \sum_{k=1}^n (1 + |\lambda_k|) (1 - |\lambda_k|)^{-1} \leq 2n(1-r)^{-1},$$

for all  $\zeta \in \mathbb{D}$ , which completes the proof.  $\square$

*Proof of Theorem C ( $p \neq 2$ , the right-hand side inequality only).* Let  $\sigma = \{\lambda_1, \dots, \lambda_n\} \subset \mathbb{D}$ . Let us remark that if  $X$  is a Banach space of holomorphic functions in  $\mathbb{D}$  containing  $H^\infty$  and if  $T : X \rightarrow H^\infty/B_\sigma H^\infty$  is the restriction map defined by

$$(5.1.1) \quad Tf = \{g \in H^\infty : f - g \in B_\sigma H^p\},$$

for every  $f \in X$ , then

$$(5.1.2) \quad \|T\|_{X \rightarrow H^\infty/B_\sigma H^\infty} = c(\sigma, X, H^\infty).$$

Now in the case  $X = H^p$ , there exists  $0 \leq \theta \leq 1$  such that  $1/p = 1 - \theta$ , and since (we use the notation of the interpolation theory between Banach spaces see [6, 17])  $[H^1, H^\infty]_\theta = H^p$  (a topological identity: the spaces are the same and the norms are equivalent (up to constants depending on  $p$  only), see [11]),

$$\|T\|_{[H^1, H^\infty]_\theta \rightarrow H^\infty/B_\sigma H^\infty} \leq (A_1 c(\sigma, H^1, H^\infty))^{1-\theta} (A_\infty c(\sigma, H^\infty, H^\infty))^\theta,$$

where  $A_1, A_\infty$  are numerical constants, and using, Lemma 5.1, the fact that  $c(\sigma, H^\infty, H^\infty) \leq 1$ , and a known interpolation Theorem (see [17], Theorem 1.9.3-(a) p.59), we find

$$\|T\|_{[H^1, H^\infty]_\theta \rightarrow H^\infty/B_\sigma H^\infty} \leq (2A_1 n(1-r)^{-1})^{1-\theta} A_\infty^\theta = (2A_1)^{1-\theta} A_\infty^\theta (n(1-r)^{-1})^{\frac{1}{p}}.$$

It remains to apply (5.1.2) with  $X = H^p$  to complete the proof.  $\square$

6. UPPER BOUNDS FOR  $C_{n,r}(L_a^2, H^\infty)$ 

Our aim here is to generalize Corollary 4.2 to the case  $X = l_a^2((k+1)^\alpha)$ ,  $\alpha \in [-1, 0]$ , the Hardy weighted spaces of all  $f = \sum_{k \geq 0} \hat{f}(k)z^k$  satisfying

$$\|f\|_X^2 := \sum_{k \geq 0} \left| \hat{f}(k) \right|^2 (k+1)^{2\alpha} < \infty.$$

Note that  $H^2 = l_a^2(1)$  and  $L_a^2 = l_a^2\left((k+1)^{-\frac{1}{2}}\right)$ . We prove the upper bound of Theorem D. The main technical tool used here is a Bernstein-type inequality for rational functions.

**6.1. Bernstein-type inequalities for rational functions.** Bernstein-type inequalities for rational functions were the subject of a number of papers and monographs (see, for instance, [2-3, 7-8, 13]). Perhaps, the stronger and closer to ours (Proposition 6.1.1) of all known results are due to K. Dyakonov [9, 10]. First, we prove Proposition 6.1.1 below, which tells that if  $\sigma = \{\lambda_1, \dots, \lambda_n\} \subset \mathbb{D}$ ,  $r = \max_j |\lambda_j|$ ,  $B = B_\sigma$  and  $f \in K_B$ , then

$$(\star) \quad \left\| f' \right\|_{H^2} \leq \alpha_{n,r} \|f\|_{H^2},$$

where  $\alpha_{n,r}$  is a constant (explicitly given in Proposition 6.1.1) depending on  $n$  and  $r$  only such that  $0 < \alpha_{n,r} \leq 3 \frac{n}{1-r}$ . Proposition 6.1 is in fact a partial case ( $p = 2$ ) of the following K. Dyakonov's result [9] (which is, in turn, a generalization of M. Levin's inequality [13] corresponding to the case  $p = \infty$ ): it is proved in [9] that the norm  $\|D\|_{K_B^p \rightarrow H^p}$  of the differentiation operator  $Df = f'$  on the star-invariant subspace of the Hardy space  $H^p$ ,  $K_B^p := H^p \cap Bz\overline{H^p}$ , (where the bar denotes complex conjugation) satisfies the following inequalities

