

Effective H^∞ interpolation constrained by Hardy and Bergman norms

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Abstract

Given a finite set σ of the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and a holomorphic function f in \mathbb{D} which belongs to a class X , we are looking for a function g in another class Y (smaller than X or incomparable with X) which minimizes the norm $\|g\|_Y$ among all functions g such that $g|_\sigma = f|_\sigma$. For $Y = H^\infty$, $X = H^p$ (the Hardy space) or $X = L_a^2$ (the Bergman space), and for the corresponding interpolation constant $c(\sigma, X, H^\infty)$, we show that $c(\sigma, X, H^\infty) \leq a\varphi_X \left(1 - \frac{1-r}{n}\right)$ where $n = \#\sigma$, $r = \max_{\lambda \in \sigma} |\lambda|$ and where $\varphi_X(t)$ stands for the norm of the evaluation functional $f \mapsto f(t)$ on the space X . The upper bound is sharp over sets σ with given n and r .

Résumé

Etant donné un ensemble fini σ du disque unité $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ et une fonction f holomorphe dans \mathbb{D} appartenant à une certaine classe X , on cherche g dans une autre classe Y (plus petite que X ou incomparable avec X) qui minimise la norme de g dans Y parmi toutes les fonctions g satisfaisant la condition $g|_\sigma = f|_\sigma$. On montre que dans le cas $Y = H^\infty$, la constante d'interpolation correspondante $c(\sigma, X, H^\infty)$ admet une majoration $c(\sigma, X, H^\infty) \leq a\varphi_X \left(1 - \frac{1-r}{n}\right)$ où $n = \#\sigma$, $r = \max_{\lambda \in \sigma} |\lambda|$ et $\varphi_X(t)$ est la norme de la fonctionnelle d'évaluation $f \mapsto f(t)$, $0 \leq t < 1$, sur l'espace X . La majoration est exacte sur l'ensemble des σ avec n et r donné.

Introduction

(1) General framework. Let $Hol(\mathbb{D})$ be the space of holomorphic functions on the unit disc \mathbb{D} . The problem considered is the following : given two Banach spaces X and Y of holomorphic functions on the unit disc \mathbb{D} , $X, Y \subset Hol(\mathbb{D})$, and a finite set $\sigma \subset \mathbb{D}$, to find the least norm interpolation by functions of the space Y for the traces $f|_\sigma$ of functions of the space X , in the worst case of f . The case $X \subset Y$ is of no interests, and so one can suppose that either $Y \subset X$ or X, Y are incomparable.

The classical interpolation problems- those of Nevanlinna-Pick (1916) and Carathéodory-Schur (1908) (see [N2] p.231 for these two problems), on the one hand and Carleson's free interpolation (1958) (see [N1] p.158) on the other hand- are of this nature. Two first are "individual", in the sens that one looks simply to compute the norms $\|f\|_{H|_\sigma^\infty}$ or $\|f\|_{H^\infty/z^n H^\infty}$ for a given f , whereas the third one is to compare the norms $\|a\|_{l^\infty(\sigma)} = \max_{\lambda \in \sigma} |a_\lambda|$ and

$$\inf (\|g\|_\infty, g(\lambda) = a_\lambda, \lambda \in \sigma).$$

Let us first explain that our problem assemblies the ones of Nevanlinna-Pick and Carathéodory-Schur.

(i) Nevanlinna-Pick interpolation problem

Given $\Lambda = (\lambda_1, \dots, \lambda_n)$ in \mathbb{D}^n and $W = (w_1, \dots, w_n) \in \mathbb{C}^n$, to find

$$C(\Lambda, W) = \inf \{ \|f\|_\infty : f(\lambda_i) = w_i, i = 1..n \}.$$

The classical answer of Pick is the following :

$$C(\Lambda, W) = \inf \left\{ c > 0 : \left(\frac{c^2 - \overline{w_i} w_j}{1 - \overline{\lambda_i} \lambda_j} \right)_{1 \leq i, j \leq n} \gg 0 \right\},$$

where for any $n \times n$ matrix M , $M \gg 0$ means that M is positive definite.

(ii) Carathéodory-Schur interpolation problem

Given $\mathcal{A} = (a_0, \dots, a_n) \in \mathbb{C}^{n+1}$, to find

$$C(\mathcal{A}) = \inf \{ \|f\|_\infty : f(z) = a_0 + a_1z + \dots + a_nz^n + \dots \}.$$

The classical answer of Schur is the following :

$$C(\mathcal{A}) = \|(T_\varphi)_n\|,$$

where T_φ is the Toeplitz operator associated with a symbol φ , $(T_\varphi)_n$ is the compression of T_φ on \mathcal{P}_n , the space of analytic polynomials of degree less or equal than n , and φ is the polynomial $\sum_{k=0}^n a_k z^k$.

Notice that the Carathéodory-Schur interpolation theorem can be seen as a particular case of the famous commutant lifting theorem of Sarason and Sz-Nagy-Foias (1968) see [N2] p.230, Theorem 3.1.11.

From a modern point of view, those two interpolation problems (i)&(ii) are unified through the following mixed problem : given

- $\sigma = \{\lambda_1, \dots, \lambda_n\} \subset \mathbb{D}$, the finite Blaschke product $B_\sigma = \prod_j b_{\lambda_j}$, where $b_\lambda = \frac{\lambda-z}{1-\lambda z}$, $\lambda \in \mathbb{D}$,
- $f \in Hol(\mathbb{D})$,

to compute or estimate

$$\|f\|_{H^\infty/BH^\infty} = \inf \{ \|g\|_\infty : f - g \in B_\sigma Hol(\mathbb{D}) \}.$$

The classical Nevanlinna-Pick problem corresponds to the case $X = Hol(\mathbb{D})$, $Y = H^\infty$, and the one of Carathéodory-Schur to the case where $\lambda_1 = \lambda_2 = \dots = \lambda_n = 0$ and $X = Hol(\mathbb{D})$, $Y = H^\infty$.

Here and everywhere below, H^∞ stands for the space (algebra) of bounded holomorphic functions in the unit disc \mathbb{D} endowed with the norm $\|f\|_\infty = \sup_{z \in \mathbb{D}} |f(z)|$. Looking at this comparison problem, say, in the form of computing/estimating the interpolation constant

$$c(\sigma, X, Y) = \sup_{f \in X, \|f\|_X \leq 1} \inf \{ \|g\|_Y : g|_\sigma = f|_\sigma \},$$

which is nothing but the norm of the embedding operator $(X|_\sigma, \|\cdot\|_{X|_\sigma}) \rightarrow (Y|_\sigma, \|\cdot\|_{Y|_\sigma})$, one can think, of course, on passing (after) to the limit- in the case of an infinite sequence $\{\lambda_j\}$ and its finite sections $\{\lambda_j\}_{j=1}^n$ - in order to obtain a Carleson type interpolation theorem $X|_\sigma = Y|_\sigma$. But not necessarily. In particular, even the classical Nevanlinna-Pick theorem (giving a necessary and sufficient condition on a function a for the existence of $f \in H^\infty$ such that $\|f\|_\infty \leq 1$ and $f(\lambda) = a_\lambda$, $\lambda \in \sigma$), does not lead immediately to Carleson's criterion for $H|_\sigma^\infty = l^\infty(\sigma)$. (Finally, a direct deduction of Carleson's theorem from Pick's result was done by P. Koosis [K] in 1999 only). Similarly, the problem stated for $c(\sigma, X, Y)$ is of interest in its own. It is a kind of "effective interpolation" because we are looking for sharp estimations or a computation of $c(\sigma, X, Y)$ for a variety of norms $\|\cdot\|_X$, $\|\cdot\|_Y$. For this paper, the following partial case was especially stimulating (which is a part of a more complicated question arising in an applied situation in [BL1] and [BL2]) : given a set $\sigma \subset \mathbb{D}$, how to estimate $c(\sigma, H^2, H^\infty)$ in terms of $n = \text{card}(\sigma)$ and $\max_{\lambda \in \sigma} |\lambda| = r$ only? (H^2 being the standard Hardy space of the disc).

Here, we consider the case of H^∞ interpolation ($Y = H^\infty$) and the following scales of Banach spaces X :

(a) $X = H^p = H^p(\mathbb{D})$, $1 \leq p \leq \infty$, the standard Hardy spaces on the disc \mathbb{D} (see [N2] p.31-p.57) of all $f \in Hol(\mathbb{D})$ satisfying

$$\sup_{0 \leq r < 1} \left(\int_{\mathbb{T}} |f(rz)|^p dm(z) \right)^{1/p} < \infty,$$

m being the Lebesgue normalized measure on \mathbb{T} .

(b) $X = l_a^2(1/\sqrt{k+1})$, the Bergman space of all $f(z) = \sum_{k \geq 0} \hat{f}(k)z^k$ satisfying

$$\sum_{k \geq 0} |\hat{f}(k)|^2 \frac{1}{k+1} < \infty.$$

An equivalent description of this space is : $X = L_a^2$, the space of holomorphic functions such that

$$\int_{\mathbb{D}} |f(z)|^2 dx dy < \infty.$$

For spaces of type (a)&(b), we show

$$c_1 \varphi_X \left(1 - \frac{1-r}{n} \right) \leq \sup \{ c(\sigma, X, H^\infty) : \#\sigma \leq n, |\lambda| \leq r, \lambda \in \sigma \} \leq c_2 \varphi_X \left(1 - \frac{1-r}{n} \right),$$

where $\varphi_X(t)$, $0 \leq t < 1$ stands for the norm of the evaluation functional $f \mapsto f(t)$ on the space X .

In order to prove the right-hand side inequality, we first use a linear interpolation:

$$f \mapsto \sum_{k=1}^n \langle f, e_k \rangle e_k,$$

where $\langle \cdot, \cdot \rangle$ means the Cauchy sesquilinear form $\langle h, g \rangle = \sum_{k \geq 0} \hat{h}(k) \overline{\hat{g}(k)}$, and $(e_k)_{k=1}^n$ is the Malmquist basis (effectively constructible) of the space $K_B = H^2 \ominus B H^2$, $B = \prod_{i=1}^n b_{\lambda_i}$ being the corresponding finite Blaschke product, $b_\lambda = \frac{\lambda-z}{1-\bar{\lambda}z}$ (see N. Nikolski, [N1] p. 117)). Next, we use the complex interpolation between Banach spaces, (see H. Triebel [Tr] Theorem 1.9.3-(a) p.59). Among the technical tools used in order to find an upper bound for $\|\sum_{k=1}^n \langle f, e_k \rangle e_k\|_\infty$ (in terms of $\|f\|_X$), the most important is a Bernstein-type inequality $\|f'\|_p \leq c_p \|B'\|_\infty \|f\|_p$ for a (rational) function f in the star-invariant subspace $K_B^p := H^p \cap \overline{BzH^p}$, $1 \leq p \leq \infty$ (for $p=2$, $K_B^2 = K_B$), generated by a (finite) Blaschke product B , (K. Dyakonov [Dya1]&[Dya2]). For $p=2$, we give an alternative proof of the Bernstein-type estimate we need and the constant c_2 we obtain is slightly better, see Section 4.

The lower bound problem is treated by using the “worst” interpolation n -tuple $\sigma = \sigma_{n,\lambda} = \{\lambda, \dots, \lambda\}$, a one-point set of multiplicity n (the Carathéodory-Schur type interpolation). The “worst” interpolation data comes from the Dirichlet kernels $\sum_{k=0}^{n-1} z^k$ transplanted from the origin to λ . We notice that spaces X of (a)&(b) satisfy the condition $X \circ b_\lambda \subset X$ which makes the problem of upper/lower bound easier.