$$c'_p \left\| B' \right\|_\infty \leq \|D\|_{K_B^p \rightarrow H^p} \leq c_p \left\| B' \right\|_\infty,$$

for every  $p$ ,  $1 \leq p \leq \infty$  where  $c_p$  and  $c'_p$  are positives constants depending on  $p$  only,  $B$  is a finite Blaschke product and  $\|\cdot\|_\infty$  means the norm in  $L^\infty(\mathbb{T})$ . For the partial case considered in Proposition 6.1.1 below, our proof is different from [9, 10]: it is based on an elementary Hilbert space construction for an orthonormal basis in  $K_B$  and the constant  $c_2$  obtained is slightly better. More precisely, it is proved in [9] that  $c'_2 = \frac{1}{36c}$ ,  $c_2 = \frac{36+c}{2\pi}$  and  $c = 2\sqrt{3\pi}$  (as one can check easily ( $c$  is not precised in [9])). It implies an inequality of type  $(\star)$  (with a constant about  $\frac{13}{2}$  instead of 3).

In [18], we discuss the ‘‘asymptotic sharpness’’ of our constant  $\alpha_{n,r}$ : we find an inequality for  $C_{n,r} = \sup \|D\|_{K_B \rightarrow H^2}$  (sup is over all  $B$  with given  $n = \deg B$  and  $r = \max_{\lambda \in \sigma} |\lambda|$ ), which is asymptotically sharp as  $n \rightarrow \infty$ . Our result in [18] is that there exists a limit  $\lim_{n \rightarrow \infty} \frac{C_{n,r}}{n} = \frac{1+r}{1-r}$  for every  $r$ ,  $0 \leq r < 1$ .

**Proposition. 6.1.1.** *Let  $B = \prod_{j=1}^n b_{\lambda_j}$ , be a finite Blaschke product (of order  $n$ ),  $r = \max_j |\lambda_j|$ , and  $f \in K_B = H^2 \ominus BH^2$ . Then for every  $n \geq 2$  and  $r \in [0, 1)$ ,*

$$\left\| f' \right\|_{H^2} \leq \alpha_{n,r} \|f\|_{H^2},$$

where  $\alpha_{n,r} = [1 + (1+r)(n-1) + \sqrt{n-2}] (1-r)^{-1}$  and in particular,

$$\left\| f' \right\|_{H^2} \leq 3 \frac{n}{1-r} \|f\|_{H^2},$$

for all  $n \geq 1$  and  $r \in [0, 1)$ .

*Proof.* Using Remark 3.1.5,  $f = P_B f = \sum_{k=1}^n (f, e_k)_{H^2} e_k$ ,  $\forall f \in K_B$ . Noticing that,

$$e'_k = \sum_{i=1}^{k-1} \frac{b'_{\lambda_i}}{b_{\lambda_i}} e_k + \bar{\lambda}_k \frac{1}{(1 - \bar{\lambda}_k z)} e_k,$$

for  $k \in [2, n]$ , we get

$$\begin{aligned} f' &= \sum_{k=2}^n (f, e_k)_{H^2} \sum_{i=1}^{k-1} \frac{b'_{\lambda_i}}{b_{\lambda_i}} e_k + \sum_{k=1}^n (f, e_k)_{H^2} \bar{\lambda}_k \frac{1}{(1 - \bar{\lambda}_k z)} e_k, \\ &= \sum_{i=1}^n \frac{b'_{\lambda_i}}{b_{\lambda_i}} \sum_{k=i+1}^{n-1} (f, e_k)_{H^2} e_k + \sum_{k=1}^n (f, e_k)_{H^2} \bar{\lambda}_k \frac{1}{(1 - \bar{\lambda}_k z)} e_k. \end{aligned}$$

Now using Cauchy-Schwarz inequality and the fact that  $e_k$  is a vector of norm 1 in  $H^2$  for  $k = 1, \dots, n$ , we get

$$\begin{aligned} \left\| \sum_{k=2}^n \bar{\lambda}_k (f, e_k)_{H^2} \frac{1}{(1 - \bar{\lambda}_k z)} e_k \right\|_{H^2} &\leq \sum_{k=1}^n |(f, e_k)_{H^2}| \left\| \bar{\lambda}_k \frac{1}{(1 - \bar{\lambda}_k z)} \right\|_{\infty} \|e_k\|_{H^2} \leq \\ &\leq \frac{1}{1-r} \sum_{k=1}^n |(f, e_k)_{H^2}| \leq \frac{1}{1-r} \left( \sum_{k=1}^n |(f, e_k)_{H^2}|^2 \right)^{\frac{1}{2}} \sqrt{n} \leq \frac{1}{1-r} \|f\|_{H^2} \sqrt{n}. \end{aligned}$$