(2) Principal results. Theorems A,C&D below in this paragraph, were already announced in the note [Z1].

Let $\sigma = \{\lambda_1, \dots, \lambda_1, \lambda_2, \dots, \lambda_2, \dots, \lambda_t, \dots, \lambda_t\}$ be a finite sequence in the unit disc, where every λ_s is repeated according its multiplicity m_s , $\sum_{s=1}^t m_s = n$ and $r = \max_{i=1..t} |\lambda_i|$. Let X, Y be Banach spaces of holomorphic functions continuously embedded into the space $Hol(\mathbb{D})$. In what follows, we systematically use the following conditions for the spaces X and Y ,

(P₁) $Hol((1+\epsilon)\mathbb{D})$ is continuously embedded into Y for every $\epsilon > 0$,

(P₂) $Pol_+ \subset X$ and Pol_+ is dense in X ,

where Pol_+ stands for the set of all complex polynomials p , $p(z) = \sum_{k=0}^N a_k z^k$,

(P₃) $[f \in X] \Rightarrow \left[z^n f \in X, \forall n \geq 0 \text{ and } \overline{\lim} \|z^n f\|^{\frac{1}{n}} \leq 1 \right]$,

(P₄) $[f \in X, \lambda \in \mathbb{D}, \text{ and } f(\lambda) = 0] \Rightarrow \left[\frac{f}{z - \lambda} \in X \right]$.

We are interested in estimating the quantity

$$c(\sigma, X, Y) = \sup_{\|f\|_X \leq 1} \inf \{ \|g\|_Y : g \in Y, g^{(j)}(\lambda_i) = f^{(j)}(\lambda_i) \forall i, j, 1 \leq i \leq t, 0 \leq j < m_i \}.$$

In order to simplify the notation, the condition

$$g^{(j)}(\lambda_i) = f^{(j)}(\lambda_i) \forall i, j, 1 \leq i \leq t, 0 \leq j < m_i$$

will also be written as

$$g|_{\sigma} = f|_{\sigma}.$$

Supposing X verifies property (P₄) and $Y \subset X$, the quantity $c(\sigma, X, Y)$ can be written as follows,

$$c(\sigma, X, Y) = \sup_{\|f\|_X \leq 1} \inf \{ \|g\|_Y : g \in Y, g - f \in B_{\sigma} X \},$$

where B_{σ} is the Blaschke product

$$B_{\sigma} = \prod_{i=1}^n b_{\lambda_i},$$

corresponding to σ , $b_{\lambda}(z) = \frac{\lambda - z}{1 - \bar{\lambda}z}$ being an elementary Blaschke factor for $\lambda \in \mathbb{D}$.

The interesting case occurs when X is larger than Y , and the sens of the issue lies in comparing $\| \cdot \|_X$ and $\| \cdot \|_Y$ when Y interpolates X on the set σ . For example, we can wonder what happens when $X = H^p$, the classical Hardy spaces of the disc or $X = L_a^p$, the Bergman spaces, etc..., and when $Y = H^{\infty}$, but also $Y = W$ the Wiener algebra (of absolutely converging Fourier series) or $Y = B_{\infty,1}^0$, a Besov algebra (an interesting case for the functional calculus of finite rank operators, in particular, those satisfying the so-called Ritt condition).

It is also important to understand what kind of interpolation we are going to study when bounding the constant $c(\sigma, X, Y)$. Namely, comparing with the Carleson free interpolation, we can say that the latter one deals with the interpolation constant defined as

$$c(\sigma, l^{\infty}(\sigma), H^{\infty}) = \sup \{ \inf (\|g\|_{\infty} : g \in H^{\infty}, g|_{\sigma} = a) : a \in l^{\infty}(\sigma), \|a\|_{l^{\infty}} \leq 1 \}.$$

We also can add some more motivations to our problem :

(a) One of the most interesting cases is $Y = H^\infty$. In this case, the quantity $c(\sigma, X, H^\infty)$ has a meaning of an intermediate interpolation between the Carleson one (when $\|f\|_{X_{|\sigma}} \asymp \sup_{1 \leq i \leq n} |f(\lambda_i)|$) and the individual Nevanlinna-Pick interpolation (no conditions on f).

(b) There is a straight link between the constant $c(\sigma, X, Y)$ and numerical analysis. For example, in matrix analysis, it is of interest to bound the norm of an H^∞ -calculus $\|f(A)\| \leq c \|f\|_\infty$, $f \in H^\infty$, for an arbitrary Banach space n -dimensional contraction A with a given spectrum $\sigma(A) \subset \sigma$. The best possible constant is $c = c(\sigma, H^\infty, W)$, so that

$$c(\sigma, H^\infty, W) = \sup_{\|f\|_\infty \leq 1} \sup \{ \|f(A)\| : A : (\mathbb{C}^n, |\cdot|) \rightarrow (\mathbb{C}^n, |\cdot|), \|A\| \leq 1, \sigma(A) \subset \sigma \},$$

where $W = \left\{ f = \sum_{k \geq 0} \hat{f}(k) z^k : \sum_{k \geq 0} |\hat{f}(k)| < \infty \right\}$ stands for the Wiener algebra, and the interior sup is taken over all contractions on n -dimensional Banach spaces. An interesting case occurs for $f \in H^\infty$ such that $f|_\sigma = \frac{1}{z|_\sigma}$ (estimation of condition numbers and the norm of inverses of $n \times n$ matrices) or $f|_\sigma = \frac{1}{\lambda - z|_\sigma}$ (for estimation of the norm of the resolvent of an $n \times n$ matrix).

We start studying general Banach spaces X and Y and give some sufficient condition under which $C_{n,r}(X, Y) < \infty$, where

$$C_{n,r}(X, Y) = \sup \{ c(\sigma, X, Y) : \#\sigma \leq n, \forall j = 1..n, |\lambda_j| \leq r \}.$$

In particular, we prove the following fact.

Theorem A. *Let X, Y be Banach spaces verifying properties (P_i) , $i = 1..4$. Then*

$$C_{n,r}(X, Y) < \infty,$$

for every $n \geq 1$ and r , $0 \leq r < 1$.

Next, we add the condition that X is a Hilbert space, and give in this case a general upper bound for the quantity $C_{n,r}(X, Y)$.

Theorem B. *Let Y be a Banach space verifying property (P_1) and $X = (H, (\cdot)_H)$ a Hilbert space satisfying properties (P_i) for $i = 2, 3, 4$. We moreover suppose that for every $0 < r < 1$ there exists $\epsilon > 0$ such that $k_\lambda \in \text{Hol}((1 + \epsilon)\mathbb{D})$ for all $|\lambda| < r$, where k_λ stands for the reproducing kernel of X at point λ , and $\bar{\lambda} \mapsto k_\lambda$ is holomorphic on $|\lambda| < r$ as a $\text{Hol}((1 + \epsilon)\mathbb{D})$ -valued function. Let $\sigma = \{\lambda_1, \dots, \lambda_1, \lambda_2, \dots, \lambda_2, \dots, \lambda_t, \dots, \lambda_t\}$ be a sequence in \mathbb{D} , where λ_s are repeated according their multiplicity m_s , $\sum_{s=1}^t m_s = n$. Then we have,*

i)

$$c(\sigma, H, Y) \leq \left(\sum_{k=1}^n \|e_k\|_Y^2 \right)^{\frac{1}{2}},$$

where $(e_k)_{k=1}^n$ stands for the Gram-Schmidt orthogonalization (in the space H) of the sequence

$$k_{\lambda_1,0}, k_{\lambda_1,1}, k_{\lambda_1,2}, \dots, k_{\lambda_1,m_1-1}, k_{\lambda_2,0}, k_{\lambda_2,1}, k_{\lambda_2,2}, \dots, k_{\lambda_2,m_2-1}, \dots, k_{\lambda_t,0}, k_{\lambda_t,1}, k_{\lambda_t,2}, \dots, k_{\lambda_t,m_t-1},$$

and $k_{\lambda,i} = \left(\frac{d}{d\lambda} \right)^i k_\lambda$, $i \in \mathbb{N}$.

ii) For the case $Y = H^\infty$, we have

$$c(\sigma, H, H^\infty) \leq \sup_{z \in \mathbb{D}} \|P_{B_\sigma}^H k_z\|_H,$$

where $P_{B_\sigma}^H = \sum_{k=1}^n (\cdot, e_k)_H e_k$ stands for the orthogonal projection of H onto $K_{B_\sigma}(H)$,

$$K_{B_\sigma}(H) = \text{span} (k_{\lambda_j, i} : 1 \leq i < m_j, j = 1, \dots, t).$$

After that, we specialize the upper bound obtained in Theorem B (ii) to the case $X = H^2$, the standard Hardy space of the disc, which can be equivalently defined as

$$H^2(\mathbb{D}) = \left\{ f = \sum_{k \geq 0} \hat{f}(k) z^k : \sum_{k \geq 0} |\hat{f}(k)|^2 < \infty \right\}.$$

Among other results, we get the following (see Proposition 2.0) : for every sequence $\sigma = \{\lambda_1, \dots, \lambda_n\}$ of \mathbb{D} ,

$$c(\sigma, H^2, H^\infty) \leq \sup_{z \in \mathbb{D}} \left(\frac{1 - |B_\sigma(z)|^2}{1 - |z|^2} \right)^{\frac{1}{2}} \leq \sqrt{2} \sup_{|\zeta|=1} |B'(\zeta)|^{\frac{1}{2}} \leq 2 \sqrt{\frac{n}{1-r}}.$$

Next, we present a slightly different approach to the interpolation constant $c(\sigma, H^2, H^\infty)$ proving an estimate in the following form:

$$c(\sigma, H^2, H^\infty) \leq \sup_{z \in \mathbb{T}} \left(\sum_{k=1}^n \frac{(1 - |\lambda_k|^2)}{|z - \lambda_k|^2} \right)^{\frac{1}{2}} \leq \left(\sum_{j=1}^n \frac{1 + |\lambda_j|}{1 - |\lambda_j|} \right)^{\frac{1}{2}} \leq \sqrt{\frac{2n}{1-r}}.$$

It is shown (in Section 6) that this estimate is sharp (over n and r). This sharpness result is treated by using the ‘‘worst’’ interpolation n -tuple $\sigma = \sigma_{n, \lambda} = \{\lambda, \dots, \lambda\}$, a one-point set of multiplicity n (the Carathéodory-Schur type interpolation). More precisely, we prove the following Theorem C, which contains the result from Corollary 2.1 and extends it to the H^p spaces, as follows.

Theorem C. *Let $1 \leq p \leq \infty$, $n \geq 1$, $r \in [0, 1)$, and λ , $|\lambda| \leq r$. We have,*

$$\frac{1}{32^{\frac{1}{p}}} \left(\frac{n}{1 - |\lambda|} \right)^{\frac{1}{p}} \leq c(\sigma_{n, \lambda}, H^p, H^\infty) \leq C_{n, r}(H^p, H^\infty) \leq A_p \left(\frac{n}{1 - r} \right)^{\frac{1}{p}},$$

where A_p is a constant depending only on p and the left hand side inequality is proved only for $p \in 2\mathbb{Z}_+$. For $p = 2$, we have $A_2 = \sqrt{2}$.

In particular, this gives yet another proof of the fact that $C_{n, r}(H^2, H^\infty) \leq a\sqrt{n}/\sqrt{1-r}$.

For the Bergman space $X = L_a^2$ we have the following Theorem D.

Theorem D. *Let $n \geq 1$, $r \in [0, 1)$, and λ , $|\lambda| \leq r$. We have,*

$$\frac{1}{32} \frac{n}{1 - |\lambda|} \leq c(\sigma_{n, \lambda}, L_a^2, H^\infty) \leq C_{n, r}(L_a^2, H^\infty) \leq 6\sqrt{2} \frac{n}{1 - r}.$$

The paper is organized as follows. In Subsection 1.1 we prove Theorem A. Theorem B is proved in Subsection 1.2. Sections 2&3 are devoted to the proof of the upper estimate of Theorem C, and Section 6 to the proofs of the lower bounds from Theorem C&D. In Section 5 we compare the method used in Sections 1, 2, 3 and 4 with those resulting from the Carleson free interpolation. Especially, we are interested in the cases of circular and radial sequences σ (see below).

1. Upper bounds for $c(\sigma, X, Y)$, as a kind of the Nevanlinna-Pick problem

1.1. General Banach spaces X and Y satisfying properties (P_i) , $i = 1 \dots 4$

In this Subsection, X and Y are Banach spaces which satisfy properties (P_i) for $i = 1 \dots 4$. We prove Theorem A which shows that in this case our interpolation constant $c(\sigma, X, Y)$ is bounded by a quantity which depends only on $n = \#\sigma$ and $r = \max_{1 \leq i \leq n} |\lambda_i|$ (and of course on X and Y). In this generality, we cannot discuss the question of sharpness of the bounds obtained. First, we prove the following lemma.

Lemma. 1.1.0. *Under (P_2) , (P_3) and (P_4) , $B_\sigma X$ is a closed subspace of X and moreover,*

$$B_\sigma X = \{f \in X : f(\lambda) = 0, \forall \lambda \in \sigma \text{ (including multiplicities)}\}.$$

Proof. Since $X \subset \text{Hol}(\mathbb{D})$ continuously, and evaluation functionals $f \mapsto f(\lambda)$ and $f \mapsto f^{(k)}(\lambda)$, $k \in \mathbb{N}^*$, are continuous on $\text{Hol}(\mathbb{D})$, the subspace

$$M = \{f \in X : f(\lambda) = 0, \forall \lambda \in \sigma \text{ (including multiplicities)}\},$$

is closed in X .

On the other hand, $B_\sigma X \subset X$, and hence $B_\sigma X \subset M$. Indeed, properties (P_2) and (P_3) imply that $h.X \subset X$, for all $h \in \text{Hol}((1 + \epsilon)\mathbb{D})$ with $\epsilon > 0$; we can write $h(z) = \sum_{k \geq 0} \widehat{h}(k)z^k$ with $|\widehat{h}(k)| \leq Cq^n$, $C > 0$ and $q < 1$. Then $\sum_{n \geq 0} \left\| \widehat{h}(k)z^k f \right\|_X < \infty$ for every $f \in X$. Since X is a Banach space we get $hf = \sum_{n \geq 0} \widehat{h}(k)z^k f \in X$.

In order to see that $M \subset B_\sigma X$, it suffices to justify that

$$[f \in X \text{ and } f(\lambda) = 0] \implies [f/b_\lambda = (1 - \bar{\lambda}z)f/(\lambda - z) \in X].$$

But this is obvious from (P_4) and the previous arguments. \square

In Definitions 1.1.1, 1.1.2, 1.1.3 and in Remark 1.1.4 below, $\sigma = \{\lambda_1, \dots, \lambda_n\}$ is a sequence in the unit disc \mathbb{D} , $B_\sigma = \prod_{i=1}^n b_{\lambda_i}$ is the finite Blaschke product corresponding to σ , where $b_\lambda = \frac{\lambda - z}{1 - \bar{\lambda}z}$ is an elementary Blaschke factor for $\lambda \in \mathbb{D}$.

Definition 1.1.1. *Malmquist family.* For $k \in [1, n]$, we set $f_k(z) = \frac{1}{1 - \lambda_k z}$, and define the family $(e_k)_{k=1}^n$, (which is known as Malmquist basis, see [N1] p.117), by

$$(1.1.1) \quad e_1 = \frac{f_1}{\|f_1\|_2} \text{ and } e_k = \left(\prod_{j=1}^{k-1} b_{\lambda_j} \right) \frac{f_k}{\|f_k\|_2},$$

for $k \in [2, n]$, where $\|f_k\|_2 = (1 - |\lambda_k|^2)^{-1/2}$.

Definition 1.1.2. *The model space K_{B_σ} .* We define K_{B_σ} to be the n -dimensional space :

$$(1.1.2) \quad K_{B_\sigma} = (B_\sigma H^2)^\perp = H^2 \ominus B_\sigma H^2.$$

Definition 1.1.3. *The orthogonal projection P_{B_σ} on K_{B_σ} .* We define P_{B_σ} to be the orthogonal projection of H^2 on its n -dimensional subspace K_{B_σ} .

Remark 1.1.4. The Malmquist family $(e_k)_{k=1}^n$ corresponding to σ is an orthonormal basis of K_{B_σ} . In particular,

$$(1.1.4) \quad P_{B_\sigma} = \sum_{k=1}^n (\cdot, e_k)_{H^2} e_k,$$

where $(\cdot, \cdot)_{H^2}$ means the scalar product on H^2 .

Lemma 1.1.5. *Let $\sigma = \{\lambda_1, \dots, \lambda_n\}$ be a sequence in the unit disc \mathbb{D} and $(e_k)_{k=1}^n$ the Malmquist family (see 1.1.1) corresponding to σ . The map $\Phi : Hol(\mathbb{D}) \rightarrow Y \subset Hol(\mathbb{D})$ defined by*

$$\Phi : f \mapsto \sum_{k=1}^n \left(\sum_{j \geq 0} \hat{f}(j) \overline{\hat{e}_k(j)} \right) e_k,$$

is well defined and has the following properties.

- (a) $\Phi|_{H^2} = P_{B_\sigma}$,
- (b) Φ is continuous on $Hol(\mathbb{D})$ for the uniform convergence on compact sets of \mathbb{D} ,
- (c) Let $\Psi = Id|_X - \Phi|_X$, then $Im(\Psi) \subset B_\sigma X$.

Proof. Indeed, the point (a) is obvious since $(e_k)_{k=1}^n$ is an orthonormal basis of K_{B_σ} and

$$\sum_{j \geq 0} \hat{f}(j) \overline{\hat{e}_k(j)} = \langle f, e_k \rangle,$$

where $\langle \cdot, \cdot \rangle$ means the Cauchy sesquilinear form $\langle h, g \rangle = \sum_{k \geq 0} \hat{h}(k) \overline{\hat{g}(k)}$. In order to check point (b), let $(f_l)_{l \in \mathbb{N}}$ be a sequence of $Hol(\mathbb{D})$ converging to 0 uniformly on compact sets of \mathbb{D} . We need to see that $(\Phi(f_l))_{l \in \mathbb{N}}$ converges to 0, for which it is sufficient to show that $\lim_l \left| \sum_{j \geq 0} \hat{f}_l(j) \overline{\hat{e}_k(j)} \right| = 0$, for every $k = 1, 2, \dots, n$. Let $\rho \in]0, 1[$, then $\hat{f}_l(j) = (2\pi)^{-1} \int_{\rho\mathbb{T}} f_l(w) w^{-j-1} dw$, for all $j, l \geq 0$. As a result,

$$\left| \sum_{j \geq 0} \hat{f}_l(j) \overline{\hat{e}_k(j)} \right| \leq \sum_{j \geq 0} \left| \hat{f}_l(j) \overline{\hat{e}_k(j)} \right| \leq (2\pi\rho)^{-1} \|f_l\|_{\rho\mathbb{T}} \sum_{j \geq 0} |\hat{e}_k(j)| \rho^{-j}.$$

Now if ρ is close enough to 1, it satisfies the inequality $1 \leq \rho^{-1} < r^{-1}$, which entails $\sum_{j \geq 0} |\hat{e}_k(j)| \rho^{-j} < +\infty$ for each $k = 1..n$. The result follows.

We now prove point (c). Using point (a), since $Pol_+ \subset H^2$ (Pol_+ standing for the set of all complex polynomials $p, p(z) = \sum_{k=0}^N a_k z^k$), we get that $Im(\Psi|_{Pol_+}) \subset B_\sigma H^2$. Now, since $Pol_+ \subset Y$ and $Im(\Phi) \subset Y$, we deduce that

$$Im(\Psi|_{Pol_+}) \subset B_\sigma H^2 \cap Y \subset B_\sigma H^2 \cap X,$$

since $Y \subset X$. Now $\Psi(p) \in X$ and satisfies $(\Psi(p))|_\sigma = 0$ (that is to say $(\Psi(p))(\lambda) = 0, \forall \lambda \in \sigma$ (including multiplicities)) for all $p \in Pol_+$. Using Lemma 1.1.0, we get that $Im(\Psi|_{Pol_+}) \subset B_\sigma X$. Now, Pol_+ being dense in X (property (P_2)), and Ψ being continuous on X (point (b)), we can conclude that $Im(\Psi) \subset B_\sigma X$. \square

Proof of Theorem A. Let $\sigma = \{\lambda_1, \dots, \lambda_n\}$ be a sequence in the unit disc \mathbb{D} and $(e_k)_{k=1}^n$ the Malmquist family (1.1.1) associated to σ . Taking $f \in X$, we set

$$g = \sum_{k=1}^n \left(\sum_{j \geq 0} \hat{f}(j) \overline{\hat{e}_k(j)} \right) e_k,$$

where the series $\sum_{j \geq 0} \hat{f}(j) \overline{\hat{e}_k(j)}$ are absolutely convergent. Indeed,

$$\hat{e}_k(j) = (2\pi i)^{-1} \int_{R\mathbb{T}} e_k(w) w^{-j-1} dw,$$

for all $j \geq 0$ and for all R , $1 < R < \frac{1}{r}$. For a subset A of \mathbb{C} and for a bounded function h on A , we define $\|h\|_A := \sup_{z \in A} |h(z)|$. As a result,

$$|\hat{e}_k(j)| \leq (2\pi R^{j+1})^{-1} \|e_k\|_{R\mathbb{T}} \quad \text{and} \quad \sum_{j \geq 0} \left| \hat{f}(j) \overline{\hat{e}_k(j)} \right| \leq (2\pi R)^{-1} \|e_k\|_{R\mathbb{T}} \sum_{j \geq 0} \left| \hat{f}(j) \right| R^{-j} < \infty,$$

since $R > 1$ and f is holomorphic in \mathbb{D} .