Further,

$$\begin{aligned} \left\| \sum_{i=1}^{n-1} \frac{b'_{\lambda_i}}{b_{\lambda_i}} \sum_{k=i+1}^n e_k (f, e_k)_{H^2} \right\|_{H^2} &\leq \sum_{i=1}^{n-1} \left\| \frac{b'_{\lambda_i}}{b_{\lambda_i}} \right\|_{\infty} \left\| \sum_{k=i+1}^n (f, e_k)_{H^2} e_k \right\|_{H^2} = \\ &= \left( \max_{1 \leq i \leq n-1} \left\| \frac{b'_{\lambda_i}}{b_{\lambda_i}} \right\|_{\infty} \right) \sum_{i=1}^{n-1} \left( \sum_{k=i+1}^n |(f, e_k)_{H^2}|^2 \right)^{\frac{1}{2}} \leq \max_i \left\| \frac{b'_{\lambda_i}}{b_{\lambda_i}} \right\|_{\infty} \sum_{i=1}^{n-1} \|f\|_{H^2}. \end{aligned}$$

Now, using

$$\left\| \frac{b'_{\lambda_i}}{b_{\lambda_i}} \right\|_{\infty} = \left\| \frac{|\lambda_i|^2 - 1}{(1 - \bar{\lambda}_i z)(\lambda_i - z)} \right\|_{\infty} \leq \frac{1 + |\lambda_i|}{1 - |\lambda_i|} \leq \frac{1+r}{1-r},$$

we get

$$\left\| \sum_{i=1}^{n-1} \frac{b'_{\lambda_i}}{b_{\lambda_i}} \sum_{k=i+1}^n (f, e_k)_{H^2} e_k \right\|_{H^2} \leq (1+r) \frac{n-1}{1-r} \|f\|_{H^2}.$$

Finally,

$$\|f'\|_{H^2} \leq [1 + (1+r)(n-1) + \sqrt{n-2}] (1-r)^{-1} \|f\|_{H^2}.$$

In particular,

$$\|f\|_{H^2} \leq (2n-2 + \sqrt{n}) (1-r)^{-1} \|f\|_{H^2} \leq 3n(1-r)^{-1} \|f\|_{H^2},$$

for all  $n \geq 1$  and for every  $f \in K_B$ . □

**6.2. An upper bound for  $c(\sigma, L_a^2, H^\infty)$ .** Here, we apply Proposition 6.1.1 to prove the right-hand side inequality of Theorem D. We first prove the following corollary.

**Corollary. 6.2.1.** *Let  $\sigma$  be a sequence in  $\mathbb{D}$ . Then,*

$$c(\sigma, l_a^2((k+1)^{-1}), H^\infty) \leq 2\sqrt{10} (n(1-r)^{-1})^{3/2}.$$

Indeed, let  $H = l_a^2((k+1)^{-N})$  and  $B = B_\sigma$  the finite Blaschke product corresponding to  $\sigma$ . Let  $\widetilde{P}_B$  be the orthogonal projection of  $H$  onto  $K_B = K_B(H^2)$ . Then  $\widetilde{P}_B|_{H^2} = P_B$ , where  $P_B$  is defined in 3.1.5. We notice that  $\widetilde{P}_B : H \rightarrow H$  is a bounded operator and the adjoint  $\widetilde{P}_B^* : H^* \rightarrow H^*$  of  $\widetilde{P}_B$  relatively to the Cauchy pairing  $\langle \cdot, \cdot \rangle$  satisfies  $\widetilde{P}_B^* \varphi = \widetilde{P}_B \varphi = P_B \varphi$ ,  $\forall \varphi \in H^* \subset H^2$ , where  $H^* = l_a^2((k+1)^N)$  is the dual of  $H$  with respect to this pairing. If  $f \in H$ , then  $|\widetilde{P}_B f(\zeta)| = |\langle \widetilde{P}_B f, k_\zeta \rangle| = |\langle f, \widetilde{P}_B^* k_\zeta \rangle|$ , where  $k_\zeta = (1 - \bar{\zeta}z)^{-1} \in H^2$  and