We now suppose that $\|f\|_X \leq 1$ and $g = \Phi(f)$, where Φ is defined in Lemma 1.1.5. Since $Hol(r^{-1}\mathbb{D}) \subset Y$, we have $g \in Y$ and using Lemma 1.1.5 point (c) we get

$$f - g = \Psi(f) \in B_\sigma X,$$

where Ψ is defined in Lemma 1.1.5, as Φ . Moreover,

$$\|g\|_Y \leq \sum_{k=1..n} |\langle f, e_k \rangle| \|e_k\|_Y.$$

In order to bound the right hand side, recall that for all $j \geq 0$ and for $R = 2/(r+1) \in]1, 1/r[$,

$$\sum_{j \geq 0} \left| \hat{f}(j) \overline{\hat{e}_k(j)} \right| \leq (2\pi)^{-1} \|e_k\|_{2(r+1)^{-1}\mathbb{T}} \sum_{j \geq 0} \left| \hat{f}(j) \right| (2^{-1}(r+1))^j.$$

Since the norm $f \mapsto \sum_{j \geq 0} \left| \hat{f}(j) \right| (2^{-1}(r+1))^j$ is continuous on $Hol(\mathbb{D})$, and the inclusion $X \subset Hol(\mathbb{D})$ is also continuous, there exists $C_r > 0$ such that

$$\sum_{j \geq 0} \left| \hat{f}(j) \right| (2^{-1}(r+1))^j \leq C_r \|f\|_X,$$

for every $f \in X$. On the other hand, $Hol(2(r+1)^{-1}\mathbb{D}) \subset Y$ (continuous inclusion again), and hence there exists $K_r > 0$ such that

$$\|e_k\|_Y \leq K_r \sup_{|z| < 2(r+1)^{-1}} |e_k(z)| = K_r \|e_k\|_{2(r+1)^{-1}\mathbb{T}}.$$

It is more or less clear that the right hand side of the last inequality can be bounded in terms of r and n only. Let us give a proof to this fact. It is clear that it suffices to estimate

$$\sup_{1 < |z| < 2(r+1)^{-1}} |e_k(z)|.$$

In order to bound this quantity, notice that

$$(1.1.6) \quad |b_\lambda(z)|^2 \leq \left| \frac{\lambda - z}{1 - \bar{\lambda}z} \right|^2 = 1 + \frac{(|z|^2 - 1)(1 - |\lambda|^2)}{|1 - \bar{\lambda}z|^2},$$

for all $\lambda \in \mathbb{D}$ and all $z \in |\lambda|^{-1}\mathbb{D}$. Using the identity (1.1.6) for $\lambda = \lambda_j$, $1 \leq j \leq n$, and $z = \rho e^{it}$, $\rho = 2(1+r)^{-1}$, we get

$$|e_k(\rho e^{it})|^2 \leq \left(\prod_{j=1}^{k-1} |b_{\lambda_j}(\rho e^{it})|^2 \right) \left| \frac{1}{1 - \bar{\lambda}_k \rho e^{it}} \right|^2 \leq \left(\prod_{j=1}^{k-1} \left(1 + \frac{(\rho^2 - 1)(1 - |\lambda_j|^2)}{1 - |\lambda_j|^2 \rho^2} \right) \right) \left(\frac{1}{1 - |\lambda_k| \rho} \right)^2,$$

for all $k = 2..n$. Expressing ρ in terms of r , we obtain

$$\|e_k\|_{2(r+1)^{-1}\mathbb{T}} \leq \frac{1}{1 - \frac{2r}{r+1}} \sqrt{2 \left(\prod_{j=1..n-1} \left(1 + \frac{2(\frac{1}{r^2} - 1)}{1 - r^2 \frac{4}{(r+1)^2}} \right) \right)} =: C_1(r, n),$$

and

$$\sum_{j \geq 0} \left| \hat{f}(j) \overline{\hat{e}_k(j)} \right| \leq (2\pi)^{-1} C_r \|e_k\|_{2(r+1)^{-1}\mathbb{T}} \|f\|_X \leq (2\pi)^{-1} C_r C_1(r, n) \|f\|_X.$$

On the other hand, since

$$\|e_k\|_Y \leq K_r \|e_k\|_{2(r+1)^{-1}\mathbb{T}} \leq K_r C_1(r, n),$$

we get

$$\|g\|_Y \leq \sum_{k=1}^n (2\pi)^{-1} C_r C_1(r, n) \|f\|_X K_r C_1(r, n) = (2\pi)^{-1} n C_r K_r (C_1(r, n))^2 \|f\|_X,$$

which proves that

$$c(\sigma, X, Y) \leq (2\pi)^{-1} n C_r K_r (C_1(r, n))^2$$

and completes the proof of Theorem A. \square

1.2. The case where X is a Hilbert space

We suppose in this Subsection that X is a Hilbert space and both X, Y satisfy properties (P_i) for $i = 1..4$. We prove Theorem B and obtain a better estimate for $c(\sigma, X, Y)$ than in Theorem A (see point (i) of Theorem B). For the case $Y = H^\infty$, (point (ii) of Theorem B), we can considerably improve this estimate. We omit an easy proof of the following lemma.

Lemma. 1.2.0. *Let $\sigma = \{\lambda_1, \dots, \lambda_1, \lambda_2, \dots, \lambda_2, \dots, \lambda_t, \dots, \lambda_t\}$ be a finite sequence of \mathbb{D} where every λ_s is repeated according to its multiplicity m_s , $\sum_{s=1}^t m_s = n$. Let $(H, (\cdot)_H)$ be a Hilbert space continuously embedded into $Hol(\mathbb{D})$ and satisfying properties (P_i) for $i = 2, 3, 4$. Then*

$$K_{B_\sigma}(H) =: H \ominus_{B_\sigma} H = span(k_{\lambda_j, i} : 1 \leq j \leq t, 0 \leq i \leq m_j - 1),$$

where $k_{\lambda, i} = \left(\frac{d}{d\lambda} \right)^i k_\lambda$ and k_λ is the reproducing kernel of H at point λ for every $\lambda \in \mathbb{D}$, i.e. $k_\lambda \in H$ and $f(\lambda) = (f, k_\lambda)_H$, $\forall f \in H$.

Proof of Theorem B. i). Let $f \in X$, $\|f\|_X \leq 1$. Lemma 1.2.0 shows that

$$g = P_{B_\sigma}^H f = \sum_{k=1}^n (f, e_k)_H e_k$$

is the orthogonal projection of f onto subspace K_{B_σ} . Function g belongs to Y because all $k_{\lambda_j, i}$ are in $Hol((1+\epsilon)\mathbb{D})$ for a convenient $\epsilon > 0$, and Y satisfies (P_1) .

On the other hand, $g - f \in B_\sigma H$ (again by Lemma 1.2.0). Moreover, using Cauchy-Schwarz inequality,

$$\|g\|_Y \leq \sum_{k=1}^n |(f, e_k)_H| \|e_k\|_Y \leq \left(\sum_{k=1}^n |(f, e_k)_H|^2 \right)^{1/2} \left(\sum_{k=1}^n \|e_k\|_Y^2 \right)^{1/2} \leq \|f\|_H \left(\sum_{k=1}^n \|e_k\|_Y^2 \right)^{1/2},$$

which proves i).

ii). If $Y = H^\infty$, then

$$|g(z)| = |(P_{B_\sigma}^H f, k_z)_H| = |(f, P_{B_\sigma}^H k_z)_H| \leq \|f\|_H \|P_{B_\sigma}^H k_z\|_H,$$

for all $z \in \mathbb{D}$, which proves ii). \square

2. Upper bounds for $C_{n,r}(H^2, H^\infty)$

In this Section, we specialize the upper estimate obtained in point (ii) of Theorem B for the case $X = H^2$, the Hardy space of the disc. Later on, we will see that this estimate is sharp at least for some special sequences σ (see Section 6). We also develop a slightly different approach to the interpolation constant $c(\sigma, H^2, H^\infty)$ giving more estimates for individual sequences $\sigma = \{\lambda_1, \dots, \lambda_n\}$ of \mathbb{D} . We finally prove the right-hand side inequality of Theorem C for the particular case $p = 2$.

Proposition. 2.0. *For every sequence $\sigma = \{\lambda_1, \dots, \lambda_n\}$ of \mathbb{D} we have*

$$(I_1) \quad c(\sigma, H^2, H^\infty) \leq \sup_{z \in \mathbb{D}} \left(\frac{1 - |B_\sigma(z)|^2}{1 - |z|^2} \right)^{1/2},$$

$$(I_2) \quad c(\sigma, H^2, H^\infty) \leq \sqrt{2} \sup_{|\zeta|=1} |B'(\zeta)|^{1/2} = \sqrt{2} \sup_{|\zeta|=1} \left| \sum_{i=1}^n \frac{1 - |\lambda_i|^2}{(1 - \bar{\lambda}_i \zeta)^2} \frac{B_\sigma(\zeta)}{b_{\lambda_i}(\zeta)} \right|^{1/2}.$$

Proof. We prove (I_1) . Applying point (ii) of Theorem B for $X = H^2$ and $Y = H^\infty$, and using

$$k_z(\zeta) = \frac{1}{1 - \bar{z}\zeta} \quad \text{and} \quad (P_{B_\sigma} k_z)(\zeta) = \frac{1 - \overline{B_\sigma(z)} B_\sigma(\zeta)}{1 - \bar{z}\zeta},$$

(see [N1] p.199), we obtain

$$\|P_{B_\sigma} k_z\|_{H^2} = \left(\frac{1 - |B_\sigma(z)|^2}{1 - |z|^2} \right)^{1/2},$$

which gives the result.

We now prove (I_2) , using (I_1) . The map $\zeta \mapsto \|P_{B_\sigma}(k_\zeta)\| = \sup \{|f(\zeta)| : f \in K_B, \|f\| \leq 1\}$, and hence the map

$$\zeta \mapsto \left(\frac{1 - |B(\zeta)|^2}{1 - |\zeta|^2} \right)^{1/2},$$

is a subharmonic function so

$$\sup_{|\zeta| < 1} \left(\frac{1 - |B(\zeta)|^2}{1 - |\zeta|^2} \right)^{1/2} \leq \sup_{|w|=1} \lim_{r \rightarrow 1} \left(\frac{1 - |B(rw)|^2}{1 - |rw|^2} \right)^{1/2}.$$

Now apply Taylor's Formula of order 1 for points $w \in \mathbb{T}$ and $u = rw$, $0 < r < 1$. (It is applicable because B is holomorphic at every point of \mathbb{T}). We get

$$(B(u) - B(w))(u - w)^{-1} = B'(w) + o(1),$$

and since $|u - w| = 1 - |u|$,

$$|(B(u) - B(w))(u - w)^{-1}| = |B(u) - B(w)|(1 - |u|)^{-1} = |B'(w) + o(1)|.$$

Now,

$$|B(u) - B(w)| \geq |B(w)| - |B(u)| = 1 - |B(u)|,$$

$$(1 - |B(u)|)(1 - |u|)^{-1} \leq (1 - |u|)^{-1} |B(u) - B(w)| = |B'(w) + o(1)|,$$

and

$$\lim_{r \rightarrow 1} ((1 - |B(rw)|)(1 - |rw|)^{-1})^{\frac{1}{2}} \leq \sqrt{|B'(w)|}.$$

Moreover,

$$B'(w) = - \sum_{i=1}^n (1 - |\lambda_i|^2) (1 - \bar{\lambda}_i w)^{-2} \prod_{j=1, j \neq i}^n b_{\lambda_j}(w),$$

for all $w \in \mathbb{T}$. This completes the proof since

$$\frac{1 - |B(rw)|^2}{1 - |rw|^2} = \frac{(1 - |B(rw)|)(1 + |B(rw)|)}{(1 - |rw|)(1 + |rw|)} \leq 2 \frac{1 - |B(rw)|}{1 - |rw|}. \quad \square$$

Corollary. 2.1. *Let $n \geq 1$ and $r \in [0, 1[$. Then,*

$$C_{n,r}(H^2, H^\infty) \leq 2 (n(1 - r)^{-1})^{\frac{1}{2}}.$$

Indeed, applying Proposition 2.0 we obtain

$$|B'(w)| \leq \left| \sum_{i=1..n} \frac{1 - |\lambda_i|^2}{(1 - |\lambda_i|)^2} \right| \leq n \frac{1 + r}{1 - r} \leq \frac{2n}{1 - r}.$$

□

Now, we develop a slightly different approach to the interpolation constant $c(\sigma, H^2, H^\infty)$.