$$|\widetilde{P}_B f(\zeta)| \leq \|f\|_H \|P_B k_\zeta\|_{H^*} \leq \|f\|_H K \left( \|P_B k_\zeta\|_{H^2}^2 + \|(P_B k_\zeta)'\|_{H^2}^2 \right)^{\frac{1}{2}},$$

where

$$K = \max \left\{ 1, \sup_{k \geq 1} (k+1)k^{-1} \right\} = 2$$

Since  $P_B k_\zeta \in K_B$ , Proposition 6.1.1 implies

$$|\widetilde{P}_B f(\zeta)| \leq \|f\|_H K \left( \|P_B k_\zeta\|_{H^2}^2 + 9(n(1-r)^{-1})^2 \|P_B k_\zeta\|_{H^2}^2 \right)^{\frac{1}{2}} \leq A(n(1-r)^{-1})^{3/2} \|f\|_H,$$

where  $A = K(2/2 + 9)^{\frac{1}{2}} = 2\sqrt{10}$ , since  $\|P_B k_\zeta\|_2 \leq \sqrt{2}(n(1-r)^{-1})^{1/2}$ , and since we can suppose  $n \geq 2$ , (the case  $n = 1$  being obvious). □

*Proof of Theorem D (the right-hand side inequality only).* The case  $\alpha = 0$  corresponds to  $X = H^2$  and has already been studied in Section 1 (we can choose  $A(0) = \sqrt{2}$ ). Let  $\sigma$  be a finite subset of  $\mathbb{D}$ . We now suppose  $\alpha \in [1, 0)$ . Setting  $\theta = -\alpha$  with  $0 < \theta \leq 1$ , we have (as in Theorem C, we use the notation of the interpolation theory between Banach spaces see [6, 17])

$$[l_a^2((k+1)^0), l_a^2((k+1)^{-1})]_{\theta, 2} = l_a^2 \left( ((k+1)^0)^{2\frac{1-\theta}{2}} ((k+1)^{-1})^{2\frac{\theta}{2}} \right) = l_a^2((k+1)^\alpha),$$

which entails, using Corollary 6.2.1 and (again) [17] Theorem 1.9.3-(a) p.59,

$$\begin{aligned} \|T\|_{l_a^2((k+1)^\alpha) \rightarrow H^\infty/B_\sigma H^\infty} &\leq (c(\sigma, l_a^2((k+1)^0), H^\infty))^{1-\theta} (c(\sigma, l_a^2((k+1)^{-1}), H^\infty))^\theta \leq \\ &\leq \left( A(0) (n(1-r)^{-1})^{\frac{1}{2}} \right)^{1-\theta} \left( A(1) (n(1-r)^{-1})^{\frac{3}{2}} \right)^\theta = A(0)^{1-\theta} A(1)^\theta (n(1-r)^{-1})^{\frac{1-\theta}{2} + \frac{3\theta}{2}}, \end{aligned}$$

where the operator  $T$  is defined in (5.1.1). It remains to use  $\theta = -\alpha$ , set  $A(\alpha) = A(0)^{1-\theta} A(1)^\theta$  and apply (5.1.2) with  $X = l_a^2((k+1)^\alpha)$ . In particular, for  $\alpha = -1/2$  we get  $(1-\theta)/2 + 3\theta/2 = 1$  and

$$A(-1/2) = A(0)^{(1-1/2)} A(1)^{1/2} = \sqrt{2}^{1/2} (2\sqrt{10})^{1/2} = \sqrt{2} \cdot 10^{\frac{1}{4}}.$$

□



7. LOWER BOUNDS FOR  $C_{n,r}(X, H^\infty)$ 

**7.1. The cases  $X = H^2$  and  $X = L_a^2$ .** Here, we consider the standard Hardy and Bergman spaces,  $X = H^2 = l_a^2(1)$  and  $X = L_a^2 = l_a^2((k+1)^{-1/2})$ , where the spaces  $l_a^2((k+1)^\alpha)$  are defined in Section 6, and the problem of lower estimates for the one-point special case  $\sigma_{n,\lambda} = \{\lambda, \lambda, \dots, \lambda\}$ , ( $n$  times)  $\lambda \in \mathbb{D}$ . Recall the definition of our constrained interpolation constant for this case

$$c(\sigma_{n,\lambda}, H, H^\infty) = \sup \{ \|f\|_{H^\infty/b_\lambda^n H^\infty} : f \in H, \|f\|_H \leq 1 \},$$

where  $\|f\|_{H^\infty/b_\lambda^n H^\infty} = \inf \{ \|f + b_\lambda^n g\|_\infty : g \in H \}$ . Our goal in this subsection is to prove the sharpness of the upper estimate stated in Theorem C ( $p = 2$ ) and in Theorem D for the quantities  $C_{n,r}(H^2, H^\infty)$  and  $C_{n,r}(L_a^2, H^\infty)$ , that is to say, to get the lower bounds of Theorem C ( $p = 2$ ) and Theorem D.

In the proof, we use properties of reproducing kernel Hilbert space on the disc  $\mathbb{D}$ , see for example [15]. Let us recall some of them adapting the general setting to special cases  $X = l_a^2((k+1)^\alpha)$ . The reproducing kernel of  $l_a^2((k+1)^\alpha)$ , by definition, is a  $l_a^2((k+1)^\alpha)$ -valued function  $\lambda \mapsto k_\lambda^\alpha$ ,  $\lambda \in \mathbb{D}$ , such that  $(f, k_\lambda^\alpha) = f(\lambda)$  for every  $f \in l_a^2((k+1)^\alpha)$ , where  $(\cdot, \cdot)$  means the scalar product  $(h, g) = \sum_{k \geq 0} \hat{h}(k) \overline{\hat{g}(k)} (k+1)^{-2\alpha}$ . Since one has  $f(\lambda) = \sum_{k \geq 0} \hat{f}(k) \lambda^k (k+1)^{2\alpha} (k+1)^{-2\alpha}$  ( $\lambda \in \mathbb{D}$ ), it follows that

$$k_\lambda^\alpha(z) = \sum_{k \geq 0} (k+1)^{2\alpha} \overline{\lambda^k} z^k, \quad z \in \mathbb{D}.$$

In particular, for the Hardy space  $H^2 = l_a^2(1)$  ( $\alpha = 0$ ), we get the Szegő kernel  $k_\lambda(z) = (1 - \overline{\lambda}z)^{-1}$  and for the Bergman space  $L_a^2 = l_a^2((k+1)^{-1/2})$  ( $\alpha = -1/2$ ), the Bergman kernel  $k_\lambda^{-1/2}(z) = (1 - \overline{\lambda}z)^{-2}$ .

We will use the previous observations for the following composed reproducing kernels (Aronszajn-deBranges, see [1, 15]): given the reproducing kernel  $k$  of  $H^2$  and  $\varphi \in \{z^N : N = 1, 2\}$ , the function  $\varphi \circ k$  is also positive definite and the corresponding Hilbert space is

$$H_\varphi = \varphi(H^2) = l_a^2\left((k+1)^{\frac{1-N}{2}}\right).$$

It satisfies the following property: for every  $f \in H^2$ ,  $\varphi \circ f \in \varphi(H^2)$  and  $\|\varphi \circ f\|_{\varphi(H^2)}^2 \leq \varphi(\|f\|_{H^2}^2)$  (see [15] p.320).

We notice in particular that

$$(7.1.1) \quad H_z = H^2 \quad \text{and} \quad H_{z^2} = L_a^2.$$

The above relation between the spaces  $H^2$ ,  $L_a^2$  and the spaces  $\varphi(H^2) = H_\varphi$  leads to establish the proof of the left-hand side inequalities stated in Theorem C (for  $p = 2$  only) and in Theorem D.

*Proof of Theorem C ( $p = 2$ ) and Theorem D (left-hand side inequalities only) .*

1) We set

$$Q_n = \sum_{k=0}^{n-1} (1 - |\lambda|^2)^{1/2} b_\lambda^k (1 - \overline{\lambda}z)^{-1}, \quad H_n = \varphi \circ Q_n \quad \text{and} \quad \Psi = bH_n, \quad b > 0.$$

Then  $\|Q_n\|_2^2 = n$ , and hence by the above Aronszajn-deBranges inequality,

$$\|\Psi\|_{H_\varphi}^2 \leq b^2 \varphi(\|Q_n\|_2^2) = b^2 \varphi(n).$$

Let  $b > 0$  such that  $b^2 \varphi(n) = 1$ .

2) Since the spaces  $H_\varphi$  and  $H^\infty$  are rotation invariant, we have  $c(\sigma_{n,\lambda}, H_\varphi, H^\infty) = c(\sigma_{\mu,n}, H_\varphi, H^\infty)$  for every  $\lambda, \mu$  with  $|\lambda| = |\mu| = r$ . Let  $\lambda = -r$ . To get a lower estimate for  $\|\Psi\|_{H_\varphi/b_\lambda^n H_\varphi}$  consider  $G$  such that  $\Psi - G \in b_\lambda^n \text{Hol}(\mathbb{D})$ , i.e. such that  $bH_n \circ b_\lambda - G \circ b_\lambda \in z^n \text{Hol}(\mathbb{D})$ .