Theorem. 2.2. *For every sequence $\sigma = \{\lambda_1, \dots, \lambda_n\}$ of \mathbb{D} ,*

$$c(\sigma, H^2, H^\infty) \leq \sup_{z \in \mathbb{T}} \left(\sum_{k=1}^n \frac{(1 - |\lambda_k|^2)}{|z - \lambda_k|^2} \right)^{1/2}$$

Proof. In order to simplify the notation, we set $B = B_\sigma$. We consider K_B (see Definition 1.1.2) and the Malmquist family $(e_k)_{k=1}^n$ corresponding to σ (see Definition 1.1.1). Now, let $f \in H^2$ and

$$g = P_B f = \sum_{k=1}^n (f, e_k)_{H^2} e_k,$$

(see Definition 1.1.3 and Remark 1.1.4). Function g belongs to H^∞ (it is a finite sum of H^∞ functions) and satisfies $g - f \in BH^2$. Applying Cauchy-Schwarz inequality we get

$$|g(\zeta)| \leq \sum_{k=1}^n |(f, e_k)_{H^2}| |e_k(\zeta)| \leq \left(\sum_{k=1}^n |(f, e_k)_{H^2}|^2 \right)^{1/2} \left(\sum_{k=1}^n \frac{(1 - |\lambda_k|^2)}{|1 - \lambda_k \zeta|^2} \right)^{1/2},$$

for all $\zeta \in \mathbb{D}$. As a result, since f is an arbitrary H^2 function, we obtain

$$c(\sigma, H^2, H^\infty) \leq \sup_{\zeta \in \mathbb{T}} \left(\sum_{k=1}^n \frac{(1 - |\lambda_k|^2)}{|\zeta - \lambda_k|^2} \right)^{1/2},$$

which completes the proof. \square

Corollary. 2.3. *For any sequence $\sigma = \{\lambda_1, \dots, \lambda_n\}$ in \mathbb{D} ,*

$$c(\sigma, H^2, H^\infty) \leq \left(\sum_{j=1}^n \frac{1 + |\lambda_j|}{1 - |\lambda_j|} \right)^{1/2}.$$

Indeed,

$$\sum_{k=1}^n \frac{(1 - |\lambda_k|^2)}{|\zeta - \lambda_k|^2} \leq \left(\sum_{k=1}^n \frac{(1 - |\lambda_k|^2)}{(1 - |\lambda_k|)^2} \right)^{1/2}$$

and the result follows from Theorem 2.2. \square

Proof of Theorem C ($p = 2$, the right-hand side inequality only). Since $1 + |\lambda_j| \leq 2$ and $1 - |\lambda_j| \geq 1 - r$ for all $j \in [1, n]$, applying Corollary 2.3 we get

$$C_{n,r}(H^2, H^\infty) \leq \sqrt{2}n^{1/2}(1 - r)^{-1/2}. \quad \square$$

Remark 2.4. As a result, we get once more the same estimate for $C_{n,r}(H^2, H^\infty)$ as in Corollary 2.1, with the constant $\sqrt{2}$ instead of 2

It is natural to wonder if it is possible to improve the bound $\sqrt{2}n^{1/2}(1 - r)^{-1/2}$. We return to this question in Section 5 below.

3. Upper bounds for $C_{n,r}(H^p, H^\infty)$, $p \geq 1$

In this Section we extend Corollary 2.1 to all Hardy spaces H^p : we prove the right-hand side inequality of Theorem C, $p \neq 2$. We first prove the following lemma.

Lemma. 3.0. *Let $n \geq 1$ and $0 \leq r < 1$. Then,*

$$C_{n,r}(H^1, H^\infty) \leq 2n(1 - r)^{-1}.$$

Proof. Let $f \in H^1$ such that $\|f\|_{H^1} \leq 1$ and let

$$g = \Phi(f) = \sum_{k=1..n} \langle f, e_k \rangle e_k,$$

where, as always, $(e_k)_{k=1}^n$ is the Malmquist basis corresponding to σ (see 1.1.1), Φ is defined in Lemma 1.1.5, and where $\langle \cdot, \cdot \rangle$ means the Cauchy sesquilinear form $\langle f, g \rangle = \sum_{k \geq 0} \hat{h}(k) \overline{\hat{g}(k)}$. That is to say that,

$$g(\zeta) = \sum_{k=1..n} \langle f, e_k \rangle e_k(\zeta) = \left\langle f, \sum_{k=1..n} e_k \overline{e_k(\zeta)} \right\rangle,$$

for all $\zeta \in \mathbb{D}$, which gives,

$$|g(\zeta)| \leq \|f\|_{H^1} \left\| \sum_{k=1..n} e_k \overline{e_k(\zeta)} \right\|_{H^\infty} \leq \left\| \sum_{k=1..n} e_k \overline{e_k(\zeta)} \right\|_{H^\infty}.$$

Since Blaschke factors have modulus 1 on the unit circle,

$$\|e_k\|_{H^\infty} \leq (1 + |\lambda_k|)^{1/2} (1 - |\lambda_k|)^{-1/2}.$$

As a consequence,

$$|g(\zeta)| \leq \sum_{k=1}^n \|e_k\|_{H^\infty} \left| \overline{e_k(\zeta)} \right| \leq \sum_{k=1}^n \|e_k\|_{H^\infty}^2 \leq \sum_{k=1}^n (1 + |\lambda_k|) (1 - |\lambda_k|)^{-1} \leq 2n(1 - r)^{-1},$$

for all $\zeta \in \mathbb{D}$, which completes the proof. \square

Proof of Theorem C ($p \neq 2$, the right-hand side inequality only). Let $\sigma = \{\lambda_1, \dots, \lambda_n\}$ be a sequence in the unit disc \mathbb{D} , $B_\sigma = \prod_{i=1}^n b_{\lambda_i}$, and $T : H^p \rightarrow H^\infty/B_\sigma H^\infty$ be the restriction map defined by

$$Tf = \{g \in H^\infty : f - g \in B_\sigma H^p\},$$

for every f . Then,

$$\|T\|_{H^p \rightarrow H^\infty/B_\sigma H^\infty} = c(\sigma, H^p, H^\infty).$$

There exists $0 \leq \theta \leq 1$ such that $1/p = 1 - \theta$, and since (we use the notation of the interpolation theory between Banach spaces see [Tr] or [Be]) $[H^1, H^\infty]_\theta = H^p$ (a topological identity : the spaces are the same and the norms are equivalent (up to constants depending on p only), see [J]),

$$\|T\|_{[H^1, H^\infty]_\theta \rightarrow H^\infty/B_\sigma H^\infty} \leq (A_1 c(\sigma, H^1, H^\infty))^{1-\theta} (A_\infty c(\sigma, H^\infty, H^\infty))^\theta,$$

where A_1, A_∞ are numerical constants, and using, Lemma 3.0, the fact that $c(\sigma, H^\infty, H^\infty) \leq 1$, and a known interpolation Theorem (see [Tr], Theorem 1.9.3-(a) p.59), we find

$$\|T\|_{[H^1, H^\infty]_\theta \rightarrow H^\infty/B_\sigma H^\infty} \leq (2A_1 n(1-r)^{-1})^{1-\theta} A_\infty^\theta = (2A_1)^{1-\theta} A_\infty^\theta (n(1-r)^{-1})^{\frac{1}{p}},$$

which completes the proof. \square

4. Upper bounds for $C_{n,r}(L_a^2, H^\infty)$

In this Section, we generalize Corollary 2.1 to the case of spaces X which contain H^2 : $X = l_a^2((k+1)^\alpha)$, $\alpha \leq 0$, the Hardy weighted spaces of all $f(z) = \sum_{k \geq 0} \hat{f}(k) z^k$ satisfying

$$\|f\|_X^2 := \sum_{k \geq 0} \left| \hat{f}(k) \right|^2 (k+1)^{2\alpha} < \infty.$$

Notice that $H^2 = l_a^2(1)$ and $L_a^2(\mathbb{D}) = l_a^2\left((k+1)^{-\frac{1}{2}}\right)$. We prove the right-hand side inequality of Theorem D and the main technical tool used in its proof is a Bernstein-type inequality for rational functions.

4.1. Bernstein-type inequalities for rational functions

Bernstein-type inequalities for rational functions were the subject of a number of papers and monographs (see, for instance, [L], [BoEr], [DeLo], [B]). Perhaps, the stronger and closer to ours (Proposition 4.1) of all known results are due to K. Dyakonov [Dya1]&[Dya2]. First, we prove Proposition 4.1 below, which tells that if $\sigma = \{\lambda_1, \dots, \lambda_n\} \subset \mathbb{D}$, $r = \max_j |\lambda_j|$, and $f \in K_{B_\sigma}$, then

$$(\star) \quad \left\| f' \right\|_{H^2} \leq \alpha_{n,r} \|f\|_{H^2},$$

where $\alpha_{n,r}$ is a constant (explicitly given in Proposition 4.1) depending on n and r only such that $0 < \alpha_{n,r} \leq \frac{5}{2} \frac{n}{1-r}$. Proposition 4.1 is in fact a partial case ($p = 2$) of the following K. Dyakonov's result [Dya1] (which is, in turn, a generalization of M. Levin's inequality [L] corresponding to the case $p = \infty$): it is proved in [Dya1] that the norm $\|D\|_{K_B^p \rightarrow H^p}$ of the differentiation operator $Df = f'$ on the star-invariant subspace of the Hardy space H^p , $K_B^p := H^p \cap \overline{BzH^p}$, (where the bar denotes complex conjugation) satisfies the following inequalities

$$c'_p \left\| B' \right\|_{\infty} \leq \|D\|_{K_B^p \rightarrow H^p} \leq c_p \left\| B' \right\|_{\infty},$$

for every p , $1 \leq p \leq \infty$ where c_p and c'_p are positives constants depending on p only, B is a finite Blaschke product and $\|\cdot\|_{\infty}$ means the norm in $L^{\infty}(\mathbb{T})$. For the partial case considered in Proposition 4.1 below, our proof is different and the constant is slightly better. More precisely, it is proved in [Dya1] that $c'_2 = \frac{1}{36c}$, $c_2 = \frac{36+c}{2\pi}$ and $c = 2\sqrt{3\pi}$ (as one can check easily (c is not precised in [Dya1])). It implies an inequality of type (\star) (with a constant about $\frac{13}{2}$ instead of $\frac{5}{2}$).

In [Z2], we discuss the ‘‘asymptotic sharpness’’ of our constant $\alpha_{n,r}$: we find an inequality for $\sup \|D\|_{K_B \rightarrow H^2} = C_{n,r}$ (\sup is over all B with given $n = \deg B$ and $r = \max_{\lambda \in \sigma} |\lambda|$), which is asymptotically sharp as $n \rightarrow \infty$. Our result in [Z2] is that there exists a limit $\lim_{n \rightarrow \infty} \frac{C_{n,r}}{n} = \frac{1+r}{1-r}$ for every r , $0 \leq r < 1$. Our method is different from [Dya1]&[Dya2] and is based on an elementary Hilbert space construction for an orthonormal basis in K_B .

Proposition. 4.1. *Let $B = \prod_{j=1}^n b_{\lambda_j}$, be a finite Blaschke product (of order n), $r = \max_j |\lambda_j|$, and $f \in K_B = H^2 \ominus BH^2$. Then for every $n \geq 2$ and $r \in [0, 1)$,*

$$\left\| f' \right\|_{H^2} \leq \alpha_{n,r} \|f\|_{H^2},$$

where $\alpha_{n,r} = [1 + (1+r)(n-1) + \sqrt{n-2}] (1-r)^{-1}$ and in particular,

$$\left\| f' \right\|_{H^2} \leq \frac{5}{2} \frac{n}{1-r} \|f\|_{H^2},$$

for all $n \geq 1$ and $r \in [0, 1)$.