3) First, we show that

$$\psi =: \Psi \circ b_\lambda = bH_n \circ b_\lambda$$

is a polynomial (of degree  $n$  if  $\varphi = z$  and  $2n$  if  $\varphi = z^2$ ) with positive coefficients. Note that

$$\begin{aligned} Q_n \circ b_\lambda &= \sum_{k=0}^{n-1} z^k \frac{(1 - |\lambda|^2)^{1/2}}{1 - \bar{\lambda} b_\lambda(z)} = (1 - |\lambda|^2)^{-\frac{1}{2}} \left( 1 + (1 - \bar{\lambda}) \sum_{k=1}^{n-1} z^k - \bar{\lambda} z^n \right) = \\ &= (1 - r^2)^{-1/2} \left( 1 + (1 + r) \sum_{k=1}^{n-1} z^k + r z^n \right) =: (1 - r^2)^{-1/2} \psi_1. \end{aligned}$$

Hence,  $\psi = \Psi \circ b_\lambda = bH_n \circ b_\lambda = b\varphi \circ \left( (1 - r^2)^{-\frac{1}{2}} \psi_1 \right)$  and

$$\varphi \circ \psi_1 = \psi_1^N(z), \quad N = 1, 2.$$

4) Next, we show that

$$\sum_{j=0}^m (\psi) =: \sum_{j=0}^m \hat{\psi}(j) \geq \begin{cases} (2\sqrt{2})^{-1} \sqrt{n(1-r)^{-1}} & \text{if } N = 1 \\ 16^{-1} n (1-r)^{-1} & \text{if } N = 2 \end{cases},$$

where  $m = n/2$  if  $n$  is even and  $m = (n+1)/2$  if  $n$  is odd.

Indeed, setting  $S_n = \sum_{j=0}^n z^j$ , we have both for  $N = 1$  and  $N = 2$

$$\sum_{j=0}^m (\psi_1^N) = \sum_{j=0}^m \left( \left( 1 + (1+r) \sum_{t=1}^{n-1} z^t + r z^n \right)^N \right) \geq \sum_{j=0}^m (S_{n-1}^N).$$

Next, we obtain

$$\begin{aligned} \sum_{j=0}^m (S_{n-1}^N) &= \sum_{j=0}^m \left( \left( \frac{1 - z^n}{1 - z} \right)^N \right) = \\ &= \sum_{j=0}^m ((1 - z)^{-N}) = \sum_{j=0}^m \left( \sum_{j \geq 0} C_{N+j-1}^j z^j \right) = \sum_{j=0}^m C_{N+j-1}^j. \end{aligned}$$

Now if  $N = 1$ , then

$$\sum_{j=0}^m C_{N+j-1}^j = m + 1 \geq \frac{n}{2},$$

whereas if  $N = 2$  then

$$\sum_{j=0}^m C_{N+j-1}^j = \frac{(m+1)(m+2)}{2} \geq \frac{(n+2)(n+4)}{8} \geq \frac{n^2}{8}.$$

Finally, since  $\sum^m (\psi) = b \sum^m (\varphi \circ \psi_1) = b (1 - r^2)^{-N/2} \sum^m (\psi_1^N)$  we get

$$\sum^m (\psi) \geq \begin{cases} (2(1-r))^{-1/2} n b / 2 & \text{if } N = 1 \\ (2(1-r))^{-1} n^2 b / 8 & \text{if } N = 2 \end{cases},$$

with  $b = \varphi(n) = \begin{cases} n^{-1/2} & \text{if } N = 1 \\ n^{-1} & \text{if } N = 2 \end{cases}$ . This gives the result claimed.

5) Now, using point 4) and denoting  $F_n = \Phi_m + z^m \Phi_m$ , where  $\Phi_k$  stands for the  $k$ -th Fejer kernel, we get

$$\begin{aligned} \|\Psi\|_{H^\infty/b\lambda H^\infty} &= \|\psi\|_{H^\infty/z^n H^\infty} \geq 2^{-1} \|\psi * F_n\|_\infty \geq 2^{-1} \sum_{j=0}^m \hat{\psi}(j) \geq \\ &\geq \begin{cases} (4\sqrt{2})^{-1} \sqrt{n(1-r)^{-1}} & \text{if } N = 1 \\ 32^{-1} n(1-r)^{-1} & \text{if } N = 2 \end{cases}. \end{aligned}$$

6) In order to conclude, it remains to use (7.1.1). □

**7.2. The case  $X = H^p$ ,  $1 \leq p \leq +\infty$ .** Here we prove the sharpness (for even  $p$ ) of the upper estimate found in Theorem C. We first prove the following lemma.