Proof. Using Remark 1.1.4, $f = P_B f = \sum_{k=1}^n (f, e_k)_{H^2} e_k$, $\forall f \in K_B$. Noticing that,

$$e'_k = \sum_{i=1}^{k-1} \frac{b'_{\lambda_i}}{b_{\lambda_i}} e_k + \bar{\lambda}_k \frac{1}{(1 - \bar{\lambda}_k z)} e_k,$$

for $k \in [2, n]$, we get

$$\begin{aligned} f' &= (f, e_1)_{H^2} e'_1 + \sum_{k=2}^n (f, e_k)_{H^2} e'_k = \\ &= (f, e_1)_{H^2} \frac{\bar{\lambda}_1}{(1 - \bar{\lambda}_1 z)} e_1 + \sum_{k=2}^n (f, e_k)_{H^2} \sum_{i=1}^{k-1} \frac{b'_{\lambda_i}}{b_{\lambda_i}} e_k + \sum_{k=2}^n (f, e_k)_{H^2} \bar{\lambda}_k \frac{1}{(1 - \bar{\lambda}_k z)} e_k, \end{aligned}$$

which gives

$$\begin{aligned} f' &= (f, e_1)_{H^2} \frac{\bar{\lambda}_1}{(1 - \bar{\lambda}_1 z)} e_1 + \sum_{k=2}^n \sum_{i=1}^{n-1} (f, e_k)_{H^2} \frac{b'_{\lambda_i}}{b_{\lambda_i}} e_k \chi_{[1, k-1]}(i) + \sum_{k=2}^n (f, e_k)_{H^2} \bar{\lambda}_k \frac{1}{(1 - \bar{\lambda}_k z)} e_k = \\ &= (f, e_1)_{H^2} \frac{\bar{\lambda}_1}{(1 - \bar{\lambda}_1 z)} e_1 + \sum_{i=1}^n \frac{b'_{\lambda_i}}{b_{\lambda_i}} \sum_{k=i+1}^{n-1} (f, e_k)_{H^2} e_k + \sum_{k=2}^n (f, e_k)_{H^2} \bar{\lambda}_k \frac{1}{(1 - \bar{\lambda}_k z)} e_k, \end{aligned}$$

where $\chi_{[1, k-1]}$ is the characteristic function of $[1, k-1]$. Now,

$$\begin{aligned} \left\| (f, e_1)_{H^2} \frac{\bar{\lambda}_1}{(1 - \bar{\lambda}_1 z)} e_1 \right\|_{H^2} &\leq |(f, e_1)_{H^2}| \left\| \frac{\bar{\lambda}_1}{(1 - \bar{\lambda}_1 z)} \right\|_{\infty} \|e_1\|_{H^2} \leq \\ &\leq \|f\|_{H^2} \|e_1\|_{H^2} \frac{1}{1-r} \|e_1\|_{H^2} \leq \|f\|_{H^2} \frac{1}{1-r}, \end{aligned}$$

using Cauchy-Schwarz inequality and the fact that e_1 is a vector of norm 1 in H^2 . By the same reason, we have

$$\begin{aligned} \left\| \sum_{k=2}^n \bar{\lambda}_k (f, e_k)_{H^2} \frac{1}{(1 - \bar{\lambda}_k z)} e_k \right\|_{H^2} &\leq \sum_{k=2}^n |(f, e_k)_{H^2}| \left\| \bar{\lambda}_k \frac{1}{(1 - \bar{\lambda}_k z)} \right\|_{\infty} \|e_k\|_{H^2} \leq \\ &\leq \frac{1}{1-r} \sum_{k=2}^n |(f, e_k)_{H^2}| \leq \frac{1}{1-r} \left(\sum_{k=2}^n |(f, e_k)_{H^2}|^2 \right)^{\frac{1}{2}} \sqrt{n-2} \leq \frac{1}{1-r} \|f\|_{H^2} \sqrt{n-2}. \end{aligned}$$

Further,

$$\begin{aligned} \left\| \sum_{i=1}^{n-1} \frac{b'_{\lambda_i}}{b_{\lambda_i}} \sum_{k=i+1}^n e_k (f, e_k)_{H^2} \right\|_{H^2} &\leq \sum_{i=1}^{n-1} \left\| \frac{b'_{\lambda_i}}{b_{\lambda_i}} \right\|_{\infty} \left\| \sum_{k=i+1}^n (f, e_k)_{H^2} e_k \right\|_{H^2} = \\ &= \left(\max_{1 \leq i \leq n-1} \left\| \frac{b'_{\lambda_i}}{b_{\lambda_i}} \right\|_{\infty} \right) \sum_{i=1}^{n-1} \left(\sum_{k=i+1}^n |(f, e_k)_{H^2}|^2 \right)^{\frac{1}{2}} \leq \max_i \left\| \frac{b'_{\lambda_i}}{b_{\lambda_i}} \right\|_{\infty} \sum_{i=1}^{n-1} \|f\|_{H^2}. \end{aligned}$$

Now, using

$$\left\| \frac{b'_{\lambda_i}}{b_{\lambda_i}} \right\|_{\infty} = \left\| \frac{|\lambda_i|^2 - 1}{(1 - \bar{\lambda}_i z)(\lambda_i - z)} \right\|_{\infty} \leq \frac{1 + |\lambda_i|}{1 - |\lambda_i|} \leq \frac{1+r}{1-r},$$

we get

$$\left\| \sum_{i=1}^{n-1} \frac{b'_{\lambda_i}}{b_{\lambda_i}} \sum_{k=i+1}^n (f, e_k)_{H^2} e_k \right\|_{H^2} \leq (1+r) \frac{n-1}{1-r} \|f\|_{H^2}.$$

Finally,

$$\|f'\|_{H^2} \leq [1 + (1+r)(n-1) + \sqrt{n-2}] (1-r)^{-1} \|f\|_{H^2}.$$

In particular,

$$\|f\|_{H^2} \leq (2n-1 + \sqrt{n-2}) (1-r)^{-1} \|f\|_{H^2} \leq 5.2^{-1} n (1-r)^{-1} \|f\|_{H^2},$$

for all $n \geq 2$ and for every $f \in K_B$. (The case $n = 1$ is obvious because $\|f'\|_{H^2} \leq (1-r)^{-1} \|f\|_{H^2}$ for every f of the form $f = (1 - \bar{\lambda}z)^{-1}$, $\lambda \in \mathbb{D}$.) \square

4.2. An upper bound for $c(\sigma, L_a^2, H^\infty)$

Corollary. 4.2. *Let σ be a sequence in \mathbb{D} . Then,*

$$c(\sigma, l_a^2((k+1)^{-1}), H^\infty) \leq 6\sqrt{2} (n(1-r)^{-1})^{3/2}.$$

Indeed, let $H = l_a^2((k+1)^{-N})$ and $B = B_\sigma$ the finite Blaschke product corresponding to σ . Let \widetilde{P}_B be the orthogonal projection of H onto $K_B = K_B(H^2)$. Then $\widetilde{P}_B|_{H^2} = P_B$, where P_B is defined in 1.1.4. We notice that $\widetilde{P}_B : H \rightarrow H$ is a bounded operator and the adjoint $\widetilde{P}_B^* : H^* \rightarrow H^*$ of \widetilde{P}_B relatively to the Cauchy pairing $\langle \cdot, \cdot \rangle$ satisfies $\widetilde{P}_B^* \varphi = \widetilde{P}_B \varphi = P_B \varphi$, $\forall \varphi \in H^* \subset H^2$, where $H^* = l_a^2((k+1)^N)$ is the dual of H with respect to this pairing. If $f \in H$, then $|\widetilde{P}_B f(\zeta)| = |\langle \widetilde{P}_B f, k_\zeta \rangle| = |\langle f, \widetilde{P}_B^* k_\zeta \rangle|$, where $k_\zeta = (1 - \bar{\zeta}z)^{-1} \in H^2$ and

$$|\widetilde{P}_B f(\zeta)| \leq \|f\|_H \|P_B k_\zeta\|_{H^*} \leq \|f\|_H K \left(\|P_B k_\zeta\|_{H^2} + \left\| (P_B k_\zeta)' \right\|_{H^2} \right),$$

where

$$K = \max \{1, \sup_{k \geq 1} (k+1)k^{-1}\} = 2$$

Since $P_B k_\zeta \in K_B$, Proposition 4.1 implies

$$|\widetilde{P}_B f(\zeta)| \leq \|f\|_H K \left(\|P_B k_\zeta\|_{H^2} + 5.2^{-1} (n(1-r)^{-1}) \|P_B k_\zeta\|_{H^2} \right) \leq A (n(1-r)^{-1})^{3/2} \|f\|_H,$$

where $A = \sqrt{2}K(1/2 + 5/2) = 6\sqrt{2}$, since $\|P_B k_\zeta\|_2 \leq \sqrt{2} (n(1-r)^{-1})^{1/2}$, and since we can suppose $n \geq 2$, (the case $n = 1$ being obvious). □

Proof of Theorem E (the right-hand side inequality only). The case $\alpha = 0$ corresponds to $X = H^2$ and has already been studied in Section 1 (we can choose $A(0) = \sqrt{2}$). We now suppose $\alpha < 0$. Let $B_\sigma = \prod_{i=1}^n b_{\lambda_i}$ and $T : l_a^2((k+1)^\alpha) \rightarrow H^\infty/B_\sigma H^\infty$ be the restriction map defined by

$$Tf = \{g \in H^\infty : f - g \in B_\sigma l_a^2((k+1)^\alpha)\},$$

for every f . Then,

$$\|T\|_{l_a^2((k+1)^\alpha) \rightarrow H^\infty/B_\sigma H^\infty} = c(\sigma, l_a^2((k+1)^\alpha), H^\infty).$$

Setting $\theta = -\alpha$ with $0 < \theta \leq 1$, we have (as in Theorem D, we use the notation of the interpolation theory between Banach spaces see [Tr] or [Be])

$$[l_a^2((k+1)^0), l_a^2((k+1)^{-1})]_{\theta, 2} = l_a^2\left(\left((k+1)^0\right)^{2\frac{1-\theta}{2}} \left((k+1)^{-1}\right)^{2\frac{\theta}{2}}\right) = l_a^2((k+1)^\alpha),$$

which entails, using Corollary 4.2 and (again) [Tr] Theorem 1.9.3-(a) p.59,

$$\begin{aligned} \|T\|_{l_a^2((k+1)^\alpha) \rightarrow H^\infty/B_\sigma H^\infty} &\leq (c(\sigma, l_a^2((k+1)^0), H^\infty))^{1-\theta} (c(\sigma, l_a^2((k+1)^{-1}), H^\infty))^\theta \leq \\ &\leq \left(A(0) (n(1-r)^{-1})^{\frac{1}{2}}\right)^{1-\theta} \left(A(1) (n(1-r)^{-1})^{\frac{3}{2}}\right)^\theta = A(0)^{1-\theta} A(1)^\theta (n(1-r)^{-1})^{\frac{1-\theta}{2} + \frac{3\theta}{2}}. \end{aligned}$$

It remains to use $\theta = -\alpha$ and set $A(\alpha) = A(0)^{1-\theta} A(1)^\theta$. In particular, for $\alpha = -1/2$ we get $(1-\theta)/2 + 3\theta/2 = 1$ and

$$A(-1/2) = A(0)^{(1-1/2)} A(1)^{1/2} = \sqrt{2}^{1/2} (6\sqrt{2})^{1/2} = 2\sqrt{3}.$$

□

5. About the links with Carleson interpolation

Recall that given a (finite) set $\sigma = \{\lambda_1, \dots, \lambda_n\} \subset \mathbb{D}$, the Carleson interpolation constant $C_I(\sigma)$ is defined by

$$C_I(\sigma) = \sup_{\|a\|_{\infty} \leq 1} \inf (\|g\|_{\infty} : g \in H^{\infty}, g|_{\sigma} = a).$$

We introduce the evaluation functionals φ_{λ} for $\lambda \in \mathbb{D}$, as well as the evaluation of the derivatives $\varphi_{\lambda,s}$ ($s = 0, 1, \dots$)

$$\varphi_{\lambda}(f) = f(\lambda), \quad f \in X, \quad \text{and} \quad \varphi_{\lambda,s}(f) = f^{(s)}(\lambda), \quad f \in X.$$

Theorem 5.1. *Let X be a Banach space, $X \subset Hol(\mathbb{D})$, and $\sigma = \{\lambda_1, \dots, \lambda_n\}$ be a sequence of distinct points in the unit disc \mathbb{D} . We have,*

$$\max_{1 \leq i \leq n} \|\varphi_{\lambda_i}\| \leq c(\sigma, X, H^{\infty}) \leq C_I(\sigma) \cdot \max_{1 \leq i \leq n} \|\varphi_{\lambda_i}\|,$$

where $C_I(\sigma)$ stands for the Carleson interpolation constant.

Theorem 5.1 tells us that, for σ with a “reasonable” interpolation constant $C_I(\sigma)$, the quantity $c(\sigma, X, H^{\infty})$ behaves as $\max_i \|\varphi_{\lambda_i}\|$. However, for “tight” sequences σ , the constant $C_I(\sigma)$ is so large that the estimate in question contains almost no information. On the other hand, an advantage of the estimate of Theorem 5.1 is that it does not contain $\#\sigma = n$ explicitly. Therefore, for well-separated sequences σ , Theorem 5.1 should give a better estimate than those of Theorem C and Theorem D.

Now, how does the interpolation constant $C_I(\sigma)$ behave in terms of the characteristics r and n of σ ? We answer this question for some particular sequences σ , see Exemples 5.2, 5.3 and 5.4.

Proof of Theorem 5.1. Let $f \in X$. By definition of $C_I(\sigma)$, there exists $g \in H^{\infty}$ such that

$$f(\lambda_i) = g(\lambda_i) \quad \forall i = 1..n \quad \text{with} \quad \|g\|_{\infty} \leq C_I(\sigma) \max_i |f(\lambda_i)| \leq C_I(\sigma) \max_i \|\varphi_{\lambda_i}\| \|f\|_X.$$

Now, taking the supremum over all $f \in X$ such that $\|f\|_X \leq 1$, we get the right-hand side inequality. The left-hand side one is clear since if $g \in H^{\infty}$ satisfies $f(\lambda_i) = g(\lambda_i) \quad \forall i = 1..n$, then $\|g\|_{\infty} \geq |g(\lambda_i)| = |f(\lambda_i)| = |\varphi_{\lambda_i}(f)|, \quad \forall i = 1..n. \quad \square$

Now, how does the interpolation constant $C_I(\sigma)$ behave in terms of the characteristics r and n of σ ? In what follows, we compare these quantities for three geometrically simple configurations : two-points sets σ , circular and radial sequences σ . The proofs of the following statements (5.2, 5.3 and 5.4) are given in [Z3].

Example. 5.2. *Two points sets. Let $\sigma = \{\lambda_1, \lambda_2\}$, $\lambda_i \in \mathbb{D}$, $\lambda_1 \neq \lambda_2$. Then,*

$$|b_{\lambda_1}(\lambda_2)|^{-1} \leq C_I(\sigma) \leq 2 |b_{\lambda_1}(\lambda_2)|^{-1},$$

and Theorem 5.1 implies

$$c(\sigma, X, H^{\infty}) \leq 2 |b_{\lambda_1}(\lambda_2)|^{-1} \max_{i=1,2} \|\varphi_{\lambda_i}\|,$$

whereas a straightforward estimate gives

$$c(\sigma, X, H^{\infty}) \leq \|\varphi_{\lambda_1}\| + \max_{|\lambda| \leq r} \|\varphi_{\lambda,1}\| (1 + |\lambda_1|),$$

where $r = \max(|\lambda_1|, |\lambda_2|)$ and the functional $\varphi_{\lambda,1}$ is defined in the beginning of Section 5. The difference is that the first upper bound blows up when $\lambda_1 \rightarrow \lambda_2$, whereas the second one is still well-bounded.

Example. 5.3. *Circular sequences. Let $0 < r < 1$ and $\sigma = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$, $\lambda_i \neq \lambda_j$, $|\lambda_i| = r$ for every i , and let $\alpha = \min_{i \neq j} |\lambda_i - \lambda_j| / (1 - r)$. Then, $\alpha^{-1} \leq C_I(\sigma) \leq 8e^{K'(1+K\alpha^{-3})}$, where $K, K' > 0$ are absolute constants. Therefore,*

$$c(\sigma, X, H^{\infty}) \leq 8e^{K'(1+K\alpha^{-3})} \max_{|\lambda|=r} \|\varphi_{\lambda}\|$$

for every r – circular set σ (an estimate does not depending on n explicitly). In particular, there exists an increasing function $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that, for any n uniformly distributed points $\lambda_1, \dots, \lambda_n$, $|\lambda_i| = r$, $|\lambda_i - \lambda_{i+1}| = 2r \sin\left(\frac{\pi}{2n}\right)$, we have

(1) $c(\sigma, H^2, H^\infty) \leq \varphi(n(1-r)r^{-1})(1-r)^{-\frac{1}{2}}$, for every n and r , $0 < r < 1$ and in particular, for $n \leq [r(1-r)^{-1}]$ we obtain

$$c(\sigma, H^2, H^\infty) \leq c(1-r)^{-\frac{1}{2}},$$

whereas our specific upper bound in Theorem C, (which is sharp over all n elements sequences σ), gives

$$c(\sigma, H^2, H^\infty) \leq c(1-r)^{-1}$$

only.

(2) $c(\sigma, L_a^2, H^\infty) \leq \varphi(n(1-r)r^{-1})(1-r)^{-1}$, for every n and r , $0 < r < 1$ and in particular, for $n \leq [r(1-r)^{-1}]$ we obtain

$$c(\sigma, L_a^2, H^\infty) \leq c(1-r)^{-1},$$

whereas our specific upper bound in Theorem D, (which, again, is sharp over all n elements sequences σ), gives

$$c(\sigma, L_a^2, H^\infty) \leq c(1-r)^{-2}$$

only.

Example. 5.4. Radial sequences. Now we consider geometric sequences on the radius of the unit disc \mathbb{D} , say on the radius $[0, 1)$. Let $0 < \rho < 1$, $p \in (0, \infty)$ and

$$\lambda_j = 1 - \rho^{j+p}, \quad j = 0, \dots, n,$$

so that the distances $1 - \lambda_j = \rho^j \rho^p$ form a geometric progression; the starting point is $\lambda_0 = 1 - \rho^p$. Let

$$r = \max_{0 \leq j \leq n} \lambda_j = \lambda_k = 1 - \rho^{n+p},$$

and $\delta = \delta(B) = \min_{0 \leq k \leq n} |B_k(\lambda_k)|$, where $B_k = b_{\lambda_k}^{-1} B$. It is known that $\delta^{-1} \leq C_I(\sigma) \leq 8\delta^{-2}$, see ([N1], p 189). So, we need to know the asymptotic behaviour of $\delta = \delta(B)$ when $n \rightarrow \infty$, or $\rho \rightarrow 1$, or $\rho \rightarrow 0$, or $p \rightarrow \infty$, or $p \rightarrow 0$.

Claim. Let $\sigma_{n,\rho,p} = \{1 - \rho^{p+k}\}_{k=1}^n$, $0 < \rho < 1$, $p > 0$. The estimate of $c(\sigma, H^2, H^\infty)$ via the Carleson constant $C_I(\sigma)$ (using Theorem 5.1) is comparable with or better than the estimates from Theorem C (for $X = H^2$) and Theorem D (for $X = L_a^2$) for sufficiently small values of ρ (as $\rho \rightarrow 0$) and/or for a fixed ρ and $n \rightarrow \infty$. In all other cases, as for $p \rightarrow \infty$ (which means $\lambda_1 \rightarrow 1$), or $\rho \rightarrow 1$, or $n \rightarrow \infty$ and $\rho \rightarrow 1$, it is worse.

Remark 5.5. More specific radial sequences are studied in [Z3]: sparse sequences σ ($\rho \rightarrow 0$, or at least $0 < \rho \leq \epsilon < 1$), condensed sequences σ ($\rho \rightarrow 1$) and long sequences ($n \rightarrow \infty$).

6. Lower bounds for $C_{n,r}(X, H^\infty)$

6.1. The cases $X = H^2$ and $X = L_a^2$

Here, we consider the standard Hardy and Bergman spaces on the disc \mathbb{D} : $X = H^2 = l_a^2(1)$ and $X = L_a^2 = l_a^2((k+1)^{-1/2})$, and the problem of lower estimates for the one point special case $\sigma_{n,\lambda} = \{\lambda, \lambda, \dots, \lambda\}$, (n times) $\lambda \in \mathbb{D}$. Recall the definition of our constrained interpolation constant for this case

$$c(\sigma_{n,\lambda}, H, H^\infty) = \sup \{ \|f\|_{H^\infty/b_\lambda^2 H^\infty} : f \in H, \|f\|_H \leq 1 \},$$

where $\|f\|_{H^\infty/b_\lambda^\alpha H^\infty} = \inf \{\|f + b_\lambda^\alpha g\|_\infty : g \in H\}$. Our goal in this Subsection is to prove the sharpness of the upper estimate from Theorem C ($p = 2$) and Theorem D for the quantities $C_{n,r}(H^2, H^\infty)$ and $C_{n,r}(L_a^2, H^\infty)$, that is to say, to get the lower bounds from Theorem C ($p = 2$) and Theorem D.

Recall that the spaces $l_a^2((k+1)^\alpha)$ are defined in Section 4.

In the proof, we use properties of reproducing kernel Hilbert space on the disc \mathbb{D} , see for example [N2]. Let us recall some of them adapting the general setting to special cases $X = l_a^2((k+1)^\alpha)$. As it is mentioned in Section 4,

$$l_a^2((k+1)^\alpha) = \left\{ f = \sum_{k \geq 0} \hat{f}(k) z^k : \|f\|^2 = \sum_{k \geq 0} |\hat{f}(k)|^2 (k+1)^{2\alpha} < \infty \right\}.$$

The reproducing kernel of $l_a^2((k+1)^\alpha)$, by definition, is a $l_a^2((k+1)^\alpha)$ -valued function $\lambda \mapsto k_\lambda^\alpha$, $\lambda \in \mathbb{D}$, such that $(f, k_\lambda^\alpha) = f(\lambda)$ for every $f \in l_a^2((k+1)^\alpha)$, where (\cdot, \cdot) means the scalar product $(h, g) = \sum_{k \geq 0} \hat{h}(k) \hat{g}(k) (k+1)^{-2\alpha}$. Since one has $f(\lambda) = \sum_{k \geq 0} \hat{f}(k) \lambda^k (k+1)^{2\alpha} (k+1)^{-2\alpha}$ ($\lambda \in \mathbb{D}$), it follows that

$$k_\lambda^\alpha(z) = \sum_{k \geq 0} (k+1)^{2\alpha} \bar{\lambda}^k z^k, \quad z \in \mathbb{D}.$$

In particular, for the Hardy space $H^2 = l_a^2(1)$ ($\alpha = 0$), we get the Szegö kernel

$$k_\lambda(z) = (1 - \bar{\lambda}z)^{-1},$$

for the Bergman space $L_a^2 = l_a^2((k+1)^{-1/2})$ ($\alpha = -1/2$) - the Bergman kernel $k_\lambda^{-1/2}(z) = (1 - \bar{\lambda}z)^{-2}$.