**Lemma. 7.2.1.** *Let  $p, q$  such that  $\frac{p}{q} \in \mathbb{Z}_+$ , then  $c(\sigma, H^p, H^\infty) \geq c(\sigma, H^q, H^\infty)^{\frac{q}{p}}$  for every sequence  $\sigma$  of  $\mathbb{D}$ .*

*Proof. Step 1.* Recalling that

$$c(\sigma, H^p, H^\infty) = \sup_{\|f\|_p \leq 1} \inf \{ \|g\|_\infty : g \in Y, g|_\sigma = f|_\sigma \},$$

we first prove that

$$c(\sigma, H^p, H^\infty) = \sup_{\|f\|_p \leq 1, f \text{ outer}} \inf \{ \|g\|_\infty : g \in Y, g|_\sigma = f|_\sigma \}.$$

Indeed, we clearly have the inequality

$$\sup_{\|f\|_p \leq 1, f \text{ outer}} \inf \{ \|g\|_\infty : g \in Y, g|_\sigma = f|_\sigma \} \leq c(\sigma, H^p, H^\infty),$$

and if the inequality were strict, that is to say

$$\sup_{\|f\|_p \leq 1, f \text{ outer}} \inf \{ \|g\|_\infty : g \in Y, g|_\sigma = f|_\sigma \} < \sup_{\|f\|_p \leq 1} \inf \{ \|g\|_\infty : g \in Y, g|_\sigma = f|_\sigma \},$$

then we could write that there exists  $\epsilon > 0$  such that for every  $f = f_i \cdot f_o \in H^p$  (where  $f_i$  stands for the inner function corresponding to  $f$  and  $f_o$  to the outer one) with  $\|f\|_p \leq 1$  (which also implies that  $\|f_o\|_p \leq 1$ , since  $\|f_o\|_p = \|f\|_p$ ), there exists a function  $g \in H^\infty$  verifying both  $\|g\|_\infty \leq (1 - \epsilon)c(\sigma, H^p, H^\infty)$  and  $g|_\sigma = f_o|_\sigma$ . This entails that  $f|_\sigma = (f_i g)|_\sigma$  and since  $\|f_i g\|_\infty = \|g\|_\infty \leq (1 - \epsilon)c(\sigma, H^p, H^\infty)$ , we get that  $c(\sigma, H^p, H^\infty) \leq (1 - \epsilon)c(\sigma, H^p, H^\infty)$ , which is a contradiction and proves the equality of Step 1.

**Step 2.** Using the result of Step 1, we get that  $\forall \epsilon > 0$  there exists an outer function  $f_o \in H^q$  with  $\|f_o\|_q \leq 1$  and such that

$$\inf \{ \|g\|_\infty : g \in Y, g|_\sigma = f_o|_\sigma \} \geq c(\sigma, H^q, H^\infty) - \epsilon.$$

Now let  $F = f_o^{\frac{q}{p}} \in H^p$ , then  $\|F\|_p^p = \|f_o\|_q^q \leq 1$ . We suppose that there exists  $g \in H^\infty$  such that  $g|_\sigma = F|_\sigma$  with

$$\|g\|_\infty < (c(\sigma, H^q, H^\infty) - \epsilon)^{\frac{q}{p}}.$$

Then, since  $g(\lambda_i) = F(\lambda_i) = f_o(\lambda_i)^{\frac{q}{p}}$  for all  $i = 1..n$ , we have  $g(\lambda_i)^{\frac{p}{q}} = f_o(\lambda_i)$  and  $g^{\frac{p}{q}} \in H^\infty$  since  $\frac{p}{q} \in \mathbb{Z}_+$ . We also have

$$\left\| g^{\frac{p}{q}} \right\|_\infty = \|g\|_\infty^{\frac{p}{q}} < (c(\sigma, H^q, H^\infty) - \epsilon)^{\frac{q}{p}},$$

which is a contradiction. As a result, we have

$$\|g\|_\infty \geq (c(\sigma, H^q, H^\infty) - \epsilon)^{\frac{q}{p}},$$

for all  $g \in H^\infty$  such that  $g|_\sigma = F|_\sigma$ , which gives

$$c(\sigma, H^p, H^\infty) \geq (c(\sigma, H^q, H^\infty) - \epsilon)^{\frac{q}{p}},$$

and since that inequality is true for every  $\epsilon > 0$ , we get the result.  $\square$