We will use the previous observations for the following composed reproducing kernels (Aronszajn-deBranges, see [N2] p.320): given the reproducing kernel k of H^2 and $\varphi \in \{z^N : N = 1, 2\}$, the function $\varphi \circ k$ is also positive definit and the corresponding Hilbert space is

$$H_\varphi = \varphi(H^2) = l_a^2\left((k+1)^{\frac{1-N}{2}}\right).$$

It satisfies the following property : for every $f \in H^2$, $\varphi \circ f \in \varphi(H^2)$ and $\|\varphi \circ f\|_{\varphi(H^2)}^2 \leq \varphi(\|f\|_{H^2}^2)$ (see [N2] p.320).

We notice in particular that

$$(6.1.0) \quad H_z = H^2 \quad \text{and} \quad H_{z^2} = L_a^2.$$

The above relation between the weighted spaces $l_a^2((k+1)^\alpha)$ and the spaces $\varphi(H^2) = H_\varphi$ leads to establish the prove of the left-hand side inequalities from Theorem C (for $p = 2$ only) and Theorem D.

Proof of Theorem C ($p = 2$) and Theorem D, (left-hand side inequalities only) .

1) We set

$$Q_n = \sum_{k=0}^{n-1} (1 - |\lambda|^2)^{1/2} b_\lambda^k (1 - \bar{\lambda}z)^{-1}, \quad H_n = \varphi \circ Q_n \quad \text{and} \quad \Psi = bH_n, \quad b > 0.$$

Then $\|Q_n\|_2^2 = n$, and hence by the above Aronszajn-deBranges inequality,

$$\|\Psi\|_{H_\varphi}^2 \leq b^2 \varphi (\|Q_n\|_2^2) = b^2 \varphi(n).$$

Let $b > 0$ such that $b^2 \varphi(n) = 1$.

2) Since the spaces H_φ and H^∞ are rotation invariant, we have $c(\sigma_{n,\lambda}, H_\varphi, H^\infty) = c(\sigma_{\mu,n}, H_\varphi, H^\infty)$ for every λ, μ with $|\lambda| = |\mu| = r$. Let $\lambda = -r$. To get a lower estimate for $\|\Psi\|_{H_\varphi/b_\lambda^n H_\varphi}$ consider G such that $\Psi - G \in b_\lambda^n \text{Hol}(\mathbb{D})$, i.e. such that $bH_n \circ b_\lambda - G \circ b_\lambda \in z^n \text{Hol}(\mathbb{D})$.

3) First, we show that

$$\psi =: \Psi \circ b_\lambda = bH_n \circ b_\lambda$$

is a polynomial (of degree n if $\varphi = z$ and $2n$ if $\varphi = z^2$) with positive coefficients. Note that

$$\begin{aligned} Q_n \circ b_\lambda &= \sum_{k=0}^{n-1} z^k \frac{(1 - |\lambda|^2)^{1/2}}{1 - \bar{\lambda} b_\lambda(z)} = (1 - |\lambda|^2)^{-\frac{1}{2}} \left(1 + (1 - \bar{\lambda}) \sum_{k=1}^{n-1} z^k - \bar{\lambda} z^n \right) = \\ &= (1 - r^2)^{-1/2} \left(1 + (1 + r) \sum_{k=1}^{n-1} z^k + r z^n \right) =: (1 - r^2)^{-1/2} \psi_1. \end{aligned}$$

Hence, $\psi = \Psi \circ b_\lambda = bH_n \circ b_\lambda = b\varphi \circ \left((1 - r^2)^{-\frac{1}{2}} \psi_1 \right)$ and

$$\varphi \circ \psi_1 = \psi_1^N(z), \quad N = 1, 2.$$

4) Next, we show that

$$\sum_{j=0}^m (\psi) =: \sum_{j=0}^m \hat{\psi}(j) \geq \begin{cases} (2\sqrt{2})^{-1} \sqrt{n(1-r)^{-1}} \text{ if } N = 1 \\ 16^{-1} n(1-r)^{-1} \text{ if } N = 2 \end{cases},$$

where $m = n/2$ if n is even and $m = (n+1)/2$ if n is odd.

Indeed, setting $S_n = \sum_{j=0}^n z^j$, we have both for $N = 1$ and $N = 2$

$$\sum_{j=0}^m (\psi_1^N) = \sum_{j=0}^m \left(\left(1 + (1+r) \sum_{t=1}^{n-1} z^t + r z^n \right)^N \right) \geq \sum_{j=0}^m (S_{n-1}^N).$$

Next, we obtain

$$\begin{aligned} \sum_{j=0}^m (S_{n-1}^N) &= \sum_{j=0}^m \left(\left(\frac{1 - z^n}{1 - z} \right)^N \right) = \\ &= \sum_{j=0}^m ((1 - z)^{-N}) = \sum_{j=0}^m \left(\sum_{j \geq 0} C_{N+j-1}^j z^j \right) = \sum_{j=0}^m C_{N+j-1}^j = \\ &= \begin{cases} m+1 \text{ if } N = 1 \\ (m+1)(m+2)/2 \text{ if } N = 2 \end{cases} \geq \begin{cases} n/2 \text{ if } N = 1 \\ (n+2)(n+4)/8 \text{ if } N = 2 \end{cases} \geq \begin{cases} n/2 \text{ if } N = 1 \\ n^2/8 \text{ if } N = 2 \end{cases}. \end{aligned}$$

Finally, since $\sum^m (\psi) = b \sum^m (\varphi \circ \psi_1) = b(1 - r^2)^{-N/2} \sum^m (\psi_1^N)$ we get

$$\sum_{j=0}^m (\psi) \geq \begin{cases} (2(1-r))^{-1/2} n b / 2 \text{ if } N = 1 \\ (2(1-r))^{-1} n^2 b / 8 \text{ if } N = 2 \end{cases},$$

with $b = \varphi(n) = \begin{cases} n^{-1/2} & \text{if } N = 1 \\ n^{-1} & \text{if } N = 2 \end{cases}$. This gives the result claimed.

5) Now, using point 4) and denoting $F_n = \Phi_m + z^m \Phi_m$, where Φ_k stands for the k -th Fejer kernel, we get

$$\begin{aligned} \|\Psi\|_{H^\infty/b\lambda H^\infty} &= \|\psi\|_{H^\infty/z^n H^\infty} \geq 2^{-1} \|\psi * F_n\|_\infty \geq 2^{-1} \sum_{j=0}^m \hat{\psi}(j) \geq \\ &\geq \begin{cases} (4\sqrt{2})^{-1} \sqrt{n(1-r)^{-1}} & \text{if } N = 1 \\ 32^{-1} n(1-r)^{-1} & \text{if } N = 2 \end{cases}. \end{aligned}$$

6) In order to conclude, it remains to use (6.1.0). □

6.2. The case $X = H^p$

Here we prove the sharpness (for even p) of the upper estimate found in Theorem C. We first prove the following lemma.

Lemma. 6.2.0 *Let p, q such that $\frac{p}{q} \in \mathbb{Z}_+$, then $c(\sigma, H^p, H^\infty) \geq c(\sigma, H^q, H^\infty)^{\frac{q}{p}}$ for every sequence σ of \mathbb{D} .*

Proof. Step 1. Recalling that

$$c(\sigma, H^p, H^\infty) = \sup_{\|f\|_p \leq 1} \inf \{ \|g\|_\infty : g \in Y, g|_\sigma = f|_\sigma \},$$

we first prove that

$$c(\sigma, H^p, H^\infty) = \sup_{\|f\|_p \leq 1, f \text{ outer}} \inf \{ \|g\|_\infty : g \in Y, g|_\sigma = f|_\sigma \}.$$

Indeed, we clearly have the inequality

$$\sup_{\|f\|_p \leq 1, f \text{ outer}} \inf \{ \|g\|_\infty : g \in Y, g|_\sigma = f|_\sigma \} \leq c(\sigma, H^p, H^\infty),$$

and if the inequality were strict, that is to say

$$\sup_{\|f\|_p \leq 1, f \text{ outer}} \inf \{ \|g\|_\infty : g \in Y, g|_\sigma = f|_\sigma \} < \sup_{\|f\|_p \leq 1} \inf \{ \|g\|_\infty : g \in Y, g|_\sigma = f|_\sigma \},$$

then we could write that there exists $\epsilon > 0$ such that for every $f = f_i \cdot f_o \in H^p$ (where f_i stands for the inner function corresponding to f and f_o to the outer one) with $\|f\|_p \leq 1$ (which also implies that $\|f_o\|_p \leq 1$, since $\|f_o\|_p = \|f\|_p$), there exists a function $g \in H^\infty$ verifying both $\|g\|_\infty \leq (1 - \epsilon)c(\sigma, H^p, H^\infty)$ and $g|_\sigma = f_o|_\sigma$. This entails that $f|_\sigma = (f_i g)|_\sigma$ and since $\|f_i g\|_\infty = \|g\|_\infty \leq (1 - \epsilon)c(\sigma, H^p, H^\infty)$, we get that $c(\sigma, H^p, H^\infty) \leq (1 - \epsilon)c(\sigma, H^p, H^\infty)$, which is a contradiction and proves the equality of Step 1.

Step 2. Using the result of Step 1, we get that $\forall \epsilon > 0$ there exists an outer function $f_o \in H^q$ with $\|f_o\|_q \leq 1$ and such that

$$\inf \{ \|g\|_\infty : g \in Y, g|_\sigma = f_o|_\sigma \} \geq c(\sigma, H^q, H^\infty) - \epsilon.$$

Now let $F = f_o^{\frac{q}{p}} \in H^p$, then $\|F\|_p^p = \|f_o\|_q^q \leq 1$. We suppose that there exists $g \in H^\infty$ such that $g|_\sigma = F|_\sigma$ with

$$\|g\|_\infty < (c(\sigma, H^q, H^\infty) - \epsilon)^{\frac{q}{p}}.$$

Then, since $g(\lambda_i) = F(\lambda_i) = f_o(\lambda_i)^{\frac{q}{p}}$ for all $i = 1..n$, we have $g(\lambda_i)^{\frac{p}{q}} = f_o(\lambda_i)$ and $g^{\frac{p}{q}} \in H^\infty$ since $\frac{p}{q} \in \mathbb{Z}_+$. We also have

$$\left\| g^{\frac{p}{q}} \right\|_\infty = \|g\|_\infty^{\frac{p}{q}} < (c(\sigma, H^q, H^\infty) - \epsilon)^{\frac{q}{p}},$$

which is a contradiction. As a result, we have

$$\|g\|_\infty \geq (c(\sigma, H^q, H^\infty) - \epsilon)^{\frac{q}{p}},$$

for all $g \in H^\infty$ such that $g|_\sigma = F|_\sigma$, which gives

$$c(\sigma, H^p, H^\infty) \geq (c(\sigma, H^q, H^\infty) - \epsilon)^{\frac{q}{p}},$$

and since that inequality is true for every $\epsilon > 0$, we get the result. \square

Proof of Theorem C (the left-hand side inequality for $p \in 2\mathbb{N}$, $p > 2$ only). We first prove the lower estimate for $c(\sigma_{n,\lambda}, H^p, H^\infty)$. Writing $p = 2(p/2)$, we apply Lemma 6.2.0 with $q = 2$ and this gives

$$c(\sigma_{n,\lambda}, H^p, H^\infty) \geq c(\sigma_{n,\lambda}, H^2, H^\infty)^{\frac{2}{p}} \geq 32^{-\frac{1}{p}} (n(1 - |\lambda|)^{-1})^{\frac{2}{p}}$$

for all integer $n \geq 1$. The last inequality is a consequence of Theorem C (left-hand side inequality) for the particular case $p = 2$ which has been proved in Subsection 6.1. \square

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