*Proof of Theorem C (the left-hand side inequality for  $p \in 2\mathbb{N}$ ,  $p > 2$  only).* We first prove the lower estimate for  $c(\sigma_{n,\lambda}, H^p, H^\infty)$ . Writing  $p = 2(p/2)$ , we apply Lemma 7.2.1 with  $q = 2$  and this gives

$$c(\sigma_{n,\lambda}, H^p, H^\infty) \geq c(\sigma_{n,\lambda}, H^2, H^\infty)^{\frac{2}{p}} \geq 32^{-\frac{1}{p}} (n(1 - |\lambda|)^{-1})^{\frac{2}{p}}$$

for all integer  $n \geq 1$ . The last inequality is a consequence of Theorem C (left-hand side inequality) for the particular case  $p = 2$  which has been proved in Subsection 7.1.  $\square$

## 8. Comparing our results with Carleson interpolation

Recall that given a (finite) set  $\sigma = \{\lambda_1, \dots, \lambda_n\} \subset \mathbb{D}$ , the Carleson interpolation constant  $C_I(\sigma)$  is defined by

$$C_I(\sigma) = \sup_{\|a\|_\infty \leq 1} \inf (\|g\|_\infty : g \in H^\infty, g|_\sigma = a).$$

We introduce the evaluation functionals  $\varphi_\lambda$  for  $\lambda \in \mathbb{D}$ , as well as the evaluation of the derivatives  $\varphi_{\lambda,s}$  ( $s = 0, 1, \dots$ )

$$\varphi_\lambda(f) = f(\lambda), \quad f \in X, \quad \text{and} \quad \varphi_{\lambda,s}(f) = f^{(s)}(\lambda), \quad f \in X.$$

**Theorem E.** *Let  $X$  be a Banach space,  $X \subset \text{Hol}(\mathbb{D})$ , and  $\sigma = \{\lambda_1, \dots, \lambda_n\}$  be a sequence of distinct points in the unit disc  $\mathbb{D}$ . We have,*

$$\max_{1 \leq i \leq n} \|\varphi_{\lambda_i}\| \leq c(\sigma, X, H^\infty) \leq C_I(\sigma) \cdot \max_{1 \leq i \leq n} \|\varphi_{\lambda_i}\|,$$

where  $C_I(\sigma)$  stands for the Carleson interpolation constant.

Theorem E (see [20] for its proof) tells us that, for  $\sigma$  with a “reasonable” interpolation constant  $C_I(\sigma)$ , the quantity  $c(\sigma, X, H^\infty)$  behaves as  $\max_i \|\varphi_{\lambda_i}\|$ . However, for “tight” sequences  $\sigma$ , the constant  $C_I(\sigma)$  is so large that the estimate in question contains almost no information. On the other hand, an advantage of the estimate of Theorem E is that it does not contain  $\text{card } \sigma = n$  explicitly. Therefore, for well-separated sequences  $\sigma$ , Theorem E should give a better estimate than those of Theorem C and Theorem D.

Now, how does the interpolation constant  $C_I(\sigma)$  behave in terms of the characteristics  $r$  and  $n$  of  $\sigma$ ? We answer this question in [20] for some particular sequences  $\sigma$ . More precisely, we compare these quantities for the cases  $X = H^2$ ,  $X = L_a^2$  and for three geometrically simple configurations: *two-points sets*  $\sigma$ , *circular* and *radial* sequences  $\sigma$ .

Let us recall that our specific upper bounds in Theorems C and D are sharp over all  $n$  elements sequences  $\sigma$ . However, we give in [20] some very special *radial* and *circular* sequences  $\sigma$  such that the estimate of  $c(\sigma, H^2, H^\infty)$  via the Carleson constant  $C_I(\sigma)$  (using Theorem E) is comparable with or

better than the estimates of Theorem C (for  $X = H^2$ ) and Theorem D (for  $X = L_a^2$ ). We also give some examples of *radial* and *circular* sequences but also of *two-points sets*, such that it is worse (i.e. for which our estimate is better). More specific *radial sequences* are studied in [20]: *sparse sequences*, *condensed sequences* and *long sequences*.

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